A SYMBOL ALGEBRA FOR PSEUDODIFFERENTIAL BOUNDARY VALUE PROBLEMS ON MANIFOLDS WITH EDGES

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We introduce a symbol algebra for pseudodifferential boundary value problems on manifolds with edges. The elements in this algebra consist of (i) a Mellin part with a holomorphic Mellin symbol near the edge, (ii) a pseudodifferential part slightly away from the edge, and (iii) a residual term, a so-called Green operator.

Introduction

Following upon earlier work [15], [16], this paper is part of a series of articles devoted to the construction of an operator-valued symbolic structure for pseudodifferential boundary value problems on manifolds with edges. Our investigations here focus on a symbol algebra for the non-smoothing part of the operators, induced by the edge-degenerate symbols in the interior. It will be completed to the full algebra by adding the smoothing elements with asymptotics treated in [16].

A wedge in our terminology is an object of the form $C \times \mathbf{R}^q$, where $C = X \times [0,\infty)/X \times \{0\}$ is an infinite cone over a smooth compact manifold with boundary, X. Following the general approach, we consider symbols which coincide with the usual elements in Boutet de Monvel's calculus away from the edge; near the edge they are described in terms of operator-valued symbols on $\mathbf{R}^q$ taking values in operators on the cone.

We show, in particular, that the approach to a wedge pseudodifferential calculus developed in [17, 18, 19] for the case of boundaryless X applies in a similar form to the case of boundary value problems. At the same time we further develop the technique of using operator-valued symbols and give a new concise description of the nonsmoothing contribution to the edge symbol algebra.

1 Pseudodifferential Boundary Value Problems

In this section we review the basic elements we need for the construction of a calculus, namely on one hand a parameter-dependent version of Boutet de Monvel's calculus based on the concept of operator-valued symbols and, on the other, the notion of wedge Sobolev spaces.

We start with the definition of parameter-dependent operator-valued symbols. The point in this construction is the special kind of estimates involving a group action. We proceed by introducing weighted Mellin Sobolev spaces, holomorphic Mellin symbols, and the associated operators. We review the definition of edge symbols, show how they can be considered as operator-valued symbols and how one can link pseudodifferential and Mellin edge operators by a process called Mellin quantization.

The exposition here is necessarily concise; all details may be found in the papers [13], [14], and, mainly, [15].

Group Actions and Operator-Valued Symbols

1.1 Operator-valued symbols. A strongly continuous group action on a Banach space E is a family $\kappa = \{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ of isomorphisms in $\mathcal{L}(E)$ such that, for $e \in E$, the mapping $\lambda \mapsto \kappa_\lambda e$ is continuous and $\kappa_\lambda \kappa_\mu = \kappa_{\lambda\mu}$.

There are constants c and M with $\|\kappa_\lambda\|_{\mathcal{L}(E)} \le c \max\{\lambda, \lambda^{-1}\}^M$.

We next fix a smooth positive function $[\cdot] : \mathbf{R}^q \to \mathbf{R}_+$ with $[\eta] = |\eta|$ for large $|\eta|$. *Peetre's inequality* states that, for each $s \in \mathbf{R}$ there is a constant C_s with

$$[\eta + \xi]^s \le C_s [\eta]^s [\xi]^{|s|}.$$

$H^s(\mathbf{R})$ is the usual Sobolev space on $\mathbf{R}$, while $H^s(\mathbf{R}_+) = \{u|_{\mathbf{R}_+} : u \in H^s(\mathbf{R})\}$ and $H^s_0(\mathbf{R}_+)$ is the set of all $u \in H^s(\mathbf{R})$ whose support is contained in $\overline{\mathbf{R}}_+$. Furthermore, $H^{s,t}(\mathbf{R}_+) = \{[r]^{-t}u : u \in H^s(\mathbf{R}_+)\}$, and $H^{s,t}_0(\mathbf{R}_+) = \{[r]^{-t}u : u \in H^s_0(\mathbf{R}_+)\}$; here r is the variable in $\mathbf{R}_+$. Finally, $\mathcal{S}(\mathbf{R}^q_+) = \{u|_{\mathbf{R}^q_+} : u \in \mathcal{S}(\mathbf{R}^q)\}$.

For all Sobolev spaces on $\mathbf{R}$ and $\mathbf{R}_+$, we will use the group action

$$(\kappa_\lambda f)(r) = \lambda^{\frac{1}{2}} f(\lambda r). \tag{1.1}$$

This action extends to distributions by $\kappa_\lambda u(\varphi) = u(\kappa_{\lambda^{-1}}\varphi)$. On $E = \mathbf{C}^l$ use the trivial group action $\kappa_\lambda = id$.

Let E, F be Banach spaces with strongly continuous group actions $\kappa, \tilde\kappa$, let $\Omega \subseteq \mathbf{R}^k$, $a \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E,F))$, and $\mu \in \mathbf{R}$. We shall write $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$, provided that, for every $K \subset\subset \Omega$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde{\kappa}_{[\eta]^{-1}} D_\eta^\alpha D_y^\beta a(y,\eta) \kappa_{[\eta]}\|_{\mathcal{L}(E,F)} \leq C[\eta]^{\mu-|\alpha|}. \tag{1.2}$$

The space $S^\mu(\Omega, \mathbf{R}^q; E, F)$ is Fréchet topologized by the choice of the best constants C. The intersection $S^{-\infty}(\Omega, \mathbf{R}^q; E, F) = \cap_\mu S^\mu(\Omega, \mathbf{R}^q; E, F)$ is independent of the choice of κ and $\tilde{\kappa}$.

The space $S^\mu(\Omega, \mathbf{R}^q; \mathbf{C}^k, \mathbf{C}^l)$ coincides with the ($l \times k$ matrix-valued) elements of Hörmander's class $S^\mu(\Omega, \mathbf{R}^q)$.

Asymptotic summation: Given a sequence $\{a_j\}$ with $a_j \in S^{\mu_j}(\Omega, \mathbf{R}^q; E, F)$ and $\mu_j \to -\infty$, there is an $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$, $\mu = \max\{\mu_j\}$ such that $a \sim \sum a_j$; a is unique modulo $S^{-\infty}(\Omega, \mathbf{R}^q; E, F)$.

A symbol $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ is said to be *classical*, if it has an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ with $a_j \in S^{\mu-j}(\Omega, \mathbf{R}^q; E, F)$ satisfying the homogeneity relation

$$a_j(y, \lambda\eta) = \lambda^{\mu-j} \tilde{\kappa}_\lambda \, a_j(y,\eta) \, \kappa_{\lambda^{-1}} \tag{1.3}$$

for all $\lambda \geq 1$, $|\eta| \geq R$ with a suitable constant R. We write $a \in S^\mu_{cl}(\Omega, \mathbf{R}^q; E, F)$. For $E = \mathbf{C}^k$, $F = \mathbf{C}^l$ we recover the standard notion.

There is an extension to projective and inductive limits: Let $\tilde{E}, \tilde{F}$ be Banach spaces with group actions. If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ and $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ are sequences of Banach spaces with the same group action, and $F = \text{proj} - \lim F_k$, $E = \text{ind} - \lim E_k$, then let

$$\begin{aligned} S^\mu(\Omega, \mathbf{R}^q; \tilde{E}, F) &= \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^q; \tilde{E}, F_k); \\ S^\mu(\Omega, \mathbf{R}^q; E, \tilde{F}) &= \text{proj} - \lim_k S^\mu(\Omega, \mathbf{R}^q; E_k, \tilde{F}); \\ S^\mu(\Omega, \mathbf{R}^q; E, F) &= \text{proj} - \lim_{k,l} S^\mu(\Omega, \mathbf{R}^q; E_k, F_l). \end{aligned}$$

Example 1.2. Let $\gamma_j : \mathcal{S}(\mathbf{R}_+) \to \mathbf{C}$ be defined by $\gamma_j f = \lim_{r \to 0^+} \partial_r^j f(r)$. Then, for all $s > j + 1/2$, we can consider γ_j as a (y,η)-independent symbol in $S^{j+1/2}(\mathbf{R}^q \times \mathbf{R}^q; H^s(\mathbf{R}_+), \mathbf{C})$.

In fact, all we have to check is that $\|\tilde{\kappa}_{[\eta]^{-1}} \gamma_j \kappa_{[\eta]}\| = O([\eta]^{j+1/2})$ for the group actions $\tilde{\kappa}$ on $\mathbf{C}$ and κ on $H^s(\mathbf{R}_+)$. Since the group action on $\mathbf{C}$ is the identity, that on $H^s(\mathbf{R}_+)$ is given by (1.2), everything follows from the observation that

$$\partial_r^j \{[\eta]^{1/2} f([\eta] r)\}|_{r=0} = [\eta]^{j+1/2} \partial_r^j f(0).$$

The following statement is obvious.

Lemma 1.3. *For $a \in S^\mu(\Omega, \mathbf{R}^q; E, F)$ and $b \in S^\nu(\Omega, \mathbf{R}^q; F, G)$, the symbol c defined by $c(y,\eta) = b(y,\eta)a(y,\eta)$ (pointwise composition of operators) belongs to $S^{\mu+\nu}(\Omega, \mathbf{R}^q; E, G)$, while $D^\alpha_\eta D^\beta_y a \in S^{\mu-|\alpha|}(\Omega, \mathbf{R}^q; E, F)$.*

Lemma 1.4. [15, Lemma 1.4] *Let $a = a(y,\eta) \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E,F))$, and suppose that $a(y,\lambda\eta) = \lambda^\mu \tilde{\kappa}_\lambda\, a(y,\eta)\, \kappa_{\lambda^{-1}}$ for all $\lambda \geq 1, |\eta| \geq R$. Then $a \in S^\mu_{cl}(\Omega, \mathbf{R}^n; E, F)$, and the symbol semi-norms for a can be estimated in terms of the semi-norms for a in $C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(E,F))$.*

Definition 1.5. Let $\Omega \subseteq \mathbf{R}^q$ be open and $a \in S^\mu(\Omega \times \Omega, \mathbf{R}^q \times \mathbf{R}^l; E, F)$. The parameter-dependent pseudodifferential operator $\operatorname{op} a$ is the operator family $\{\operatorname{op} a(\lambda) : \lambda \in \mathbf{R}^l\}$ defined by

$$(\operatorname{op} a(\lambda) f)(y) = \int e^{i(y-\tilde{y})\eta} a(y, \tilde{y}, \eta, \lambda) f(\tilde{y}) d\tilde{y} đ\eta, \tag{1.4}$$

$f \in C^\infty_0(\Omega, E), y \in \Omega$. This reduces to $(\operatorname{op} a(\lambda) f)(y) = \int e^{iy\eta} a(y,\eta)\hat{f}(\eta) đ\eta$ for symbols that are independent of y'. Here, $\hat{f}(\eta) = \mathcal{F}_{y\to\eta} f(\eta) = \int e^{-iy\eta} f(y) dy$ is the vector-valued Fourier transform of f, and $đ\eta = (2\pi)^{-q} d\eta$.

Definition 1.6. Let E, κ be as in 1.1, $q \in \mathbf{N}, s \in \mathbf{R}$. The *wedge Sobolev space* $\mathcal{W}^s(\mathbf{R}^q, E)$ is the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi E$ in the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q, E)} = \left(\int [\eta]^{2s} \|\kappa_{[\eta]^{-1}} \mathcal{F}_{y\to\eta} u(\eta)\|^2_E d\eta \right)^{\frac{1}{2}}.$$

It is a subset of $\mathcal{S}'(\mathbf{R}^q, E)$. There are a few straightforward generalizations: If $\{E_k\}$ is a sequence of Banach spaces, $E_{k+1} \hookrightarrow E_k$, $E = \text{proj} - \lim E_k$, and the group action coincides on all spaces, we let $\mathcal{W}^s(\mathbf{R}^q, E) = \text{proj} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k)$. Similarly we treat inductive limits. For $\Omega \subseteq \mathbf{R}^q$ open we shall write $u \in \mathcal{W}^s_{comp}(\Omega, E)$, if there is a function $\varphi \in C^\infty_0(\Omega)$ such that $u = \varphi u$, and say $u \in \mathcal{W}^s_{loc}(\Omega, E)$, if $u \in \mathcal{D}'(\Omega, E)$ and $\varphi u \in \mathcal{W}^s(\mathbf{R}^q, E)$ for all $\varphi \in C^\infty_0(\mathbf{R}^q)$.

1.7 Elementary properties of wedge Sobolev spaces (see [9]).
(a) $\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^{q+1}_+)$.
(b) $\mathcal{W}^s(\mathbf{R}^q, H^s_0(\mathbf{R}_+)) = H^s_0(\mathbf{R}^{q+1}_+)$.
(c) $\mathcal{W}^s(\mathbf{R}^q, \mathbf{C}) = H^s(\mathbf{R}^q)$, using the trivial group action $\kappa_\lambda = id$.

Theorem 1.8. [19, Section 3.2.1] *Let a be as in Definition 1.5. Then*

$$\operatorname{op} a(\lambda) : \mathcal{W}^s_{comp}(\Omega, E) \longrightarrow \mathcal{W}^{s-\mu}_{loc}(\Omega, F)$$

is bounded for every $\lambda \in \mathbf{R}^l$. If a is independent of y and $\tilde{y}$, then we may omit the subscripts 'comp' and 'loc'. The mapping op : *symbol* $\mapsto$ *operator is continuous in the corresponding topologies for all $s \in \mathbf{R}$.*

Definition 1.9. Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The direct sum $E \oplus F$ is Fréchet and has the closed subspace $\mathcal{N} = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space $E + F := E \oplus F/\mathcal{N}$.

1.10 Boutet de Monvel's Algebra. Let X be an n-dimensional C^∞ manifold with boundary Y, embedded in an n-dimensional manifold G without boundary, all not necessarily compact. In the following we shall denote by X the open interior of X, while $\overline{X}$ denotes the closure. Let $V_1, V_2, \ldots,$ be vector bundles over G and let $W_1, W_2, \ldots,$ be vector bundles over Y.

