The Chiral de Rham Complex and Positivity of the Equivariant Signature of the Loop Space

V. Gorbounov, F. Malikov[1]

Abstract

In this note we show that the positivity property of the equivariant signature of the loop space, first observed in [MS1] in the case of the even-dimensional projective spaces, is valid for Picard number 2 toric varieties. A new formula for the equivariant signature of the loop space in the case of a toric spin variety is derived.

0. Introduction The equivariant signature of the loop space, an example of an elliptic type genus, is, in particular, a rule which assigns to a manifold X a power series in q:

$$X \mapsto sign(q, \mathcal{L}X) = b_0 + b_1 q + b_1 q^2 + \cdots .$$

The intuition behind its definition (a well-known theorem in fact, see sect. 2) is that $sign(q, \mathcal{L}X)$ is the equivariant index of a certain elliptic operator and therefore the coefficients b_j are integers, not necessarily positive since they are equal to the difference between the dimensions of two vector spaces. The cohomological computations of [MS1] fairly unexpectedly showed that in the case of the even-dimensional complex projective space $\mathbb{P}^{2n}$ the series $sign(q, \mathcal{L}\mathbb{P}^{2n})$ has positive coefficients. Moreover, there is a graded vector space naturally associated to $\mathbb{P}^{2n}$ so that b_j essentially equals the dimension of the j-th homogeneous component. Let us formulate this result more precisely.

Associated with any smooth manifold X there is a sheaf of vertex algebras Ω_X^{ch}, the chiral de Rham complex [MSV]. It is graded:

$$\Omega_X^{ch} = \oplus_{j\geq 0}(\Omega_X^{ch})_j,$$

and so are its cohomology groups. The following equality of formal power series was proven in [MS1] (cf. Theorem 4.1 below)

$$sign(q, \mathcal{L}\mathbb{P}^{2n}) = 2\sum_{j=0}^{\infty} \mathrm{dim} H^0(\mathbb{P}^{2n}, (\Omega^{ch}_{\mathbb{P}^{2n}})_j)q^j - 1. \tag{0.1}$$

One can say that the vector space $H^0(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}})$, a vertex algebra in fact, provides a realization of $sign(q, \mathcal{L}\mathbb{P}^{2n})$.

The present note came out of the discussion of an earlier version of [MS1], and (0.1) appearing in the final version of [MS1] is a result of this discussion. We further use the Borisov-Libgober formula for the equivariant signature of the loop space in the case of a toric variety [BL] to extend the positivity result to the Picard number 2 toric varieties. (Recall that the projective spaces exhaust the class of Picard number 1 smooth toric varieties.) The very nature of this calculation does not

[1]partially supported by an NSF grant

allow us to conclude whether there is a graded vector space such that it is naturally associated to the manifold in question and realizes this signature. The existence of such a vector space, be it a vertex algebra or not, remains an open question.

A similar computation combined with Witten's rigidity theorem gives the following formula for the equivariant signature of the loop space in the case of a spin toric variety X of complex dimension d (cf. Theorem 6.2):

$$sign(q, \mathcal{L}X) = signX \frac{\epsilon^{-\frac{d}{4}}}{2^d}, \tag{0.2}$$

where $signX$ is the signature of X and ϵ a well-known modular form defined by (2.2) below.

The r.h.s. of this formula is not unfamiliar to vertex algebra specialists. Let V be the vertex algebra of $2d$ free bosons coupled to $2d$ free fermions. V is naturally graded:

$$V = \oplus_{j=0}^{\infty} V_j$$

so that

$$\sum_{j=0}^{\infty} \dim V_j q^j = \epsilon^{-\frac{d}{4}}. \tag{0.3}$$

Having compared (0.1) to (0.2-0.3) one may perhaps be so bold as to conjecture that the vertex algebra V is isomorphic to $H^0(X, \Omega_X^{ch})$. If correct this conjecture will be a natural extension of the realization result (0.1) to the spin-case.

Acknowledgments. We are grateful to F.Hirzebruch and D.Zagier for their interest in this work. Part of this work was completed when we were visiting the Max-Planck-Institut für Mathematik in Bonn. We are grateful to the Institut for the outstanding working conditions.

