Classification of $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ actions on surfaces

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Abstract. In this article we classify the fixed point free actions of the groups $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ (where p is a prime integer) on compact oriented surfaces.

1 Introduction

Finite abelian group actions on surfaces by autohomeomorphisms constitutes a classical subject, see the articles [N], [E], [J1], [J2], [Na1], [S], [Z], [CN]. In our article we present a direct and complete way to deal with the topological classification of fixed point free $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ actions, where p is a prime integer ($\mathbb{Z}_{p^s}^m = \mathbb{Z}_{p^s} \oplus \overset{m}{\cdots} \oplus \mathbb{Z}_{p^s}$ and $\mathbb{Z}_{p^s} = \mathbb{Z}/p^s\mathbb{Z}$). Our results show some incorrections in [J1] (compare, for example, Proposition 4.6 in [J1] with our Corollary 15).

An important part of our work is the classification of skew symmetric forms on $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ (Theorem 11), since a fixed point free action of $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ on a surface provides an alternating or skew symmetric bilinear form on $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$. We define a signature to classify such skew symmetric forms (Definition 12) and that signature classify the actions on surfaces of a given genus (Theorem 13), considering the weak equivalence of actions (in the terminology of Edmonds [E]). We deal with the strong equivalence in Theorem 8 and the realization of the possible actions is in Theorem 9.

Since G is a finite group, given an action $(\widetilde{S}, f)$ of G, it is possible to construct an analytic structure on S such that $f(G)$ is a group of automorphisms as Riemann surface (see [K]). Hence all the actions considered in this paper appear as automorphism group actions of complex algebraic curves. One of the motivations for our study is the description of the set of connected components in the moduli space M of pairs (C, G), where C is a complex algebraic curve and $G \cong \mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ is a group of fixed point free automorphisms of C. According to [Na2] the description of connected components of M is reduced to the description of topological classes of pairs $(\widetilde{S}, K)$ where K is a group of

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autohomeomorphisms of $\widetilde{S}$ and K is isomorphic to $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$. We consider that $(\widetilde{S}, K)$ and $(\widetilde{S}', K')$ are equivalent if there exist a homeomorphism $\varphi : \widetilde{S} \to \widetilde{S}'$ such that $K' = \varphi \circ K \circ \varphi^{-1}$. These equivalence classes are in one to one correspondence with classes of weak equivalence.

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2 Algebraic preliminaries.

Let us consider the standard lattice $\mathbb{Z}^{2g} = \mathbb{Z} \oplus \overset{2g}{\cdots} \oplus \mathbb{Z}$ with the standard basis $(e_i) = ((0, ..., 1^{(i)}, ...0))$. We define the skew symmetric bilinear form $(.,.) : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$, by $(e_i, e_j) = \delta_{i+j,2g+1}$ for $i < j$.

We consider also the group $\mathbb{Z}_{p^2}^{2g} = \mathbb{Z}_{p^2} \oplus \overset{2g}{\cdots} \oplus \mathbb{Z}_{p^2}$ where p is a prime and $\mathbb{Z}_{p^2} = \mathbb{Z}/p^2\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, ..., \overline{p^2-1}\}$. Let $\varphi : \mathbb{Z}^{2g} \to \mathbb{Z}_{p^2}^{2g}$ be the natural projection defined by $\varphi(e_i) = \overline{e_i}$, where $\overline{e_i} = (0, ..., \overline{1}^{(i)}, ...0)$. Then we have a skew symmetric bilinear form $(.,.)_{p^2} : \mathbb{Z}_{p^2}^{2g} \times \mathbb{Z}_{p^2}^{2g} \to \mathbb{Z}_{p^2}$ defined by $(\overline{e_i}, \overline{e_j}) = (\varphi(e_i), \varphi(e_j))_{p^2} = (e_i, e_j) \bmod p^2$. We shall denote by $\widehat{\mathbb{Z}}_{p^2}^{2g}$ the group $\mathbb{Z}_{p^2}^{2g}$ provided with the bilinear form $(.,.)_{p^2}$.