Given $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$, we denote by $\mathcal{B}^{\mu,d}(X; \mathbf{R}^q)$ the Fréchet space of parameter-dependent elements of order μ and type d in Boutet de Monvel's calculus, acting between vector bundles in the usual way:

$$A(\lambda) : \begin{array}{c} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array}. \tag{1.5}$$

We write $\mathcal{B}_{cl}^{\mu,d}(X; \mathbf{R}^q)$ for the subspace of classical operators. The elements of $\mathcal{B}_{cl}^{\mu,d}(X; \mathbf{R}^q)$ have two principal symbols, namely the interior principal pseudodifferential symbol $\sigma_\psi^\mu(A)$ and the (operator-valued) boundary symbol $\sigma_\partial^\mu(A)$.

For an introduction to the parameter-dependent version of Boutet de Monvel's calculus see [13, Section 2]; short accounts were given in [14] and [15]. In [14], the principal boundary symbol was denoted $\sigma_\wedge^\mu$. The elements of Boutet de Monvel's calculus form an algebra in the following sense:

Proposition 1.11. [11, Section 2.3.3.2] *Let $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^q)$, $B \in \mathcal{B}^{\mu',d'}(X; \mathbf{R}^q)$, and $\alpha, \beta \in \mathbf{C}$. Then*
(a) $\alpha A + \beta B \in \mathcal{B}^{\mu'',d''}(X; \mathbf{R}^q)$ *for* $\mu'' = \max\{\mu, \mu'\}$, $d'' = \max\{d, d'\}$.
(b) $A \circ B \in \mathcal{B}^{\mu'',d''}(X; \mathbf{R}^q)$ *for* $\mu'' = \{\mu + \mu'\}$, $d'' = \max\{\mu' + d, d'\}$.
We assume here that the vector bundles A and B act on are such that the addition and composition make sense.

Example 1.12. The Dirichlet problem $\begin{pmatrix} \Delta \\ \gamma_0 \end{pmatrix}$ is an operator in Boutet de Monvel's calculus of order 2 and type 1. In fact, the Laplacian Δ is a differential operator of order 2, while according to Example 1.2, the operator of evaluation at the boundary, γ_0, is an operator-valued symbol in $S^{1/2}(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}; H^s(\mathbf{R}_+), \mathbf{C})$, provided $s > 1/2$; it is well-known to be of type 1. The Dirichlet problem is independent of any parameter, but since it is a *differential* boundary value problem, we may also consider it as a parameter-dependent element. Since the order of γ_0 only is $1/2$, we may even replace γ_0 by $\Lambda\gamma_0$, where Λ is a (parameter-dependent) order reduction of order $3/2$, and still have order 2.

Here, the vector bundle W_1 is zero, while V_1, V_2, W_2 can be taken trivial one-dimensional.

Wedge Sobolev Spaces

We use the notation G, X, Y of 1.10, but from now on we assume G, X, and Y to be compact. Let $G^\wedge = G \times \mathbf{R}_+$, $X^\wedge = X \times \mathbf{R}_+$, $Y^\wedge = Y \times \mathbf{R}_+$.

1.13 Parameter-dependent order reductions on G. For each $\mu \in \mathbf{R}$ there is a pseudodifferential operator Λ^μ with local parameter-dependent elliptic symbols of order μ, depending on the parameter $\tau \in \mathbf{R}$, such that

$$\Lambda^\mu(\tau) : H^s(G, V) \to H^{s-\mu}(G, V)$$

is an isomorphism for all τ.

One can construct such an operator for example starting from symbols of the form $[(\xi, \tau, C)]^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n_\xi; \mathbf{R}_\tau)$ with a large constant $C > 0$ and patching them together to an operator on the manifold G.

Definition 1.14. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$. The Mellin transform Mu of a $C_0^\infty(\mathbf{R}_+)$-function u is

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt. \tag{1.6}$$

M extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{1/2})$. Of course, (1) also makes sense for functions with values in a Fréchet space E. The fact that $Mu|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ for $u \in C_0^\infty(\mathbf{R}_+)$ motivates the definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+, E).$$

For a Hilbert space E, the inverse of M_γ is given by $(M_\gamma^{-1}h)(z) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z}h(z)dz$.

1.15 Totally characteristic Sobolev spaces. [13, Section 3.1] (a) Let $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ be a family of parameter-dependent order reductions as in 1.13. For

$s, \gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(G^\wedge)$ is the closure of $C_0^\infty(G^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(G^\wedge)} = \left\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z) Mu(z)\|^2_{L^2(G)} \, |dz| \right\}^{1/2}. \tag{1.7}$$

Recall that n is the dimension of X and G. The space $\mathcal{H}^{s,\gamma}(G^\wedge)$ is independent of the particular choice of the order reducing family.

(b) For $s = l \in \mathbf{N}$ we obtain the alternative description

$$u \in \mathcal{H}^{l,\gamma}(G^\wedge) \quad \text{iff} \quad t^{n/2-\gamma}(t\partial_t)^k Du(x,t) \in L^2(G^\wedge)$$

for all $k \leq l$ and all differential operators D of order $\leq l - k$ on G, cf. [19, Section 2.1.1, Proposition 2].

(c) We let $\mathcal{H}^{s,\gamma}(X^\wedge) = \{f|_{X^\wedge} : f \in \mathcal{H}^{s,\gamma}(G^\wedge)\}$, endowed with the quotient norm: $\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \inf\{\|f\|_{\mathcal{H}^{s,\gamma}(G^\wedge)} : f \in \mathcal{H}^{s,\gamma}(G^\wedge), f|_{X^\wedge} = u\}$.

(d) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$, where the subscript '*loc*' refers to the t-variable only. Moreover, $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2} L^2(X^\wedge)$.

(e) $\mathcal{H}^{0,0}(X^\wedge)$ has a natural inner product

$$(u, v)_{\mathcal{H}^{0,0}(X^\wedge)} = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)} \, dz.$$

(f) If φ is the restriction to $X^\wedge$ of a function in $\mathcal{S}(G \times \mathbf{R}) = \mathcal{S}(\mathbf{R}, C^\infty(G))$, then the operator M_φ of multiplication by φ, $M_\varphi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$, is bounded for all $s, \gamma \in \mathbf{R}$, and the mapping $\varphi \mapsto M_\varphi$ is continuous in the corresponding topology.

1.16 The spaces H^s_{cone}. Let $\{G_j\}_{j=1}^J$ be a finite covering of G by open sets, $\kappa_j : G_j \to U_j$ the coordinate maps onto bounded open sets in $\mathbf{R}^n$, and $\{\varphi_j\}_{j=1}^J$ a subordinate partition of unity. The maps κ_j induce a push-forward of functions and distributions: For a function u on G_j

$$(\kappa_{j*}u)(x) = u(\kappa_j^{-1}(x)), \quad x \in U_j; \tag{1.8}$$

for a distribution u ask that $(\kappa_{j*}u)(\varphi) = u(\varphi \circ \kappa_j)$, $\varphi \in C_0^\infty(U_j)$. For $j = 1, \ldots, J$, consider the diffeomorphism

$$\chi_j : U_j \times \mathbf{R} \to \{(x[t], t) : x \in U_j, t \in \mathbf{R}\} =: C_j \subset \mathbf{R}^{n+1}$$

given by $\chi_j(x,t) = (x[t], t)$. Its inverse is $\chi_j^{-1}(y,t) = (y/[t], t)$. For $s \in \mathbf{R}$ we define $H^s_{cone}(G \times \mathbf{R})$ as the set of all $u \in H^s_{loc}(G \times \mathbf{R})$ such that, for

$j = 1, \ldots, J$, the push-forward $(\chi_j \kappa_j)_*(\varphi_j u)$, which may be regarded as a distribution on $\mathbf{R}^{n+1}$ after extension by zero, is an element of $H^s(\mathbf{R}^{n+1})$. The space $H^s_{cone}(G \times \mathbf{R})$ is endowed with the corresponding Hilbert space topology. We let

$$H^s_{cone}(X^\wedge) \quad = \quad \{u|_{X \times \mathbf{R}_+} : u \in H^s_{cone}(G \times \mathbf{R})\}.$$

For more details see Schrohe&Schulze [14, Section 4.2]. The subscript "cone" is motivated by the fact that, away from zero, these are the Sobolev spaces for an infinite cone with center at the origin and cross-section X. In particular, the space $H^s_{cone}(S^n \times \mathbf{R}_+)$ coincides with $H^s(\mathbf{R}^{n+1} \setminus \{0\})$ outside a neighborhood of zero.

Definition 1.17. For $s, \gamma \in \mathbf{R}$ and $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(r) \equiv 1$ near $r = 0$, let

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{u \in \mathcal{D}'(X^\wedge) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)\, u \in H^s_{cone}(X^\wedge)\}. \qquad (1.9)$$

The definition is independent of ω by 1.15(f). We endow $\mathcal{K}^{s,\gamma}(X^\wedge)$ with the Banach space topology $\|u\|_{\mathcal{K}^{s,\gamma}(X^\wedge)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega)u\|_{H^s_{cone}(X^\wedge)}$. In fact, this is a Hilbert topology with the inner product inherited from $\mathcal{H}^{s,\gamma}$ and H^s_{cone}. By 1.15(d), $\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = t^{-n/2}L^2(X^\wedge)$.

Theorem 1.18. *For $s > 1/2$ and $\gamma \in \mathbf{R}$ the restriction $\gamma_0 u = u|_{Y^\wedge}$ of u to $Y^\wedge$ induces a continuous operator $\gamma_0 : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)$.*

By r denote the normal coordinate in a neighborhood of Y. Then the operators $\gamma_j : u \mapsto \partial_r^j u|_{Y^\wedge}$ define continuous mappings $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-j-1/2,\gamma-1/2}(Y^\wedge)$.

This can be deduced from the trace theorem for the usual Sobolev spaces. The shift in the weight $\gamma \mapsto \gamma - 1/2$ is due to the fact that $\dim Y = n - 1$.

The lemma, below, is lenghty but straigthforward to prove.

Lemma 1.19. *A strongly continuous group action κ_λ can be defined on $\mathcal{K}^{s,\gamma}(X^\wedge)$ by*

$$(\kappa_\lambda f)(x,t) = \lambda^{\frac{n+1}{2}} f(x, \lambda t), \quad f \in \mathcal{K}^{s,\gamma}(X^\wedge),\ s \geq 0.$$

This action is unitary on $\mathcal{K}^{0,0}(X^\wedge)$. It naturally extends to distributions in $\mathcal{K}^{s,\gamma}(X^\wedge)$, $s, \gamma \in \mathbf{R}$.

Remark 1.20. The definitions of the spaces $\mathcal{H}^{s,\gamma}$ and $\mathcal{K}^{s,\gamma}$ also make sense for functions and distributions taking values in a vector bundle V. We shall then write $\mathcal{H}^{s,\gamma}(X^\wedge, V)$ and $\mathcal{K}^{s,\gamma}(X^\wedge, V)$, respectively. In later constructions we will often have to deal with direct sums $\mathcal{K}^{s,\gamma}(X^\wedge, V) \oplus \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge, W)$ for vector bundles V and W over X and Y, respectively. On these spaces we use the group action $\kappa_\lambda(u, v) = (\lambda^{\frac{n+1}{2}} u(\cdot, \lambda\cdot), \lambda^{\frac{n}{2}} v(\cdot, \lambda\cdot))$.

Proposition 1.21. [15, Theorem 2.12] *For all $s > 1/2$, the restriction operator γ_0 induces a continuous map*

$$\gamma_0 : \mathcal{W}^s(\mathbf{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^{s-1/2}(\mathbf{R}^q, \mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)).$$

Proposition 1.22. [15, Proposition 2.13] *Let $\varphi \in \mathcal{S}(\overline{X}^\wedge \times \mathbf{R}^q)$. Then the operator of multiplication by φ furnishes a bounded operator on $\mathcal{W}^s(\mathbf{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ for all $s, \gamma \in \mathbf{R}$. Its norm depends continuously on the semi-norms for φ in $\mathcal{S}(\overline{X}^\wedge \times \mathbf{R}^q)$.*

Operator-Valued Mellin Symbols

Convention: In the following we fix $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$. Whenever we write $\omega, \tilde{\omega}, \omega_1, \ldots,$ without further specification or refer to a function as a *cut-off function* we mean an element of $C_0^\infty(\overline{\mathbf{R}}_+)$ which is equal to one near the origin.

Definition 1.23. (a) $M_O^{\mu,d}(X; \mathbf{R}^q)$ is the space of all $a \in \mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X; \mathbf{R}^q))$ such that, for all $c_1 < c_2$ in $\mathbf{R}$,

$$a(\beta + i\tau) \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^q \times \mathbf{R}_\tau), \tag{1.10}$$

uniformly for all $\beta \in [c_1, c_2]$. We call the elements of $M_O^{\mu,d}(X; \mathbf{R}^q)$ holomorphic Mellin symbols of order μ and type d. We are assuming that the vector bundles $a(z)$ is acting on are independent of z.

The topology of $M_O^{\mu,d}(X; \mathbf{R}^q)$ is given by the semi-norm systems for the topology of $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X; \mathbf{R}^q))$ and, for families $\{a_\beta : \beta \in \mathbf{R}\}$, the topology of uniform convergence on compact subsets of $\mathbf{R}_\beta$ in $\mathcal{B}^{\mu,d}(X; \mathbf{R}^q \times \mathbf{R}_\tau)$. Clearly, $M_O^{\mu,d}(X; \mathbf{R}^q)$ is a Fréchet space with this topology.

(b) $M_{O,cl}^{\mu,d}(X; \mathbf{R}^q)$ is the corresponding space with $\mathcal{B}^{\mu,d}(X; \mathbf{R}^q)$ replaced by $\mathcal{B}_{cl}^{\mu,d}(X; \mathbf{R}^q)$.