1. Genera. In our brief review of the relevant genera we shall follow the book [HBJ].

Let R be a commutative ring, Ω^* the cobordism ring. Recall that $\Omega^* \otimes \mathbb{Q} = \mathbb{Q}[P^2, P^4, ...]$, where P^{2n} is the cobordism class of the complex projective space of dimension $2n$. According to Hirzebruch, a *genus* is a ring homomorphism

$$g : \Omega^* \otimes \mathbb{Q} \to R. \tag{1.1}$$

An invertible even formal power series $Q(x) = a_0 + a_1 x^2 + \dots$ with coefficients in R defines a genus, to be denoted g_Q, as follows. Let $x_1, \dots x_n$ be formal variables and p_i the i-th elementary symmetric function in x_i^2. Then

$$Q(x_1)Q(x_2)\dots Q(x_n) = a_0^n(1 + K_1(p_1) + K_2(p_1, p_2) + \dots)$$

where $K_i(p_1, \dots, p_i)$ is a (uniquely determined) homogeneous polynomial of degree $2i$. For a manifold X of dimension $4n$ define

$$g_Q(X) = a_0^n K_n(p_1, \dots p_n)[X], \tag{1.2}$$

where $p_i(M)$, $i = 1, ..., n$, is the i-th the Pontryagin class of X and $[X]$ is the fundamental class of X. In fact the rule $Q(x) \to g_Q$ sets up a 1-1 correspondence.

2. The elliptic genus and the equivariant signature of the loop space. The notion of an elliptic genus is due to Ochanine[O]. A genus is called elliptic if it is associated to a series $Q(x)$ such that:

if $f(x) = \frac{x}{Q(x)}$, then

$$(f')^2 = 1 - 2\delta f^2 + \epsilon f^4, \tag{2.1}$$

for some parameters δ and ϵ.

All solutions to (2.1) can be constructed as follows. Fix a lattice L in $\mathbb{C}$ with generators ω_1, ω_2 and let $\mathfrak{p}_L(z)$ be the corresponding Weierstraßfunction. Then

$$f(x) = \frac{1}{\sqrt{\mathfrak{p}_L(x) - \mathfrak{p}_L(\omega_1/2)}}$$

is a solution to (2.1) with

$$\delta = -\frac{2}{3}\mathfrak{p}_L(\omega_1/2),$$

$$\epsilon = [\mathfrak{p}_L(\omega_1/2) - \mathfrak{p}_L(\omega_2/2)][\mathfrak{p}_L(\omega_1/2) - \mathfrak{p}_L((\omega_1 + \omega_2)/2].$$

Therefore each elliptic genus equals (cf. (1.2))

$$g_{x\sqrt{\mathfrak{p}_L(x) - \mathfrak{p}_L(\omega_1/2)}}(.),$$

for some L and ω_1, and $g_{x\sqrt{\mathfrak{p}_L(x) - \mathfrak{p}_L(\omega_1/2)}}(.)$, as a function of L and ω_1, can be considered the universal elliptic genus.

Introducing the modular parameter $q = \exp(2\pi i \omega_2/\omega_1)$ one sees that all the defined expressions are naturally identified with functions of q. In particular, δ and ϵ become the standard generators of the ring $M_*(2)$ of modular forms on $\Gamma_0(2)$ so that $\delta \in M_2(2)$, $\epsilon \in M_4(2)$. Needed for (2.6) below is the following formula for ϵ:

$$\epsilon = (2\prod_{n=1}^{\infty} \frac{(1+q^n)^2}{(1-q^n)^2})^{-4}, \tag{2.2}$$

Likewise, for any X, $g_{x\sqrt{\mathfrak{p}_L(x) - \mathfrak{p}_L(\omega_1/2)}}(X)$ is a function of q. We shall emphasize this by changing (and unburdening) the notation as follows:

$$och(q, X) = g_{x\sqrt{\mathfrak{p}_L(x) - \mathfrak{p}_L(\omega_1/2)}}(X).$$

In fact, for a manifold X of real dimension $4k$, $och(q, X)$ is a weight $2k$ polynomial in δ counted with weight 2 and ϵ counted with weight 4. Therefore $och(q, X)$ is a weight $2k$ modular form on $\Gamma_0(2)$.

