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Analytical Properties of Elliptic Currents

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Abstract

We observe analytical properties of generating currents of elliptic affine algebras, which are defined, due to M.Jimbo et al. by precise twisting of quantum affine algebra in "new realization". In particular, an explicit formula for the universal R-matrix is presented.

1 Introduction

Affine Lie algebras $\hat{\mathfrak{g}}$, unlike other members of Kac-Moody algebras, admit a loop realization, that $\hat{\mathfrak{g}}$ can be identified as $\mathfrak{g}\otimes[t,t^{-1}]\oplus\mathbb{C}c$, where c is the central element of $\hat{\mathfrak{g}}$. With the loop realization, we can define the so-called current operators in the form :

$$x(z)=\sum_{n\in\mathbb{Z}}x\otimes t^n z^{-n},$$

where $x\in\mathfrak{g}$ and z is a formal variable. The current operators have since play effective and fundamental role in representation theory of affine Lie algebras and in its applications. In particular, they provide a bridge to connect algebraic structure of affine Lie algebra to various analytic structures in mathematics and theoretical physics. For quantum affine algebras, the Drinfeld loop realization [D], as a Hopf algebra in terms of current operators, is the loop realization that is closest to the loop realization of affine Lie algebras above, which we call the quantized current algebras. The main distinctions of the quantized current algebras and the standard quantum affine algebras, which are connected through a infinite twist [KT], are in coalgebra structures and topological structures, In [DK], we study the Hopf algebra structure for the quantized current algebras and construct the universal R-matrix for admissible category of representations that are compatible with the topological structure of the quantized current algebras, where we use the Braid group action in terms of current operators and a new ordered exponent of current operators. The basic idea in [DK] comes from a transplant of an basic idea in physics, which can be "translated" into mathematics as: the basic structure of a current algebra is determined by the analytic properties of the products of the current operators acting on the admissible category of representations, in particular, the locations of the poles of the product of current operators.

In this paper, we use the similar idea to study elliptic algebras. By elliptic algebras in general, we mean a current algebra, whose structure are given in term of elliptic functions. There has been intensive study recently on elliptic algebras, which is basically due to the motivation to solve physical models [F], [JKOS], [EF], [DI], [K]. In [JKOS], it is shown that elliptic algebras can be derived from certain twisting of the

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quantized current algebras. In this paper, we use a different but simpler twisting, namely the twisting coming from the Heisenberg algebra inside the quantized current algebra, Accordingly [DK1], this leads us to new elliptic algebras. The main part of the paper, along the similar line of [DK], is devoted to studying the structure of the elliptic algebras, essentially in terms of the analytic structure of the elliptic algebra.

In [JKOS2], a fact of necessity to extend the elliptic algebras by certain zero mode operators was observed. There are certain algebraic reasons why any infinite dimensional algebra of affine type or their deformations should be extended by this sort of operators. This follows also from considering integrable lattice models of IRF type [LP], where elliptic algebras described in [JKOS2] appears naturally. This in turn leads to appearing of general dynamical Yang-Baxter equation. In the present paper we consider the elliptic algebras which are not extended by zero mode operators and which depend only one additional elliptic parameter associated with the central element of Cartan subalgebra and related to DYB equation where the only shifts by the central elements are presented.

The paper is arranged as follows. In Section 2, we present the basic results in [DK], where we explain the structure of quantized current algebra in terms of the analytic property of current operators. In Section 3, we describe twisting by the elements from Heisenberg subalgebra of the quantized current algebra and define elliptic current operators. they are generating functions of elliptic current algebra. In the last section, we study analytical and algebraic properties of elliptic current operators We define the corresponding braid group action on the elliptic algebra , study the analytic property of the products of the currents and finally construct the universal R-matrix in term of the current operators similar to that in [DK].

2 Braid group action in quantized current algebras

We present in this section the construction borrowed from [DK].

Let $\mathfrak{g}$ be a simple Lie algebra of the simply laced type[1]. Consider the algebra generated by the central element c, gradation operator d and by the modes of the currents $e_{\alpha_i}(z)$, $f_{\alpha_i}(z)$ and $\psi^{\pm}_{\alpha_i}(z)$,

$$e_{\alpha_i}(z)=\sum_{k\in\mathbb{Z}} e_{\alpha_i,k}z^{-k},\qquad f_{\alpha_i}(z)=\sum_{k\in\mathbb{Z}} f_{\alpha_i,k}z^{-k},$$

$$\psi^{\pm}_{\alpha_i}(z)=k^{\pm 1}_{\alpha_i}\exp\left(\pm(q-q^{-1})\sum_{n>0}a_{\alpha_i,\pm n}z^{\mp n}\right), \tag{2.1}$$

where α_i are positive simple roots of the algebra $\mathfrak{g}$.

The generating elements of the quantum current algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ satisfy the following defining relations (α,β are simple roots of $\mathfrak{g}$):

$$q^d x(z) q^{-d}=x(qz),\quad \text{for}\quad x=e,f,\psi^{\pm}\ , \tag{2.2}$$

$$(z-q^{(\alpha,\beta)}w)e_\alpha(z)e_\beta(w)=e_\beta(w)e_\alpha(z)(q^{(\alpha,\beta)}z-w)\ , \tag{2.3}$$

$$(z-q^{-(\alpha,\beta)}w)f_\alpha(z)f_\beta(w)=f_\beta(w)f_\alpha(z)(q^{-(\alpha,\beta)}z-w)\ , \tag{2.4}$$

$$\frac{(q^{c/2}z-q^{(\alpha,\beta)}w)}{(q^{(\alpha,\beta)+c/2}z-w)}\psi^+_\alpha(z)e_\beta(w)=e_\beta(w)\psi^+_\alpha(z)\ , \tag{2.5}$$

$$\psi^-_\alpha(z)e_\beta(w)=\frac{(q^{(\alpha,\beta)-c/2}z-w)}{(q^{-c/2}z-q^{(\alpha,\beta)}w)}e_\beta(w)\psi^-_\alpha(z)\ , \tag{2.6}$$

$$\frac{(q^{-c/2}z-q^{-(\alpha,\beta)}w)}{(q^{-(\alpha,\beta)-c/2}z-w)}\psi^+_\alpha(z)f_\beta(w)=f_\beta(w)\psi^+_\alpha(z)\ , \tag{2.7}$$

[1]We restrict ourselves to the simply laced case only to simplify the presentation below. The general case can be treated along the same lines

$$\psi^-_\alpha(z)f_\beta(w)=\frac{(q^{-(\alpha,\beta)+c/2}z-w)}{(q^{c/2}z-q^{-(\alpha,\beta)}w)}f_\beta(w)\psi^-_\alpha(z)\ , \tag{2.8}$$

$$\frac{(z-q^{(\alpha,\beta)-c}w)(z-q^{-(\alpha,\beta)+c}w)}{(q^{(\alpha,\beta)+c}z-w)(q^{-(\alpha,\beta)-c}z-w)}\psi^+_\alpha(z)\psi^-_\beta(w)=\psi^-_\beta(w)\psi^+_\alpha(z)\ , \tag{2.9}$$

$$\psi^\pm_\alpha(z)\psi^\pm_\beta(w)=\psi^\pm_\beta(w)\psi^\pm_\alpha(z)\ , \tag{2.10}$$

$$[e_\alpha(z),f_\beta(w)]=\frac{\delta_{\alpha,\beta}}{q-q^{-1}}\left(\delta(z/q^cw)\psi^+_\alpha(zq^{-c/2})-\delta(zq^c/w)\psi^-_\alpha(wq^{-c/2})\right) \tag{2.11}$$

and the Serre relations:

$$e_\alpha(w_1)e_\alpha(w_2)e_\beta(z)-[2]_qe_\alpha(w_1)e_\beta(z)e_\alpha(w_2)+e_\beta(z)e_\alpha(w_1)e_\alpha(w_2)+\ w_1\leftrightarrow w_2=0\ , \tag{2.12}$$

$$f_\alpha(w_1)f_\alpha(w_2)f_\beta(z)-[2]_qf_\alpha(w_1)f_\beta(z)f_\alpha(w_2)+f_\beta(z)f_\alpha(w_1)f_\alpha(w_2)+\ w_1\leftrightarrow w_2=0 \tag{2.13}$$

for any α,β , $(\alpha,\beta)=-1$. Here $\delta(z)=\sum_{k\in\mathbb{Z}}z^k$.

The formulas (2.5)–(2.9) can be rewritten also using the generators $a_{\alpha,n}$ introduced by (2.1):

$$[a_{\alpha,n},e_\beta(z)]=\frac{[(\alpha,\beta)n]}{n}q^{-c|n|/2}z^ne_\beta(z)\ , \tag{2.14}$$

$$[a_{\alpha,n},f_\beta(z)]=-\frac{[(\alpha,\beta)n]}{n}q^{c|n|/2}z^nf_\beta(z)\ , \tag{2.15}$$

$$[a_{\alpha,n},a_{\beta,m}]=\delta_{n+m,0}\frac{[(\alpha,\beta)n][cn]}{n}\ . \tag{2.16}$$

2.1 The automorphisms and composed current

To define the rest of the generators of the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ which coincides by definition with universal enveloping algebra of modes of the simple currents and describe some basis of Cartan-Weyl type for $\mathcal{U}_q(\hat{\mathfrak{g}})$ we need to introduce certain completion of it, see [DK1] or [M]. Here we use the completion described in [M].

We call V to be an admissible representation of the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ if for $\forall v\in V$ there exists an integer number N such that $xv=0$ for $x=e_{\alpha_i,n},f_{\alpha_i,n},a_{\alpha_i,n}$ with $n>N$. Let us say that the elements $e_{\alpha_i,n}$, $f_{\alpha_i,n}$, $a_{\alpha_i,n}$ have degree n . Let $U_{\pm s}$ be subalgebras in $\mathcal{U}_q(\hat{\mathfrak{g}})$ generated by the elements of degree bigger than s or smaller than $-s$ for $s\geq 0$. Then we can introduce the system of neighborhoods

$$U_{kl}=\sum_{\substack{r\geq k\\ s\geq l}}U_{-r}\,\mathcal{U}_q(\hat{\mathfrak{g}})\,U_s \tag{2.17}$$

Then the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ becomes a topological algebra with the fundamental system of neighborhoods and denote by $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ its completion with respect to this topology. The completed algebra $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ as well as original $\mathcal{U}_q(\hat{\mathfrak{g}})$ is acting in the admissible representations.