Example 1.24. Let $A_k \in \mathcal{B}^{\mu-k,d}(X)$, $k = 0, \ldots, \mu$, be *differential* boundary value problems. Then $a(z) = \sum_{k=0}^{\mu} A_k z^k \in M_O^{\mu,d}(X; \mathbf{R}^q)$.

1.25 Mellin symbols and operators. Let $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$. For each fixed $(t, t', z) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \Gamma_{1/2-\gamma}$, we have a boundary value problem

$$f(t, t', z) : \begin{matrix} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{matrix} \rightarrow \begin{matrix} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{matrix}$$

in Boutet de Monvel's calculus.
For $u \in C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(\overline{Y}^\wedge, W_1) = C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$ we define the Mellin operator $\mathrm{op}_M^\gamma f$ by

$$\mathrm{op}_M^\gamma(f)u(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}}\int_0^\infty (t/t')^{-z} f(t,t',z)u(t')dt'/t'dz.$$

If f is independent of t', then $\mathrm{op}_M^\gamma(f)u(t) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z} f(t,z)Mu(z)dz$.
It is easy to see the continuity of

$$\mathrm{op}_M^\gamma(f): \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array}.$$

For $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$ we obtain a bounded extension

$$\omega_1 \mathrm{op}_M^\gamma(f)\omega_2: \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array} \tag{1.11}$$

provided $s > d - 1/2$. A proof is given in [14, Proposition 2.1.5].

Definition 1.26. In the following we shall use the abbreviation

$$\mathcal{K}_j^{s,\gamma} = \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_j) \oplus \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_j) \quad j = 1, 2, \ldots .$$

The following proposition follows immediately from Proposition 1.11.

Proposition 1.27. *Given $\mu, \mu' \in \mathbf{Z}$ and $d, d' \in \mathbf{N}$, let $\mu'' = \mu + \mu'$ and $d'' = \max\{\mu' + d, d'\}$. Then there is a continuous multiplication*

$$M_O^{\mu,d}(X;\mathbf{R}^q) \times M_O^{\mu',d'}(X;\mathbf{R}^q) \to M_O^{\mu'',d''}(X;\mathbf{R}^q)$$

given by the pointwise composition in Boutet de Monvel's calculus: $(a,b) \mapsto c$ with $c(z,\eta) = a(z,\eta) \circ b(z,\eta)$.

1.28 Theorem: Operator-valued Mellin symbols. ([15, Corollary 3.9]) *Let $\gamma \in \mathbf{R}$, $\Omega \subseteq \mathbf{R}^q$, and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q))$. Recall that $[\cdot]$ is a smooth positive function on $\mathbf{R}^q$ coinciding with $|\cdot|$ outside a neighborhood of zero and define*

$$a(y,\eta) = \omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^\gamma(f(t,y,z,t\eta))\omega_2(t[\eta]).$$

By (1.11) *this furnishes an* $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$*-valued function* a *on* $\Omega \times \mathbf{R}^q$ *for all* $s > d - 1/2$. *Moreover,* $a \in S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$.
We then deduce from Theorem 1.8 that the operator

$$\operatorname{op} a : \mathcal{W}^s_{comp}(\Omega, \mathcal{K}_1^{s,\gamma}) \to \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}_2^{s-\mu,\gamma-\mu})$$

is continuous for all $s > d - 1/2$.

Lemma 1.29. [15, Lemma 3.11] *We use the above notation and let* $\beta \in \mathbf{R}$. *Then*

$$\omega_1(t[\eta])\operatorname{op}_M^\gamma(f(t,y,z,t\eta))\omega_2(t[\eta])t^\beta = \omega_1(t[\eta])t^\beta \operatorname{op}_M^{\gamma-\beta}(T^{-\beta}f(t,y,z,t\eta))\omega_2(t[\eta]).$$

In case f *even is an element in* $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$ *we also have*

$$\omega_1(t[\eta])\operatorname{op}_M^\gamma(f(t,y,z,t\eta))\omega_2(t[\eta])t^\beta = \omega_1(t[\eta])t^\beta \operatorname{op}_M^{\gamma}(T^{-\beta}f(t,y,z,t\eta))\omega_2(t[\eta]).$$

Here we consider both sides as operators on $C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}))$; $T^{-\beta}$ *is the translation operator defined by* $T^{-\beta}f(t,y,z,t\eta) = f(t,y,z-\beta,t\eta)$.

Mellin Quantization and Kernel Cut-Off

Definition 1.30. A symbol $p = p(t,y,\tau,\eta)$ in $C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau \times \mathbf{R}^q_\eta))$ is called *edge-degenerate,* if there is a symbol $\tilde{p}$ in $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau \times \mathbf{R}^q_\eta))$ with $p(t,y,\tau,\eta) = \tilde{p}(t,y,t\tau,t\eta)$.

Given an edge-degenerate symbol we can find a Mellin symbol which induces the same operator up to a smoothing perturbation and vice versa. This is the contents of the following assertion, proven in [15, Theorems 3.17, 3.19].

Theorem 1.31. *Let* $p \in C^\infty(\mathbf{R}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R} \times \mathbf{R}^q_\eta))$ *be edge-degenerate and* $\gamma \in \mathbf{R}$. *Then there is an* $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q_\eta))$ *with*

$$\operatorname{op}_t p(t,y,\tau,\eta) = \operatorname{op}_M^\gamma f_\gamma(t,y,i\tau,t\eta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q)). \quad (1.12)$$

Conversely, given $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma} \times \mathbf{R}^q_\eta))$, *there is an edge-degenerate boundary value problem* p *such that relation* (1.12) *holds.*
The same statement holds for classical symbols, i.e., for $\mathcal{B}^{\mu,d}$ *replaced by* $\mathcal{B}^{\mu,d}_{cl}$.
Kernel cut-off is a simple way to switch from an arbitrary Mellin symbol to a holomorphic Mellin symbol, up to a smoothing error. The proof of the theorem, below, was given in [15, Theorems 3.20, 3.21].

Theorem 1.32. *Let* $f \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q))$*, choose* $\varphi \in C_0^\infty(\mathbf{R}_+)$ *and* $\psi \in C_0^\infty(\mathbf{R}_+)$ *with* $\psi(\rho) \equiv 1$ *near* $\rho = 1$*. Then the operator-valued function* f_φ *defined by*

$$f_\varphi(t,y,z,\eta) = M_{\rho\to z}\varphi(\rho)M^{-1}_{1/2,\zeta\to\rho}f(t,y,\zeta,\eta)$$

is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$*, while* $f_{1-\psi}$ *defined by*

$$f_{1-\psi}(t,y,z,\eta) = M_{\rho\to z}(1-\psi(\rho))M^{-1}_{1/2,\zeta\to\rho}f(t,y,\zeta,\eta)$$

is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{-\infty,d}(X;\Gamma_0 \times \mathbf{R}^q))$*. Moreover, the mapping* $(\varphi, f) \mapsto f_\varphi$ *is separately continuous* $C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q)) \to C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$*, and* $(\psi, f) \mapsto f_{1-\psi}$ *is separately continuous* $C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X;\Gamma_0 \times \mathbf{R}^q)) \to C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{-\infty,d}(X;\Gamma_0 \times \mathbf{R}^q))$*.*

Notice that upon starting with a holomorphic Mellin symbol, kernel cut-off with a function ψ satisfying $\psi(\rho) \equiv 1$ near $\rho = 1$ produces the same symbol up to an error which is regularizing and holomorphic:

Theorem 1.33. ([15, Theorems 3.29]) *Given* $h \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$ *and* $\psi \in C_0^\infty(\mathbf{R}_+)$ *with* $\psi(\rho) \equiv 1$ *near* $\rho = 1$*, the difference* $h - h_\psi$ *is an element of* $C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{-\infty,d}(X;\mathbf{R}^q))$*.*

2 Operator-Valued Edge Symbols

In this section we shall first analyze the behavior of edge-degenerate pseudodifferential operators on cone Sobolev spaces, then we shall focus on Green symbols with trivial asymptotics.

Parameter-Dependent Boundary Value Problems on Cone Sobolev Spaces

Theorem 2.1. *For* $p \in C^\infty(\Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}^{1+q}_{\tau,\eta})$ *and an excision function* $\zeta \in C^\infty(\mathbf{R}^q)$ *vanishing near zero and equal to* 1 *near infinity define*

$$a(y,\eta) = \zeta(\eta)(1-\omega(t[\eta])\mathrm{op}_t(t^{-\mu}p(y,t\tau,t\eta))(1-\omega_1(t[\eta])).$$

Then $a \in S^\mu_{cl}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ *whenever* $s > d - 1/2$*. The symbol estimates for* a *depend continuously on those for* p*.*

The subscript t with op indicates that the action is with respect to this variable only. Note that p is assumed to be independent of t; the covariable associated with t is τ.

2.2 Outline. The proof of the theorem is rather long and the full details will be given elsewhere. In order to keep the exposition transparent let us sketch the following steps leading to the conclusion. For simplicity let us assume that V_1 and V_2 are trivial one-dimensional while W_1, W_2 vanish.

Step 1. Suppose we know that, for $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^{1+q})$ and fixed $\eta \in \mathbf{R}^q \setminus \{0\}$,

$$(1-\omega(t[\eta]))\mathrm{op}_t(t^{-\mu}A(t\tau,t\eta))(1-\omega_1(t[\eta])) \in \mathcal{L}(H^s_{cone}(X^\wedge), H^{s-\mu}_{cone}(X^\wedge)), \quad (2.1)$$

$s > d - 1/2$, and that the associated mapping is continuous. Whenever $|\eta|$ is large, a is homogeneous in η of degree μ in the sense of (1.3). Indeed, let $|\eta|$ be so large that $[\eta] = |\eta|$ and let $u \in C_0^\infty(\mathbf{R}_+, C^\infty(X))$. Then, in the notation of 2.1,

$$\begin{aligned} & \kappa_\lambda\{a(y,\eta)\kappa_{\lambda^{-1}}u\}(t) && (2.2)\\ = \; & \kappa_\lambda\{\lambda^{-(n+1)/2}(1-\omega(t|\eta|))t^{-\mu} \\ & \times \int e^{i(t-t')\tau}p(y,t\tau,t\eta)(1-\omega(t'|\eta|))u(\lambda^{-1}t')dt'd\tau\}(t) \\ = \; & (1-\omega(t|\lambda\eta|))(\lambda t)^{-\mu}\int e^{i(t-s')\lambda\tau}p(y,t\lambda\tau,t\eta)(1-\omega_1(s'|\lambda\eta|))u(s')\lambda ds'd\tau \\ = \; & \lambda^{-\mu}a(y,\lambda\eta)u(t). \end{aligned}$$

If (2.1) holds then we know that, for each fixed choice of y and $\eta \neq 0$, the operator $a(y,\eta)$ is an element of $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$. Via the excision function ζ we also cover the case $\eta = 0$; we obtain that $a \in C^\infty(\Omega\times\mathbf{R}^q, \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu}))$, since the mapping from $\mathcal{B}^{\mu,d}(X;\mathbf{R}^{1+q})$ to $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ given via (2.1) is continuous. So Lemma 1.4 gives the assertion.

Step 2. We are now reduced to showing (2.1). Since the normal derivative induces a bounded operator $H^s_{cone}(X^\wedge) \to H^{s-1}_{cone}(X^\wedge)$, linearity allows us to assume $d = 0$. A parameter-dependent element $A \in \mathcal{B}^{\mu,0}(X;\mathbf{R}^{1+q})$ is given as a finite sum of terms induced by local symbols supported arbitrarily close to the diagonal, plus a term which is an integral operator with a smooth kernel over $\overline{X} \times \overline{X}$, rapidly decreasing with respect to (τ,η), cf. 1.10.

Step 3. Suppose $r \in \mathcal{S}(\mathbf{R}^{1+q}, C^\infty(\overline{X} \times \overline{X}))$. Then the formula

$$K_\eta u(x,t) = \int_{\mathbf{R}}\int_{\mathbf{R}_+}\int_X e^{i(t-t')\tau}t^{-\mu}r(t\tau,t\eta,x,x')u(x',t')dx'dt'd\tau \quad (2.3)$$

defines an element K_η of $\mathcal{L}(H^s_{cone}(X^\wedge), H^{s'}_{cone}(X^\wedge))$ for each choice of $s, s' \in \mathbf{R}$, depending smoothly on $\eta \neq 0$. In order to see this, consider the integral kernel

$k_\eta(x,t,x',t') = \int e^{i(t-t')\tau} r(t\tau, t\eta, x, x')d\tau$, reduce the task to the L^2-case, and apply Schur's lemma.

Step 4. Next we consider the local terms. Let U be a coordinate neighborhood for X, and let $q = q(x, x', \xi, \tau, \eta) \in S^\mu_{tr}(U \times U, \mathbf{R}^{n+1+q}_{\xi,\tau,\eta})$ be a pseudodifferential symbol with the transmission property. Boundedness on cone Sobolev spaces corresponds to boundedness on the usual Sobolev spaces under via the push-forward under the mapping $(x,t) \mapsto (x[t], t)$. We may compute explicitly the push-forward on the symbol level. For $s > -1/2$ we then obtain

$$\{\mathrm{op}^+_{x,t} t^{-\mu} q(x, x', \xi, t\tau, t\eta) : \eta \in \mathbf{R}^q\} \subseteq \mathcal{L}(H^s_{cone}(X^\wedge), H^{s-\mu}_{cone}(X^\wedge)),$$

depending smoothly on η.

A corresponding result holds for the singular Green part: Let $\tilde{U} \subseteq \mathbf{R}^{n-1}$ be open and $g \in S^\mu(\tilde{U} \times \tilde{U}, \mathbf{R}^{n+q}_{\tilde{\xi},\tau,\eta}; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$. Then we get a family of bounded operators

$$\{\mathrm{op}_{\tilde{x},t} t^{-\mu} g(\tilde{x}, \tilde{x}', \tilde{\xi}, t\tau, t\eta) : \eta \in \mathbf{R}^q\} \subseteq \mathcal{L}(H^s_{cone}(X^\wedge), H^{s-\mu}_{cone}(X^\wedge))$$

depending smoothly on η.