We will be more interested in a closely related genus, $sign(q, \mathcal{L}X)$, proposed by Witten and called the formal equivariant signature of the loop space [HBJ]. By definition

$$sign(q, \mathcal{L}X) = g_Q, \tag{2.3a}$$

where

$$Q(x) = x\frac{1+e^{-x}}{1-e^{-x}}\Pi_{n=1}^{\infty}\frac{(1+q^n e^{-x})(1+q^n e^{x})}{(1-q^n e^{-x})(1-q^n e^{x})}. \qquad (2.3b)$$

This genus can be equivalently defined to be the signature of X twisted by the bundle

$$W = \otimes_{n=1}^{\infty} S_{q^n} TX_{\mathbb{C}} \otimes_{n=1}^{\infty} \Lambda_{q^n} TX_{\mathbb{C}}, \qquad (2.4)$$

where we habitually set

$$\Lambda_t E = \sum_{i=0}^{\infty} t^i \Lambda^i E, \qquad (2.5a)$$

$$S_t E = \sum_{i=0}^{\infty} t^i S^i E. \qquad (2.5b)$$

Yet another possibility is to define $sign(q, \mathcal{L}X)$ to be the index of the elliptic operator $d + d^*$ acting on the global sections of the bundle (2.4).

We conclude this section by noting that $sign(q, \mathcal{L}X)$ is related to the elliptic genus as follows:

$$sign(q, \mathcal{L}X) = och(q, X)\epsilon^{-\frac{k}{2}}, \qquad (2.6)$$

where ϵ is defined by (2.2) and $dimX = 4k$.

3. The chiral de Rham complex and the equivariant signature of the loop space. Defined in [MSV] for any smooth complex manifold X of complex dimension n there is a sheaf $\Omega_X^{ch}(X)$ of vertex algebras over X. Morally, $\Omega_X^{ch}(X)$ is a semi-infinite de Rham complex on a "small" loop space with coefficients in distributions supported on the submanifold of analytically contractible loops. This vague assertion was made a theorem in [KV] after overcoming considerable technical difficulties.

$\Omega_X^{ch}(X)$ is not a sheaf of $\mathcal{O}_X$-modules but it possesses a filtration such that the associated graded sheaf $gr\Omega_X^{ch}$ is. In fact, $gr\Omega_X^{ch}$ is associated to a holomorphic vector bundle and this vector bundle was explicitly described in [BL] as follows:

$$gr\Omega_X^{ch} = \otimes_{n=1}^{\infty}\{S_{q^n}TX \otimes S_{q^n}TX^* \otimes \Lambda_{yq^{n-1}}T^*X \otimes \Lambda_{y^{-1}q^n}TX\}, \qquad (3.1)$$

where we use the notation introduced in (2.5a,b). Thus $gr\Omega_X^{ch}(X)$ is bi-graded so that the component of weight (i,j) is the coefficient of $q^j y^i$. In fact, the sheaf Ω_X^{ch} is itself bi-graded :

$$\Omega_X^{ch} = \oplus_{i\in\mathbb{Z}, j\geq 0}(\Omega_X^{ch})_j^i, \qquad (3.2)$$

and this bi-grading descends to the graded object. In particular, Ω_X^i, i.e. the sheaf of holomorphic i-forms , canonically identifies with $(\Omega_X^{ch})_0^i$:

$$\Omega_X^i \xrightarrow{\sim} (\Omega_X^{ch})_0^i, \qquad (3.3.)$$

which partially justifies the name " a chiral de Rham complex".

