Small Divisors and the Construction of Stable Manifolds for Nonlinear Klein–Gordon Equations on $\mathbb{R}_0^+ \times \mathbb{R}$. *

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A.5 Zum Gradienten

Wir zeigen: Sei $U \subseteq \mathbb{R}^n$ offen und zu je zwei Punkten P und Q gebe es einen Weg $c : [a,b] \to U$ mit $c(a) = P$ und $c(b) = Q$. Weiter seien $f_1, f_2 : U \to \mathbb{R}$ zwei Funktionen mit $\operatorname{grad} f_1 = \operatorname{grad} f_2$. Dann gibt es ein $k \in \mathbb{R}$ mit $f_1 = f_2 + k$.

Beweis. Wir definieren eine Funktion $h : U \to \mathbb{R}$ durch $h := f_1 - f_2$. Dann gilt wegen der Voraussetzung $\operatorname{grad} h = 0$. Zu zeigen ist, das $h \equiv k$ für ein $k \in \mathbb{R}$ gilt. Dazu betrachten wir zwei beliebige Punkte $P, Q \in U$ und einen Weg $c : [a,b] \to U$ in U zwischen P und Q. Die Kettenregel liefert wieder

$$(h \circ c)'(t) = \langle \operatorname{grad} h(c(t)), c'(t) \rangle.$$

Wegen $\operatorname{grad} h = 0$ ist nun $\langle \operatorname{grad} h(c(t)), c'(t) \rangle = 0$, also $h \circ c$ konstant. Daraus folgt, daß $h(P) = h(c(a)) = h(c(b)) = h(Q)$ gilt. Also ist für beliebige Punkte $P, Q \in U$ gezeigt, daß $h(P) = h(Q)$ gilt, also ist h konstant.

A.6 Stammfunktionen zu Vektorfeldern

Es sei U eine Teilmenge des $\mathbb{R}^n$ und $F : U \to \mathbb{R}^n$ ein gegebenes Vektorfeld. Eine Funktion $\phi : U \to \mathbb{R}$ mit $\operatorname{grad} \phi = F$ heißt *Stammfunktion* zu F.

Uns interessiert nun, wann es solch eine Stammfunktion gibt. Dazu betrachten wir zunächst folgenden Spezialfall:

- Es sei $n = 2$ und F gegeben durch die Funktionen $f, g : U \to \mathbb{R}$. Nehmen wir an, es gebe eine Funktion ϕ mit $\operatorname{grad} \phi = F$, also $F = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$. Dies bedeutet, daß gerade $f = \frac{\partial \phi}{\partial x_1}$ und $g = \frac{\partial \phi}{\partial x_2}$ ist. Dann ist aber $\frac{\partial f}{\partial x_2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_2}$ und $\frac{\partial g}{\partial x_1} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}$. Wenn nun ϕ von der Klasse C^1 ist, dann ist $\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1}$, also gilt dann $\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_1}$.

Analog zeigt man allgemein: Wenn es ein $\phi \in C^1$ mit $F = \operatorname{grad} \phi$ gibt, dann gilt $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ für alle i, j.

Wir können uns nun fragen: Wenn umgekehrt $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$ gilt, gibt es dann eine Stammfunktion ?

Die Antwort liefert der folgende

Satz. *Es sei U ein Rechteck im $\mathbb{R}^n$, d.h ein kartesisches Produkt von offenen Intervallen in $\mathbb{R}$. Weiter sei $F = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} : U \to \mathbb{R}^n$ mit $f_i : U \to \mathbb{R}$ eine differenzierbare Funktion mit $\frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j}$. Dann gibt es eine Stammfunktion $\phi : U \to \mathbb{R}$ mit $F = \operatorname{grad} \phi$.*

Beweis. Wir führen den Beweis für den Fall $n = 2$ mit $f_1 = f$ und $f_2 = g$. Im allgemeinen Fall schließt man ganz analog.

Abstract. In this expository article we analyze a new method which was recently devised to construct smooth local stable manifolds for certain infinite–dimensional dynamical systems. Some of those systems are associated with nonlinear Klein–Gordon equations in one space dimension. They are characterized by the fact that the corresponding linearized flow has a spectrum which may consist of either a purely continuous part, or of a continuum of eigenvalues, or of the union of a point spectrum with a continuous spectrum in which the point spectrum is everywhere dense. The major difficulty to overcome in our construction is thereby a small divisor problem; we resolve it without using KAM – or related techniques. A corollary to our main result is the existence of spatially localized time–quasi-periodic classical solutions to nonlinear Klein–Gordon equations on $\mathbb{R}_0^+ \times \mathbb{R}$. The set of Fourier exponents of such solutions possesses a preassigned integer basis consisting of finitely many rationally independent frequencies.