Let $Sp(2g, \mathbb{Z})$ and $Sp_{p^2}(2g, \mathbb{Z}_{p^2})$ be the subgroups of the automorphisms groups of $\mathbb{Z}^{2g}$ and $\mathbb{Z}_{p^2}^{2g}$ that preserve the bilinear forms $(.,.)$ and $(.,.)_{p^2}$ respectively. The natural projection $\varphi : \mathbb{Z}^{2g} \to \mathbb{Z}_{p^2}^{2g}$ induces a homomorphism $\varphi_* : Sp(2g, \mathbb{Z}) \to Sp_{p^2}(2g, \mathbb{Z}_{p^2})$ such that $\varphi_*(f) \circ \varphi = \varphi \circ f$ for all $f \in Sp(2g, \mathbb{Z})$.

The following result is known:

Lemma 1 *(See Lemma 4.1, pg 177 of [E])* $\varphi_*(Sp(2g, \mathbb{Z})) = Sp_{p^2}(2g, \mathbb{Z}_{p^2})$.

We shall also need the following result:

Theorem 2 *(Witt's Theorem, see Theorem 3.1, pg. 175 of [E]) Let G, G' be subgroups of $\widehat{\mathbb{Z}}_{p^2}^{2g}$ both subgroups isomorphic to $\mathbb{Z}_{p^2}^r$ and $\psi : G \to G'$ be an isomorphism such that $(\psi(a), \psi(b))_{p^2} = (a, b)_{p^2}$ for all $a, b \in G$. Then there is an automorphism $\widetilde{\psi} \in Sp_{p^2}(2g, \mathbb{Z}_{p^2})$ such that $\widetilde{\psi}$ restricted to G is ψ.*

Corollary 3 *Let G, G' be subgroups of $\widehat{\mathbb{Z}}_{p^2}^{2g}$ and $\psi : G \to G'$ be an isomorphism such that $(\psi(a), \psi(b))_{p^2} = (a, b)_{p^2}$ for all $a, b \in G$. Then there is an automorphism $\widetilde{\psi} \in Sp_{p^2}(2g, \mathbb{Z}_{p^2})$ such that $\widetilde{\psi}$ restricted to G is ψ.*

3 Strong classification of fixed point free orientation preserving actions of $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ on surfaces

Let G be a group isomorphic to $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$, where p is a prime integer. An (orientation preserving) action of G on an orientable surface of genus $\widetilde{g}$ is a pair $(\widetilde{S}, f)$, where $\widetilde{S}$ is a closed (compact without boundary) oriented surface with genus $\widetilde{g}$, and f is a exact representation of G in the group of orientation preserving autohomeomorphisms of $\widetilde{S}$. We shall consider fixed point free actions.

Definition 4 *(Strong equivalence). Two actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are called strongly equivalent if there is a homeomorphism, $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$, sending the orientation of $\widetilde{S}$ to the orientation of $\widetilde{S}'$ and such that $f'(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, for all $h \in G$.*

We are interested in finding all strong equivalence classes of actions of $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$

We denote by $S = \widetilde{S}/f(G)$ and by $\varphi = \varphi(f) : \widetilde{S} \to S$ the natural projection. We shall consider the case when $f(h)$ has no fixed points for any $h \in G - \{id\}$, i. e. the action of $(\widetilde{S}, f)$ is fixed point free. Then the projection $\varphi(f) : \widetilde{S} \to S$ is an unbranched covering with deck transformation group $f(G)$.