Consider the Euler character $\mathrm{Eu}(\Omega_X^{ch})(q,y)$, which by definition is a formal Laurent power series in q, y such that the coefficient of $y^i q^j$ is the Euler characteristic of the component $(\Omega_X^{ch})_j^i$. Define $\mathrm{Eu}(\Omega_X^{ch})(q)$ to be $\mathrm{Eu}(\Omega_X^{ch})(q,1)$ Applications of Ω_X^{ch} to elliptic genera are based on the following observation due to [BL]:

$$\mathrm{Eu}(\Omega_X^{ch})(q) = sign(q, \mathcal{L}X). \tag{3.4}$$

Let us prove (3.4) for the sake of completeness. Since the Euler characteristic does not change under the passage to the graded object, we can write

$$\mathrm{Eu}(\Omega_X^{ch})(q) = \mathrm{Eu}(gr\Omega_X^{ch})(q) = \int_X ch(gr\Omega_X^{ch})tdX, \tag{3.5}$$

where the 2nd equality follows from the Riemann-Roch Theorem, and tdX is the Todd genus of X. We now compute $ch(gr\Omega_X^{ch})$ by using (3.1) and the multiplicativity of ch to the effect that

$$ch(gr\Omega_X^{ch}) = \prod_{n=1}^{\infty}\{ch(S_{q^n}TX)ch(S_{q^n}TX^*)ch(\Lambda_{q^{n-1}}T^*X)ch(\Lambda_{q^n}TX)\}. \tag{3.6}$$

As is well known for a k dimensional manifold X

$$ch\Lambda_{q^{n-1}}TX = \Pi_{i=1}^k(1+q^{n-1}e^{x_i}), \quad ch\Lambda_{q^{n-1}}TX^* = \Pi_{i=1}^k(1+q^{n-1}e^{-x_i}),$$

$$chS_{q^n}TX = \Pi_{i=1}^k\frac{1}{(1-q^ne^{x_i})}, \quad chS_{q^n}TX^* = \Pi_{i=1}^k\frac{1}{(1-q^ne^{-x_i})},$$

$$tdX = \Pi_{i=1}^k\frac{x_i}{1-e^{-x_i}}.$$

Plugging these in the right hand side of (3.6) we readily see that the integrand in the right hand side of (3.5) is $Q(x_1)Q(x_2)\cdots Q(x_k)$, where $Q(x)$ is the series (2.3b). Therefore (3.4) is identical to the definition (1.2) with $Q(x)$ given by (2.3b). $\square$

Note that $\mathrm{Eu}(\Omega_X^{ch})(0)$ is equal to the signature of the manifold X.

4. Positivity of the equivariant signature of the loop space.

Introduce the formal character

$$chH^i(X, \Omega_X^{ch}) = \sum_{j=0}^{\infty}\mathrm{dim}H^i(X, (\Omega_X^{ch})_j)q^j,$$

where we ignore the upper-index grading, cf. (3.2). (It follows easily from (3.1) that the dimensions entering the formal character are all finite.)

Theorem 4.1. The following character formula is valid

$$2chH^0(\mathbb{P}^{2n}, \Omega_{\mathbb{P}^{2n}}^{ch}) - 1 = sign(q, \mathcal{L}\mathbb{P}^{2n}).$$

Corollary 4.2. The series $sign(q, \mathcal{L}\mathbb{P}^{2n})$ has positive coefficients and the coefficients against positive powers of q are even.

Corollary 4.2 is an immediate consequence of Theorem 4.1 whereas Theorem 4.1 is a simple consequence of certain properties of the chiral de Rham complex over projective spaces discovered in [MS1]. We shall list these properties here and reproduce from [MS1] a simple computation leading to Theorem 4.1.

The definition of the Euler characteristic and (3.5) give

$$sign(q, \mathcal{L}\mathbb{P}^{2n}) = \sum_{i=0}^{2n} (-1)^i chH^i(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}}). \tag{4.1}$$

We now make use of the following

Theorem [MS1]. The natural embedding of sheaves $\Omega^*_{\mathbb{P}^n} \hookrightarrow \Omega^{ch}_{\mathbb{P}^n}$ due to (3.3) provides an isomorphism

$$H^i(\mathbb{P}^n, \Omega^*_{\mathbb{P}^n}) \xrightarrow{\sim} H^i(\mathbb{P}^n, \Omega^{ch}_{\mathbb{P}^n}).$$

for $0 < i < n$.