1. Introduction and Outline. There are at least two well known techniques to construct stable manifolds of dynamical systems around an equilibrium point. The first one is the geometric method of Hadamard ([1], [2]), while the second one is the celebrated fixed point method of Liapounov and Perron ([3] – [6]). Both have been widely used by many authors in various contexts, which includes the stability theory of nonlinear parabolic and hyperbolic partial differential equations ([7] – [16]). In both cases a detailed knowledge of the spectrum of the linearized flow is necessary. Typically, the spectral information is used to show that the nonlinear flow behaves locally as its linearized counterpart, at least in the stable and unstable directions. Moreover, in most applications the success of the two methods has mainly been the fact that the spectrum of the linearized flow possesses nice separation properties. However, there are important classes of problems for which the spectral properties of the linearization are not nice from the point of view of dynamical system theory. It is the purpose of this short expository article to describe such a class of problems, and to show that one can nevertheless still construct appropriate stable

manifolds for them.

Consider for instance nonlinear Klein–Gordon equations of the form

$$u_{tt}(x,t) = u_{xx}(x,t) - g(u(x,t)) \tag{1.1}$$

where $(x,t) \in \mathbb{R}_0^+ \times \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Equations of the form (1.1) occur in various contexts, such as in the description of wave propagation phenomena in superconductors, ferromagnets and nonlinear optics (with $g(u) = \sin(u)$ or $g(u) = \sin(u) + \sin(u/2)$) , in the theory of dislocation of crystals (for example with $g(u) = u - u^2 - u^3$) and in the classical modelling of certain phenomena in field theory (with $g(u) = u^3 - u$) ([17] – [21]) . Suppose now that we fix an $N \in \mathbb{N}^+ \cap [2,\infty)$, and that we preassign a rationally independent set of frequencies $\{\omega_1,\dots,\omega_N\} \subset \mathbb{R}/\{0\}$. In view of the above applications of equation (1.1) , it is then important and natural to ask whether there exist suitable restrictions concerning the nonlinearity g , in such a way that equation (1.1) possesses classical solutions of the form

$$u(x,t) \sim \sum_{k \in \mathbb{Z}^N} u_k(x) \exp[i\Omega_k t] \tag{1.2}$$

(Throughout this article, the tilda refers to the formal Fourier series associated with the function on the left–hand side). In relation (1.2) , we require that the Fourier series eventually converges in some appropriate topology, that $x \to u_k(x)$ decays exponentially rapidly for every k , and that $\Omega_k = \sum_{\ell=1}^{N} k_\ell \omega_\ell$ for some $k_\ell \in \mathbb{Z}$. In other words, we are looking for spatially localized classical solutions to equation (1.1) which are time–quasiperiodic, with the property that the corresponding set of Fourier exponents be generated by $\{\omega_1,\dots\omega_N\}$ over $\mathbf{Z}$. From the mathematical viewpoint this problem has had a relatively short history, and we refer the reader to ([22] – [25]) for a description of its physical origins in the periodic case. One may reasonably ask whether there is any connection at all between the above problem and the construction of stable manifolds in the presence of small divisors. The answer is affirmative for the following reason: there is first a very natural way to go about solving the problem of existence for solutions of the form (1.2) to equation (1.1) . It amounts to exploiting the deep connections between almost–periodic functions and periodic functions of several variables, as was first brought about in the beautiful works of Bohr [26] . For $j = 1, \dots N$, let $\tau_j = 2\pi|\omega_j|^{-1}$ be the periods associated with each one of the frequencies ω_j , and consider the N–dimensional torus $T_N = \prod_{j=1}^{N} \mathbb{R}/\tau_j\mathbf{Z}$. Consider then the infinite–dimensional dynamical system

$$\begin{aligned} U_x(x,t_1,\dots t_N) &= V(x,t_1,\dots t_N) \\ V_x(x,t_1,\dots t_N) &= \Box_N U(x,t_1,\dots t_N) + g(U(x,t_1,\dots t_N)) \end{aligned} \tag{1.3}$$

on $\mathbb{R}_0^+ \times T_N$, where we have defined

$$\Box_N U(x,t_1\dots t_N) = \sum_{i,j=1}^{N} U_{t_i t_j}(x,t_1\dots t_N) \tag{1.4}$$