The convenient basis of the generators of the algebra $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ which corresponds to simple and composed roots of the algebra $\mathfrak{g}$ can be defined using following automorphisms of the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Let η be a continuous anti-automorphism of $\mathcal{U}_q(\hat{\mathfrak{g}})$ [M] defined by

$$\eta\left(e_{\alpha_i}(z)\right)=e_{\alpha_i}(z^{-1}),\quad \eta\left(f_{\alpha_i}(z)\right)=f_{\alpha_i}(z^{-1}),\quad \eta\left(\psi^\pm_{\alpha_i}(z)\right)=\psi^\mp_{\alpha_i}(q^cz^{-1}),\quad \eta\left(c\right)=c,\quad \eta\left(d\right)=d. \tag{2.18}$$

For any simple root α let T_α and T^{-1}_α be the following maps [DK]:

$$T_\alpha e_\alpha(z)=f_\alpha(q^{-c}z)\psi^+_\alpha(q^{-c/2}z)^{-1},\quad T_\alpha f_\alpha(z)=\psi^-_\alpha(q^{-c/2}z)^{-1}e_\alpha(q^{-c}z),\quad T_\alpha\psi^\pm_\alpha(z)=\psi^\pm_\alpha(z)^{-1}\ , \tag{2.19}$$

and for the simple β , $(\alpha,\beta)=-1$

$$T_\alpha\psi^\pm_\beta(z)=\psi^\pm_\alpha(qz)\psi^\pm_\beta(q^2z), \tag{2.20}$$

$$T_\alpha e_\beta(z) = \int \frac{dw}{w}\left(e_\alpha(w)e_\beta(q^2z) - q^{-(\alpha,\beta)}\frac{1-q^{(\alpha,\beta)-2}w/z}{1-q^{-(\alpha,\beta)-2}w/z}e_\beta(q^2z)e_\alpha(w)\right)\ , \tag{2.21}$$

$$T_\alpha f_\beta(z) = \int \frac{dw}{w}\left(-f_\beta(q^2z)f_\alpha(w) + q^{(\alpha,\beta)}\frac{1-q^{-(\alpha,\beta)+2}z/w}{1-q^{(\alpha,\beta)+2}z/w}f_\alpha(w)f_\beta(q^2z)\right)\ , \tag{2.22}$$

where the operation $\int\frac{dw}{w}$ means the coefficient at w^0 of the Laurent series in r.h.s. (2.21) and (2.22). The automorphisms T_α^{-1} are defined by the composition

$$T_\alpha^{-1} = \eta\circ T_\alpha\circ\eta \tag{2.23}$$

and the rest of Drinfeld current operators for simple roots is stable.

It is convenient to denote the action of the automorphisms $T_\alpha^{\pm1}$ on the currents $e_\beta(z)$ and $f_\beta(z)$ for simple roots α and β such that $(\alpha,\beta)=-1$ by

$$T_\alpha e_\beta(z)\equiv e_{\alpha+\beta}(z),\quad T_\alpha f_\beta(z)\equiv f_{\alpha+\beta}(z),\quad T_\alpha^{-1}e_\beta(z)\equiv \check{e}_{\alpha+\beta}(z),\quad T_\alpha^{-1}f_\beta(z)\equiv \check{f}_{\alpha+\beta}(z) \tag{2.24}$$

and understand these currents as composed currents which correspond to the composed roots $\alpha+\beta$. To observe that automorphism T_α^{-1} is an inverse to T_α we should use the commutation relations (2.11).

Note that the action of the automorphisms T_α and T_α^{-1} produces the normal ordered currents with respect to the rule that all the modes $a_{\alpha,n}$ with $n>0$ are on the right, the modes $a_{\alpha,n}$ with $n<0$ on the left and in the infinite sums appearing in (2.21) and (2.22) the modes in the products $e_{\alpha,n}e_{\beta,m}$ are ordered according to $n<m$ but finite number of terms.

Let

$$w_0 = s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_N}} \tag{2.25}$$

be a reduced decomposition of the longest element w_0 of the Weyl group of Lie algebra $\mathfrak{g}$. Construct the set of positive roots $\gamma_1,\ldots,\gamma_N$ by the following prescription

$$\gamma_N = \alpha_{i_N}$$

$$\gamma_{N-1} = s_{\alpha_{i_N}}\alpha_{i_{N-1}}$$

$$\gamma_{N-2} = s_{\alpha_{i_N}}s_{\alpha_{i_{N-1}}}\alpha_{i_{N-2}}$$

and so on. For example, in case of the algebra $\mathfrak{g}=sl_3$ we have two simple roots α and β, the longest element $w_0=s_\alpha s_\beta s_\alpha$ and abore prescription

$$\gamma_3=\alpha,\quad \gamma_2=s_\alpha\beta=\alpha+\beta,\quad \gamma_3=s_\alpha s_\beta\alpha=s_\alpha(\alpha+\beta)=\beta$$

produces all positive roots of the algebra $\mathfrak{g}=sl_3$.

The (linear independent) PBW basis in the algebra $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ consists of the monomials of the form

$$\prod_{i=1}^N f_{\gamma_i,m_1}f_{\gamma_i,m_2}\ldots f_{\gamma_i,m_l}\prod_{i=1}^N a_{\gamma_i,p_1}a_{\gamma_i,p_2}\ldots a_{\gamma_i,p_s}\prod_{i=1}^N e_{\gamma_i,k_1}e_{\gamma_i,k_2}\ldots e_{\gamma_i,k_n}\ , \tag{2.26}$$

where each product is ordered with respect to linear order in the set of the roots γ_i, $i=1,\ldots,N$, attached to the reduced decomposition (2.25) and with respect to the indices of the generators, that is,

$$\gamma_1<\ldots\gamma_N,\quad m_1\le m_2\le\ldots\le m_l,\quad p_1\le p_2\le\ldots\le p_s,\quad k_1\le k_2\le\ldots\le k_n. \tag{2.27}$$

Any non-ordered monomial from the algebra $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ can be expressed using formulas like (2.41) as a series over the monomials of the type (2.26), convergent in topology (2.17).

It is clear that composed root generators act on the admissible representations in an admissible way. For example, for two simple roots α and β consider element v in the admissible representation V such that generators $e_{\alpha,m}$ and $e_{\beta,m}$ annihilate v for $m>M$. Then it is clear that composed root generators $e_{\alpha+\beta,m}$ will annihilate this element for $m>2M$.

On the other hand, it is possible to express the monomials from $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$ via convergent series over degree ordered monomials

The composed root generators are introduced exactly for the reason to have root ordering in the PBW basis. It is clear, that it is impossible to construct PBW basis using only generators which corresponds to the simple roots. Among the elements of such a 'basis' we will have

$$b_{\alpha_1,n_1}b_{\alpha_2,n_2}b_{\alpha_1,n_3}$$

with $n_1\leq n_2\leq n_3$, and $b=e,f,a$ without ordering of roots.

2.2 Universal $\mathcal{R}$-matrices

The algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a Hopf algebra with the comultiplication, antipode and counit maps:

$$\Delta^{(1)}e_\alpha(z)=e_\alpha(z)\otimes 1+\psi^-_\alpha(zq^{c_1/2})\otimes e_\alpha(zq^{c_1}) \ , \tag{2.28a}$$

$$\Delta^{(1)}f_\alpha(z)=1\otimes f_\alpha(z)+f_\alpha(zq^{c_2})\otimes \psi^+_\alpha(zq^{c_2/2}) \ , \tag{2.28b}$$

$$\Delta^{(1)}\psi^\pm_\alpha(z)=\psi^\pm_\alpha(zq^{\pm c_2/2})\otimes\psi^\pm_\alpha(zq^{\mp c_1/2}) \ , \tag{2.28c}$$

$$\begin{gathered}S^{(1)}\left(\psi^\pm_\alpha(z)\right)=\psi^\pm_\alpha(z)^{-1},\quad S^{(1)}\left(e_\alpha(z)\right)=-\psi^-_\alpha\left(zq^{-c/2}\right)^{-1}e_\alpha(zq^{-c}),\\ S^{(1)}\left(f_\alpha(z)\right)=-f_\alpha(zq^{-c})\psi^+_\alpha\left(zq^{-c/2}\right)^{-1},\quad S^{(1)}(c)=-c,\quad a^{(1)}(d)=-d \ ,\end{gathered} \tag{2.28d}$$

$$\epsilon\left(\psi^\pm_\alpha(z)\right)=1,\quad \epsilon(x)=0,\quad \text{for}\quad x=c,\ d,\ e_\alpha(z),\ f_\alpha(z) \ . \tag{2.28e}$$

One can observe that the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ can be obtained using quantum double construction [D, DK] starting from the subalgebra generated by the modes of the currents $f_{\alpha_i}(z)$ and $\psi^+_{\alpha_i}(z)$, $i=1,\ldots,r$, where r is a rank of $\mathfrak{g}$.

Let $\gamma_1,\ldots,\gamma_N$ be the system of positive roots of the algebra $\mathfrak{g}$ described at the end of the previous subsection. Let $e_{\gamma_1}(u)$, $e_{\gamma_2}(u)$, $\ldots$, $e_{\gamma_N}(u)$ and $f_{\gamma_1}(u)$, $f_{\gamma_2}(u)$, $\ldots$, $f_{\gamma_N}(u)$ be the composed root currents given by the prescription:

$$\begin{gathered}e_{\gamma_N}(u)=e_{\alpha_{i_N}}(u),\qquad f_{\gamma_N}(u)=f_{\alpha_{i_n}}(u) \ ,\\ e_{\gamma_{N-1}}(u)=T_{\alpha_{i_N}}e_{\alpha_{i_{N-1}}}(u),\quad f_{\gamma_{N-1}}(u)=T_{\alpha_{i_N}}f_{\alpha_{i_{N-1}}}(u) \ ,\\ e_{\gamma_{N-2}}(u)=T_{\alpha_{i_N}}T_{\alpha_{i_{N-1}}}e_{\alpha_{i_{N-2}}}(u),\quad f_{\gamma_{N-2}}(u)=T_{\alpha_{i_N}}T_{\alpha_{i_{N-1}}}f_{\alpha_{i_{N-2}}}(u)\end{gathered}$$

and so on. For any simple or composed root γ_i we define the tensor:

$$\overline{\mathcal{R}}_{\gamma_i}=\mathcal{P}\ \overrightarrow{\exp}_{q^2}\left((q^{-1}-q)\int f_{\gamma_i}(z)\otimes e_{\gamma_i}(z)\frac{dz}{z}\right) \ , \tag{2.29}$$

where the formal operation $\int\frac{dz}{z}$ means the same as in (2.21) and (2.22) and ordered exponent is defined by series

$$\begin{gathered}\mathcal{P}\ \overrightarrow{\exp}_{q^2}\left((q^{-1}-q)\int f_{\gamma_i}(z)\otimes e_{\gamma_i}(z)\frac{dz}{z}\right)=1+\\ +\sum_{n>0}\frac{(q^{-1}-q)^n}{n!}\int\frac{dz_n}{z_n}\int\frac{dz_{n-1}}{z_{n-1}}\cdots\int\frac{dz_1}{z_1}f_{\gamma_i}(z_1)f_{\gamma_i}(z_2)\ldots f_{\gamma_i}(z_n)\otimes\\ \otimes\prod_{j<k}\frac{q^2z_j-z_k}{z_j-q^2z_k}e_{\gamma_i}(z_n)e_{\gamma_i}(z_{n-1})\ldots e_{\gamma_i}(z_1) \ .\end{gathered} \tag{2.30}$$

To get n th term of the series (2.30) in terms of generators of current algebra we should expand the rational functions in the domain of the analyticity of the product $e_{\gamma_i}(z_n)\ldots e_{\gamma_i}(z_1)$, namely with respect to the powers of $q^{-2}z_j/z_k$, $j<k$ and pick up the coefficient in front of $z_1^0z_2^0\cdots z_n^0$.

Let

$$\mathcal{K} = q^{-t} q^{\frac{-c\otimes d - d\otimes c}{2}} \prod_{n>0} \exp\left(-n(q-q^{-1}) \sum_{i,j=1}^{r} d_{i,j}^{(n)} a_{\alpha_i,n} \otimes a_{\alpha_j,-n} \right) q^{\frac{-c\otimes d - d\otimes c}{2}} \tag{2.31}$$

be a tensor constructed from the Cartan part of the algebra, where $t = \sum h_i \otimes h^i$ is an invariant in tensor square of Cartan subalgebra $\mathbf{h}\otimes\mathbf{h}$ and $d_{i,j}^{(n)}$ is an inverse matrix to

$$b_{i,j}^{(n)} = [n(\alpha_i,\alpha_j)]_q \ .$$

By the quantum double construction one can prove that the tensor

$$\mathcal{R} = \mathcal{K}\cdot\overline{\mathcal{R}} = \mathcal{K}\cdot \overrightarrow{\prod}_{1\le i\le N} \overline{\mathcal{R}}_{\gamma_i} \tag{2.32}$$

is the universal R-matrix for the algebra quantized current algebra with comultiplication $\Delta^{(1)}$.