This assertion reduces (4.1) to

$$sign(q, \mathcal{L}\mathbb{P}^{2n}) = chH^0(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}}) + chH^{2n}(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}}) - 1. \tag{4.2}$$

To conclude it remains to notice that due to the chiral Poincaré duality [MS2]

$$chH^0(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}}) = chH^{2n}(\mathbb{P}^{2n}, \Omega^{ch}_{\mathbb{P}^{2n}}).$$

□

Remark The vector space $H^0(\mathbb{P}^n, \Omega^{ch}_{\mathbb{P}^n})$ is a vertex algebra. Thanks to Theorem 4.1, the known [HBJ] modular properties of the equivariant signature of the loop space say that $chH^0(\mathbb{P}^n, \Omega^{ch}_{\mathbb{P}^n})$ is a modular function when $n = 0 \bmod 4$. The modular properties of characters have been the hallmark of vertex algebra theory, see for example [Z].

5. Extending the positivity result

5.1 It seems natural to ask in what generality the positivity result (Corollary 4.2) holds true. Since according to [BR] every cobordism class contains a non-singular toric variety, not every non-singular toric variety X has positive $sign(q, \mathcal{L}X)$. We will show nevertheless that apart from projective spaces there is a class of toric varieties with this property. To do so, we will have to rely on the calculations from [BL] instead of sheaves of vertex algebras.

When talking about toric varieties we shall keep to the following notation. Let e_i, $1 \le i \le d$, be the standard basis of $\mathbb{Z}^d$; thus the j-th component of e_i is δ_{ij}. Define the inner product

$$\mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}, \; x, y \mapsto x \cdot y$$

by the requirement that

$$e_i \cdot e_j = \delta_{ij}.$$

This identifies $\mathbb{Z}^d$ and hence $\mathbb{R}^d$ with their duals.

By Σ we denote a *complete, regular* fan in $\mathbb{R}^d$. This means, in particular, that each cone $C^* \in \Sigma$ is spanned by part of a basis of $\mathbb{Z}^d$, and we denote this spanning set by $|C^*|$.

Associated to Σ there is a smooth compact toric variety of complex dimension d to be denoted X_Σ or simply X if no confusion is likely to arise.

The following formula holds true [BL]

$$sign(q, \mathcal{L}X) = \sum_{m \in \mathbb{Z}^d} \sum_{C^* \in \Sigma} (-1)^{codim C^*} \Big(\prod_{n \in |C^*|} \frac{1}{1 + q^{m \cdot n}} \Big) \epsilon^{-d/4}. \tag{5.1}$$

Note that to make sense out of this expression one has to expand each factor

$$\frac{1}{1 + q^{m \cdot n}}$$

at $q = 0$ and then convince oneself that the sum of thus arising power series with respect to $m \in \mathbb{Z}^d$ makes sense as a formal power series.

Remark The formula in Theorem 5.5 in [BL] where we borrowed (5.1) contains an extra factor $(-1)^{d/2}$. We drop it so as to conform to the standard notation.

5.2 Now we extend the result of Theorem 4.1 to a larger class of toric variates. Recall that the Picard number of a toric variety is the difference between the dimension of the variety and the number of one dimensional cones. Each smooth Picard number one toric variety is a projective space. The Picard number two toric varieties, which can be viewed as generalized Hirzebruch's surfaces, were classified in [Kl]. Let us formulate this result.

We give ourselves a triple of integers d, s, r such that $1 < d$, $1 < s < d + 1$, $r = d - s + 1$, and an increasing sequence of non-negative integers $a_1, \ldots a_r$. Define the following vectors in $\mathbb{Z}^d$:

$$u_i = e_i, 0 < i < r + 1;$$

$$u_{r+1} = -\sum_{i=1}^{r} u_i;$$

$$v_j = e_{r+j}, 0 < j < s;$$

$$v_s = \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s-1} v_j;$$

We set $U = \{u_1, \ldots, u_{r+1}\}$, $V = \{v_1, \ldots, v_s\}$ and let $C^*_{ij} \subset \mathbb{R}^d$, $0 < i < r + 2$, $0 < j < s + 1$, be the cone spanned by $U \cup V \setminus \{u_i, v_j\})$. One checks that there is a uniquely determined regular complete fan such that $\{C^*_{ij},\ 0 < i < r + 2, 0 < j < s + 1\}$ is the set of d-dimensional cones. We denote this fan by $\sum_d(a_1, \ldots a_r)$ and the corresponding toric variety by $X_d(a_1, \ldots a_r)$.