We claim that the problem of constructing solutions of the form (1.2) to Problem (1.1) is then essentially reduced to constructing a stable manifold for the dynamical system (1.3) , in an appropriate Banach space of multiperiodic functions on T_N . In fact, suppose that

$$U(x,t_1,\dots t_N) \sim \sum_{k\in Z^N} U_k(x)\exp\left[i\sum_{j=1}^{N}\omega_j k_j t_j\right] \tag{1.5}$$

and

$$V(x,t_1,\dots t_N) \sim \sum_{k\in Z^N} U_k'(x)\exp\left[i\sum_{j=1}^{N}\omega_j k_j t_j\right] \tag{1.6}$$

denotes a solution pair lying on such a manifold; by definition, it then exists globally for every x and exhibits a behaviour with exponential decay. Moreover, the function given by (1.5) is formally a solution to the partial differential equation

$$\square_N U(x,t_1,\dots t_N) = U_{xx}(x,t_1,\dots t_N) - g(U(x,t_1\dots t_N)) \tag{1.7}$$

since (1.7) is formally equivalent to system (1.3) . Consider then the section of the function (1.5) by the main diagonal on T_N , namely

$$u(x,t) = U(x,t,\dots t) \sim \sum_{k\in Z^N} U_k(x)\exp[i\Omega_k t] \tag{1.8}$$

It is then clear that u formally solves equation (1.1) , since the formal Fourier series of $(x,t) \to u_{tt}(x,t)$ and of $(x,t) \to \square_N U(x,t_1\dots t_N)|_{t_1=t_2=\dots=t_N=t}$ are the same, namely

$$u_{tt}(x,t) = \square_N U(x,t_1\dots t_N)|_{t_1=t_2=\dots=t_N=t} \sim \sum_{k\in Z^N} -\Omega_k^2\, U_k(x)\exp[i\Omega_k t] \tag{1.9}$$

In the remaining part of this article it is our intention to show that the above ideas can be implemented rigorously. In Section 1 , we prove that the spectrum of the linearization of (1.3) around the desired equilibrium solution consists of the union of a point spectrum and of a continuous spectrum in which the point spectrum is everywhere dense. This fact is due to the rational independence of the ω_j's , and lies at the very source of the small divisor difficulty. We are nevertheless able to formulate a stable manifold theorem for system (1.3) , whose proof we briefly outline. We finally state a corollary concerning the existence of solutions of the form (1.2) to equation (1.1) . Section 3 is devoted to the discussion of an open problem.

2. A Stable Manifold Theorem for System (1.3) in the Presence of Small Divisors.

Consider equation (1.1) or system (1.3) ; we shall assume that the nonlinearity g satisfies the following hypothesis:

(G) $g: \mathbb{R} \to \mathbb{R}$ is entire analytic and there exists $u_0 \in \mathbb{R}$ such $g(u_0) = 0$ and $g'(u_0) > 0$.

Now write $\mathscr{C}(T_N, \mathbb{C})$ for the Banach algebra of all complex continuous functions on T_N equipped with the uniform norm. In order to investigate system (1.3) and control the small divisors associated with the corresponding spectral analysis, it is necessary to introduce a family of Banach spaces on T_N which carry stronger topologies than that of $\mathscr{C}(T_N, \mathbb{C})$. This motivates the following

Definition 2.1 . Let $s \in \mathbb{R}_0^+$; we define $B^{(s)}(T_N, \mathbb{C})$ as the set of all $U \in \mathscr{C}(T_N, \mathbb{C})$ such that the corresponding multiple Fourier series

$$U(t_1, \ldots t_N) \sim \sum_{k \in \mathbb{Z}^N} U_k \exp\left[i \sum_{j=1}^{N} \omega_j k_j t_j\right] \tag{2.1}$$

converges in the sense that

$$\|U\|_s = \sum_{k\in Z^N} |U_k| \, (1 + |\Omega_k|^s) < \infty \tag{2.2}$$

where $\Omega_k = \sum_{j=1}^{N} k_j\omega_j$.