Let us consider $\pi_1(S)$ as the group of deck transformations of the universal covering of S. Then we have:

$$\omega(\widetilde{S}, f) : \pi_1(S) \to \pi_1(S)/\pi_1(\widetilde{S}) = f(G) \xrightarrow{f^{-1}} G.$$

The resulting epimorphism $\omega(\widetilde{S}, f) : \pi_1(S) \to G$ is the monodromy epimorphism of the covering $\varphi(f) : \widetilde{S} \to S$. The epimorphism $\omega(\widetilde{S}, f) : \pi_1(S) \to G$, induces the epimorphism $\theta_{p^2}(\widetilde{S}, f) : H_1(S, \mathbb{Z}_{p^2}) \to G$, since G is abelian.

Conversely, given an epimorphism $\theta_{p^2} : H_1(S, \mathbb{Z}_{p^2}) \to G$, there is an action $(\widetilde{S}, f)$ such that $\theta_{p^2} = \theta_{p^2}(\widetilde{S}, f)$. To obtain $\widetilde{S}$ it is enough to consider the monodromy $\omega : \pi_1(S) \to H_1(S, \mathbb{Z}_{p^2}) \xrightarrow{\theta_{p2}} G$ and then $\widetilde{S} = U/\ker\omega$, where U is the universal covering of S and the action of G is given by $G = \pi_1(S)/\ker\omega$.

Definition 5 *Let S and S' be two surfaces. Two epimorphisms $\theta : H_1(S, \mathbb{Z}_{p^2}) \to G$ and $\theta' : H_1(S', \mathbb{Z}_{p^2}) \to G$ are called strongly equivalent if there is an orientation preserving homeomorphism $\psi : S \to S'$ inducing an isomorphism $\psi_{p^2} : H_1(S, \mathbb{Z}_{p^2}) \to H_1(S', \mathbb{Z}_{p^2})$ such that $\theta = \theta' \circ \psi_{p^2}$.*

Using covering space theory we have that:

Theorem 6 *Two actions* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *are strongly equivalent if and only if the epimorphisms* $\theta_{p^2}(\widetilde{S}, f)$ *and* $\theta_{p^2}(\widetilde{S}', f')$ *are strongly equivalent.*

Definition 7 *Let* $(\widetilde{S}, f)$ *be an action of* G, $S = \widetilde{S}/f(G)$, *and* $\theta = \theta_{p^2}(\widetilde{S}, f) : H_1(S, \mathbb{Z}_{p^2}) \to G$ *be the epimorphism defined by the action* $(\widetilde{S}, f)$. *Let us consider the spaces of homomorphisms* $G^* = \{e : G \to \mathbb{Z}_{p^2}\}$ *and* $H^1(S, \mathbb{Z}_{p^2}) = \{e : H_1(S, \mathbb{Z}_{p^2}) \to \mathbb{Z}_{p^2}\}$. *Then* θ *generates a monomorphism* $\theta^* = \theta^*(\widetilde{S}, f) : G^* \to H^1(S, \mathbb{Z}_{p^2})$. *The intersection form* $(.,.)_{p^2} = (.,.)^S_{p^2}$ *on* $H_1(S, \mathbb{Z}_{p^2})$ *induces an isomorphism* $i : H^1(S, \mathbb{Z}_{p^2}) \to H_1(S, \mathbb{Z}_{p^2})$ *defined by* $(a,.) \to a$ *and a form* $(.,.)_{(\widetilde{S},f)} : G^* \times G^* \to \mathbb{Z}_{p^2}$ *such that* $(a,b)_{(\widetilde{S},f)} = (i \circ \theta^*(a), i \circ \theta^*(b))_{p^2}$.

Theorem 8 *Two actions* $(\widetilde{S}, f)$ *and* $(\widetilde{S}', f')$ *of the group* G *are strongly equivalent if and only if* $\widetilde{S}$ *and* $\widetilde{S}'$ *have the same genus and* $(.,.)_{(\widetilde{S},f)} = (.,.)_{(\widetilde{S}',f')}$.