Theorem 5.1 [Kl] Every compact smooth toric variety of complex dimension d with $d+2$ generators is isomorphic to precisely one of the varieties $X_d(a_1, \dots a_r)$.

Theorem 5.2

1. For d even

$$sign(q, \mathcal{L}\mathbb{P}^d) = \sum_{m \in \mathbb{Z}^d} \frac{2}{(1+q^{-m_1 \dots -m_d}) \prod_{i=1}^d (1+q^{m_i})} \epsilon^{-d/4}, \tag{5.2}$$

where $m = (m_1 \dots m_d)$. The series in the RHS has positive coefficients.

2. For a smooth toric variety X of even complex dimension d with Picard number 2, $sign(q, \mathcal{L}X)$ is 0 if s is even. Otherwise

$$sign(q, \mathcal{L}X) = \sum_{m \in \mathbb{Z}^d} \frac{2 + 2q^{m \cdot (v_1 + \dots v_s)}}{\prod_{i=1}^{r+1} (1+q^{m \cdot u_i}) \prod_{j=1}^s (1+q^{m \cdot v_j})} \epsilon^{-d/4} \tag{5.3}$$

The series in the RHS has positive coefficients.

Remark. Statement 1. of Theorem 5.2 is an alternative to the chiral de Rham complex approach of sect. 4. But note that formula (5.2) is of a different nature than that in Theorem 4.1, and the comparison of (5.2) and Theorem 4.1 may give rise to non-trivial combinatorial identities.

Proof. Both of the statements follow from (5.1) and the explicit description of the fans of the toric varieties in question.

1. Let $k_1 = e_1, \dots, k_d = e_d, k_{d+1} = -e_1 \dots - e_d$. The fan defining $\mathbb{P}^d$ consists of the cones spanned by all proper subsets of the set $\{k_1, \dots, k_{d+1}\}$.

Now observe that (5.1) rewrites as follows:

$$sign(q, \mathcal{L}X) = \epsilon^{-d/4} \sum_{m \in M} < m, \sum_{C^* \in \Sigma} (-1)^{codim C^*} (\prod_{k \in |C^*|} \frac{1}{1+q^k}) >, \tag{5.4}$$

where q^k is an element of the group ring of $\mathbb{Z}^d$ and $< m, q^k >= q^{m \cdot k}$.

The expression

$$\sum_{C^* \in \Sigma} (-1)^{codim C^*} (\prod_{k \in |C^*|} \frac{1}{1+q^k}), \tag{5.5}$$

(appearing in the r.h.s. of (5.4)) in the case of the projective space simplifies as follows:

$$\sum_B (-1)^{d - \#B} \frac{1}{\prod_{k \in B} (1+q^k)}, \tag{5.6}$$

where the summation is performed over all *proper* subsets $B \subsetneq \{k_1, \dots, k_{d+1}\}$ and $\#B$ is the number of elements in B. Converting (5.6) to the common denominator we obtain

$$\frac{\sum_B (-1)^{d-\#B} \prod_{k \in \bar{B}} (1+q^k)}{\prod_{i=1}^{d+1} (1+q^{k_i})}, \tag{5.7}$$

where $\bar{B}$ denotes the complement of B. The multiple use of the binomial identuty

$$\sum_{i=0}^{n}(-1)^i C_n^i = 0 \text{ if } n > 0 \tag{5.8}$$

allows us to collect the like terms in the numerator of (5.7). The result is

$$\frac{1+(-1)^d q^{k_1+\ldots+k_{n+1}}}{\prod_{i=1}^{d+1}(1+q^{k_i})}. \tag{5.9}$$

Since $k_1 + k_2 + \cdots + k_{d+1} = 0$, it is zero if d is odd (as it should !) and

$$\frac{2}{\prod_{i=1}^{d+1}(1+q^{k_i})} \tag{5.10}$$

otherwise. Plugging (5.10) in the r.h.s. of (5.4) and performing pairing with $m \in \mathbb{Z}^d$ we obtain (5.2), as desired. It is clear that every term in the denominator of (5.2) cancels against the appropriate term in $\epsilon^{-d/4}$. Indeed, for a fixed $m \in \mathbb{Z}^d$ the denominator of (5.2) contains at most $d+1$ factors $(1+q^n)$ for each $n \in \mathbb{Z}$ whereas $\epsilon^{-d/4}$, see (2.2), contains $2d$ such factors in the numerator. Therefore all the coefficients in the series $sign(q, \mathcal{L}\mathbb{P}^d)$ are positive.