In relations (2.1) and (2.2) , the Fourier coefficients U_k are given as usual by

$$U_k = \prod_{j=1}^{N} \tau_j^{-1} \int_0^{\tau_1} dt_1 \dots \int_0^{\tau_N} dt_N \, U(t_1 \dots t_N) \exp\left[-i \sum_{j=1}^{N} \omega_j k_j t_j\right] \tag{2.3}$$

It is then clear that $B^{(s)}(T_N, \mathbb{C})$ becomes a complex Banach space with respect to the usual pointwise operations and the norm (2.2) . We then define the phase space associated with system (1.3) as $B(\mathbb{C}) = B^{(1)}(T_N,\mathbb{C}) \oplus B^{(0)}(T_N,\mathbb{C})$, and we write $(U,V) \longrightarrow \|(U,V)\|_{1,0} = \|U\|_1 + \|V\|_0$ for the corresponding norm. Finally, we write $B(\mathbb{R}) = B^{(1)}(T_N,\mathbb{R}) \oplus B^{(0)}(T_N,\mathbb{R})$ for the real component in $B(\mathbb{C})$. The main result of this article is then the following

Theorem 2.1 . Assume that g satisfies hypothesis (G) ; for every $\nu \in [0, \sqrt{g'(u_0)})$, define $r_{u_0}(\nu) = (g'(u_0) - \nu^2)^{1/2}$. Then there exist constants $k \in [1,\infty)$, $\varepsilon_0 \in (0,\infty)$, a Banach subspace M^ν of $B(\mathbb{R})$ and, for every $\varepsilon \in (0,\varepsilon_0]$, an open spherical neighborhood $\mathcal{N}_{\varepsilon/2k}$ of radius $(2k)^{-1}\varepsilon$ centered at the origin of $B(\mathbb{R})$, such that the following conclusions hold:

(1) For every $\eta \in \mathscr{N}_{\varepsilon/2k} \cap M^{\nu}$, there exists a unique $(\tilde{U}(\eta),\tilde{V}(\eta)) \in \mathscr{N}_{\varepsilon}$, and a unique $x \longrightarrow (U(x,\eta), V(x,\eta)) \in \mathscr{C}^{(1)}(\mathbb{R}_0^+, B(\mathbb{R}))$ satisfying the Cauchy problem

$$\begin{aligned} U_x(x,\eta)(t_1,\dots t_N) &= V(x,\eta)(t_1,\dots t_N), \ (x,t_1\dots t_N) \in \mathbb{R}_0^+ \times T_N \\ V_x(x,\eta)(t_1,\dots t_N) &= \Box_N U(x,\eta)(t_1,\dots t_N) + g(U(x,\eta)(t_1,\dots t_N)) \end{aligned} \tag{2.4}$$

$$\begin{aligned} U(0,\eta)(t_1\dots t_N) &= \tilde{U}(\eta)(t_1\dots t_N) + u_0 \\ U_x(0,\eta)(t_1\dots t_N) &= \tilde{V}(\eta)(t_1\dots t_N) \end{aligned} \tag{2.5}$$

(2) The exponential decay estimate

$$\|(U(x,\eta) - u_0, V(x,\eta))\|_{1,0} \leq \varepsilon \exp[-r_{u_0}(\nu)x] \tag{2.6}$$

holds for every $x \in \mathbb{R}_0^+$.

(3) The set

$$\mathscr{M}_{\varepsilon}^{\nu} = \left\{ (\tilde{U}(\eta), \tilde{V}(\eta)), \eta \in \mathscr{N}_{\varepsilon/2k} \cap M^{\nu} \right\} \tag{2.7}$$

is in fact a $\mathscr{C}^{(1)}$–Banach manifold in $B(\mathbb{R})$, tangent to M^{ν} at the origin.

<u>Remarks.</u> (1) Theorem 2.1 is a typical stable manifold statement: it asserts that for a very special set of initial data in $B(\mathbb{R})$, namely those on the manifold $\mathscr{M}_{\varepsilon}^{\nu}$, there exists a

unique classical solution to Problem (1.3) which exists globally in $x \in \mathbb{R}_0^+$, and which enjoys the exponential decay estimate (2.6). Moreover, the rate of exponential decay is explicitly known: it is the quantity $r_{u_0}(\nu) = (g'(u_0) - \nu^2)^{1/2}$. We see here a posteriori the necessity of having assumed $g'(u_0) > 0$. In many physical applications one refers to $g'(u_0) > 0$ as the positive mass term, a terminology borrowed from relativistic quantum mechanics and quantum field theory ([20],[21]).