Green Symbols with Trivial Asymptotics

In the following let Ω denote an open set in $\mathbf{R}^q$, $\mu \in \mathbf{Z}$, and $d \in \mathbf{N}$, while $\mathbf{g}$ is the weight datum $\mathbf{g} = (\gamma + n/2, \delta + n/2, (-k, 0])$; here $\gamma, \delta \in \mathbf{R}$, $0 < k \in \mathbf{N}$.

Definition 2.3. Given $\gamma \in \mathbf{R}$ and the integer k in the weight datum $\mathbf{g}$ we let $\mathcal{S}^\gamma_O(X^\wedge)$ denote the space of all functions f on $X^\wedge$ such that, for all $c < k$ and every cut-off function ω, we have $\omega f \in \mathcal{H}^{\infty,\gamma+c}(X^\wedge)$ and $(1-\omega)f \in \mathcal{S}(X^\wedge)$. Similarly, for $f \in \mathcal{S}^{\gamma-1/2}_O(Y^\wedge)$ we require that $\omega f \in \mathcal{H}^{\infty,\gamma-1/2+c}(Y^\wedge)$ and $(1-\omega)f \in \mathcal{S}(Y^\wedge)$. The notation carries over to functions taking values in the vector bundles $V_1, V_2, \ldots,$ over $\overline{X}$ and $W_1, W_2, \ldots,$ over Y. Following Definition 1.26 we now set

$$\mathcal{S}^\gamma_{j,O} = \mathcal{S}^{\gamma+\frac{n}{2}}_O(X^\wedge, V_j) \oplus \mathcal{S}^{\gamma+\frac{n-1}{2}}_O(Y^\wedge, W_j), \quad j = 1, 2, \ldots. \tag{2.4}$$

The spaces $\mathcal{S}^\gamma_{j,O}$ are Fréchet spaces with the canonical topology of a non-direct sum of Fréchet spaces. Moreover, it is easily seen that they may be written as projective limits of suitable Hilbert spaces, cf. e.g. [16, Lemma 1.20].

Definition 2.4. (a) $\mathcal{R}^{\mu,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ is the space of all operator-valued symbols

$$g = g(y, y', \eta) \in \bigcap_{s > -1/2} S^\mu_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}^{s,\gamma}_1, \mathcal{K}^{s-\mu,\delta}_2)$$

with the following property: For each $s > -1/2$, the symbol g yields an element of $S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{S}^{\delta}_{2,O})$, while the pointwise formal adjoint g^*, defined by $g^*(y, y'\eta) = g(y, y', \eta)^*$, yields an element of $S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_2^{s,-\delta}, \mathcal{S}^{-\gamma}_{1,O})$.

(b) $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ is the space of all operator-valued symbols

$$g \in \bigcap_{s > d-1/2} S^{\mu}_{cl}(\Omega \times \Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\delta})$$

which can be written in the form $g = g_0 + \sum_{j=1}^d g_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & 0 \end{bmatrix}$ with $g_j \in \mathcal{R}^{\mu-j,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$. The matrix refers to the decompositions of the spaces as in (2.4). The space $\mathcal{R}^{\mu,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ clearly is a Fréchet space, $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ is topologized as a non-direct sum of Fréchet spaces.

Definition 2.4 is a special case of [16, Definition 2.2]. For the present purposes we need neither the asymptotics nor the trace/potential contributions from the boundary. We collect a few basic results, see [16, Proposition 2.4, 2.5, 2.6].

Proposition 2.5. *Let* $g_1 \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ *and* $g_2 \in \mathcal{R}^{\mu',d'}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$. *Then*

(a) $D^{\alpha}_{\eta} D^{\beta}_{y,y'} g_1 \in \mathcal{R}^{\mu-|\alpha|,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

(b) *The pointwise composition* $g_1 g_2$ *is an element of* $\mathcal{R}^{\mu+\mu',d'}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

(c) *If* $d = 0$, *then the pointwise adjoint is an element of* $\mathcal{R}^{\mu,0}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

(d) *Given* $g_j \in \mathcal{R}^{\mu_j,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ *with* $\mu_0 > \mu_1 \ldots \to -\infty$, *there is a* $g \in \mathcal{R}^{\mu_0,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ *with* $g \sim \sum_{j=0}^{\infty} g_j$.

(e) *For* $\nu_1, \nu_2 \in \mathbf{N}$ *we have* $t^{\nu_2} g_1 t^{\nu_1} \in \mathcal{R}^{\mu-\nu_1-\nu_2,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

(f) *For* $\varphi \in \mathcal{S}(\overline{\mathbf{R}}_+)$ *the symbols* φg_1, $g_1 \varphi$, $\varphi(\cdot[\eta]) g_1$, *and* $g_1 \varphi(\cdot[\eta])$ *all are elements of* $\mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

In (e) we understand t^{ν_1} as the operator of multiplication by the diagonal matrix diag $\{t^{\nu_1}, t^{\nu_1}\}$; a similar interpretation applies to t^{ν_2} and φ in (f), while $\varphi(\cdot[\eta])$ is the corresponding η-dependent multiplier.

The following theorem is immediate from Theorem 1.8. It motivates the definition of the corresponding space of operators.

Theorem 2.6. *Let* $g \in \mathcal{R}^{\mu,d}_G(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$. *Then*

$$\operatorname{op} g : \mathcal{W}^s_{comp}(\Omega, \mathcal{K}_1^{s,\gamma}) \to \mathcal{W}^{s-\mu}_{loc}(\Omega, \mathcal{K}_2^{\infty,\delta})$$

is continuous for all $s > d - 1/2$. *In fact the result also holds for* δ *replaced by* $\delta + k - \varepsilon$, *whenever* $\varepsilon > 0$ *and* k *is the integer in the weight datum.*

Definition 2.7. $Y_G^{\mu,d}(\Omega \times X^\wedge, \mathbf{g})_{O,O}$ is the space of all operators of the form

$$G = \operatorname{op} g + \sum_{j=0}^{d} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}, \tag{2.5}$$

where $g \in \mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$, and, for each $j = 0, \ldots, d$, $s > -1/2$, and $c < k$, the operators G_j and their formal adjoints G_j^* yield continuous maps

$$G_j \ : \ \mathcal{W}^s_{comp}(\Omega, \mathcal{K}_1^{s,\gamma}) \to \mathcal{W}^\infty_{loc}(\Omega, \mathcal{K}_2^{\infty,\delta+c}) \quad \text{and} \tag{2.6}$$

$$G_j^* \ : \ \mathcal{W}^s_{comp}(\Omega, \mathcal{K}_2^{s,-\delta}) \to \mathcal{W}^\infty_{loc}(\Omega, \mathcal{K}_1^{\infty,-\gamma+c}). \tag{2.7}$$

We let $Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O} = \bigcap_\mu Y_G^{\mu,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$.

Remark 2.8. $Y_G^{\mu,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$ is a Fréchet space with the topologies inherited from $\mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ and from properties (2.5), (2.6), and (2.7).

Proposition 2.9. *Let $g \in \mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$. Then there exists a left symbol $g_L = g_L(y,\eta)$ such that*

$$\operatorname{op} g \equiv \operatorname{op} g_L \mod Y^{-\infty}(X^\wedge \times \Omega, \mathbf{g})_{O,O}.$$

Similarly there is a right symbol $g_R = g_R(y',\eta)$ such that

$$\operatorname{op} g \equiv \operatorname{op} g_R \mod Y^{-\infty}(X^\wedge \times \Omega, \mathbf{g})_{O,O}.$$

We have the asymptotic expansions

$$g_L(y,\eta) \ \sim \ \sum_\alpha \frac{1}{\alpha!} D_{y'}^\alpha \partial_\eta^\alpha g(y,y',\eta)|_{y'=y}, \tag{2.8}$$

$$g_R(y',\eta) \ \sim \ \sum_\alpha \frac{(-1)^{|\alpha|}}{\alpha!} D_y^\alpha \partial_\eta^\alpha g(y,y',\eta)|_{y=y'}, \tag{2.9}$$

Proof. Proceed just as in the standard case. □

Corollary 2.10. *Let $\varphi_1, \varphi_2 \in C_0^\infty(\Omega)$ with* $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 = \emptyset$, *and let $G \in Y_G^{\mu,d}(\Omega \times X^\wedge, \mathbf{g})_{O,O}$. Then $\varphi_1 G \varphi_2 \in Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$.*
Here we consider φ_1 and φ_2 as the operators of multiplication by the corresponding functions.

Proof. Let $G = \operatorname{op} g + G_0$ with $G_0 \in Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$ and $g \in \mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$. Then the mapping properties show that $\varphi_1 G_0 \varphi_2 \in Y^{-\infty,d}(X^\wedge \times$

$\Omega, \mathbf{g})_{O,O}$, while $\varphi_1[\mathrm{op}\, g]\varphi_2 = \mathrm{op}\, \tilde{g}$ with $\tilde{g}(y, y', \eta) = \varphi_1(y) g(y, y', \eta)\varphi_2(y')$. We conclude from Proposition 2.9, in particular (2.8), that $\varphi_1[\mathrm{op}\, g]\varphi_2 \in Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$. □

Theorem 2.11. *Let* $\mathbf{g}_1 = (\gamma + n/2, \delta + n/2, (-k, 0]), \mathbf{g}_2 = (\delta + n/2, \sigma + n/2, (-k, 0])$, *and* $\mathbf{g}_3 = (\gamma + n/2, \sigma + n/2, (-k, 0])$, *be weight data. Choose* $\varphi \in C_0^\infty(\Omega)$. *Since* φ *maps* $\mathcal{W}^s_{loc}(\Omega, \mathcal{K}_j^{s,\gamma})$ *to* $\mathcal{W}^s_{comp}(\Omega, \mathcal{K}_j^{s,\gamma})$ *for every choice of* s *and* γ, *the composition* $(G_2, G_1) \mapsto G_2\varphi G_1$ *is defined; it induces a continuous mapping*

$$Y_G^{\mu,d}(X^\wedge \times \Omega, \mathbf{g}_2)_{O,O} \times Y_G^{\mu',d'}(X^\wedge \times \Omega, \mathbf{g}_1)_{O,O} \to Y_G^{\mu+\mu',d'}(X^\wedge \times \Omega, \mathbf{g}_3)_{O,O}$$

and has continuous restrictions

$$\begin{aligned} Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g}_2)_{O,O} \times Y_G^{\mu',d'}(X^\wedge \times \Omega, \mathbf{g}_1)_{O,O} &\to Y^{-\infty,d'}(X^\wedge \times \Omega, \mathbf{g}_3)_{O,O}, \\ Y_G^{\mu,d}(X^\wedge \times \Omega, \mathbf{g}_2)_{O,O} \times Y^{-\infty,d'}(X^\wedge \times \Omega, \mathbf{g}_1)_{O,O} &\to Y^{-\infty,d'}(X^\wedge \times \Omega, \mathbf{g}_3)_{O,O}. \end{aligned}$$

Proof. The mapping properties of the elements in $Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$ immediately yield the last two relations. So we may assume that $G_j = \mathrm{op}\, g_j, j = 1, 2$ with $g_1 \in \mathcal{R}_G^{\mu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}_1)_{O,O}$ and $g_2 \in \mathcal{R}_G^{\mu,d}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}_2)_{O,O}$. In view of Corollary 2.10 we find a left symbol g_L for g_1 and right symbol g_R for $\varphi(y) g_2(y, y', \eta)$. Then $g_L g_R \in \mathcal{R}_G^{\mu+\mu',d'}(\Omega \times \Omega \times \mathbf{R}^q, \mathbf{g}_3)_{O,O}$ by Proposition 2.5, and

$$[\mathrm{op}\, g_2]\varphi[\mathrm{op}\, g_1] \equiv \mathrm{op}\, g_L g_R \mod Y^{-\infty,d'}(X^\wedge \times \Omega, \mathbf{g}_3)_{O,O}.$$

□

Definition 2.12. Let $G = \mathrm{op}\, g + G_0 \in Y_G^{\mu,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$ with a left symbol $g \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ and $G_0 \in Y^{-\infty,d}(X^\wedge \times \Omega, \mathbf{g})_{O,O}$. Moreover let $g \sim \sum_{j=0}^\infty g_{\mu-j}$ be the asymptotic expansion of g into homogeneous terms. Then we define the *edge symbol of* G, or, also of g, by

$$\sigma_\wedge^\mu(G) = \sigma_\wedge^\mu(g) = g_\mu,$$

the homogeneous principal symbol of g.

3 The Symbol Algebra near the Edge

Proposition 3.1. *Let* $p \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}^{\mu,d}(X; \mathbf{R}^{1+q}))$. *Define*

$$a(y, \eta) = \omega(t)(1 - \omega_1(t[\eta]))\mathrm{op}_t(t^{-\mu} p(t, t', y, t\tau, t\eta))(1 - \omega_2(t[\eta]))\tilde{\omega}(t).$$

Then $a \in S^{\mu}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$.

Proof. In view of the factors ω and $\tilde{\omega}$ we may assume that p vanishes for large t, t'. We shall first show that, for every fixed choice of (y, η), the operator

$$\omega(t)(1-\omega_1(t[\eta]))\mathrm{op}_t(t^{-\mu}p(t,t',y,t\tau,t\eta))(1-\omega_2(t[\eta]))\tilde{\omega}(t) \qquad (3.1)$$

is an element of $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ and that, moreover, this operator depends smoothly on (y,η). It is no restriction to suppose that the vector bundles V_1 and V_2 are trivial one-dimensional, while W_1 and W_2 vanish.

Since η is fixed, we may also assume that p vanishes for small $t, t' > 0$ and the task reduces to showing that the operator in (3.1) belongs to $\mathcal{L}(H^s_{cone}(X^\wedge), H^{s-\mu}_{cone}(X^\wedge))$.