2. Proof of the second statement is similar. Theorem 5.1 combined with (5.1) gives the following analogue of (5.4):

$$sign(q, \mathcal{L}X) = \epsilon^{-d/4} \sum_{m \in M} < m, \sum_{I,J} \frac{(-1)^{d-\#I-\#J}}{\prod_{i\in I}(1+q^{u_i})\prod_{j\in J}(1+q^{v_j})} >, \tag{5.11}$$

where I and J are *proper* subsets of $\{1, \ldots, r+1\}$ and $\{1, \ldots, s\}$ respectively and the vectors u_i, v_j are those defined in the beginning of 5.2.

Converting the sum $\sum_{I,J}$ in (5.11) to the common denominator we obtain

$$\sum_{I,J} \frac{(-1)^{d-\#I-\#J}\prod_{i\in \bar{I}}(1+q^{u_i})\prod_{j\in \bar{J}}(1+q^{v_j})}{\prod_{i=1}^{r+1}(1+q^{u_i})\prod_{j=1}^{s}(1+q^{v_j})} \tag{5.12}$$

Now observe that the numerator of (5.12) is the product of two factors analogous to the numerator of (5.7) – one is the numerator of (5.7) with B replaced with $\{u_i,\ i \in I\}$, another is also with B replaced with $\{v_j,\ j \in J\}$. Therefore identity (5.8), which allowed us to pass from (5.7) to (5.9), allows us to analogously rewrite (5.12) as follows:

$$\frac{(1+(-1)^r q^{u_1+u_2+\cdots+u_{r+1}})(1+(-1)^{s-1}q^{v_1+v_2+\cdots+v_s})}{\prod_{i=1}^{r+1}(1+q^{u_i})\prod_{j=1}^{s}(1+q^{v_j})}. \tag{5.13}$$

Since by definition (see the beginning of 5.1)

$$\sum_{i=1}^{r+1} u_i = 0,$$

(5.13) vanishes if r is odd and equals

$$\frac{2(1+q^{v_1+v_2+\cdots+v_s})}{\prod_{i=1}^{r+1}(1+q^{u_i})\prod_{j=1}^{s}(1+q^{v_j})} \tag{5.14}$$

otherwise. (Note that $(-1)^{s-1}$ has disappeared because, d being even, the relation $d=r+s-1$ forces r and s to have different parity.)

Plugging (5.14) in the r.h.s. of (5.11) and performing pairing with $m\in\mathbb{Z}^d$ we obtain desired formula (5.3).

To show that series (5.3) has positive coefficients we observe that for a fixed $m\in\mathbb{Z}^d$ the denominator of (5.3) contains at most $r+s+1$, that is, $d+2$ factors $(1+q^n)$ for each $n\in\mathbb{Z}$ whereas $\epsilon^{-d/4}$, see (2.2), contains $2d$ such factors in the numerator. Therefore, having carried out the cancellations we make (5.3) into a sum of power series with positive coefficients. $\square$

6. Toric spin varieties

Recall Witten's rigidity theorem [W] proved in [BT]. Let a torus T^n act on a manifold X. This action lifts to an action on the holomorphic bundle $gr\Omega_X^{ch}$, see (3.1) Therefore each cohomology group $H^i(X, gr\Omega_X^{ch})$ becomes a T^n-module – a direct sum of the torus characters $t\mapsto t^m$, $t\in T^n$, $m\in\mathbb{Z}^n$, in fact. Formula (3.4) then implies that $sign(q,\mathcal{L}X)$ is a formal sum of the torus characters and one can think of $sign(q,\mathcal{L}X)$ as a function of $t\in T^n$ with values in $\mathbb{C}[[q]]$.