Note that one can define the second Drinfeld coalgebraic structures for the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ which we will distinguish from the first one by the superscript $\Delta^{(2)}$:

$$\Delta^{(2)} e_\alpha(z) = e_\alpha(z)\otimes 1 + \psi_\alpha^+(zq^{-c_1/2})\otimes e_\alpha(zq^{-c_1}) \ , \tag{2.33a}$$

$$\Delta^{(2)} f_\alpha(z) = 1\otimes f_\alpha(z) + f_\alpha(zq^{-c_2})\otimes \psi_\alpha^-(zq^{-c_2/2}) \ , \tag{2.33b}$$

$$\Delta^{(2)} \psi_\alpha^\pm(z) = \psi_\alpha^\pm(zq^{\pm c_2/2})\otimes \psi_\alpha^\pm(zq^{\mp c_1/2}) \ , \tag{2.33c}$$

$$\begin{gathered} S^{(2)}\left(\psi_\alpha^\pm(z)\right) = \psi_\alpha^\pm(z)^{-1}, \quad S^{(2)}\left(e_\alpha(z)\right) = -\psi_\alpha^-\left(zq^{c/2}\right)^{-1} e_\alpha(zq^c), \\ S^{(2)}\left(f_\alpha(z)\right) = -f_\alpha(zq^c)\psi_\alpha^+\left(zq^{c/2}\right)^{-1}, \quad S^{(2)}(c) = -c, \quad a^{(1)}(d) = -d \end{gathered} \tag{2.33d}$$

and counit map is the same as in (2.28e). In this case the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ can be obtained using quantum double construction starting from Hopf subalgebra generated by $e_\alpha(z)$ and $\psi_\alpha^+(z)$.

The comultiplication $\Delta^{(2)}$ is related to $\Delta^{(1)}$ by the twist over tensor $\mathcal{K}$

$$\Delta^{(2)} = \mathcal{K}^{-1}\Delta^{(2)}_{\rm op}\mathcal{K} \ . \tag{2.34}$$

The corresponding universal $\mathcal{R}$-matrix is

$$\overleftarrow{\prod}_{1\le i\le N} \left(\overline{\mathcal{R}}_{\gamma_i}\right)_{21} \mathcal{K} \ .$$

2.3 The analytical properties of the products of the currents

We are in position now to translate some of the construction of the previous section into analytical language.

The action of the current generators into highest weight modules is well defined if the appropriate ordering in the monomial is fixed. For example, the generators given by the infinite sum of the monomials $e_{\gamma_1,n}e_{\gamma_2,m}$ have well defined action on highest weight modules if $n\le m$ for all but finite values of the summation indices n and m. This ordering results to the fact that the product of the current $e_\alpha(z)e_\beta(w)$ is an analytical function of the variables z and w in the domain $|z|\gg|w|$, in particular, as follows from commutation relations for $|z|>|q^{(\alpha,\beta)}w|$.

The commutation relation (2.3) should be understood then as the rule for the analytical continuation of the product $e_\alpha(z)e_\beta(w)$ from the region of analyticity $|z|>|q^{(\alpha,\beta)}w|$ into the region $|z|<|q^{(\alpha,\beta)}w|$, where this analytical continuation is equal to $\frac{q^{(\alpha,\beta)}z-w}{z-q^{(\alpha,\beta)}w}e_\beta(z)e_\alpha(w)$. So the product $e_\alpha(z)e_\beta(w)$ can be continued to $(\mathbb{C}^*)^2$ as a meromorphic function with simple pole at the point $z=q^{(\alpha,\beta)}w$. The same for the product $f_\beta(w)f_\alpha(z)$ which can be analytically continued from the domain $|w|>|q^{(\alpha,\beta)}z|$ to $(\mathbb{C}^*)^2$ and has the pole at the same point.

Fix now the simple roots α and β such that $(\alpha,\beta)=-1$. Let us calculate the residue of the functions $e_\alpha(w)e_\beta(q^2z)$ and $f_\beta(q^2z)f_\alpha(w)$ at the point $w=qz$. We can easily verify that these residues coincide precisely with the action of the automorphisms T_α onto currents $e_\beta(z)$ and $f_\beta(z)$. We have

$$\begin{aligned}
e_{\alpha+\beta}(z) =: e_\alpha(qz)e_\beta(q^2z) := \operatorname*{res}_{w=qz} e_\alpha(w)e_\beta(q^2z) = \frac{1}{2\pi i}\oint_{w \text{ around } qz} e_\alpha(w)e_\beta(q^2z)\frac{dw}{w} = \\
= \frac{1}{2\pi i}\left(\oint_{C_\infty} e_\alpha(w)e_\beta(q^2z)\frac{dw}{w} - \oint_{C_0}\frac{q^{-1}w-q^2z}{w-qz}e_\beta(q^2z)e_\alpha(w)\frac{dw}{w}\right) ,
\end{aligned} \tag{2.35}$$

$$\begin{aligned}
f_{\alpha+\beta} =: f_\beta(q^2z)f_\alpha(qz) := \operatorname*{res}_{w=qz} f_\beta(q^2z)f_\alpha(w) = \frac{1}{2\pi i}\oint_{w \text{ around } qz} f_\beta(q^2z)f_\alpha(w)\frac{dw}{w} = \\
= \frac{1}{2\pi i}\left(-\oint_{C_0} f_\beta(q^2z)f_\alpha(w)\frac{dw}{w} + \oint_{C_\infty}\frac{q^{-1}w-q^2z}{w-qz}f_\alpha(w)f_\beta(q^2z)\frac{dw}{w}\right) .
\end{aligned} \tag{2.36}$$

In order to get two terms in the r.h.s. of (2.35) and (2.36) we split the integral over small circle around the point $w=qz$ into the difference of two contour integrals such that the contour C_0 is around zero and the point $w=qz$ is outside the contour while C_∞ is a contour close to infinity, which includes zero and the point $w=qz$. The rational functions in the r.h.s. of (2.35) and (2.36) appeared due to the analytical continuation of the products $e_\alpha(w)e_\beta(q^2z)$ and $f_\beta(q^2z)f_\alpha(w)$ into the corresponding domains.

One can calculate also the residues of the 'functions' $e_\beta(q^{-2}z)e_\alpha(w)$ and $f_\alpha(w)f_\beta(q^{-2}z)$ at the points $w=q^{-1}z$ where these products have the simple poles. Therefore, we will have the action of the automorphisms T_α^{-1} onto the currents $e_\beta(z)$ and $f_\beta(z)$.

$$\begin{aligned}
\check{e}_{\alpha+\beta}(z) =: e_\beta(q^{-2}z)e_\beta(q^{-1}z) := \operatorname*{res}_{w=q^{-1}z} e_\beta(q^{-2}z)e_\alpha(w) = \frac{1}{2\pi i}\oint_{w \text{ around } q^{-1}z} e_\beta(q^{-2}z)e_\beta(w)\frac{dw}{w} = \\
\frac{1}{2\pi i}\left(-\oint_{C'_0} e_\beta(q^{-2}z)e_\alpha(w)\frac{dw}{w} + \oint_{C'_\infty}\frac{qw-q^{-2}z}{w-q^{-1}z}e_\alpha(w)e_\beta(q^{-2}z)\frac{dw}{w}\right) ,
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
\check{f}_{\alpha+\beta}(z) =: f_\alpha(q^{-1}z)f_\beta(q^{-2}z) := \operatorname*{res}_{w=q^{-1}z} f_\alpha(w)f_\beta(q^{-2}z) = \frac{1}{2\pi i}\oint_{w \text{ around } q^{-1}z} f_\alpha(w)f_\beta(q^{-2}z)\frac{dw}{w} = \\
\frac{1}{2\pi i}\left(\oint_{C'_\infty} f_\alpha(w)f_\beta(q^{-2}z)\frac{dw}{w} - \oint_{C'_0}\frac{qw-q^{-2}z}{w-q^{-1}z}f_\beta(q^{-2}z)f_\alpha(w)\frac{dw}{w}\right) ,
\end{aligned} \tag{2.38}$$

where ${C'}_0$ is a contour around zero such that the point $w=q^{-1}z$ is outside the contour while ${C'}_\infty$ is a contour close to infinity, which includes zero and the point $w=q^{-1}z$.

Now we would like to demonstrate that the definition of the composed currents (2.21), (2.22) or their analytical counterparts (2.35), (2.36) and (2.37), (2.38) produces for the modes of the composed currents the normal ordered expressions in the sense explained above. For example, let us consider the n th mode of the composed current $e_{\alpha+\beta}(z)$ given by (2.21). Expanding the rational function in geometrical progression with respect to the powers of $q^{-1}w/z$ we obtain

$$e_{\alpha+\beta,n} = q^{-2n}\left(e_{\alpha,0}e_{\beta,n} - qe_{\beta,n}e_{\alpha,0} - (q-q^{-1})\sum_{m>0}q^m e_{\beta,n-m}e_{\alpha,m}\right) . \tag{2.39}$$

The r.h.s. in (2.39) contains only finite number of non-ordered terms.

Using the commutation relations for the modes of simple currents

$$\begin{aligned}
e_{\alpha,k+1}e_{\beta,m} - q^{-1}e_{\alpha,k}e_{\beta,m+1} = q^{-1}e_{\beta,m}e_{\alpha,k+1} - e_{\beta,m+1}e_{\alpha,k} \\
f_{\alpha,k+1}f_{\beta,m} - qf_{\alpha,k}f_{\beta,m+1} = qf_{\beta,m}f_{\alpha,k+1} - f_{\beta,m+1}f_{\alpha,k}
\end{aligned} \tag{2.40}$$

we can rewrite the defining relations (2.21) and (2.22) in many equivalent ways

$$\begin{aligned} e_{\alpha+\beta,n} &= q^{-2n+p}\left(e_{\alpha,p}e_{\beta,n-p} - qe_{\beta,n-p}e_{\alpha,p} - (q-q^{-1})\sum_{m>0} q^m e_{\beta,n-m-p}e_{\alpha,m+p}\right) \\ f_{\alpha+\beta,n} &= q^{2n-p}\left(-f_{\beta,n-p}f_{\alpha,p} + q^{-1}f_{\alpha,p}e_{\beta,n-p} - (q-q^{-1})\sum_{m>0} q^{-m} f_{\alpha,p-m}e_{\alpha,n-m+p}\right) \end{aligned} \tag{2.41}$$

for arbitrary integer p. This results in the analytical form of (2.21) and (2.22):

$$e_{\alpha+\beta}(q^{-2}z)\delta(qw/z) = e_\alpha(w)e_\beta(z) - \frac{q-w/z}{1-qw/z}e_\beta(z)e_\alpha(w), \quad |qw|<|z| \tag{2.42}$$

$$f_{\alpha+\beta}(q^{-2}z)\delta(qw/z) = -f_\beta(z)f_\alpha(w) + \frac{q^{-1}-z/w}{1-q^{-1}z/w}f_\alpha(w)f_\beta(z), \quad |z|<|qw| \tag{2.43}$$

and inequalities signify the direction of the expansion of the rational functions in the r.h.s. One can think about the relations (2.42) and (2.43) as follows. One starts from the analytical in the domain $|w|>|q^{-1}z|$ function $e_\alpha(w)e_\beta(z)$ and try the analytical continuation with respect the variable w to the domain $|w|<|q^{-1}z|$. It is clear that at this process one will pick up the δ-function with coefficient being the residue of the function $e_\alpha(w)e_\beta(z)$ at the point $w=q^{-1}z$. The formulas (2.42) and (2.43) are analogues of the simple formula

$$\frac{1}{1-z} = \frac{z^{-1}}{z^{-1}-1} + \delta(z)$$

which describe the analytical continuation of the analytical in the domain $|z|<1$ function $(1-z)^{-1}$ into the domain $|z|>1$.