The proof is a direct adaptation to our case of the proof of Theorem 8 in [CN]. It is necessary to use the Corollary 3 and the Lemma 1 in the above Section instead of the Witt's Theorem in [A, 3.9] and the Theorems VII.20, VII.21 of [Ne] (as used in [CN]).

Theorem 9 *Let* $G \cong \mathbb{Z}^m_{p^2} \oplus \mathbb{Z}^n_p$, $(.,.) : G \times G \to \mathbb{Z}_{p^2}$ *be an alternating bilinear form. Then there exists an action* $(\widetilde{S}, f)$ *such that* $(.,.) = (.,.)_{(\widetilde{S},f)}$ *and the genus of* $\widetilde{S}/f(G)$ *is* g *if and only if* $G \hookrightarrow \widehat{\mathbb{Z}}^{2g}_{p^2}$.

Proof. First we construct the action from the form and the numerical conditions. We have a basis of G, $(x_i, (i = 1, ..., m))$. Now we consider a surface S of genus g and applying the fact that Witt problems are solvable in $\mathbb{Z}^{2g}_{p^2}$, (see [E, pg. 175]) we construct a basis of $H_1(S, \mathbb{Z}_{p^2})$, $(\chi_i, (i = 1, ..., m), \nu_i, (i = m + 1, ..., 2g))$, such that $(\chi_i, \chi_j) = (x_i, x_j)$. Then we construct the epimorphism $\theta : H_1(S, \mathbb{Z}_{p^2}) \to G$, defined by $\theta(\chi_i) = x_i$, if $i \leq m$, $\theta(\nu_i) = 0$. Then the epimorphism θ defines a homomorphism $H_1(S, \mathbb{Z}) \to H_1(S, \mathbb{Z}_{p^2}) \to G$ and thus a regular covering $\widetilde{S} \to S$ with automorphism group G. The action of G on $\widetilde{S}$ satisfies $(.,.)_{(\widetilde{S},f)} = (.,.)$.

Conversely if there is an action $(\widetilde{S}, f)$ such that $(.,.) = (.,.)_{(\widetilde{S},f)}$ it is obvious, by the construction of $(.,.)_{(\widetilde{S},f)}$, that $G \hookrightarrow \mathbb{Z}^{2g}_{p^2}$. ∎

4 Weak classification of fixed point free orientation preserving actions of $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ on surfaces

Definition 10 *(Weak equivalence) Let $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ be two actions of a group G. We shall say that $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are weakly equivalent if there is a homeomorphism $\widetilde{\psi} : \widetilde{S} \to \widetilde{S}'$ and an automorphism $\alpha \in Aut(G)$ such that $f' \circ \alpha(h) = \widetilde{\psi} \circ f(h) \circ \widetilde{\psi}^{-1}$, $h \in G$.*

We need an algebraic theorem of classification of skew-symmetric bilinear forms on $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$.

Theorem 11 *Let V be $\mathbb{Z}_{p2}^m \oplus \mathbb{Z}_p^n$ and $\xi : V \times V \to \mathbb{Z}_{p2}$ be a skew-symmetric bilinear form. Then there exists a factorization $V = V_{2,2|2} \oplus V_{2,2|1} \oplus V_{2,2|0} \oplus V_{2,1|1} \oplus V_{1,1|1} \oplus V_{1,1|0}$ such that $V_{s,s|t} \cong \mathbb{Z}_{p^s}^{v_{s.s|t}}$, $V_{2,1|1} \cong (\mathbb{Z}_{p2} \oplus \mathbb{Z}_p)^{v_*}$, $v_* \leq 1$, $Ker(\xi) = V_{2,2|0} \oplus V_{1,1|0}$, $\xi(V_{s_a,s_b|t}, V_{k_a,k_b|l}) = 0$ for $(s_a, s_b|t) \neq (k_a, k_b|l)$ and $v_{2,2|0} = 0$ if $v_{1,1|1} > 0$.*

Moreover we can construct the factorization such that the restriction ξ on $V_{s_a,s_b|2}$ and $V_{s_a,s_b|1}$ is nondegenerated, and $V_{s,s|2}$, $V_{s,s|1}$ have bases $\{a_{s|t}^i, b_{s|t}^i | i = 1, ..., k_{s|t}\}$ such that $\xi(a_{s|t}^i, b_{s|t}^j) = p^{2-t}\delta_{ij}$.