Theorem 6.1 [BT] If X is a spin manifold equipped with an action of a torus T^n, then $sign(q,\mathcal{L}X)$ is a constant function of $t\in T^n$.

Theorem 6.2 For a toric spin variety X of complex dimension d,

$$sign(q,\mathcal{L}X) = signX\frac{\epsilon^{-\frac{d}{4}}}{2^d}.$$

Proof. Any toric variety X carries the natural action of a torus; hence $sign(q,\mathcal{L}X)$ is a formal sum of the torus characters. Formula (5.1) sharpens accordingly [BL]:

$$sign(q,\mathcal{L}X) = \sum_{m\in M} t^m \sum_{C^*\in\Sigma}(-1)^{codimC^*}\Big(\prod_{i=1,\ldots dimC^*}\frac{1}{1+q^{m\cdot n_i}}\Big)\epsilon^{-d/4}. \tag{6.1}$$

Theorem 6.1 implies that only t^0 may appear in the r.h.s. of (6.1) with non-zero coefficient. Therefore

$$sign(q,\mathcal{L}X) = C\epsilon^{-d/4} \tag{6.2}$$

for some constatnt C. To compute C recall that

$$sign(0,\mathcal{L}X) = signX.$$

Therefore, having specialized (6.2) to $q=0$ we obtain

$$signX = C2^d,$$

and Theorem 6.2 follows. $\square$

Remarks.

1) According to [HS] $sign(q, \mathcal{L}X) = sign X$ if X is a homogeneous space and a spin manifold at the same time. Therefore Theorem 6.2 is an extension of this result to spin toric varieties.

2) Theorem 6.2 suggests that perhaps the vertex algebra of $2d$ bosons coupled to $2d$ fermions provides a natural realization of $sign(q, \mathcal{L}X)$ in the case of a toric spin manifold as discussed in greater detail in the introduction .

References

[BL] L. Borisov, A. Libgober, Elliptic Genera and Applications to Mirror Symmetry, Inv. Math. **140** (2000), 453-485;.

[BT] R. Bott, C. Taubes, On the rigidity theorems of Witten. J. Amer. Math. Soc. 2 (1989), no. 1, 137–186.

[D] A. N. Dessai, Elliptic genera, positive curvature and symmetry, preprint, 2002.

[GMS] V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differential operators **II** and **III** math AG/0003170, math AG/0005201.

[HBJ] F. Hirzebruch, Th. Berger, R. Jung, Manifolds and modular forms. With appendices by Nils-Peter Skoruppa and by Paul Baum. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. xii+211 pp.

[HS] F. Hirzebruch, P. Slodowy, Elliptic genera, involutions, and homogeneous spin manifolds. Geom. Dedicata 35 (1990), no. 1-3, 309–343.

[K] V. Kac, Vertex algebras for beginners, Second Edition, University Lecture Series, **10**, American Mathematical Society, Providence, Rhode Island, 1998.

[Kl] P. Kleinschidt, A classification of toric varieties with few generators, Aerquationes Mathematicae 35 (1988) 254-266.

[KV] M. Kapranov, E. Vasserot, Vertex algebras and the formal loop space, math.AG/0107143.

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, *Comm. Math. Phys.*, **204** (1999), 439-473.

[MS1] F. Malikov, V. Schechtman, Deformations of chiral algebras and quantum cohomology of toric varieties, to appear in Comm. Math. Phys.

[MS2] F. Malikov, V. Schechtman, Chiral Poincaré duality, Math. Res. Lett. vol. **6** (1999), 533-546.

[O] S. Ochanine, Sur les genres multiplicatifs définis par des intégrales elliptiques. (French) [On multiplicative genera defined by elliptic integrals] Topology 26 (1987), no. 2, 143–151.

[W] E. Witten, The index of the Dirac operator in loop space, in: Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), 161-181, Lect. Notes in Math. **1326** (1988).

[Z] Y. Zhu, Modular invariance of characters of vertex operators algebras, J. Amer. Math. Soc. **9** (1996), no 1, 237–302.