(2) In spite of the formal equivalence between system (1.3) and the partial differential equation (1.7), we may not assert that the function $(x,t_1,\dots t_N) \longrightarrow U(x,\eta)(t_1,\dots t_N)$ of Theorem 2.1 provides a classical solution to the second–order equation (1.7). All that we can assert is that the function $x \longrightarrow (U(x,\eta), V(x,\eta))$ provides a classical solution to the first–order system (1.3). However, we shall see that the section of $(x,t_1\dots t_N) \longrightarrow U(x,\eta)(t_1\dots t_N)$ by the main diagonal on the torus does provide a classical $\mathscr{C}^{(2)}$–solution of the original equation (1.1) (compare with the statement of Corollary 2.1 below).

(3) The exponential decay estimate (2.6) implies that $(U(x,\eta), U_x(x,\eta))$ converges strongly to the equilibrium $(u_0,0)$ in $B(\mathbb{R})$. We shall see exactly what this means for equation (1.1) in Corollary 2.1.

Sketch of the Proof of Theorem 2.1. Without restricting the generality we may assume that $u_0 = 0$; the linearized system associated with (1.3) is then

$$\begin{aligned} U_x(x)(t_1\dots t_N) &= V(x)(t_1\dots t_N), \quad (x,t_1\dots t_N) \in \mathbb{R}_0^+ \times T_N \\ V_x(x)(t_1\dots t_N) &= \Box_N U(x,t_1\dots t_N) + g'(0)\, U(x)(t_1\dots t_N) \end{aligned} \tag{2.8}$$

so that the infinitesimal generator of the linearized flow is formally given by

$$L(U,V) = (V,\ \square_N U + g'(0)U) \tag{2.9}$$

If we realize L as a linear operator on the dense domain $D(L) = B^{(2)}(T_N,\mathbb{C}) \times B^{(1)}(T_N,\mathbb{C})$ in $B(\mathbb{C})$, we then obtain

Statement (A). L generates a $\mathscr{C}^{(0)}$–group $\{W(x)\}_{x\in\mathbb{R}}$ on $B(\mathbb{C})$; moreover, the spectrum of L is

$$\sigma(L) = \{\lambda \in \mathbb{C}\colon \lambda^2 = g'(0) - \Lambda^2,\ \Lambda \in \mathbb{R}\} \tag{2.10}$$

and consists of the union of the countable point spectrum

$$\sigma_p(L) = \{\lambda \in \mathbb{C}\colon \lambda^2 = g'(0) - \Omega_k^2,\ k \in \mathbb{Z}^N\} \tag{2.11}$$

with the continuous spectrum

$$\sigma_c(L) = \sigma(L)/\sigma_p(L) \tag{2.12}$$

Moreover, $\sigma_p(L)$ is everywhere dense in $\sigma(L)$.

The proof that L generates a $\mathscr{C}^{(0)}$–group (and not merely a semigroup) follows from standard arguments (this of course means that (2.8) can be solved for $(x,t,\ldots t_N) \in \mathbb{R} \times T_N$, and not merely for $(x,t_1\ldots t_N) \in \mathbb{R}_0^+ \times T_N$). It is also straightforward to show that $\sigma_p(L)$ is the point spectrum of L; however, since $\{\omega_1,\ldots\omega_N\}$ is rationally independent, the set $\{\mu \in \mathbb{R} : \mu = \Omega_k,\ k \in \mathbb{Z}^N\}$ is everywhere dense in $\mathbb{R}$ by Kronecker's theorem of number theory [27] . It follows that $\sigma_p(L)$ cannot be the whole spectrum of L . In fact, independent arguments show that the resolvent set of L is $\rho(L) = \mathbb{C} - \sigma(L)$, so that the closure of $\sigma_p(L)$ in the complex plane is $\overline{\sigma}_p(L) = \sigma(L)$. Finally, the fact that the complementary set of $\sigma_p(L)$ given by (2.12) identifies with the continuous spectrum follows from an elementary Fourier analysis of the resolvent operator in $B(\mathbb{C})$. ■

Now let $|||\cdot|||_\infty$ denote the usual operator norm on $B(\mathbb{C})$. The following statement is the cornerstone of the entire construction.