We know from [13, Lemma 4.2.2] that $H^s_{cone}(X) \hookrightarrow [t]^{\nu} H^s(X^\wedge)$ for $\nu = -n/2 + \max\{0, s+1\}$ while $[t]^{-n/2}H^{s-\mu}(X^\wedge) \hookrightarrow H^{s-\mu}_{cone}(X^\wedge)$. Here, $H^s(X^\wedge)$ consists of the restrictions of elements in the usual Sobolev space $H^s(X \times \mathbf{R})$ to $X^\wedge$. The powers of t need not worry us, since the symbol has compact support on $\mathbf{R}_+$ in both t and t'. So all we have to show is that we obtain an element of $\mathcal{L}(H^s(X^\wedge), H^{s-\mu}(X^\wedge))$. This, however, is an immediate consequence of the usual boundedness result for elements in Boutet de Monvel's calculus.

In addition, we know that the mapping that associates operators to symbols is continuous with respect to the parameters, hence we conclude that

$$a \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})). \qquad (3.2)$$

Next we apply Theorem 2.1. Pick an excision function ζ. Since

$$\omega(t)p(t,t',\tau,\eta)\tilde{\omega}(t') \in C^\infty(\overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\overline{\mathbf{R}}_+)\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\mathbf{R}^{1+q})$$

we may write $\omega(t)p(t,t',\tau,\eta)\tilde{\omega}(t') = \sum_{j=0}^{\infty} \lambda_j \varphi_j(t)\psi_j(t')p_j(y,\tau,\eta)$ with $\{\lambda_j\} \in l^1$ and null sequences $\{p_j\}$ in $C^\infty(\Omega, \mathcal{B}^{\mu,d}(X;\mathbf{R}^{1+q}))$, $\{\varphi_j\}, \{\psi_j\}$ in $C^\infty(\overline{\mathbf{R}}_+)$. Since we may multiply from the left and the right by cut-off functions without changing the operator, we may assume that $\{\varphi_j\}, \{\psi_j\}$ are null sequences in $C_0^\infty(\overline{\mathbf{R}}_+)$. Let

$$a_j(y,\eta) = \zeta(\eta)(1-\omega_1(t[\eta]))\mathrm{op}_t(t^{-\mu}p_j(y,t\tau,t\eta))(1-\omega_2(t[\eta])).$$

By Theorem 2.1 the a_j form a null sequence in $S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$. What about the operators of multiplication by φ_j and ψ_j? We shall consider them as operator-valued symbols independent of y and η and show that they form null sequences in $S^0(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s-\mu,\gamma-\mu}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ and $S^0(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_1^{s,\gamma})$, respectively: Clearly, multiplication by φ_j is bounded on $\mathcal{K}_2^{s-\mu,\gamma-\mu}$; the operator norm can be estimated via the semi-norms in $\mathcal{S}(\mathbf{R}_+)$, cf. 1.15(f). So $\varphi_j \in C^\infty(\Omega \times \mathbf{R}^q, \mathcal{K}_2^{s-\mu,\gamma-\mu}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$; moreover, $\kappa_{[\eta]^{-1}}\varphi_j\kappa_{[\eta]} = \varphi_j([\eta]^{-1}\cdot)$. Since the semi-norms of $\varphi_j([\eta]^{-1}\cdot)$ in $\mathcal{S}(\mathbf{R}_+)$ can be estimated uniformly in terms of those for φ_j, we obtain the desired statement for φ_j; for ψ_j an analogous argument applies.

This shows that $\zeta a \in S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$. Together with relation (3.2) the proof is complete. □

Convention: In the following we fix $\mu \in \mathbf{Z}$, $d, k \in \mathbf{N}$, $\gamma \in \mathbf{R}$, and the weight data $\mathbf{g} = (\gamma + n/2, \gamma + n/2 - \mu, (-k, 0])$.

3.2 The symbol algebra. Given $\tilde{h} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_{O,cl}^{\mu,d}(X; \mathbf{R}^q))$, $\tilde{p} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}_{cl}^{\mu,d}(X; \mathbf{R}^{q+1}))$ let

$$h(t, y, z, \eta) = \tilde{h}(t, y, z, t\eta) \quad \text{and} \quad p(t, y, \tau, \eta) = \tilde{p}(t, y, t\tau, t\eta)$$

be the corresponding edge-degenerate symbols. We assume additionally that $\tilde{h}$ and $\tilde{p}$ induce the same operators in the interior modulo smoothing terms:

$$\mathrm{op}_M^\gamma(h)(y, \eta) = \mathrm{op}_t(p)(y, \eta) \mod C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{R}^{1+q}_{\tau,\eta})). \tag{3.3}$$

This is possible by Mellin quantization, see Theorem 1.31. Here $\mathrm{op}_M^\gamma(h)(y, \eta)$ is the operator resulting from $\mathrm{op}_M^\gamma h(t, y, z, \eta)$, while $\mathrm{op}\,(p)(y, \eta) = \mathrm{op}_t p(t, y, \tau, \eta)$. Next we let $g \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$, and let $\omega, \tilde{\omega}, \omega_1, \omega_2, \omega_3$ be cut-off functions satisfying $\omega_1\omega_2 = \omega_1, \omega_1\omega_3 = \omega_3$. We shall consider the operator-valued symbols of the form

$$\begin{aligned} a(y, \eta) \;=\; & \omega(t)\,\{\omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^\gamma(h)(y, \eta)\omega_2(t[\eta]) \\ & + (1 - \omega_1(t[\eta]))t^{-\mu}\mathrm{op}\,(p)(y, \eta)(1 - \omega_3(t[\eta]))\}\,\tilde{\omega}(t) + g(y, \eta). \end{aligned} \tag{3.4}$$

Here we interpret $\omega, \tilde{\omega}$, $\omega_j(\cdot[\eta])$ and $t^{-\mu}$ as operators of multiplication by the corresponding functions. It follows from Theorem 1.28, Proposition 3.1, and Definition 2.4 that indeed $a(y, \eta) \in S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ for all $s > d - 1/2$. In the following we shall see that the symbols of this type form an algebra under pointwise composition. This requires some preliminary work.

Lemma 3.3. *Let $c = c(y, \eta) \in C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge; \mathbf{R}^q))$, $\varphi, \psi \in C_0^\infty(\mathbf{R}_+)$. Then*

$$\varphi(t[\eta])c\psi(t[\eta]) \in \mathcal{R}_G^{-\infty,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}.$$

Here we consider $\varphi(t[\eta]), \psi(t[\eta])$ operators of multiplication by $\varphi(\cdot[\eta]), \psi(\cdot[\eta])$.

Proof. For simplicity of the notation let us assume that V_1, V_2 are trivial one-dimensional while W_1, W_2 vanish. Since φ, ψ commute with the normal derivative on X we may assume that $d = 0$. The assumption implies that $c(y,\eta)$ is an integral operator on $X^\wedge$ with a kernel $k(y,\eta; x, t, x', t') \in C^\infty(\Omega)\hat{\otimes}_\pi \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi C^\infty(\overline{X}^\wedge \times \overline{X}^\wedge)$. So $\varphi(t[\eta])c\psi(t[\eta])$ has the kernel $\varphi(t[\eta])$ $k(y,\eta;x,t,x',t')\psi(t'[\eta])$. For each fixed y, η we therefore obtain an element in $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ provided $s > -1/2$. The operator $\kappa_{[\eta]^{-1}}\varphi(t[\eta])\, c\,\psi(t[\eta])\kappa_{[\eta]}$ has the kernel $\varphi(t)k(y,\eta;x,t/[\eta],x',t'/[\eta])\psi(t')[\eta]^{-1}$.
Its operator norm clearly is $O([\eta]^{-K})$ for arbitrary K. The same is true for derivatives with respect to y and η, so

$$\varphi(t[\eta])c\psi(t[\eta]) \in S^{-\infty}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu}), \quad s > -1/2.$$

Considering once more the kernel, the fact that φ and ψ belong to $C_0^\infty(\mathbf{R}_+)$ implies that, for fixed y and η, the operators $\varphi(t[\eta])c(y,\eta)\psi(t[\eta])$ map $\mathcal{K}_1^{s,\gamma}$ to $\mathcal{S}_{2,O}^{\gamma-\mu}$, while the adjoint maps $\mathcal{K}_2^{s,\mu-\gamma}$ to $\mathcal{S}_{2,O}^{-\gamma}$. As before, the operator seminorms are $O([\eta]^{-K})$ for arbitrary K. Hence $\varphi(t[\eta])c\psi(t[\eta]) \in S^{-\infty}(\Omega, \mathbf{R}^q;$ $\mathcal{K}_1^{s,\gamma}, \mathcal{S}_{2,O}^{\gamma-\mu})$, while its adjoint belongs to $S^{-\infty}(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s}, \mathcal{S}_{1,O}^{-\gamma})$. □

Lemma 3.4. *Let $p, \tilde{p}, h$ be as in 3.2 and $\varphi \in C_0^\infty(\mathbf{R}_+)$. Then*
(a) *If* $\operatorname{supp}\omega \cap \operatorname{supp}\varphi = \emptyset$ *then*

$$\begin{aligned} g_1(y,\eta) &= \varphi(t[\eta])t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\omega(t[\eta]) \quad \text{and} \\ g_2(y,\eta) &= \omega(t[\eta])t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\varphi(t[\eta]) \end{aligned}$$

are elements of $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.
(b) *If* $\operatorname{supp}(1-\omega) \cap \operatorname{supp}\varphi = \emptyset$ *then*

$$\begin{aligned} g_3(y,\eta) &= \omega_1(t)(1-\omega(t[\eta]))\mathrm{op}\,(t^{-\mu}p)(y,\eta)\varphi(t[\eta])\omega_2(t) \quad \text{and} \\ g_4(y,\eta) &= \omega_1(t)\varphi(t[\eta])\mathrm{op}\,(t^{-\mu}p)(y,\eta)(1-\omega(t[\eta]))\omega_2(t) \end{aligned}$$

are elements of $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.
(c) *Let ζ be an excision function. If* $\operatorname{supp}(1-\omega) \cap \operatorname{supp}\varphi = \emptyset$ *and $\tilde{p}$ is independent of t, then*

$$\begin{aligned} g_5(y,\eta) &= \zeta(\eta)(1-\omega(t[\eta]))\mathrm{op}\,(t^{-\mu}p)(y,\eta)\varphi(t[\eta]) \quad \text{and} \\ g_6(y,\eta) &= \zeta(\eta)\varphi(t[\eta])\mathrm{op}\,(t^{-\mu}p)(y,\eta)(1-\omega(t[\eta])) \end{aligned}$$

are elements of $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.

Proof. Since the normal derivative on X commutes with multiplication by functions of $t[\eta]$, we may assume that $d = 0$.
(a) We know that g_1 and g_2 are elements of $S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$. Fix y, η, and let us show that

$$g_1(y,\eta), g_2(y,\eta) \ : \ \mathcal{K}_1^{s,\gamma} \to \mathcal{S}_{2,O}^{\gamma-\mu}; \tag{3.5}$$

$$g_1(y,\eta)^*, g_2(y,\eta)^* \ : \ \mathcal{K}_2^{s,\mu-\gamma} \to \mathcal{S}_{1,O}^{-\gamma}. \tag{3.6}$$

Let us first consider $g_1(y,\eta)$. In view of the fact that φ and ω have disjoint support, we may replace h by a Mellin symbol of arbitrarily negative order. Hence $g_1(y,\eta) : \mathcal{K}_1^{s,\gamma} \to \mathcal{K}_2^{\infty,\gamma-\mu}$. Moreover, let k be the integer in the weight datum $\mathbf{g}$, and write $g_1(y,\eta) = t^k(t^{-k}g_1(y,\eta))$. For fixed η, the function $t^{-k}\varphi(t[\eta])$ is in $C_0^\infty(\mathbf{R}_+)$. Thus $g_1(y,\eta)$ satisfies relation (3.5). For $g_2(y,\eta)$ we know as before that it maps $\mathcal{K}_1^{s,\gamma}$ to $\mathcal{K}_2^{\infty,\gamma-\mu}$. We recall that

$$\mathrm{op}_M^\gamma(h)t^k = t^k \mathrm{op}_M^\gamma(T^{-k}h).$$

Together with the fact that $t^{-k}\varphi(t[\eta]) \in C_0^\infty(\mathbf{R}_+)$, we get relation (3.5). The relations in (3.6) follow by duality from those in (3.5).
Next we show that g_1 and g_2 are classical symbols in $S_{cl}^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{S}_{2,O}^{\gamma-\mu})$ while their adjoints belong to $S_{cl}^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s,\mu-\gamma}, \mathcal{S}_{1,O}^{-\gamma})$. For arbitrary $N \in \mathbf{N}$,

$$\begin{aligned} \tilde{h}(t,y,z,\eta) &= \sum_{j=0}^{N-1} \frac{t^j}{j!}\partial_t^j \tilde{h}(0,y,z,\eta) + t^N \tilde{h}_N(t,y,z,\eta) \\ &= \sum_{j=0}^{N-1} t^j \tilde{h}_j(y,z,\eta) + t^N \tilde{h}_N(t,y,z,\eta), \end{aligned}$$

with the obvious notation and $\tilde{h}_N \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_O^{\mu,d}(X; \mathbf{R}^q))$. Let $h_j(\ldots,\eta) = \tilde{h}_j(\ldots,t\eta)$, and denote by g_{kj} the symbols g_k, $k = 1, 2$, with h replaced by h_j. For $j = 0, \ldots, N-1$, we see that g_{kj} is homogeneous of degree μ in η in the sense of (1.3); the computation is analogous to that in (2.2). The above consideration shows that $g_{kj}(y,\eta)$ is an element of $\mathcal{L}_{12} = \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{S}_{2,O}^{\gamma-\mu})$, while $g_{kj}(y,\eta)^*$ belongs to $\mathcal{L}_{21} = \mathcal{L}(\mathcal{K}_2^{s,\mu-\gamma}, \mathcal{S}_{1,O}^{-\gamma})$. Moreover, the operator semi-norms in $\mathcal{L}_{12}$ and $\mathcal{L}_{21}$ depend continuously on the symbol semi-norms for the h_j; those in turn vary smoothly with y, η. By Lemma 1.4, $g_{kj} \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$, so $t^j g_{kj} \in \mathcal{R}_G^{\mu-j,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$.
In order to complete the proof let us show that, for $k = 1, 2$, and $s > -1/2$,

$$g_{kN} \in S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{S}_{2,O}^{\gamma-\mu}), \quad g_{kN}^* \in S^\mu(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s,\mu-\gamma}, \mathcal{S}_{1,O}^{-\mu}); \tag{3.7}$$

the factor t^N will then improve to order to $\mu - N$. Any possible non-classical contribution therefore has to be negligible. Consider g_{kN} first, starting with the case where $\tilde{h}_N$ is independent of t; then the assertion follows by homogeneity and Lemma 1.4. In case $\tilde{h}_N$ depends on t we may assume it to vanish for large t due to the multiplication by ω_1. Since $\tilde{h}_N \in C^\infty(\overline{\mathbf{R}}_+)\hat{\otimes}_\pi C^\infty(\Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$ we can write $\tilde{h}_N(t,y,z,\eta) = \sum_{j=0}^\infty \lambda_j \varphi_j(t)\tilde{g}_j(y,z,\eta)$ with null sequences $\{\varphi_j\}$ in $C_0^\infty(\overline{\mathbf{R}}_+)$, $\{\tilde{g}_j\}$ in $C^\infty(\Omega, M_O^{\mu,d}(X;\mathbf{R}^q))$ and $\{\lambda_j\} \in l^1$. Multiplication by φ_j is bounded on $\mathcal{S}_{2,O}^{\gamma-\mu}$; the semi-norms can be estimated in terms of semi-norms for φ_j in $C_0^\infty(\overline{\mathbf{R}}_+)$. Hence we get (3.7) from the t-independent case. For g_{kN}^* we argue in the same way.