On the other hand it was proved in [DK] that so defined composed currents satisfy the commutation relations

$$e_\alpha(z)e_{\alpha+\beta}(w) = \frac{qz-w}{z-qw}e_{\alpha+\beta}(w)e_\alpha(z)\ , \quad |z|<|qw|\ , \tag{2.44}$$

$$e_{\alpha+\beta}(z)e_\beta(w) = \frac{q^2z-q^{-2}w}{qz-q^{-1}w}e_\beta(w)e_{\alpha+\beta}(z)\ , \quad |z|<|q^{-2}w|\ , \tag{2.45}$$

$$f_\beta(z)f_{\alpha+\beta}(w) = \frac{q^{-2}z-q^2w}{q^{-1}z-qw}f_{\alpha+\beta}(w)f_\alpha(z)\ , \quad |z|<|q^2w|\ , \tag{2.46}$$

$$f_{\alpha+\beta}(z)f_\alpha(w) = \frac{q^{-1}z-w}{z-q^{-1}w}f_\alpha(w)f_{\alpha+\beta}(z)\ , \quad |z|<|q^{-1}w|\ , \tag{2.47}$$

which are equivalent to the Serre relations (2.12) and (2.13).

Using analogous analytical considerations we can rewrite the path ordered exponent entering the universal $\mathcal{R}$-matrix as ordered contour integrals

$$\begin{gathered} \mathcal{P}\,\overrightarrow{\exp}_{q^2}\left(\frac{(q^{-1}-q)}{2\pi i}\oint f_{\gamma_i}(z)\otimes e_{\gamma_i}(z)\frac{dz}{z}\right) = 1+ \\ +\sum_{n>0}\frac{(q^{-1}-q)^n}{n!(2\pi i)^n}\oint_{C_n}\frac{dz_n}{z_n}\oint_{C_{n-1}}\frac{dz_{n-1}}{z_{n-1}}\cdots\oint_{C_1}\frac{dz_1}{z_1}f_{\gamma_i}(z_1)f_{\gamma_i}(z_2)\ldots f_{\gamma_i}(z_n)\otimes \\ \otimes e_{\gamma_i}(z_1)e_{\gamma_i}(z_2)\ldots e_{\gamma_i}(z_n), \end{gathered}$$

where the contours C_k enclose an origin and are in a region where the points $z_i=q^2z_j$ are outside the contours C_i and the points $z_i=q^{-2}z_j$ are inside the contours C_i. The form of the contours is dictated by the analytical properties of the function $f_\gamma(z)\otimes e_\gamma(z)$ (see details in [DK]).

3 Twisting of the quantized current algebra

3.1 General principles of twisting

Here we present main facts concerning quasi-Hopf algebras [D2] relevant for the following discussion.

Definition 3.1 *A quasi-triangular quasi-Hopf algebra is a set* $(\mathcal{A},\Delta,\varepsilon,S,\Phi,\mathcal{R},\alpha,\beta)$ *such that* $\mathcal{A}$ *is a unital associative algebra over field* $\mathbb{C}$ *,* Δ *and* ε *are comultiplication and counit homomorphisms of the form* $\Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ *and* $\varepsilon:\mathcal{A}\to\mathbb{C}$ *,* S *is antipod, that is an antiautomorphism* $S:\mathcal{A}\to\mathcal{A}$ *,* $\Phi\in\mathcal{A}^{\otimes 3}$ *and* $\mathcal{R}\in\mathcal{A}^{\otimes 2}$ *are an invertible elements called coassociator and universal* R *-matrix respectively,* $\alpha,\beta\in\mathcal{A}$ *are some element such that following axioms are satisfied:*

$$(\mathrm{id}\otimes\Delta)\Delta(x)=\Phi(\Delta\otimes\mathrm{id})\Delta(x)\Phi^{-1},\quad\forall\ x\in\mathcal{A}\ , \tag{3.1a}$$

$$(\mathrm{id}\otimes\mathrm{id}\otimes\Delta)\Phi\cdot(\Delta\otimes\mathrm{id}\otimes\mathrm{id})\Phi=(1\otimes\Phi)\cdot(\mathrm{id}\otimes\Delta\otimes\mathrm{id})\Phi\cdot(\Phi\otimes 1)\ , \tag{3.1b}$$

$$(\varepsilon\otimes\mathrm{id})\circ\Delta=(\mathrm{id}\otimes\varepsilon)\circ\Delta=\mathrm{id}\ , \tag{3.1c}$$

$$(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\Phi=1\ , \tag{3.1d}$$

$$S(x^{(1)})\alpha x^{(2)}=\varepsilon(x)\alpha,\quad x^{(1)}\beta S(x^{(2)})=\varepsilon(x)\beta,\quad\forall\ x\in\mathcal{A}\ , \tag{3.1e}$$

$$\Phi^{(1)}\ \beta\ S(\Phi^{(2)})\ \alpha\ \Phi^{(3)}=1,\quad \textit{where}\quad \Phi=\Phi^{(1)}\otimes\Phi^{(2)}\otimes\Phi^{(3)}\ , \tag{3.1f}$$

$$\Delta^{\mathrm{op}}(x)=\mathcal{R}\ \Delta(x)\ \mathcal{R}^{-1},\quad\forall\ x\in\mathcal{A}\ , \tag{3.1g}$$

$$(\Delta\otimes\mathrm{id})\mathcal{R}=\Phi_{312}\mathcal{R}_{13}\Phi^{-1}_{132}\mathcal{R}_{23}\Phi_{123}\ , \tag{3.1h}$$

$$(\mathrm{id}\otimes\Delta)\mathcal{R}=\Phi^{-1}_{231}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}_{123}\ , \tag{3.1i}$$

where Δ^{op} *is opposite comultiplication and we used standard notations to signify the positions of the elements in the tensor products.*

Definition 3.2 *Let* $\mathcal{F}\in\mathcal{A}^{\otimes 2}$ *is an invertible element such that* $(\mathrm{id}\otimes\varepsilon)\mathcal{F}=(\varepsilon\otimes\mathrm{id})\mathcal{F}=1$ *. Then to a given quasi-triangular quasi-Hopf algebra* $(\mathcal{A},\Delta,\varepsilon,S,\Phi,\mathcal{R},\alpha,\beta)$ *we can associate the twisted algebra* $(\mathcal{A},\Delta^{\mathcal{F}},\varepsilon,S,$ $\Phi^{\mathcal{F}},\mathcal{R}^{\mathcal{F}},\alpha^{\mathcal{F}},\beta^{\mathcal{F}})$ *where*

$$\Delta^{\mathcal{F}}(x)=\mathcal{F}\ \Delta(x)\ \mathcal{F}^{-1},\quad\forall\ x\in\mathcal{A}\ , \tag{3.2a}$$

$$\Phi^{\mathcal{F}}=(\mathcal{F}_{23}(\mathrm{id}\otimes\Delta)\mathcal{F})\,\Phi\,(\mathcal{F}_{12}(\Delta\otimes\mathrm{id})\mathcal{F})^{-1}\ , \tag{3.2b}$$

$$\mathcal{R}^{\mathcal{F}}=\mathcal{F}_{21}\ \mathcal{R}\ \mathcal{F}^{-1}_{12}\ , \tag{3.2c}$$

$$\alpha^{\mathcal{F}}=S(\mathcal{F}^{-\prime})\ \alpha\ \mathcal{F}^{-\prime\prime},\quad \beta^{\mathcal{F}}=\mathcal{F}^{\prime}\ \beta\ S(\mathcal{F}^{\prime\prime})\ , \tag{3.2d}$$

where $\mathcal{F}=\mathcal{F}^{\prime}\otimes\mathcal{F}^{\prime\prime}$ *and* $\mathcal{F}^{-1}=\mathcal{F}^{-\prime}\otimes\mathcal{F}^{-\prime\prime}$ *.*

It is clear that the Hopf algebras, those which have trivial coassociator $\Phi=1\otimes 1\otimes 1$, are stable under the twisting if element $\mathcal{F}$ satisfying the cocycle condition:

$$\mathcal{F}_{12}(\Delta\otimes 1)\mathcal{F}=\mathcal{F}_{23}(1\otimes\Delta)\mathcal{F}\ . \tag{3.3}$$

If element $\mathcal{F}(\lambda)$ depend on the element λ of abelian subalgebra $\mathfrak{h}$ and satisfies instead of (3.3) the shifted cocycle condition

$$\mathcal{F}_{12}(\lambda)(\Delta\otimes 1)\mathcal{F}(\lambda)=\mathcal{F}_{23}(\lambda+h_1)(1\otimes\Delta)\mathcal{F}(\lambda) \tag{3.4}$$

then the twisting of the Hopf algebra $\mathcal{A}$ by this element brought it to the quasi-Hopf algebra with coassociator $\Phi(\lambda)$ and R-matrix $\mathcal{R}(\lambda)$ which satisfy the properties

$$\Phi(\lambda)=\mathcal{F}_{23}(\lambda)\mathcal{F}^{-1}_{23}(\lambda+h_1)\ , \tag{3.5a}$$

$$(\Delta(\lambda)\otimes\mathrm{id})\mathcal{R}(\lambda)=\Phi_{312}(\lambda)\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda+h_1)\ , \tag{3.5b}$$

$$(\mathrm{id}\otimes\Delta(\lambda))\mathcal{R}(\lambda)=\mathcal{R}_{13}(\lambda+h_2)\mathcal{R}_{12}(\lambda)\Phi_{123}^{-1}(\lambda)\ . \tag{3.5c}$$

As a corollary a dynamical Yang-Baxter equation for the R-matrix $\mathcal{R}(\lambda)$ holds

$$\mathcal{R}_{12}(\lambda+h_3)\mathcal{R}_{13}(\lambda)\mathcal{R}_{23}(\lambda+h_1)=\mathcal{R}_{23}(\lambda)\mathcal{R}_{13}(\lambda+h_2)\mathcal{R}_{12}(\lambda)\ . \tag{3.6}$$

In the formulas above the notation $\lambda+h_1$ etc. means that if element $\lambda\in\mathfrak{h}$ has decomposition $\lambda=\sum_i\lambda^ih_i$ where h_i is a linear independent basis in $\mathfrak{h}$, then $\lambda+h_1$ means $\sum_i(\lambda^i+h_1^i)h_i$. In the considerations below we will consider the case when subalgebra $\mathfrak{h}$ is one dimensional and correspond to the central element, so we have only one parameter which we call τ and which admits the shifts by the central element c.

3.2 Twisting of the quantized current algebra

Let us return to the current algebra from the first section which is a Hopf algebra with coalgebraic structure given by the formulas (2.28). In terms of the Drinfeld generators $a_{\alpha,n}$ this coalgebraic structure is given by

$$\Delta\, a_{\alpha,n}=a_{\alpha,n}\otimes q^{-\frac{c|n|}{2}}+q^{\frac{c|n|}{2}}\otimes a_{\alpha,n},\quad \forall n\neq 0\ , \tag{3.7}$$

where we have identified the maps $\Delta^{(1)}=\Delta^{(2)}=\Delta$ as follows from comparing (2.28c) and (2.33c). ¿From the general principles of twisting we can see that due to the special factorization properties of the universal $\mathcal{R}$-matrix (2.32) the Cartan part of this matrix is twisted while the nonabelian part $\overline{\mathcal{R}}$ is dressed by the twisting element. This observation allows to guess that the solution of the shifted cocycle condition can be found using only the Cartan Drinfeld's generators $a_{\alpha,n}$, $n\in\mathbb{Z}$.