Statement (B) . For $\nu \in [0, \sqrt{g'(0)})$ as in Theorem 2.1 , the linear operator

$$P^\nu(U,V) = \sum_{\substack{k \in \mathbb{Z}^N \\ \Omega_k^2 \in [0,\nu^2]}} \left[1/2 \left[U_k - \frac{V_k}{(g'(0) - \Omega_k^2)^{1/2}} \right], 1/2 \left[V_k - (g'(0) - \Omega_k^2)^{1/2} U_k \right] \right] \times$$

$$\times \exp\left[i \sum_{j=1}^{N} \omega_j k_j t_j \right] \tag{2.13}$$

is a projection operator on $B(\mathbb{C})$. Moreover, for $r_{u_0}(\nu)$ as in Theorem 2.1 (with $u_0 = 0$) , and for the linearized flow $\{W(x)\}_{x \in \mathbb{R}}$, there exists $c_1 > 0$ such that the estimate

$$|||W(x)P^\nu|||_\infty \leq c_1 \exp[-r_0(\nu)x] \tag{2.14}$$

holds for every $x \in \mathbb{R}_0^+$, and there exists $c_2 > 0$ such that the estimate

$$|||W(x)(I - P^\nu)|||_\infty \leq c_2 \exp[-r_0(\nu)x] \tag{2.15}$$

holds for every $x \in \mathbb{R}_0^-$. In relation (2.15), I denotes the identity operator on $B(\mathbb{C})$.

The proof of Statement (B) follows from the fact that the action of $\{W(x)\}_{x\in\mathbb{R}}$ on the trigonometric polynomials can be determined explicitely, and from a careful analysis of the frequency domain to control the small divisors around the resonant value $\sqrt{g'(0)}$. In fact, we can already see such small divisors lurking in expression (2.13) since, because of the linear independence of $\{\omega_1,...\omega_N\}$ over $\mathbb{Q}$, Ω_k can become arbitrarily close to $\sqrt{g'(0)}$. Expression (2.13) thus provides an a posteriori meaning for the parameter $\nu \in [0, \sqrt{g'(0)})$ of Theorem 2.1 : it is a cut–off parameter which prevents Ω_k^2 from getting arbitrarily close to $g'(0)$. This and the very special topology of $B(\mathbb{C})$ induced by the norm (2.2) then allow one to prove the basic estimates (2.14) and (2.15). We refer the reader to [13] for complete details. ∎

Remark. Although system (1.3) is eventually analyzed for $(x,t_1...t_N) \in \mathbb{R}_0^+ \times T_N$, it is essential that $\{W(x)\}_{x\in\mathbb{R}}$ be a group for Statement (B) to hold; indeed, notice that x is non positive in relation (2.15), so that $\{W(x)\}$ may blow up exponentially on $\operatorname{Ran}(I-P^\nu)$. This fact is crucial to the construction of the manifold $\mathcal{M}_\varepsilon^\nu$ of Theorem 2.1. On the other hand, $\{W(x)\}_{x\in\mathbb{R}}$ decays exponentially on $M^\nu = \operatorname{Ran} P^\nu$, and in fact leaves this suspace globally invariant. It is then natural to call M^ν the linearized stable manifold associated with systems (1.3) and (2.8). M^ν is of course the Banach subspace alluded to in the statement of Theorem 2.1.

From statements (A) and (B) we can then obtain the conclusion of Theorem 2.1 upon using a suitable refinement of Perron's fixed point method along with the technique of exponentially weighted Banach spaces of maps developed in [11]. ∎

Remark. The reason for which we assumed the analyticity of g in Theorem 2.1 is that it is not possible to represent equation (1.1) as a well–defined dynamical system on $B(\mathbb{C})$ if g satisfies only weaker differentiability properties. This has to do with the fact that $B^{(0)}(T_N,\mathbb{C})$, the right–component of $B(\mathbb{C})$, is the Banach algebra of all functions $U \in \mathscr{C}(T_N,\mathbb{R})$ which possess an absolutely and uniformly convergent Fourier series (take relation (2.2) with $s = 0$) . In order to make system (1.3) well defined on $B(\mathbb{C})$, the analyticity is then forced upon us by the Wiener–Lévy theorem of harmonic analysis [28] . However, it is naturally conceivable that one can prove a stable manifold theorem such as Theorem 2.1 with only weaker differentiability properties for g in choosing a different phase space. How one can possibly do that and simultaneously solve the above small divisor problem remains an open problem.

Theorem 2.1 implies the following result for equation (1.1) through the section method described earlier.