(b) is proven in the same spirit. First treat the t-independent case, then apply a Taylor expansion into powers of t.

(c) The symbols are homogeneous of degree μ in η for large $|\eta|$. For every fixed choice of (y,η), $\eta \neq 0$, we see, similarly as in the proof of Theorem 2.1, that the operator $g_5(y,\eta)$ is an element of $\mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{S}_{2,O}^{\gamma-\mu})$. Assuming without loss of generality that $d=0$, the adjoint for the same reason is an element of $\mathcal{L}(\mathcal{K}_2^{s,\mu-\gamma}, \mathcal{S}_{1,O}^{-\gamma})$. An application of Lemma 1.4 completes the argument for g_5; the one for g_6 is analogous. □

Proposition 3.5. *We use the notation of* 3.2, *and define a as in* (3.4). *Now we choose cut-off functions $\tilde\omega_1, \tilde\omega_2, \tilde\omega_3$, with $\tilde\omega_1\tilde\omega_2 = \tilde\omega_1, \tilde\omega_1\tilde\omega_3 = \tilde\omega_3$, and define b by replacing in equation* (3.4) *the ω_j by $\tilde\omega_j, j = 1,2,3$. Then $a - b \in \mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$*

Proof. Since we might compare to a third operator, we can assume that $\tilde\omega_1\omega_2 = \tilde\omega_1$ and $\tilde\omega_3\omega_1 = \tilde\omega_3$. Write $A = \omega(t)\mathrm{op}_M^\gamma(h)\tilde\omega(t), B = \omega(t)\mathrm{op}_t(t^{-\mu}p)\tilde\omega(t)$. In the following we shall omit the variables $(t[\eta])$ with the $\omega_j, \tilde\omega_j$, and denote congruence modulo $\mathcal{R}_G^{\mu,d}(\Omega \times \mathbf{R}^q\mathbf{g})_{O,O}$ by $\equiv$. Then

$$\begin{aligned} a - b &= \omega_1 A\omega_2 + (1-\omega_1)B(1-\tilde\omega_3) + (1-\omega_1)B(\omega_3 - \tilde\omega_3) \\ &\quad - \tilde\omega_1 A\omega_2 - (1-\tilde\omega_1)B(1-\tilde\omega_3) + \tilde\omega_1 A(\omega_2 - \tilde\omega_2) \\ &\equiv (\omega_1 - \tilde\omega_1)A\omega_2 - (\omega_1 - \tilde\omega_1)B(1-\tilde\omega_3) \\ &= (\omega_1 - \tilde\omega_1)\{A\omega_2(1-\tilde\omega_3) + A\omega_2\tilde\omega_3 - B\omega_2(1-\tilde\omega_3) - B(1-\omega_2)(1-\tilde\omega_3)\} \\ &\equiv (\omega_1 - \tilde\omega_1)\{A\omega_2(1-\tilde\omega_3) - B\omega_2(1-\tilde\omega_3)\} \equiv 0. \end{aligned}$$

Here the first two congruences are due to Lemma 3.4, since $\operatorname{supp}(\omega_3 - \tilde\omega_3) \cap \operatorname{supp}(1-\omega_1) = \operatorname{supp}\tilde\omega_1 \cap \operatorname{supp}(\omega_2 - \tilde\omega_2) = \operatorname{supp}(\omega_1 - \tilde\omega_1) \cap \operatorname{supp}\tilde\omega_3 = \operatorname{supp}(\omega_1 - \tilde\omega_1) \cap \operatorname{supp}(1-\omega_2) = \emptyset$. Note that $(1-\omega_2)(1-\tilde\omega_3) = 1-\omega_2$. The final congruence is due to Lemma 3.3 together with (3.3). □

Lemma 3.6. *Let a be as in* (3.4). *Then*

$$a(y,\eta) - t^{-\mu}\mathrm{op}\,(p)(y,\eta) \in C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q_\eta)).$$

Proof. It follows from (3.3) that

$$a(y,\eta) - t^{-\mu}\mathrm{op}_t(p)(y,\eta) - g(y,\eta) \in C^\infty(\Omega, \mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q)).$$

For arbitrary K we write $g = t^{-K}t^K g \in t^{-K}\mathcal{R}_G^{\mu-K,d}(\Omega\times\mathbf{R}^q,\mathrm{g})_{O,O}$. Hence g induces an element in $C^\infty(\Omega,\mathcal{B}^{-\infty,d}(X^\wedge;\mathbf{R}^q))$, and the proof is complete. □

3.7 Symbols. We use the notation of 3.2; moreover, we let $h_0(t,y,z,\eta) = \tilde{h}(0,y,z,t\eta)$, $p_0(t,y,z,\eta) = \tilde{p}(0,y,t\tau,t\eta)$, and recall that $\sigma^\mu_\wedge(g)$ is the principal edge symbol of g as introduced in Definition 2.12. For $y\in\Omega, \eta\neq 0$ we define *the principal edge symbol* $\sigma^\mu_\wedge(a)$ of a as the operator

$$\begin{aligned}\sigma^\mu_\wedge(a)(y,\eta) &= \omega_1(t|\eta|)t^{-\mu}\mathrm{op}^\gamma_M(h_0)(y,\eta)\omega_2(t|\eta|) \\ &\quad + (1-\omega_1(t|\eta|))\mathrm{op}\,(t^{-\mu}p_0)(y,\eta)(1-\omega_3(t|\eta|)) + \sigma^\mu_\wedge(g).\end{aligned} \tag{3.8}$$

By Theorem 2.1, $\sigma^\mu_\wedge(a)(y,\eta)\in\mathcal{L}(\mathcal{K}_1^{s,\gamma},\mathcal{K}_2^{s-\mu,\gamma-\mu})$. We obtain the relation

$$\sigma^\mu_\wedge(a)(y,\lambda\eta) = \lambda^\mu\kappa_\lambda\sigma^\mu_\wedge(a)(y,\eta)\kappa_{\lambda^{-1}}, \quad y\in\Omega,\eta\neq 0,\lambda>0.$$

According to Lemma 3.6 we also have for a the symbol $p = p(t,y,\tau,\eta) \in C^\infty(\mathbf{R}_+\times\Omega,\ B^{\mu,d}_{cl}(X^\wedge;\mathbf{R}^{q+1}))$. This enables us to associate to a also the interior principal pseudodifferential symbol $\sigma^\mu_\psi(a)$ and the principal boundary symbol $\sigma^\mu_\partial(a)$, both being defined as the corresponding terms for p in the sense of 1.10:

$$\sigma^\mu_\psi(a) = \sigma^\mu_\psi(p), \quad \text{and} \quad \sigma^\mu_\partial(a) = \sigma^\mu_\partial(p).$$

For each $\eta\neq 0$ we can associate to $\sigma^\mu_\wedge(a)$ the symbol p_0 which again has a principal pseudodifferential symbol, namely $\sigma^\mu_\psi(\sigma^\mu_\wedge(a)) = \sigma^\mu_\psi(p_0)$, and a principal boundary symbol, namely $\sigma^\mu_\partial(\sigma^\mu_\wedge(a)) = \sigma^\mu_\partial(p_0)$.

3.8 Facts from the cone calculus. The space $C^{\mu,d}_{M+G}(X^\wedge,\mathrm{g})_{O,O}$ consists of all operators of the form

$$A = \omega\, t^{-\mu}\sum_{j=0}^{k-1} t^j\mathrm{op}^\gamma_M(h_j)\tilde{\omega} + G,$$

where $\omega, \tilde{\omega}$ are cut-off functions, $h_j \in M_O^{-\infty,d}(X)$, and G is a Green operator in $C_G^d(X^\wedge, \mathbf{g})_{O,O}$, in other words, $G = G_0 + \sum_{j=1}^d G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & 0 \end{bmatrix}$ with $G_j \in \bigcap_{s>-1/2} \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{K}_2^{s-\mu,\gamma-\mu})$ having, for all $s > -1/2$ and $j = 0, \ldots, d$, continuous extensions

$$G_j : \mathcal{K}_1^{s,\gamma} \to \mathcal{S}_{2,O}^{\gamma-\mu} \quad \text{and} \quad G_j^* : \mathcal{K}_2^{s,\mu-\gamma} \to \mathcal{S}_{1,O}^{-\gamma}. \tag{3.9}$$

A classical element A of order μ and type d in the cone calculus belongs to $C_{M+G}^{\mu,d}(X^\wedge, \mathbf{g})_{O,O}$ if and only if the interior symbol is regularizing.
The so-called conormal symbols $\sigma^{\mu-j}(A) = h_j$ are uniquely determined; h_j is the coefficient of t^j in a Taylor expansion of an arbitrary Mellin symbol for A at $t = 0$. The conormal symbols obey the composition rule

$$\sigma_M^{\mu+\mu'-j}(AB) = \sum_{p+q=j} \left[T^{\mu'-q}\, \sigma_M^{\mu-p}(A)\right] \sigma_M^{\mu'-q}(B).$$

For details see [13, 3.3.1, 4.3.1, 4.3.7, 4.3.10] and [14, 3.1.27, 3.1.29(c)].

3.9 Compositions. Consider two symbols $a, \tilde{a}$ in the sense of 3.2:

$$\begin{aligned} a(y,\eta) \;=\; & \omega(t)\, \{\omega_1(t[\eta]) t^{-\mu} \mathrm{op}_M^{\gamma}(h)(y,\eta)\omega_2(t[\eta]) \\ & + (1-\omega_1(t[\eta]))\mathrm{op}_t(t^{-\mu}p)(y,\eta)(1-\omega_3(t[\eta]))\}\, \tilde{\omega}(t) + g(y,\eta), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \tilde{a}(y,\eta) \;=\; & \omega_4(t) \left\{\tilde{\omega}_1(t[\eta]) t^{-\mu'} \mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)\tilde{\omega}_2(t[\eta]) \right. \\ & \left. + (1-\tilde{\omega}_1(t[\eta]))\mathrm{op}_t(t^{-\mu'}q)(y,\eta)(1-\tilde{\omega}_3(t[\eta]))\right\} \tilde{\omega}_4(t) + \tilde{g}(y,\eta). \end{aligned} \tag{3.11}$$

For a we use the notation of 3.2, while $\tilde{a}$ has corresponding properties. Explicity,

(i) $f(t,y,z,\eta) = \tilde{f}(t,y,z,t\eta)$ and $\tilde{f} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_{O,cl}^{\mu',d'}(X;\mathbf{R}^q))$;

(ii) $q(t,y,\tau,\eta) = \tilde{q}(t,y,t\tau,t\eta)$ and $\tilde{q} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}_{cl}^{\mu',d'}(X^\wedge;\mathbf{R}^{q+1}))$;

(iii) the compatibility condition is satisfied:

$$\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta) \equiv \mathrm{op}_t(q)(y,\eta) \bmod C^\infty(\Omega, \mathcal{B}^{-\infty,d''}(X^\wedge;\mathbf{R}_{\tau,\eta}^{1+q})).$$

(iv) We assume that $\tilde{a}$ is associated with the weight datum $\mathbf{g}_1 = (\gamma - \mu + n/2, \gamma - \mu - \mu' + n/2, (-k, 0])$ and acts between vector bundles V_2, V_3 over $\overline{X}$ and W_2, W_3 over Y;

(v) $\tilde{g} \in \mathcal{R}_G^{\mu',d'}(\Omega \times \mathbf{R}^q, \mathbf{g}_1)_{O,O}$.

(vi) In order to simplify the computation we shall assume that $\omega_4\tilde{\omega}_4 = \omega_4$, $\tilde{\omega}_4\omega = \tilde{\omega}_4$, and $\omega\tilde{\omega} = \omega$. This is no restriction since f, h, q, and p depend on t. We then know that $\tilde{a} \in S^{\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s,\gamma-\mu}, \mathcal{K}_3^{s-\mu',\gamma-\mu-\mu'})$ for $s > d' - 1/2$, so that we may form $b(y,\eta) = \tilde{a}(y,\eta)a(y,\eta)$ in the sense of operator-valued symbols and get $b = \tilde{a}a \in S^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_3^{s-\mu-\mu',\gamma-\mu-\mu'})$.