Fix a complex number τ such that $\tau\neq\pm k^{-1}$ and $\tau+c\neq\pm k^{-1}$ for integer $k>0$, where by c we mean the value of the central element c on the representation of $\hat{\mathcal{U}}_q(\hat{\mathfrak{g}})$. For such values of the parameter τ we have

Lemma 3.1 *The shifted cocycle condition (3.4) has two unique solution fixed by the comultiplication (3.7), the commutation relations (2.16) and by the requirement that the dressed elliptic currents satisfy the commutation relation (4.4)–(4.10):*

$$\mathcal{F}^{(1)}(\tau)=\exp\left(\sum_{\substack{n>0\\ \alpha,\beta}}\frac{nd^{(n)}_{\alpha,\beta}q^{\tau n}}{[(\tau+c_1)n]}\ \left(q^{\frac{cn}{2}}a_{\alpha,-n}\otimes q^{\frac{cn}{2}}a_{\beta,n}\right)\right)\ , \tag{3.8}$$

$$\mathcal{F}^{(2)}(\tau)=\exp\left(-\sum_{\substack{n>0\\ \alpha,\beta}}\frac{nd^{(n)}_{\alpha,\beta}q^{\tau n}}{[(\tau+c_1)n]}\ \left(q^{\frac{cn}{2}}a_{\alpha,n}\otimes q^{\frac{cn}{2}}a_{\beta,-n}\right)\right)\ . \tag{3.9}$$

Definition 3.3 *By the dressed elliptic currents $f_{\gamma_i}(z;\tau)$ and $e_{\gamma_i}(z;\tau)$ we understand the currents which obtained by the dressing of the tensor $f_{\gamma_i}(z)\otimes e_{\gamma_i}(z)$ with twisting elements $\mathcal{F}^{(1)}(\tau)$ or $\mathcal{F}^{(2)}(\tau)$. For example,*

$$f_{\gamma_i}(z;\tau)\otimes e_{\gamma_1}(z;\tau+c_1)=\mathcal{F}^{(i)}(\tau)\ f_{\gamma_i}(z)\otimes e_{\gamma_1}(z)\left(\mathcal{F}^{(i)}(\tau)\right)^{-1},\quad i=1,2\ . \tag{3.10}$$

It is convenient to define the elliptic current by this formula as shifted by the central element in the first tensor product $c_1=c\otimes 1$.

For all elements of the algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ let us define the twisted comultiplications

$$\Delta^{(1)}_\tau(x)=\mathcal{F}^{(1)}(\tau)\ \Delta^{(1)}(x)\ \left(\mathcal{F}^{(1)}(\tau)\right)^{-1},\quad\text{and}\quad \Delta^{(2)}_\tau(x)=\mathcal{F}^{(2)}(\tau)\ \Delta^{(2)}(x)\ \left(\mathcal{F}^{(2)}(\tau)\right)^{-1}. \tag{3.11}$$

We have the following

Proposition 3.1 *The elliptic currents defined by (3.10) with* $i=1$ *and for simple root* α *can be written in normal ordered form*

$$f_\alpha(z;\tau)=u^-_\alpha(zq^{-\frac{c}{2}};\tau)\,f_\alpha(z),\quad e_\alpha(z;\tau)=e_\alpha(z)\,u^+_\alpha(zq^{-\frac{c}{2}};\tau), \tag{3.12}$$

where

$$u^-_\alpha(z;\tau)=\exp\left(\sum_{n>0}\frac{q^{(\tau+c)n}}{[(\tau+c)n]}\,a_{\alpha,-n}z^n\right),\qquad u^+_\alpha(z;\tau)=\exp\left(-\sum_{n>0}\frac{q^{\tau n}}{[\tau n]}\,a_{\alpha,n}z^{-n}\right). \tag{3.13}$$

The comultiplication of these currents with respect to $\Delta^{(1)}_\tau$ *is given by*

$$\Delta^{(1)}_\tau e_\alpha(z;\tau)=e_\alpha(z;\tau)\otimes 1+k_\alpha^{-1}\psi_\alpha(zq^{-\tau};\tau)\otimes e_\alpha(zq^{c_1};\tau+c_1)\ , \tag{3.14a}$$

$$\Delta^{(1)}_\tau f_\alpha(z;\tau)=1\otimes f_\alpha(z;\tau+c_1)+f_\alpha(zq^{c_2};\tau)\otimes k_\alpha\psi_\alpha(zq^{\tau+c_2};\tau+c_1)\ , \tag{3.14b}$$

$$\Delta^{(1)}_\tau c=c\otimes 1+1\otimes c,\quad \Delta^{(1)}_\tau k^{\pm1}_\alpha=k^{\pm1}_\alpha\otimes k^{\pm1}_\alpha,\quad \Delta^{(1)}_\tau\psi_\alpha(z)=\psi_\alpha(z;\tau)\otimes\psi_\alpha(z;\tau+c_1)\ , \tag{3.14c}$$

where the elliptic Cartan current $\psi_\alpha(z;\tau)$ *is given by the normal ordered expression*

$$\begin{aligned}\psi_\alpha(z;\tau)&=k_\alpha^{-1}u^-_\alpha(zq^{-\tau-\frac{3c}{2}};\tau)\,\psi^+_\alpha(zq^{-\tau-\frac{c}{2}})\,u^+_\alpha(zq^{-\tau-\frac{c}{2}};\tau)=\\&=k_\alpha u^-_\alpha(zq^{\tau+\frac{c}{2}};\tau)\,\psi^-_\alpha(zq^{\tau+\frac{c}{2}})\,u^+_\alpha(zq^{\tau-\frac{c}{2}};\tau)=\\&=\exp\left(\sum_{n>0}\frac{q^{-cn/2}}{[(\tau+c)n]}\,a_{\alpha,-n}z^n\right)\exp\left(-\sum_{n>0}\frac{q^{cn/2}}{[\tau n]}\,a_{\alpha,n}z^{-n}\right).\end{aligned} \tag{3.15}$$

Analogously, the elliptic currents defined by (3.10) for $i=2$ *can be written in the normal ordered form*

$$f_\alpha(z;\tau)=f_\alpha(z)\,v^+_\alpha(zq^{\frac{c}{2}};\tau)\ ,\quad e_\alpha(z;\tau)=v^-_\alpha(zq^{\frac{c}{2}};\tau)\,e_\alpha(z)\ , \tag{3.16}$$

where

$$v^-_\alpha(z;\tau)=\exp\left(\sum_{n>0}\frac{q^{\tau n}}{[\tau n]}\,a_{\alpha,-n}z^n\right),\qquad v^+_\alpha(z;\tau)=\exp\left(-\sum_{n>0}\frac{q^{(\tau+c)n}}{[(\tau+c)n]}\,a_{\alpha,n}z^{-n}\right). \tag{3.17}$$

These currents satisfy the comultiplication rules

$$\Delta^{(2)}_\tau e_\alpha(z;\tau)=e_\alpha(z;\tau)\otimes 1+k_\alpha\psi_\alpha(zq^{\tau};\tau)\otimes e_\alpha(zq^{-c_1};\tau+c_1)\ , \tag{3.18a}$$

$$\Delta^{(2)}_\tau f_\alpha(z;\tau)=1\otimes f_\alpha(z;\tau+c_1)+f_\alpha(zq^{-c_2};\tau)\otimes k_\alpha^{-1}\psi_\alpha(zq^{-\tau-c_2};\tau+c_1)\ , \tag{3.18b}$$

$$\Delta^{(2)}_\tau c=c\otimes 1+1\otimes c,\quad \Delta^{(2)}_\tau k^{\pm1}_\alpha=k^{\pm1}_\alpha\otimes k^{\pm1}_\alpha,\quad \Delta^{(2)}_\tau\psi_\alpha(z)=\psi_\alpha(z;\tau)\otimes\psi_\alpha(z;\tau+c_1)\ , \tag{3.18c}$$

with respect to the $\Delta^{(2)}_\tau$ *and the Cartan current is defined as follows*

$$\begin{aligned}\psi_\alpha(z;\tau)&=k_\alpha^{-1}u^-_\alpha(zq^{-\tau-\frac{3c}{2}};\tau)\,\psi^+_\alpha(zq^{-\tau-\frac{c}{2}})\,u^+_\alpha(zq^{-\tau-\frac{c}{2}};\tau)=\\&=k_\alpha u^-_\alpha(zq^{\tau+\frac{c}{2}};\tau)\,\psi^-_\alpha(zq^{\tau+\frac{c}{2}})\,u^+_\alpha(zq^{\tau-\frac{c}{2}};\tau)\\&=\exp\left(\sum_{n>0}\frac{q^{cn/2}}{[\tau n]}\,a_{\alpha,-n}z^n\right)\exp\left(-\sum_{n>0}\frac{q^{-cn/2}}{[(\tau+c)n]}\,a_{\alpha,n}z^{-n}\right).\end{aligned} \tag{3.19}$$

4 Analytical properties of elliptic current algebra

The goal of this section is to describe the analytical properties of the elliptic currents and proof of the fact that so defined elliptic current algebra conserve the notion of highest weight representations of the original current algebra. In all the formulas below we understand the elliptic currents $e_\alpha(z;\tau)$, $f_\alpha(z;\tau)$, $\psi_\alpha(z;\tau)$ given by formulas (3.12) and (3.15) or (3.16) and (3.19).

4.1 The commutation relations

We consider the algebra $\mathcal{U}_{q,\tau}(\hat{\mathfrak{g}})$ generated by the elements

$$\{c,\ d,\ k_\alpha^{\pm 1},\ e_{\alpha,n}(\tau),\ f_{\alpha,n}(\tau),\ \psi_{\alpha,n}(\tau)\},\quad n\in\mathbb{Z},\quad \tau\in\mathbb{C},\quad |q^\tau|\neq 1,\quad |q^{\tau+c}|\neq 1 \tag{4.1}$$

which are decompositions of the elliptic currents into series with respect to the spectral parameter:

$$e_\alpha(z;\tau)=\sum_{n\in\mathbb{Z}}e_{\alpha,n}(\tau)z^{-n},\quad f_\alpha(z;\tau)=\sum_{n\in\mathbb{Z}}f_{\alpha,n}(\tau)z^{-n}, \tag{4.2a}$$

$$\psi_\alpha(z;\tau)=\sum_{n\in\mathbb{Z}}\psi_{\alpha,n}(\tau)z^{-n}=\exp\left(\sum_{n>0}\frac{a_{\alpha,-n}(\tau)}{[\tau n]}z^n\right)\exp\left(-\sum_{n>0}\frac{a_{\alpha,n}(\tau)}{[\tau n]}z^{-n}\right)\ , \tag{4.2b}$$

where we have introduced by (4.2b) the elliptic version of Drinfeld generators $a_{\alpha,n}(\tau)$. The value of the parameter τ are restricted to the condition formulated before Lemma 3.1.