Corollary 2.1. The hypotheses and the definitions concerning $\{\omega_1,...\omega_N\}$, g , u_0 , ν , and $r_{u_0}(\nu)$ are the same as in Theorem 2.1 . Let $(\tilde{U}(\eta), \tilde{V}(\eta)) \in \mathscr{M}_\varepsilon^\nu$ and write $\mathfrak{u}(\eta)$ (resp. $\mathfrak{v}(\eta)$) for the section of $\tilde{U}(\eta)$ (resp. $\tilde{V}(\eta)$) by the main diagonal on T_N , that is

$$t \longrightarrow \mathfrak{u}(\eta)(t) = \tilde{U}(\eta)(t,...t) \tag{2.16}$$

and

$$t \longrightarrow \mathfrak{v}(\eta)(t) = \tilde{V}(\eta)(t,...t) \tag{2.17}$$

Then there exists a function $(x,t) \longrightarrow u(x,t,\eta) \in \mathscr{C}^{(2)}(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R})$ which satisfies the Cauchy problem

$$\left\{\begin{array}{l} u_{tt}(x,t) = u_{xx}(x,t) - g(u(x,t)),\ (x,t) \in \mathbb{R}_0^+ \times \mathbb{R} \\ u(0,t) = \bar{u}(\eta)(t) + u_0 \\ u_x(0,t) = \bar{v}(\eta)(t) \end{array}\right\} \tag{2.18}$$

Moreover, the exponential decay estimates

$$\sup_{t\in\mathbb{R}} |u_t(x,t,\eta)| + \sup_{t\in\mathbb{R}} |u(x,t,\eta) - u_0| \leq \varepsilon \exp[-r_{u_0}(\nu)x] \tag{2.19}$$

$$\sup_{t\in\mathbb{R}} |u_x(x,t,\eta)| \leq \varepsilon \exp[-r_{u_0}(\nu)x] \tag{2.20}$$

hold for every $x \in \mathbb{R}_0^+$. Finally, $t \longrightarrow u(x,t,\eta)$ is quasiperiodic for every $x \in \mathbb{R}_0^+$, is not constant for every $x \in \mathbb{R}^+$ if $t \longrightarrow \bar{u}(\eta)(t)$ is not constant, and the set $\{\omega_1,...\omega_N\}$ provides an integer basis for the set of its Fourier exponents.

<u>Sketch of the Proof of Corollary 2.1.</u> Let $x \longrightarrow (U(x,\eta), V(x,\eta))$ be the unique classical solution to Problem (2.4) – (2.5) corresponding to the initial datum $(\bar{U}(\eta),\bar{V}(\eta))$. We can then prove that the section $(x,t) \longrightarrow u(x,t,\eta) = U(x,\eta)(t,...,t)$ possesses all of the required properties. ∎

<u>Remark.</u> A technique similar to that used to prove Theorem 2.1 can be used to deal with dynamical systems whose linearization has a continuous spectrum or a continuum of eigenvalues. Such situations occur for instance in some questions of parabolic stability [10] , or in the construction of spatially localized time–almost–periodic solutions to equation (1.1)

when the basic frequencies $\{\omega_1,...\omega_N\}$ are not a priori specified [11] .
We devote the last section of this article to the discussion of an open problem.

3. On the Structure of Spatially Localized Time–Quasiperiodic Solutions to Equation (1.1) . In Corollary 2.1 , we have shown that at least some exponentially localized time—quasiperiodic solutions to equation (1.1) can be generated by taking the diagonal section of multiperiodic solutions to system (1.3) . In this context, there is an interesting question motivated by a beautiful structure theorem of Bohr ([26]),([27]) ; that theorem asserts that every Bohr almost–periodic function can in fact be viewed as the diagonal section of some appropriate multiperiodic function, which may of course depend on infinitely many variables when the given almost–periodic function is not quasiperiodic. In particular, it is then natural to ask whether every exponentially localized time–quasiperiodic solution of small norm to equation (1.1) with preassigned basis $\{\omega_1,...\omega_N\}$ is the diagonal section of some multiperiodic solution to (1.3) , or the diagonal section of some multiperiodic solution to some appropriately constructed dynamical system on $\mathbb{R}_0^+ \times T_N$. We have been able to find neither a proof nor a counterexample. Of course, we can also formulate the above question in more general terms. We leave it to the imagination of the reader to formulate such questions.

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