We shall now show that b has a decomposition analogous to that of a and $\tilde{a}$ in (3.10),(3.11) associated with the weight datum $\mathbf{g}_2 = (\gamma + n/2, \gamma - \mu - \mu' + n/2, (-k, 0])$. In fact we shall do the following:

(vii) First we define r by $t^{-\mu-\mu'}r \sim t^{-\mu'}q \#_t t^{-\mu}p$, where $\#_t$ is the Leibniz product with respect to t, τ. We shall see in Lemma 3.10, below, that then

(viii) $r(t,y,\tau,\eta) = \tilde{r}(t,y,t\tau,t\eta)$ for suitable $\tilde{r} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, \mathcal{B}_{cl}^{\mu+\mu',d''}(X; \mathbf{R}^{1+q}))$, $d'' = \max\{\mu + d', d\}$. Moreover, $\tilde{r}$ will be independent of t for large t provided this is the case for $\tilde{p}$ and $\tilde{q}$.

(ix) By Mellin quantization with respect to the weight γ define $k(t,y,z,\eta) = \tilde{k}(t,y,z,t\eta)$ with $\tilde{k} \in C^\infty(\overline{\mathbf{R}}_+ \times \Omega, M_{O,cl}^{\mu'',d''}(X; \mathbf{R}^q))$.

(x) By construction the compatibility condition holds:

$$\mathrm{op}_M^\gamma(k)(y,\eta) - \mathrm{op}_t(r)(y,\eta) \in C^\infty(\Omega, \mathcal{B}^{-\infty,d''}(X^\wedge; \mathbf{R}^{1+q})).$$

(xi) For fixed (y,η) we may consider the difference

$$\omega_4(t)\left\{t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta) - t^{\mu-\mu'}\mathrm{op}_M^\gamma(k)(y,\eta)\right\}\tilde{\omega}(t). \quad (3.12)$$

Here, ω_4, $\tilde{\omega}$ are the functions in the definition of $\tilde{a}$ and a respectively. Since $\tilde{k}$ arose as the Mellin quantization of the op_t-composition, the (full) interior symbol of this operator is regularizing. So the difference is an element of $C_{M+G}^{\mu+\mu',d''}(X^\wedge, \mathbf{g}_2)_{O,O}$. Since the symbols involved have the arguments $(t,z,t\eta)$ and the conormal symbols are just the Taylor coefficients at $t = 0$, they are of the form $h_j(z,\eta) = \sum_{|\alpha|\le j} h_{j,\alpha}\eta^\alpha$, $j = 0,\ldots,k-1$, with $h_{j,\alpha} \in M_O^{-\infty,d''}(X)$. We replace $\tilde{k}$ by $\tilde{k} + \sum_{j=0}^{k-1}\sum_{|\alpha|\le j} h_{j,\alpha}\eta^\alpha t^{j-|\alpha|}s(\eta)$. Here $s \in \mathcal{S}(\mathbf{R}^q)$ is an arbitrary function with $s(\eta) \equiv 1$ for η near zero. Since $k(t,z,\eta) = \tilde{k}(t,z,t\eta)$, the Taylor coefficients of k are such that all conormal symbols for the difference (3.12) vanish. The change in $\tilde{k}$ is an element of $C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d''}(X; \mathbf{R}^q))$, hence the compatibility condition in relation (x) remains satisfied. Note also that a change in the cut-off functions ω_4 and $\tilde{\omega}$ in (3.12) results in an error which is, for each fixed (y,η), an element of $C_G^{d''}(X^\wedge, \mathbf{g}_2)_{O,O}$; in that sense the construction is independent of the choice of the cut-off.

(xii) We then let

$$\begin{aligned} c(y,\eta) &= \omega_4\Big\{\omega_1(t[\eta])t^{-\mu-\mu'}\mathrm{op}_M^\gamma k(y,\eta)\omega_2(t[\eta]) \\ &\quad +(1-\omega_1(t[\eta]))\mathrm{op}\,(t^{-\mu-\mu'}r)(y,\eta)(1-\omega_3(t[\eta]))\Big\}\tilde{\omega}. \end{aligned}$$

By construction, this is an element of the symbol algebra introduced in 3.2.
(xiii) We will then show that $c-\tilde{a}a$ is an element of $\mathcal{R}_G^{\mu+\mu',d''}(\Omega\times\mathbf{R}^q,\mathbf{g}_2)_{O,O}$. The details can be found in Propositions 3.11 and 3.12, below. Apart from the technical facts the proof then is complete.

Lemma 3.10. *Let $\tilde{p}\in C^\infty(\overline{\mathbf{R}}_+\times\Omega,\mathcal{B}_{cl}^{\mu,d}(X;\mathbf{R}^{q+1}))$ and $\tilde{q}\in C^\infty(\overline{\mathbf{R}}_+\times\Omega,\mathcal{B}_{cl}^{\mu',d'}(X;\mathbf{R}^{q+1}))$. Then all the homogeneous terms in the asymptotic expansion of*

$$t^{-\mu'}\mathrm{op}\,\tilde{q}(t,y,t\tau,t\eta)\,\#_t\,t^{-\mu}\mathrm{op}\,\tilde{p}(t,y,t\tau,t\eta) \qquad (3.13)$$

have the form $t^{-\mu-\mu'}\tilde{r}_l(t,y,t\tau,t\eta)$ with $\tilde{r}_l\in C^\infty(\overline{\mathbf{R}}_+\times\Omega,\ \mathcal{B}_{cl}^{\mu+\mu'-l,d''}(X;\mathbf{R}^{1+q}))$, $d''=\max\{\mu+d',d\}$. In particular, we may sum these terms asymptotically in $C^\infty(\overline{\mathbf{R}}_+\times\Omega,\mathcal{B}_{cl}^{\mu+\mu',d''}(X;\mathbf{R}^{1+q}))$.

Proof. Let $\tilde{p}_j$ and $\tilde{q}_k$, $j,k=0,1,\ldots$, denote the homogeneous terms in the asymptotic expansions of $\tilde{p}$ and $\tilde{q}$, respectively. The terms in the asymptotic expansion for (3.13) are of the form

$$\partial_\tau^m\{\tilde{q}_k(t,y,t\tau,t\eta)\}D_t^m\{\tilde{p}_j(t,y,t\tau,t\eta), \qquad (3.14)$$

hence the assertion follows by iteration from the fact that, for $m=1$, the product in (3.14) is

$$\begin{aligned} &(\partial_\tau\tilde{q}_k)(t,y,t\tau,t\eta)\{(tD_t\tilde{p}_j)(t,y,t\tau,t\eta) \\ &\quad +t\tau(D_\tau\tilde{p}_j)(t,y,t\tau,t\eta)+\sum_{\nu=1}^{q}t\eta_\nu(D_{\eta_\nu}\tilde{p}_j)(t,y,t\tau,t\eta)\}. \end{aligned}$$

□

We shall deal in Proposition 3.11, below, with the compositions involving g and $\tilde{g}$. We know already from Proposition 2.5 that

$$\tilde{g}g\in\mathcal{R}_G^{\mu+\mu',d}(\Omega\times\mathbf{R}^q,\mathbf{g}_2)_{O,O},$$

so this term needs no special attention.

Proposition 3.11. *We use the notation introduced in* 3.9. *The following compositions furnish elements of* $\mathcal{R}_G^{\mu+\mu',d}(\Omega \times \mathbf{R}^q, \mathbf{g}_2)_{O,O}$:

(a) $\{\tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)\tilde{\omega}_2(t[\eta])\}g$

(b) $\tilde{g}\{\omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^{\gamma}(h)\omega_2(t[\eta])\}$

(c) $\{\omega_4(t)(1-\tilde{\omega}_1(t[\eta]))\mathrm{op}\,(t^{-\mu'}q)(1-\tilde{\omega}_3(t[\eta]))\tilde{\omega}_4(t)\}g$

(d) $\tilde{g}\omega(t)\{(1-\omega_1(t[\eta]))\mathrm{op}\,(t^{-\mu}p)(1-\omega_3(t[\eta]))\tilde{\omega}(t)\}$

The same statement holds for $\{\omega_4(t)\tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)\tilde{\omega}_2(t[\eta])\tilde{\omega}_4(t)\}g$ *and* $\tilde{g}\{\omega(t)\omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^{\gamma}(h)\omega_2(t[\eta])\tilde{\omega}(t)\}$ *by Proposition* 2.5(f).

Proof. (a) Let $F = \tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)\tilde{\omega}_2(t[\eta])$. Suppose first that $\tilde{f}$ is independent of t. Then F is homogeneous of degree μ in η for large $|\eta|$, hence an element of $S_{cl}^{\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s-\mu,\gamma-\mu}, \mathcal{K}_3^{s,\gamma-\mu-\mu'})$ for all $s > d' - 1/2$. By linearity we may assume $d' = 0$. Hence

$$Fg \in S_{cl}^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_3^{s-\mu-\mu',\gamma-\mu-\mu'}) \tag{3.15}$$

whenever $s > -1/2$, noting that $g(y,\eta)$ maps into $\mathcal{K}_2^{\infty,\gamma-\mu}$. We want to show

$$Fg \in S_{cl}^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{S}_{3,O}^{\gamma-\mu-\mu'}). \tag{3.16}$$

Close to zero, the space $\mathcal{S}_{3,O}^{\gamma-\mu-\mu'}$ coincides with $\bigcap_{\varepsilon>0}\mathcal{K}_3^{\infty,\gamma-\mu-\mu'+k-\varepsilon}$, where k is the integer in $\mathbf{g}_2$. For arbitrary $\varepsilon > 0$ use Lemma 1.29 and write

$$Fg = F([\eta]t)^{k-\varepsilon}([\eta]t)^{\varepsilon-k}g = ([\eta]t)^{k-\varepsilon}F_\varepsilon([\eta]t)^{\varepsilon-k}g$$

with $F_\varepsilon = \tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(T^{\varepsilon-k}f)\tilde{\omega}_2(t[\eta])$. Multiplication by $([\eta]t)^{\varepsilon-k}$ is an element of $S_{cl}^0(\Omega, \mathbf{R}^q; \mathcal{S}_{2,O}^{\gamma-\mu}, \mathcal{K}_2^{\infty,\gamma-\mu})$; the symbol $([\eta]t)^{k-\varepsilon}\omega_1(t[\eta])$ belongs to $S_{cl}^0(\Omega, \mathbf{R}^q; \mathcal{K}_3^{\infty,\gamma-\mu-\mu'}, \mathcal{K}_3^{\infty,\gamma-\mu+k-\varepsilon})$. Hence

$$Fg \in \bigcap_\varepsilon S_{cl}^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_3^{\infty,\gamma-\mu-\mu'+k-\varepsilon}). \tag{3.17}$$

Next choose ω_5 with $\tilde{\omega}_1\omega_5 = \tilde{\omega}_1$, so that $\omega_5(t[\eta])F = F$. Multiplication by $\omega_5(t[\eta])$ is an element of $S_{cl}^0(\Omega, \mathbf{R}^q; \mathcal{K}_3^{s,\gamma-\mu-\mu'+k-\varepsilon}, [t]^{-l}\mathcal{K}_3^{s,\gamma-\mu-\mu'+k-\varepsilon})$ for arbitrary l and s, so (3.16) follows from (3.17).

Our next task is the relation

$$(Fg)^* \in S_{cl}^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_3^{s,\mu+\mu'-\gamma}, \mathcal{S}_{1,O}^{-\gamma}). \tag{3.18}$$

Since the normal derivative composed with g furnishes an element of $\mathcal{R}_G^{\mu+1,d}(\Omega \times \mathbf{R}^q, \mathbf{g})_{O,O}$ we may also assume that $d = 0$. Then $(Fg)^* = g^*F^*$, and (3.18) is immediate. We therefore know that $Fg \in \mathcal{R}_G^{\mu+\mu',d}(\Omega \times \mathbf{R}^q, \mathbf{g}_2)_{O,O}$.

It remains to consider the case where $\tilde{f}$ depends on t. Then we use a Taylor expansion: For $N \in \mathbf{N}$,

$$\tilde{f}(t,y,z,\eta) = \sum_{j=0}^{N-1} \frac{t^j}{j!}\partial_t^j \tilde{f}(0,y,z,\eta) + t^N \tilde{f}_N(t,y,z,\eta).$$

We let $F_j = \tilde{\omega}_1(t[\eta])\frac{1}{j!}t^{-\mu'}\operatorname{op}_M^{\gamma-\mu}\partial_t^j\tilde{f}(0,y,z,t\eta)\tilde{\omega}_2(t[\eta])$. From the above result for the t-independent case we know that $F_jg \in \mathcal{R}_G^{\mu+\mu',d}(\Omega \times \mathbf{R}^q; \mathbf{g}_2)$. Applying Proposition 2.5(e), we conclude that $t^jF_jg \in \mathcal{R}^{\mu+\mu'-j,d}(\Omega \times \mathbf{R}^q, \mathbf{g}_2)$. We therefore obtain the beginning of an asymptotic expansion. Finally we let

$$F_N = \tilde{\omega}_1(t[\eta])t^{-\mu'}\operatorname{op}_M^{\gamma-\mu}(\tilde{f}_N(t,y,z,t\eta))\tilde{\omega}_2(t[\eta]).$$

We can now proceed just like in the t-independent case, except for the fact that

$$F_N \in S^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_2^{s,\gamma-\mu}, \mathcal{K}_3^{\gamma-\mu-\mu'})$$

is not obviously a classical symbol. Hence we get relations (3.15), (3.16), (3.18) with F replaced by F_N and the subscript "cl" omitted. The crucial point now is that we still have the factor t^N. It lowers all orders by N. Hence the possible non-classical contribution is of arbitrarily negative order and therefore negligible. So (a) is proven.