Using formulas (2.2)–(2.13) we present the commutation relations of elliptic currents in the form:

$$q^d x(z)q^{-d}=x(qz),\quad \text{for}\quad x=e_\alpha(z,\tau),\ f_\alpha(z,\tau),\ \psi_\alpha(z,\tau)\ , \tag{4.3}$$

$$k_\alpha^{\pm1}e_\beta(z;\tau)=q^{\pm(\alpha,\beta)}e_\beta(z;\tau)k_\alpha^{\pm1},\quad k_\alpha^{\pm1}f_\beta(z;\tau)=q^{\mp(\alpha,\beta)}f_\beta(z;\tau)k_\alpha^{\pm1},\quad k_\alpha^{\pm1}\psi_\beta(z;\tau)=\psi_\beta(z;\tau)k_\alpha^{\pm1}\ , \tag{4.4}$$

$$\begin{aligned}&(z-q^{(\alpha,\beta)}w)\ \Upsilon\left(q^\tau w/z;(\alpha,\beta),\tau\right)e_\alpha(z;\tau)e_\beta(w;\tau)=\\&=(q^{(\alpha,\beta)}z-w)\ \Upsilon\left(q^\tau z/w;(\alpha,\beta),\tau\right)e_\beta(w;\tau)e_\alpha(z;\tau)\ ,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&(z-q^{-(\alpha,\beta)}w)\ \Upsilon\left(q^{\tau+c}w/z;-(\alpha,\beta),\tau+c\right)f_\alpha(z;\tau)f_\beta(w;\tau)=\\&=(q^{-(\alpha,\beta)}z-w)\ \Upsilon\left(q^{\tau+c}z/w;-(\alpha,\beta),\tau+c\right)f_\beta(w;\tau)f_\alpha(z;\tau)\ ,\end{aligned} \tag{4.6}$$

$$\Upsilon\left(w/z;(\alpha,\beta),\tau\right)\psi_\alpha(z;\tau)e_\beta(w;\tau)=\ \Upsilon\left(z/w;(\alpha,\beta),\tau\right)e_\beta(w;\tau)\psi_\alpha(z;\tau)\ , \tag{4.7}$$

$$\Upsilon\left(w/z;-(\alpha,\beta),\tau+c\right)\psi_\alpha(z;\tau)f_\beta(w;\tau)=\ \Upsilon\left(z/w;-(\alpha,\beta),\tau+c\right)f_\beta(w;\tau)\psi_\alpha(z;\tau)\ , \tag{4.8}$$

$$\frac{\Upsilon\left(q^\tau w/z;(\alpha,\beta),\tau\right)}{\Upsilon\left(q^{\tau+c}w/z;(\alpha,\beta),\tau+c\right)}\psi_\alpha(z;\tau)\psi_\beta(w;\tau)=\frac{\Upsilon\left(q^\tau z/w;(\alpha,\beta),\tau\right)}{\Upsilon\left(q^{\tau+c}z/w;(\alpha,\beta),\tau+c\right)}\psi_\beta(w;\tau)\psi_\alpha(z;\tau)\ , \tag{4.9}$$

$$[e_\alpha(z;\tau),f_\beta(w;\tau)]=\frac{\delta_{\alpha,\beta}}{q-q^{-1}}\left(\delta(q^{-c}z/w)k_\alpha\psi_\alpha(zq^\tau;\tau)-\delta(q^cz/w)k_\alpha^{-1}\psi_\alpha(zq^{-\tau};\tau)\right)\ . \tag{4.10}$$

For $(\alpha,\beta)=-1$ the currents $e_\alpha(z)$ and $f_\alpha(z)$ satisfy deformed Serre relations:

$$\sum_{k=1}^{3}\frac{(-1)^k}{[k-1]![3-k]!}\prod_{i<j}^{3}\ \Upsilon\left(q^\tau\tilde{w}_i/\tilde{w}_j;(\tilde{\alpha}_i,\tilde{\alpha}_j),\tau\right)\prod_{i=1}^{3}e_{\tilde{\alpha}_i}(\tilde{w}_i;\tau)+\cdots=0 \tag{4.11a}$$

$$\sum_{k=1}^{3}\frac{(-1)^k}{[k-1]![3-k]!}\prod_{i<j}^{3}\ \Upsilon\left(q^{\tau+c}\tilde{w}_j/\tilde{w}_i;(\tilde{\alpha}_i,\tilde{\alpha}_j),\tau+c\right)^{-1}\prod_{i=1}^{3}f_{\tilde{\alpha}_i}(\tilde{w}_i;\tau)+\cdots=0 \tag{4.11b}$$

where in sums over k we should understand $\tilde{\alpha}_i=\alpha$ for $i\neq k$, $\tilde{\alpha}_k=\beta$, $\tilde{w}_i=w_i$ for $i<k$, $\tilde{w}_k=z$, $\tilde{w}_i=w_{i-1}$ for $i>k$ and the dots in (4.11) means symmetrization with respect to the spectral parameters w_1 and w_2.

In these formulas we denoted by $\Upsilon\left(z;a,b\right)$ the function

$$\Upsilon\left(z;a,b\right)=\exp\left(\sum_{n>0}\frac{z^n}{n}\frac{[an]}{[bn]}\right) \tag{4.12}$$

which always is understood as power series with respect to the parameter z and has different presentation in terms of infinite products depending whether $|q^b|<1$ or $|q^b|>1$. The commutation relations of the type (4.4)–(4.11b) have been considered in [DI] for arbitrary function $\Upsilon\left(z;a,b\right)$.

The commutation relations (4.7)–(4.9) can be also written in terms of the elliptic Drinfeld's generators $a_{\alpha,n}(\tau)$. To make these commutation relations more symmetric it is natural to introduce another set of the Drinfeld's generators $a'_{\alpha,n}(\tau)$ related to $a_{\alpha,n}(\tau)$ ones as follows:

$$a'_{\alpha,n}(\tau) = \frac{[(\tau+c)n]}{[\tau c]} a_{\alpha,n}(\tau) \ , \tag{4.13}$$

$$[a_{\alpha,n}(\tau), e_\beta(z;\tau)] = \frac{[(\alpha,\beta)n]}{n} z^n e_\beta(z) \ , \tag{4.14}$$

$$[a'_{\alpha,n}(\tau), f_\beta(z;\tau)] = -\frac{[(\alpha,\beta)n]}{n} z^n f_\beta(z) \ , \tag{4.15}$$

$$[a_{\alpha,n}(\tau), a_{\beta,m}(\tau)] = \delta_{n+m,0} \frac{[(\alpha,\beta)n][cn]}{n} \frac{[\tau n]}{[(\tau+c)n]} \ . \tag{4.16}$$

4.2 Reductions to quantized current algebra

Let us denote the values of the parameter τ such that $q^\tau = 0$ and $q^{-\tau} = 0$, τ_0 and τ'_0 respectively. We can see that at the value $\tau = \tau_0$ the elliptic algebra $\mathcal{U}_{q,\tau}(\hat{\mathfrak{g}})$ becomes current algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$. The Cartan currents at $\tau = \tau_0$ becomes:

$$k_\alpha \psi_\alpha(zq^{\tau_0};\tau_0) = \psi^+_\alpha(zq^{-c/2}), \qquad k^{-1}_\alpha \psi_\alpha(zq^{-\tau_0};\tau_0) = \psi^-_\alpha(zq^{c/2}) \tag{4.17}$$

or in terms of the Drinfeld generators $a_{\alpha,n}(\tau)$

$$a_{\alpha,n}(\tau_0) = a_{\alpha,n} q^{c|n|/2} \ . \tag{4.18}$$

On the other hand the elliptic algebra $\mathcal{U}_{q,\tau}(\hat{\mathfrak{g}})$ degenerates at $\tau = \tau'_0$ into current algebra $\mathcal{U}_{q^{-1}}(\hat{\mathfrak{g}})$. The Drinfeld's generators in this case are related to the current ones as follows:

$$a_{\alpha,n}(\tau'_0) = a_{\alpha,n} q^{-c|n|/2} \ . \tag{4.19}$$

4.3 The automorphisms

The elliptic current algebra defined in the previous subsection possesses the inner automorphisms[2] η , T_α and T^{-1}_α , such that η is an anti-automorphism

$$\eta\left(e_{\alpha_i}(z,\tau)\right) = e_{\alpha_i}(z^{-1},\tau), \quad \eta\left(f_{\alpha_i}(z,\tau)\right) = f_{\alpha_i}(z^{-1},\tau), \quad \eta\left(\psi_{\alpha_i}(z,\tau)\right) = \psi_{\alpha_i}(z^{-1},), \quad \eta\left(k_\alpha\right) = k^{-1}_\alpha \tag{4.20}$$

and T_α and T^{-1}_α are automorphisms defined by the formulas

$$\begin{gathered} T_\alpha k_\alpha = k^{-1}_\alpha, \quad T_\alpha \psi_\alpha(z;\tau) = {:}\psi^{-1}_\alpha(z;\tau){:}, \\ T_\alpha e_\alpha(z;\tau) = {:}f_\alpha(zq^{-c};\tau)\psi^{-1}_\alpha(zq^\tau;\tau)k^{-1}_\alpha{:}\ , \quad T_\alpha f_\alpha(z;\tau) = {:}k_\alpha \psi^{-1}_\alpha(zq^{-\tau-c};\tau)e_\alpha(zq^{-c};\tau){:}\ , \end{gathered} \tag{4.21}$$

and for the simple β such that $(\alpha,\beta) = -1$:

$$\begin{gathered} T_\alpha k_\beta = k_\alpha k_\beta, \quad T_\alpha \psi_\beta(z;\tau) = {:}\psi_\alpha(qz;\tau)\psi_\beta(q^2z;\tau){:}\ , \\ T_\alpha e_\beta(z;\tau) = e_{\alpha+\beta}(z;\tau)\ , \quad T_\alpha f_\beta(z;\tau) = f_{\alpha+\beta}(z;\tau)\ , \end{gathered} \tag{4.22}$$

where we denoted by $e_{\alpha+\beta}(z;\tau)$ and $f_{\alpha+\beta}(z;\tau)$ the following series:

$$\begin{aligned} e_{\alpha+\beta}(z) = \int \frac{dw}{w} \Big(& \Upsilon\left(q^{2+\tau}z/w;(\alpha,\beta),\tau\right) e_\alpha(w;\tau)e_\beta(q^2z;\tau) - \\ & - q\ \Upsilon\left(q^{-2-\tau}w/z;(\alpha,\beta),\tau\right) e_\beta(q^2z;\tau)e_\alpha(w;\tau)\Big)\ , \end{aligned} \tag{4.23}$$

[2]We denoted them by the same letters as in the case of quantum current algebras.

$$f_{\alpha+\beta}(z) = \int \frac{dw}{w} \Big(- \Upsilon\left(q^{\tau+c-2}w/z; -(\alpha,\beta), \tau+c\right) f_\beta(q^2z;\tau) f_\alpha(w;\tau) + \\ + q^{-1}\, \Upsilon\left(q^{2-\tau-c}z/w; -(\alpha,\beta), \tau+c\right) f_\alpha(w;\tau) f_\beta(q^2z;\tau)\Big) \ . \tag{4.24}$$

The symbol $\int \frac{dw}{w}$ means again picking up the coefficient in front of w^0 and the function $\Upsilon(z;a,b)$ is understood as power series with respect to spectral parameter z.