The dimensions of $V_{s_a,s_b|t}$ are uniquely defined by ξ and thus form a complete system of invariants for ξ.

Proof. For $s, t \in \{1, 2\}$ call a set $\{a^i, b^i | i = 1, ..., k\}$ (s|t)-symplectic system, if it consist of elements of order p^s and $\xi(a^i, b^j) = p^{(2-t)}\delta_{ij}$.

Let $B_{2|2} = \{a_{2|2}^i, b_{2|2}^i | i = 1, ..., k_{2|2}\}$ be a (2|2)-symplectic system in V with maximal cardinality. Let $V_{2,2|2}$ be the submodule generated by $B_{2|2}$. We define $W_{2,2|2} = \{v \in V | \xi(v, V_{2,2|2}) = 0\}$. Then $V = V_{2,2|2} \oplus W_{2,2|2}$ and moreover $p\xi(v_1, v_2) = 0$ for $v_1, v_2 \in W_{2,2|2}$.

Let $B_{2|1} = \{a_{2|1}^i, b_{2|1}^i | i = 1, ..., k_{2|1}\}$ be a (2|1)-symplectic system in $W_{2,2|2}$ with maximal cardinality. Let $V_{2,2|1}$ be the submodule generated by $B_{2|1}$. Now we define $W_{2,2|1} = \{v \in W_{2,2|2} | \xi(v, V_{2,2|1}) = 0\}$. Then $W_{2,2|2} = V_{2,2|1} \oplus W_{2,2|1}$ and moreover $\xi(v_1, v_2) = 0$ if $v_1, v_2 \in W_{2,2|1}$ are elements of order p^2.

We define $W_{2,1|1} = W_{2,2|1}$ if $W_{2,2|1} - W_{2,2|1} \cap Ker(\xi)$ does not contain elements of order p^2. In opposite case there exist an element a of order p^2 and an element b of order p such that $\xi(a, b) = p$. In this case $W_{2,1|1} = \{v \in W_{2,2|1} | \xi(v, a) = \xi(v, b) = 0\}$. Note that there is no possible the existence of a_1, a_2 of order p^2 and b_1 of order p such that $\xi(a_1, a_2) = 0$, $\xi(a_i, b_1) = p\delta_{i1}$. In such a case we can add $\{a_1, a_2 + b_1\}$ to the system $B_{2|1}$ and this fact contradicts the maximality of $B_{2|1}$.

Let $B_{1|1} = \{a^i_{1|1}, b^i_{1|1} | i = 1, ..., k_{2|1}\}$ be a (1|1)-symplectic system in $W_{2,1|1}$ with maximal cardinality. Let $V_{1,1|1}$ be the submodule generated by $B_{1|1}$. We define $W_{1,1|1} = \{v \in W_{2,1|1} | \xi(v, V_{1,1|1}) = 0\}$. Then $W_{2,1|1} = V_{1,1|1} \oplus W_{1,1|1}$ and $W_{1,1|1} = Ker(\xi) \cong V_{2,2|0} \oplus V_{1,1|0}$ where $V_{s,s|0} \cong \mathbb{Z}^{v_{s|0}}_{p^s}$.