The proof of (b) is virtually the same as that of (a). Finally (c) and (d) follow in an analogous way. Here, the mapping is nice near $t = 0$; we only have to take a closer look for large t. Write $(1-\omega_1(t[\eta])) = t^{-k}(1-\omega_1(t[\eta]))t^k$. Noting that $[t, \operatorname{op}_t q] = -D_\tau q$, we may commute powers of t to the right, where we can make use of the mapping properties of g. □

Proposition 3.12. *We use the notation of* 3.9. *Then*

$$c - \tilde{a}a \in \mathcal{R}_G^{\mu+\mu',d'}(\Omega \times \mathbf{R}^q, \mathbf{g}_2)_{O,O}.$$

Proof. In order to avoid notational complications let us assume that $d = d' = d'' = 0$.

Step 1. *The pointwise consideration.* We know from the cone calculus that for fixed y and η, the operator

$$\omega(t)(1-\omega_1(t[\eta]))\left\{t^{-\mu}\operatorname{op}_M^\gamma(h)(y,\eta) - t^{-\mu}\operatorname{op}_t(p)(y,\eta)\right\}(1-\omega_3(t[\eta]))\tilde{\omega}(t)$$

by 3.9(xi) is an element of $C_G^0(X^\wedge, \mathbf{g})_{O,O}$. Hence we can write

$$a(y,\eta) = \omega(t)t^{-\mu}\operatorname{op}_M^\gamma(h)(y,\eta)\tilde{\omega}(t) + G_1 \tag{3.19}$$

with $G_1 \in C_G^0(X^\wedge, g)_{O,O}$. Similarly, $\tilde{a}(y,\eta) = \omega_4(t)t^{-\mu}\mathrm{op}_M^\gamma(f)(y,\eta)\tilde{\omega}_4(t) + G_2$ for some G_2 in $C_G^0(X^\wedge, g_1)_{O,O}$. Denoting congruence modulo $C_G^0(X^\wedge, g_2)_{O,O}$ by $\equiv$ and using that $\tilde{\omega}_4\omega = \tilde{\omega}_4$, we have

$$\begin{aligned}\tilde{a}(y,\eta)a(y,\eta) &\equiv (\omega_4 t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)\tilde{\omega}_4)(\omega t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\tilde{\omega})\\ &\equiv \omega_4 t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\tilde{\omega}\\ &= \omega_4 t^{-\mu-\mu'}\mathrm{op}_M^\gamma(k)(y,\eta)\tilde{\omega} \equiv c(y,\eta);\end{aligned}$$

the last identity stems from 3.9(xi); the second congruence is due to the fact that

$$\omega_4 t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)(1-\tilde{\omega}_4)t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\tilde{\omega} \in C_G^0(X^\wedge, g_2)_{O,O},$$

and the last congruence is the analog of (3.19) for $c(y,\eta)$. The continuous dependence of the operators on the symbols shows that the construction is smooth in y and η, hence

$$c - \tilde{a}a \in C^\infty(\Omega\times\mathbf{R}^q, \mathcal{L}(\mathcal{K}_1^{s,\gamma}, \mathcal{S}_{3,O}^{\gamma-\mu-\mu'})). \tag{3.20}$$

Similarly,

$$(c-\tilde{a}a)^* \in C^\infty(\Omega\times\mathbf{R}^q, \mathcal{L}(\mathcal{K}_3^{s,\mu+\mu'-\gamma}, \mathcal{S}_{1,O}^{-\gamma})). \tag{3.21}$$

Step 2. The case of t-independent symbols. Assume next that the symbols $\tilde{h}, \tilde{p}$ and $\tilde{f}, \tilde{q}$ involved in the definition of a and $\tilde{a}$ are independent of t. By construction, this then is true for $\tilde{r}$. Employing the formula for the asymptotic expansion of $\tilde{k}$, [15, Proposition 3.14] or [14, Theorem 2.4.13], also $\tilde{k}$ is independent of t before the modification in 3.9(xi). Using the notation of 3.9, the resulting change in k is a finite linear combination of terms of the form $h_{j,\alpha}t^{j-\alpha}(t\eta)^\alpha s(t\eta)$, hence homogeneous of degree $|\alpha| - j$ in the sense of (1.3) for large $|\eta|$. Choose excision functions ζ_1 and ζ_2 and abbreviate

$$\begin{aligned}a_0 &= \omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^\gamma(h)(y,\eta)\omega_2(t[\eta])\\ &\quad+(1-\omega_1(t[\eta]))\mathrm{op}_t(t^{-\mu}p)(y,\eta)(1-\omega_3(t[\eta]));\\ \tilde{a}_0 &= \tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)\tilde{\omega}_2(t[\eta])\\ &\quad+(1-\tilde{\omega}_1(t[\eta]))\mathrm{op}_t(t^{-\mu'}q)(y,\eta)(1-\tilde{\omega}_3(t[\eta]));\\ c_0 &= \omega_1(t[\eta])t^{-\mu-\mu'}\mathrm{op}_M^\gamma(k)(y,\eta)\omega_2(t[\eta])\\ &\quad+(1-\omega_1(t[\eta]))\mathrm{op}\,(t^{-\mu-\mu'}r)(y,\eta)(1-\omega_3(t[\eta])).\end{aligned}$$

We first consider the difference $\zeta_1\tilde{a}_0\zeta_2 a_0 - \zeta_1\zeta_2 c_0$. By 1.28 and 2.1, this is an element of $S^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{K}_3^{s-\mu-\mu',\gamma-\mu-\mu'})$ for $s > -1/2$. Moreover, it is a finite sum of terms that are homogeneous in η for large $|\eta|$ in the sense of (1.3). Hence it is classical.

We have to show that

$$\zeta_1\zeta_2\{(\omega_4\tilde{a}_0\tilde{\omega}_4)(\omega a_0\tilde{\omega}) - \omega_4 c_0\tilde{\omega}_4\} \in \mathcal{R}_G^{\mu+\mu',0}(\Omega\times\mathbf{R}^q, \mathrm{g}_2)_{O,O}. \tag{3.22}$$

Since we have homogeneity, it suffices to prove that

$$\zeta_1\zeta_2\{\tilde{a}_0a_0 - c_0\} \in S^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_1^{s,\gamma}, \mathcal{S}_{3,O}^{\gamma-\mu-\mu'}) \text{ and} \tag{3.23}$$

$$\zeta_1\zeta_2\{\tilde{a}_0a_0 - c_0\}^* \in S^{\mu+\mu'}(\Omega, \mathbf{R}^q; \mathcal{K}_3^{s,\mu+\mu'-\gamma}, \mathcal{S}_{1,O}^{-\gamma}). \tag{3.24}$$

Indeed, suppose this holds. Then $\zeta_1\zeta_2\{(\omega_4\tilde{a}_0)(a_0\tilde{\omega}) - \omega_4c_0\tilde{\omega}\}$ is an element of $\mathcal{R}_G^{\mu+\mu',0}(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$. Moreover, we argue that $\zeta_1\zeta_2\omega_4\tilde{a}_0(1-\tilde{\omega}_4)a_0\tilde{\omega} \in \mathcal{R}_G^{-\infty,0}(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$: In view of the fact that $\operatorname{supp}(1-\tilde{\omega}_4)\cap\operatorname{supp}\omega_4 = \emptyset$ we may replace the Mellin symbol h by a symbol $t^N h_N$, $N\in\mathbf{N}$ with $h_N(t,z,\eta) = \tilde{h}_N(t,z,t\eta)$ and $\tilde{h}_N \in C^\infty(\overline{\mathbf{R}}_+\times\Omega, M_{O,cl}^{\mu-N}(X;\mathbf{R}^q))$ without changing the operator and then apply Theorem 1.28. So we deduce (3.22) from (3.20) and (3.21).

Next we focus on (3.23). Choose cut-off functions $\omega_5,\omega_6,\omega_7$ with

$$(1-\omega_3)\omega_5 = 0,\ (1-\omega_6)\omega_1 = 0\ , (1-\tilde{\omega}_3)\omega_7 = 0, \tag{3.25}$$

$$\operatorname{supp}(\omega_6-\omega_7)\cap\operatorname{supp}\omega_5 = \emptyset. \tag{3.26}$$

This is possible, provided ω_5 and ω_7 have support in a sufficiently small neighborhood of zero, while $\omega_7\omega_5 = \omega_5$. In particular $(1-\tilde{\omega}_3)\omega_5 = 0$. Without loss of generality we also assume that $\tilde{\omega}_1\omega_3 = \omega_3$.

In the following few lines of computation we denote by $\equiv$ congruence modulo $\mathcal{R}_G^{\mu+\mu',0}(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$. Abbreviate $M_{\tilde{a}} = \tilde{\omega}_1(t[\eta])t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)\tilde{\omega}_2(t[\eta])$, $M_a = \omega_1(t[\eta])t^{-\mu}\mathrm{op}_M^{\gamma}(h)(y,\eta)\tilde{\omega}_2(t[\eta])$, and $M_c = \omega_1(t[\eta])t^{-\mu-\mu'}\mathrm{op}_M^{\gamma}(k)(y,\eta)$ $\omega_2(t[\eta])$. Also omit, just for the moment, the argument $(t[\eta])$ of the cut-off functions for better legibility. The first two equalities, below, are immediate from (3.25).

$$\begin{aligned}
&\zeta_1\zeta_2\{\tilde{a}_0a_0 - c\}\omega_5 = \zeta_1\zeta_2\{\tilde{a}_0M_a - M_c\}\omega_5\\
&= \zeta_1\zeta_2\{\tilde{a}_0\omega_6M_a - M_c\}\omega_5 \equiv \zeta_1\zeta_2\{\tilde{a}_0\omega_7M_a - M_c\}\omega_5\\
&= \zeta_1\zeta_2\{M_{\tilde{a}}\omega_7M_a - M_c\}\omega_5\\
&\equiv \zeta_1\zeta_2\{\tilde{\omega}_1t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)t^{-\mu}\mathrm{op}_M^{\gamma}(h)(y,\eta)\omega_3 - M_c\}\omega_5\\
&\equiv \zeta_1\zeta_2\{\omega_1t^{-\mu'}\mathrm{op}_M^{\gamma-\mu}(f)(y,\eta)t^{-\mu}\mathrm{op}_M^{\gamma}(h)(y,\eta)\omega_3 - M_c\}\omega_5 \equiv 0.
\end{aligned}$$

The first congruence follows from (3.26) together with Lemma 3.4(a). For the second we use the same lemma in connection with the fact that $\tilde{\omega}_2(1-\omega_7)\omega_1$ is a function in $C^\infty(\overline{\mathbf{R}}_+)$ whose support is disjoint to that of ω_5. The third congruence comes from replacing $\tilde{\omega}_1$ by ω_1; this is justified again by the lemma together with the fact that $\operatorname{supp}(\omega_1-\tilde{\omega}_1)\cap\operatorname{supp}\omega_3=\emptyset$. The final congruence is slightly more subtle: By construction, the expression between the braces is, for fixed (y,η), an element of $C^0_G(X^\wedge,\mathrm{g}_2)_{O,O}$. We may therefore first employ (3.9) in order to obtain the pointwise mapping properties required for elements in $\mathcal{R}^{\mu+\mu',0}_G(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$ and then homogeneity in connection with Lemma 1.4 for the conclusion.

What about $\zeta_1\zeta_2\{\tilde{a}_0a_0-c\}(1-\omega_5(t[\eta]))$? We may change ω_1, $\tilde{\omega}_1,\ldots$, at the expense of an operator in $\mathcal{R}^{\mu+\mu',0}_G(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$. Invoking Lemma 3.4(c) we can therefore show – just as above – that the term in question is congruent modulo $\mathcal{R}^{\mu+\mu',0}_G(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$ to

$$\zeta_1\zeta_2(1-\omega_1)\{\operatorname{op}(t^{-\mu'}q)\operatorname{op}(t^{\mu}p)-\operatorname{op}(t^{-\mu-\mu'}r)\}(1-\omega_3)(1-\omega_5); \qquad (3.27)$$

both (y,η) and $(t[\eta])$ have been omitted. The pseudodifferential operator between the braces is regularizing, hence given by an integral operator with a kernel that is rapidly decreasing in $t\eta$. For small $|\eta|$, the excision function vanishes. For η away from zero, it yields an element in $\mathcal{L}(\mathcal{K}^{s,\gamma}_1,\mathcal{S}^{\gamma-\mu-\mu'}_{3,O})$ for fixed (y,η); moreover, the estimates in the sense of (1.2) are $O((t[\eta])^{-N})=O([t]^{-N})$ for arbitrary N. Hence (3.27) defines an element of $S^{-N}(\Omega,\mathbf{R}^q;\mathcal{K}^{s,\gamma}_1,\mathcal{S}^{\gamma-\mu}_{3,O})$ for each N. Applying a similar argument to the adjoint, we conclude that (3.27) is a symbol in $\mathcal{R}^{-\infty,0}_G(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$.

So the case of t-independent symbols is proven.

Step 3. The t-dependent case. In case the symbols do depend on t, we use a Taylor expansion up to order N. According to the above consideration the polynomial part furnishes elements in $\mathcal{R}^{\mu+\mu',0}_G(\Omega\times\mathbf{R}^q,\mathrm{g}_2)_{O,O}$. So we can confine ourselves to the case where the symbols have compact support in t and we have an additional factor t^N. As in the proof of Proposition 3.11 the resulting term then induces an element

$$g_N\in S^{-N}(\Omega,\mathbf{R}^q;\mathcal{K}^{s,\gamma}_1,\mathcal{K}^{s,\gamma-\mu-\mu'}_3),\quad s>-1/2$$

with the additional properties

$$\begin{aligned} g_N &\in S^{-N}(\Omega,\mathbf{R}^q;\mathcal{K}^{s,\gamma}_1,\mathcal{S}^{\gamma-\mu-\mu'}_{3,O}),\\ g_N^* &\in S^{-N}(\Omega,\mathbf{R}^q;\mathcal{K}^{s,\mu+\mu'-\gamma}_3,\mathcal{S}^{-\gamma}_{1,O}). \end{aligned}$$

Since N is arbitrary, this completes the proof. □

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