The normal ordered in the formulas (4.21) and (4.22) signify that all the modes $a_{\alpha,n}(\tau)$ of the Cartan current $\psi_\alpha(z)$ with $n>0$ are placed on the right and with $n<0$ on the left in the corresponding product. For example, the notation $:\psi_\alpha^{-1}(z;\tau):$ means

$$:\psi_\alpha^{-1}(z;\tau): = \exp\left(-\sum_{n>0} \frac{a_{\alpha,-n}(\tau)}{[\tau n]} z^n \right) \exp\left(\sum_{n>0} \frac{a_{\alpha,n}(\tau)}{[\tau n]} z^{-n} \right) \ . \tag{4.25}$$

The inverse to T_α automorphism is defined by the composition

$$T_\alpha^{-1} = \eta \circ T_a \circ \eta \ . \tag{4.26}$$

As well as in the case of quantum current algebra the action of the inverse automorphisms T_a^{-1} on the currents $e_\beta(z;\tau)$ and $f_\beta(z;\tau)$ is defined by the series

$$\check{e}_{\alpha+\beta}(z) = T_\alpha^{-1} e_\beta(z;\tau) = \int \frac{dw}{w} \Big(- \Upsilon\left(q^{\tau+2}w/z; (\alpha,\beta), \tau\right) e_\beta(q^{-2}z;\tau) e_\alpha(w;\tau) + \\ + q\, \Upsilon\left(q^{-\tau-2}z/w; (\alpha,\beta), \tau\right) f_\alpha(w;\tau) f_\beta(q^{-2}z;\tau)\Big) \ , \tag{4.27}$$

$$\check{f}_{\alpha+\beta}(z) = T_\alpha^{-1} f_\beta(z;\tau) = \int \frac{dw}{w} \Big(\Upsilon\left(q^{\tau+c-2}z/w; -(\alpha,\beta), \tau+c\right) f_\alpha(w;\tau) f_\beta(q^{-2}z;\tau) - \\ - q^{-1}\, \Upsilon\left(q^{2-\tau-c}w/z; -(\alpha,\beta), \tau+c\right) f_\beta(q^{-2}z;\tau) f_\alpha(w;\tau)\Big) \ . \tag{4.28}$$

4.4 The analytical properties of quantum elliptic currents

The commutation relations above encoded the analytical structure of the product of the currents. They are given by following

Lemma 4.1 *(i) For $|q^\tau|<1$ the product $e_\alpha(z;\tau)e_\beta(w;\tau)$ has simple poles at the points $z=q^{(\alpha,\beta)}q^{2\tau k}w$, $k\ge 0$ and simple zeros at the points $z=q^{-(\alpha,\beta)}q^{2\tau k}w$, $k>0$;*

(ii) For $|q^\tau|>1$ the product $e_\alpha(z;\tau)e_\beta(w;\tau)$ has simple poles at the points $z=q^{-(\alpha,\beta)}q^{-2\tau k}w$, $k\ge 0$ and simple zeros at the points $z=q^{(\alpha,\beta)}q^{-2\tau k}w$, $k>0$;

(iii) For $|q^{\tau+c}|<1$ the product $f_\alpha(z;\tau)f_\beta(w;\tau)$ has simple poles at the points $z=q^{-(\alpha,\beta)}q^{2(\tau+c)k}w$, $k\ge 0$ and simple zeros at the points $z=q^{(\alpha,\beta)}q^{2(\tau+c)k}w$, $k>0$;

(iv) For $|q^{\tau+c}|>1$ the product $f_\alpha(z;\tau)f_\beta(w;\tau)$ has simple poles at the points $z=q^{(\alpha,\beta)}q^{-2(\tau+c)k}$, $k\ge 0$ and simple zeros at the points $z=q^{-(\alpha,\beta)}q^{-2(\tau+c)k}w$, $k>0$.

This Lemma states that depending on whether $|q^\tau|<1$ or $|q^\tau|>1$ the product $e_\alpha(z;\tau)e_\beta(w;\tau)$ is analytical function in the domains $|z|>|q^{(\alpha,\beta)}w|$ and $|z|<|q^{-(\alpha,\beta)}w|$ respectively. Analogously, the product $f_\alpha(z;\tau)f_\beta(w;\tau)$ is analytical function in the domains $|z|>|q^{-(\alpha,\beta)}w|$ and $|z|<|q^{(\alpha,\beta)}w|$ for the conditions $|q^{\tau+c}|<1$ and $|q^{\tau+c}|>1$ respectively.

As well as in case of quantized current algebras we can calculate the residues of the products of the elliptic currents. We will demonstrate below that the residues at the points $z=q^{\pm(\alpha,\beta)}w$ coincide with the action of automorphisms T_α and T_α^{-1} on the currents $e_\beta(z;\tau)$ and $f_\beta(z;\tau)$ while the residues at others points are equal to the values of the corresponding product in their domains of analyticity.

For the simplicity we consider a particular case of $|q^\tau|<1$ and calculate the residue of the product $e_\alpha(w;\tau)\times$ $e_\beta(q^2z;\tau)$ for $(\alpha,\beta)=-1$ at the point $w=qz$ (see Fig.1, where $\circ$ signify the poles and $\bullet$ the zeros).

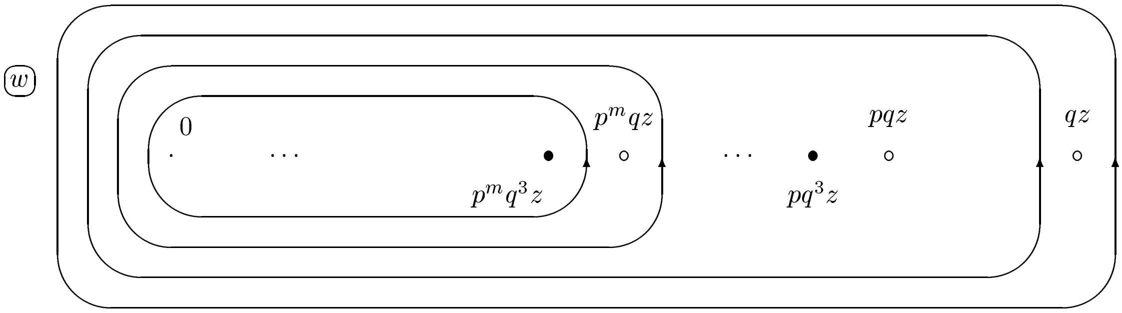

Fig. 1. The form of the contours in the complex plane w , $p=q^{2\tau}$.

$$
\begin{aligned}
\operatorname*{res}_{w=qz} e_\alpha(w)e_\beta(q^2z) = \oint_{w \text{ around } qz} \frac{dw}{2\pi i w} e_\alpha(w)e_\beta(q^2z) = \\
= \oint_{|w|>|qz|} \frac{dw}{2\pi i w}\, e_\alpha(w)e_\beta(q^2z) - q\oint_{|q^{2\tau+1}z|<|w|<|qz|} \frac{dw}{2\pi i w}\, \frac{\Theta_{q^{2\tau}}(q^{-3}w/z)}{\Theta_{q^{2\tau}}(q^{-1}w/z)} e_\beta(q^2z)e_\alpha(w) = \\
= \prod_{k>0} \frac{(1-q^{2k\tau+2})}{(1-q^{2k\tau})} e_{\alpha+\beta}(z)
\end{aligned}
\tag{4.29}
$$

where for $|p|<1$ $\Theta_p(z)$ denote multiplicative θ -function given by infinite product

$$
\Theta_p(z) = \prod_{k=0}^{\infty} (1-zp^k)(1-z^{-1}p^{k+1})(1-p^{k+1}). \tag{4.30}
$$

As we can see from (4.29) the residue of the product $e_\alpha(w)e_\beta(q^2z)$ in the point $w=qz$ yields the composed elliptic current given by the automorphism (4.23). The rest of the composed elliptic currents are given by the analogous residues in the point $w=qz$ for $f_{\alpha+\beta}$ and $w=q^{-1}z$ for $\check{e}_{\alpha+\beta}$ and $\check{f}_{\alpha+\beta}$.

Let us calculate now the residues of the product $e_\alpha(w;\tau)e_\beta(q^2z;\tau)$ in the points $w=q^{2m\tau+1}z$ for integer $m>0$. To do this we consider the operator valued function

$$
E(w,z) = \prod_{k=1}^{m} \frac{w-q^{2\tau(k-1)+1}z}{w-q^{2\tau k+3}z} e_\alpha(w)e_\beta(q^2z). \tag{4.31}
$$

It is obvious that this function is analytical in the domain $|w|>|q^{2\tau m+1}z|$ and its residue at the point $w=q^{2\tau m+1}z$ coincide with the residue which we are interesting in up to some constant. This residue can be presented as difference of two contour integrals around origin $w=0$ (see Fig. 1.) such that first contour contain the point $w=q^{2\tau m+1}z$ and the variable of integration in the second integral restricted to the domain $|q^{2\tau(m+1)+1}z|<|w|<|q^{2\tau m+1}z|$. In this domain the function $E(w,z)$ is replaced by its analytical continuation given by the commutation relation (4.5)

$$
q\frac{\Theta_{q^{2\tau}}(q^{-3}w/z)}{\Theta_{q^{2\tau}}(q^{-1}w/z)} \prod_{k=1}^{m} \frac{w-q^{2\tau(k-1)+1}z}{w-q^{2\tau k+3}z} e_\beta(q^2z)e_\alpha(w). \tag{4.32}
$$

We can now move the contour of the integration in the second integral through the pole at the point $w=q^{2\tau m+1}z$, calculate the residue of (4.32) and using once more the commutation relation (4.5) in the domain $|w|>|q^{2\tau m+1}z|$ we conclude with the answer:

$$
\operatorname*{res}_{w=q^{2\tau m+1}z} e_\alpha(w)e_\beta(q^2z) = q^{2m}(q^{-1}-q)\prod_{k>0} \frac{(1-q^{2\tau k+2})(1-q^{2\tau k-2})}{(1-q^{2\tau k})^2} e_\beta(q^2z)e_\alpha(q^{2\tau m+1}z). \tag{4.33}
$$

To make sure that our analytical considerations are correct we can calculate this residue directly using the dressing formulas for elliptic currents (3.12). To proceed with this calculation and for $|q^\tau|<1$ it is convenient to express the dressing factor $u^+_\alpha(z;\tau)$ in terms of operators

$$\tilde\psi^+_\alpha(z)=k^{-1}_\alpha\psi^+_\alpha(z)=\exp\left((q-q^{-1})\sum_{n>0}a_{\alpha,n}z^{-n}\right) \tag{4.34}$$

as follows

$$u^+_\alpha(z;\tau)=\prod_{k>0}\tilde\psi^+_\alpha(zq^{-2\tau k}) \tag{4.35}$$

Then the calculation of the residue is

$$\begin{aligned}
qz(q^{2\tau m}-1)\mathop{\mathrm{res}}_{w=q^{2\tau m+1}z} e_\alpha(w;\tau)e_\beta(q^2z;\tau) &= \mathop{\mathrm{res}}_{w=q^{2\tau m+1}z}(w-qz)e_\alpha(w;\tau)e_\beta(q^2z;\tau)=\\
=\mathop{\mathrm{res}}_{w=q^{2\tau m+1}z}(w-qz)e_\alpha(w)e_\beta(q^2z)\prod_{k>0}\frac{q^{-1}w-q^{2\tau k+2}z}{qw-q^{2\tau k+2}z}\,&\tilde\psi^+_\alpha\left(wq^{-2\tau k-c/2}\right)\tilde\psi^+_\beta\left(zq^{2-2\tau k-c/2}\right)=\\
=\mathop{\mathrm{res}}_{w=q^{2\tau m+1}z}(q^{-1}w-q^2z)e_\beta(q^2z)e_\alpha(w)\prod_{k>0}\frac{q^{-1}w-q^{2\tau k+2}z}{qw-q^{2\tau k+2}z}\,&\tilde\psi^+_\beta\left(zq^{2-2\tau k-c/2}\right)\tilde\psi^+_\alpha\left(wq^{-2\tau k-c/2}\right)=\\
=\mathop{\mathrm{res}}_{w=q^{2\tau m+1}z}(q^{-1}w-q^2z)e_\beta(q^2z;\tau)e_\alpha(w;\tau)&\prod_{k>0}\frac{w-q^{2\tau k+3}z}{w-q^{2\tau k+1}z}\,\frac{z-q^{2\tau k-3}w}{z-q^{2\tau k-1}w}=\\
=qz(q^{2\tau m}-1)q^{2m}(q^{-1}-q)&\prod_{k>0}\frac{(1-q^{2\tau k+2})(1-q^{2\tau k-2})}{(1-q^{2\tau k})^2}e_\beta(q^2z)e_\alpha(q^{2\tau m+1}z)
\end{aligned} \tag{4.36}$$

which obviously yields the same answer as (4.33).