Now we shall prove that $v_{2,2|0} = 0$ if $v_{1,1|1} > 0$. Assume that $v_{1,1|1} > 0$. If $W(2, 2|1)$ contain two linear independent elements v_1, v_2 of order p^2 then either $\xi(v_1 + a^1_{1,1|1}, v_2 + b^1_{1,1|1}) \neq 0$ or $\xi(v_1 + a^1_{1,1|1}, v_2) \neq 0$ or $\xi(v_1, v_2 + b^1_{1,1|1}) \neq 0$ contradicting the maximal cardinality of $B_{2|1}$. Assume that $W(2, 2|1)$ contains an element v of order p^2, hence either $\xi(v, b^1_{1,1|1}) \neq 0$ or $\xi(v + a^1_{1,1|1}, b^1_{1,1|1}) \neq 0$. Thus $v_* = 1$ and $v_{2,2|0} = 0$.

The dimensions of $V_{s_a,s_b|t}$ are defined by n, m , n', m' and m" where $Ker(\xi) \cong \mathbb{Z}^{m'}_{p2} \oplus \mathbb{Z}^{n'}_p$, $Ker(p\xi) \cong \mathbb{Z}^{m"}_{p2} \oplus \mathbb{Z}^n_p$. From the factorization follows that $m = v_{2,2|2} + v_{2,2|1} + v_{2,2|0} + v_*$, $m' = v_{2,2|0}$, $m" = v_{2,2|0} + v_{2,2|1} + v_*$, $n = v_{1,1|1} + v_{1,1|0} + v_*$, $n' = v_{1,1|0}$. Moreover $v_* = 0$ if $m - m'$ is even and $v_* = 1$ if $m - m'$ odd. ■

Definition 12 *Let V be $\mathbb{Z}^m_{p^2} \oplus \mathbb{Z}^n_p$ and $\xi : V \times V \to \mathbb{Z}_{p^2}$ be a skew-symmetric bilinear form. Let $V = V_{2,2|2} \oplus V_{2,2|1} \oplus V_{2,2|0} \oplus V_{2,1|1} \oplus V_{1,1|1} \oplus V_{1,1|0}$ be the factorization of V produced by ξ and given in the above Theorem. We shall call signature of ξ to the ordered set $(v_{2,2|2}, v_{2,2|1} v_{2,2|0}, v_*, v_{1,1|1}, v_{1,1|0})$*

The next Theorem solves the problem of weak classification of fixed point free actions of $\mathbb{Z}^m_{p^2} \oplus \mathbb{Z}^n_p$ on surfaces:

Theorem 13 *Let $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ be two actions of a group $G \cong \mathbb{Z}^m_{p^2} \oplus \mathbb{Z}^n_p$. Let $(.,.)_{(\widetilde{S},f)}$ and $(.,.)_{(\widetilde{S}',f')}$ be the skew symmetric bilinear forms induced by the two actions. Then the actions $(\widetilde{S}, f)$ and $(\widetilde{S}', f')$ are weakly equivalent if and only if the forms $(.,.)_{(\widetilde{S},f)}$ and $(.,.)_{(\widetilde{S}',f')}$ have the same signature and $\widetilde{S}$, $\widetilde{S}'$ have the same genus.*

Proof. Let us call $S = \widetilde{S}/f(G)$ and $S' = \widetilde{S}'/f'(G)$. Let $\theta^*(\widetilde{S}, f)$ and $\theta^*(\widetilde{S}', f')$ be the epimorphisms defined by the two actions, $\widetilde{G}$ be the image of G^* in $H_1(S, \mathbb{Z}_{p^2})$ by $\theta^*(\widetilde{S}, f)$ and $\widetilde{G}'$ be the image of G^* in $H_1(S', \mathbb{Z}_{p^2})$ by $\theta^*(\widetilde{S}', f')$.

Let $(.,.)_{(\widetilde{S},f)}$ and $(.,.)_{(\widetilde{S}',f')}$ be the skew symmetric bilinear forms induced by the two actions. Assume that $(.,.)_{(\widetilde{S},f)}$ and $(.,.)_{(\widetilde{S}',f')}$ have the same signature then there exists an isomorphism $\psi : \widetilde{G}' \to \widetilde{G}$ such that $(\psi(a), \psi(b))_{(\widetilde{S}',f')} = (a, b)_{(\widetilde{S},f)}$.