We result of this calculations can be interpreted as follows. Although the product of elliptic currents has many poles only those at the shifted diagonal $w=q^{\pm(\alpha,\beta)}z$ produces new generators of the algebra which are gathered into composed currents. The definitions of these composed roots can be written in the analytical language.

$$\begin{aligned}e_{\alpha+\beta}(q^{-1}z;\tau)\delta(qz/w)=\ &\Upsilon\left(q^\tau w/z;-1,\tau\right)e_\alpha(z;\tau)e_\beta(w;\tau)-\\ &-q\ \Upsilon\left(q^{-\tau}z/w;-1,\tau\right)e_\beta(w;\tau)e_\alpha(z;\tau)\ ,\end{aligned} \tag{4.37a}$$

$$\begin{aligned}f_{\alpha+\beta}(q^{-1}z;\tau)\delta(q^{-1}z/w)=\ &\Upsilon\left(q^{\tau+c}w/z;1,\tau+c\right)f_\alpha(z;\tau)f_\beta(w;\tau)-\\ &-q^{-1}\ \Upsilon\left(q^{-(\tau+c)}z/w;1,\tau+c\right)f_\beta(w;\tau)f_\alpha(z;\tau)\ ,\end{aligned} \tag{4.37b}$$

$$\begin{aligned}\check e_{\alpha+\beta}(qz;\tau)\delta(q^{-1}z/w)=\ &q\ \Upsilon\left(q^{-\tau}w/z;-1,\tau\right)e_\alpha(z;\tau)e_\beta(w;\tau)-\\ &-\ \Upsilon\left(q^{\tau}z/w;-1,\tau\right)e_\beta(w;\tau)e_\alpha(z;\tau)\ ,\end{aligned} \tag{4.37c}$$

$$\begin{aligned}\check f_{\alpha+\beta}(qz;\tau)\delta(qz/w)=\ &q^{-1}\ \Upsilon\left(q^{-(\tau+c)}w/z;1,\tau+c\right)f_\alpha(z;\tau)f_\beta(w;\tau)-\\ &-\ \Upsilon\left(q^{\tau+c}z/w;1,\tau+c\right)f_\beta(w;\tau)f_\alpha(z;\tau)\ .\end{aligned} \tag{4.37d}$$

Analogously to the case of the current algebra we can prove that the relations

$$\Upsilon\left(q^\tau w/z;1,\tau\right)e_\alpha(z;\tau)e_{\alpha+\beta}(w;\tau)=q^{-1}\ \Upsilon\left(q^{-\tau}z/w;1,\tau\right)e_{\alpha+\beta}(w;\tau)e_\alpha(z;\tau)\ , \tag{4.38a}$$

$$\Upsilon\left(q^{\tau-3}w/z;1,\tau\right)e_{\alpha+\beta}(z;\tau)e_\beta(w;\tau)=q^{-1}\ \Upsilon\left(q^{-\tau+3}z/w;1,\tau\right)e_\beta(w;\tau)e_{\alpha+\beta}(z;\tau)\ , \tag{4.38b}$$

$$\Upsilon\left(q^{\tau+c+3}w/z;1,\tau+c\right)f_\beta(z;\tau)f_{\alpha+\beta}(w;\tau)=q\ \Upsilon\left(q^{-\tau-c-3}z/w;1,\tau+c\right)f_{\alpha+\beta}(w;\tau)f_\alpha(z;\tau)\ , \tag{4.38c}$$

$$\Upsilon\left(q^{\tau+c}w/z;1,\tau+c\right)f_{\alpha+\beta}(z;\tau)f_\alpha(w;\tau)=q\ \Upsilon\left(q^{-\tau-c}z/w;1,\tau+c\right)f_\alpha(w;\tau)f_{\alpha+\beta}(z;\tau)\ . \tag{4.38d}$$

are equivalent to deformed Serre relations (4.11). In formulas (4.37) and (4.38) all the functions $\Upsilon(z;a,b)$ should be understood as power series with respect to variable z.

Describing the commutation relations of the algebra $\mathcal{U}_{q,\tau}(\hat{\mathfrak{g}})$ and its analytical properties we did not use the explicitly elliptic functions, because the formulation which uses these functions is not universal. In particular, the explicit form of the relations depends whether $q^\tau > 1$ or $q^\tau < 1$. Moreover, using the elliptic functions to define elliptic current algebras requires to add the additional information on the analytical properties of currents. For example, for $q^\tau < 1$ the commutation relation (4.5) can be rewritten in the form

$$\Theta_{q^{2\tau}}\left(q^{(\alpha,\beta)}w/z\right) e_\alpha(z;\tau)e_\beta(w;\tau) = q^{(\alpha,\beta)}\Theta_{q^{2\tau}}\left(q^{-(\alpha,\beta)}w/z\right) e_\beta(w;\tau)e_\alpha(z;\tau)\ , \tag{4.39}$$

but we should say additionally to (4.39) that the product $e_\alpha(z;\tau)e_\beta(w;\tau)$ has zeros in the points $z = q^{-(\alpha,\beta)}q^{2\tau k}w$, $k>0$. Otherwise the commutation relations (4.39) yields the algebra which is not compatible with the category of the highest weight representations. On the other hand the elliptic functions can be used to rewrite the equalities like (4.37) as equalities of the meromorphic functions. For example, (4.37a) can be written in the form

$$e_\alpha(z;\tau)e_\beta(w;\tau) = q^{(\alpha,\beta)}\frac{\Theta_{q^{2\tau}}\left(q^{-(\alpha,\beta)}w/z\right)}{\Theta_{q^{2\tau}}\left(q^{(\alpha,\beta)}w/z\right)}e_\beta(w;\tau)e_\alpha(z;\tau) + \delta(qz/w)e_{\alpha+\beta}(q^{-1}z;\tau)\prod_{k>0}\frac{1-q^{2\tau k+2}}{1-q^{2\tau k}}\ .$$

4.5 Coalgebraic structures and universal $\mathcal{R}$-matrix

Using Lemma 4.1 we have

Proposition 4.1 *The product of two-tensors $G(z) = f_\gamma(z;\tau)\otimes e_\gamma(z;\tau+c_1)$, $G(z)G(w)$ has the poles only in the points $z = q^{\pm 2}w$, for all positive roots γ of simply laced algebra $\mathfrak{g}$.*

Define the composed currents using the same decomposition of the longest element of the Weyl group (2.25) and the automorphisms (4.21)–(4.24). Then the universal $\mathcal{R}$-matrix which correspond to the comultiplication (3.14) is given by

$$\mathcal{R}(\tau) = \mathcal{K}(\tau)\prod_{1\le i\le N}^{\rightarrow}\overline{\mathcal{R}}_{\gamma_i}(\tau) \tag{4.40}$$

where the elliptic versions of the tensors $\overline{\mathcal{R}}_\gamma(\tau)$ and $\mathcal{K}(\tau)$ reads as follows:

$$\overline{\mathcal{R}}_{\gamma_i}(\tau) = \mathcal{P}\,\overrightarrow{\exp}_{q^2}\left(\frac{(q^{-1}-q)}{2\pi i}\oint f_{\gamma_i}(z;\tau)\otimes e_{\gamma_i}(z;\tau+c_1)\frac{dz}{z}\right), \tag{4.41}$$

$$\mathcal{K}(\tau) = \mathcal{F}^{(1)}_{21}(\tau)\ \mathcal{K}\ \left(\mathcal{F}^{(1)}(\tau)\right)^{-1} \tag{4.42}$$

and the contours in the definition of the $\mathcal{P}$-ordered exponent are the same as in (2.30) which is result of the Proposition 4.1.

The universal $\mathcal{R}$-matrix which correspond to this comultiplication (3.18) is given by

$$\prod_{1\le i\le N}^{\rightarrow}\left(\overline{\mathcal{R}}_{\gamma_i}(\tau)\right)_{21}\cdot\tilde{\mathcal{K}}(\tau) \tag{4.43}$$

where $\overline{\mathcal{R}}_\gamma(\tau)$ is given by (4.41) and $\tilde{\mathcal{K}}(\tau) = \mathcal{F}^{(2)}_{21}(\tau)\ \mathcal{K}\ \left(\mathcal{F}^{(2)}(\tau)\right)^{-1}$.

From the general principles of twisting one can observe that the $\mathcal{R}$-matrix $\mathcal{R}(\tau)$ satisfy the dynamical Yang-Baxter equation

$$\mathcal{R}_{12}(\tau+c_3)\mathcal{R}_{13}(\tau)\mathcal{R}_{23}(\tau+c_1) = \mathcal{R}_{23}(\tau)\mathcal{R}_{13}(\tau+c_2)\mathcal{R}_{12}(\tau) \tag{4.44}$$

Moreover, because the factorization property of the universal $\mathcal{R}$-matrix (2.32) associated to the Drinfeld's comultiplication in current algebra (2.28) the Cartan part of dynamical $\mathcal{R}$-matrix also satisfy the same equation

$$\mathcal{K}_{12}(\tau+c_3)\mathcal{K}_{13}(\tau)\mathcal{K}_{23}(\tau+c_1) = \mathcal{K}_{23}(\tau)\mathcal{K}_{13}(\tau+c_2)\mathcal{K}_{12}(\tau) \tag{4.45}$$

The coassociator can be easily calculated from (3.5a) and is equal to

$$\Phi^{(1)}(\tau) = \exp\left(\sum_{\substack{n>0\\ \alpha,\beta}} \frac{n d^{(n)}_{\alpha,\beta}[c_1 n]}{[(\tau+c_2)n][(\tau+c_1+c_2)n]}\ \left(1\otimes q^{-\frac{cn}{2}} a_{\alpha,-n}\otimes q^{\frac{cn}{2}} a_{\beta,n}\right)\right) \tag{4.46}$$

$$\Phi^{(2)}(\tau) = \exp\left(-\sum_{\substack{n>0\\ \alpha,\beta}} \frac{n d^{(n)}_{\alpha,\beta}[c_1 n]}{[(\tau+c_2)n][(\tau+c_1+c_2)n]}\ \left(1\otimes q^{-\frac{cn}{2}} a_{\alpha,n}\otimes q^{\frac{cn}{2}} a_{\beta,-n}\right)\right) \tag{4.47}$$

for comultiplication (3.14) and (3.18) respectively. It is interesting to get a closed formulas for the elements α and β (see Definition 3.1).

To get the dynamical YB equation which involves general element of Cartan subalgebra, not only the central one, we should expand the algebra by some additional operators, which where postulated in [JKOS]. We prefer to get those operators from natural algebraic construction on the level of quantum affine algebra such that the universal dynamical $\mathcal{R}$-matrices which includes shifts onto general element of the Cartan subalgebra can be obtained by the same twisting procedure described in this paper. In particular this may give an universal explanation of finite-dimensional elliptic groups described in [BBB, Fe]. The meaning of this operators is very important in application of quantum elliptic algebras into integrable models.

Another problem which deserve further investigation is factorization problem of universal elliptic $\mathcal{R}$-matrices (4.40), (4.43) and twisting of Drinfeld's comultiplication to the physical one. Solving this problem allows to obtain systematically the Boltzmann weights of lattice integrable model of ABF type [ABF].

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