Assume now that $\widetilde{S}$, $\widetilde{S}'$ have the same genus, then $g(S) = g(S')$ (the actions are fixed point free). Using Corollary 3, there is an isomorphism $\widetilde{\psi}$:

$H^1(S', \mathbb{Z}_{p^2}) \to H^1(S, \mathbb{Z}_{p^2})$ giving by restriction ψ and sending the intersection form of $H^1(S', \mathbb{Z}_{p^2})$ to the intersection form of $H^1(S, \mathbb{Z}_{p^2})$. By Lemma 1 and [MKS, page 178], there exists a homeomorphism $\varphi : S \to S'$ inducing $\widetilde{\psi}$ on cohomology. Then by Theorem 8, the actions $(\widetilde{S}, f)$ and $(\widetilde{S}, \varphi^{-1} \circ f' \circ \varphi)$ are strongly equivalent. The isomorphism ψ, defines an automorphism of G giving the weak equivalence between $(\widetilde{S}, f)$ and $(\widetilde{S'}, f')$. ■

Note that, if $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ acts on a surface of genus $\widetilde{g}$, then the genus of the quotient surface is $g = \frac{\widetilde{g}-1}{p^{2m+n}} + 1$. The following corollary is one of the main results of this paper:

Corollary 14 *Weak equivalence classes of fixed point free $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ actions on surfaces of genus $\widetilde{g}$ are in bijection with the signatures of subgroups of $\widehat{\mathbb{Z}}_{p^2}^{2g}$.*

Proof. Theorem 13 tells us that each signature determines a weak equivalence class. By Theorem 9 each signature of a subgroup of $\widehat{\mathbb{Z}}_{p^2}^{2g}$ defines a non-empty weak equivalence class. ■

Corollary 15 *Weak equivalence classes of fixed point free $\mathbb{Z}_{p^2}^m \oplus \mathbb{Z}_p^n$ actions on surfaces of genus $\widetilde{g}$ are in bijection with the ordered set of integers $(v_{2,2|2}, v_{2,2|1}, v_{2,2|0}, v_*, v_{1,1|1}, v_{1,1|0})$ such that $v_{2,2|2}, v_{2,2|1}, v_{1,1|1}$ are even, $v_* < 2$, $v_{2,2|0} = 0$ if $v_{1,1|1} > 0$ and $m = v_{2,2|2} + v_{2,2|1} + v_{2,2|0} + v_*$, $n = v_{1,1|1} + v_{1,1|0} + v_*$.*

Example. We shall construct an action of $\mathbb{Z}_4^5 \oplus \mathbb{Z}_2^4$ on a surface of genus $2^{16} + 1$ (with quotient surface of genus 5) and inducing a bilinear form with signature $(2, 2, 0, 1, 2, 1)$. Let e_1, e_2, e_3, e_4, e_5 be a base of $\mathbb{Z}_4^5$ and f_1, f_2, f_3, f_4 be a base of $\oplus\mathbb{Z}_2^4$ Let S be a surface of genus 5 and $(a_i, b_i; i = 1, ..., 5)$ be a symplectic base of $H^1(S, \mathbb{Z}_4)$. The action is defined by $\theta : H^1(S, \mathbb{Z}_4) \to \mathbb{Z}_4^5 \oplus \mathbb{Z}_2^4$, where $\theta(a_1) = e_1, \theta(b_1) = e_2, \theta(a_2) = e_3, \theta(b_2) = e_4, \theta(a_3) = e_5, \theta(b_3) = f_1, \theta(a_4) = f_2, \theta(b_4) = f_3, \theta(a_5) = f_4, \theta(b_5) = 0$

The above example shows also the way of geometrical realization of a representation of each weak equivalence class of action. This gives a proof of Corollary 15.

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