Distribution of zeros of twisted motivic L-functions

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0. INTRODUCTION.

Let M be a pure motive over $\mathbb{Q}$ of weight $w = w(M)$ and of the global conductor $c(M)$. Also given Dirichlet character χ one can define the twisted motive M_χ; let $L(M_\chi, s)$ denote the corresponding L-function.

Let $\mathcal{C}$ be the set of real quadratic characters χ of conductors $c(\chi)$ which are relatively prime to $c(M)$. Let

$$\mathcal{C}(D) = \{\chi \in \mathcal{C} : \ \ c(\chi) \leq D\}.$$

and

$$m_\chi(\alpha) = \operatorname*{ord}_{s=\alpha} L(M_\chi, s).$$

The aim of this paper is to investigate the average (over the characters from $\mathcal{C}(D)$) order of zeros ρ of $L(M_\chi, s)$ and the regularities in their distribution. Our results generalize those of [P-P] and [P] proved for the Hecke L-functions. We assume the good (conjectural) analytic properties of the motivic L-functions $L(M_\chi, s)$ (see section 1). On the other hand we require the suitable mean value estimate for $L(M_\chi, s)$ which we can prove only for the motives of rank $d(M) \leq 2$ using the argument similar as in [H-B].

The starting point of the proof is the Riemann-Weil explicit formula in the form proved by Mestre [Me] (see also [Mor]). Using directly Mestre's device we could obtain the nontrivial results only assuming the Generalized Riemann Hypothesis for the corresponding L-functions (cf. also [B] and [G]). We avoid this assumption by proving the corresponding density estimate for the zeros of $L(M_\chi, s)$ (see section 3). On the other hand we use the modified test function which transmits the investigation of zeros around the critical point $\frac{w+1}{2}$ to those around any, fixed point $s_0 = \frac{w+1}{2} + it_0$ on the critical line (see [C-P]).

The required mean value estimate has the following form

$$\sum_{\chi \in \mathcal{C}(D)} \left| L\left(M_\chi, \frac{w+1}{2} + it\right)\right|^2 \ll_\varepsilon \left(D(|t|+1)^A\right)^{1+\varepsilon}, \tag{0.1}$$

where $A = A(M) \geq 1$ is some positive constant and $\varepsilon > 0$ is arbitrarily small. Let us remark that in the case of the classical Dirichlet L-functions ($d(M) = 1$) the estimate (0.1) is true with D^{ε} replaced by a power of $\log D$ (see [J]). In this case Theorem 1 below holds true with any $\delta = \delta(D) \geq \frac{\log\log D}{\log D}$. To formulate the main result we put

$$R_{\chi}(s_0, \delta) = \sum_{|\rho - s_0| \leq \delta} m_{\chi}(\rho)$$

Under the above assumptions we have

Theorem 1. *There exist some absolute, positive constants c_1 and c_2 such that for any, sufficiently small $\delta > 0$, we have*

$$\sum_{\chi \in \mathcal{C}(D)} R\chi(s_0, \delta) \leq c_1 \delta D \log D, \tag{0.2}$$

$$\sum_{\chi \in \mathcal{C}(D)} R\chi(s_0, \delta) \geq c_2 \delta D \log D, \tag{0.3}$$

$$\sum_{\chi \in \mathcal{C}(D)} R\chi(0, \delta) = o(D \log D). \tag{0.4}$$

as D tends to infinity.

All the results of this paper could be stated (and proved) without using any motivic language. Almost all we need to know about motives concerns the conjectural description of their L- functions (see section 1).

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1. PRELIMINARIES ON MOTIVIC L-FUNCTIONS.

We will define a pure motive M over $\mathbb{Q}$ of weight $w = w(M)$ and rank $d = d(M)$, by specifying Betti, de Rham and l-adic realisations (for each prime l)

$$H_B(M), \quad H_{DR}(M), \quad H_l(M)$$

which are, respectively, vector spaces over $\mathbb{Q}$, $\mathbb{Q}$ and $\mathbb{Q}_l$ of dimension d, and which are endowed with the additional structures and comparison isomorphisms [CP-R,D]. In particular $H_B(M)$ admits an involution ρ_B, $H_l(M)$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, and there is a Hodge decomposition into $\mathbb{C}$-vector spaces

$$H_B(M) \otimes \mathbb{C} = \oplus_{i+j=w} H^{i,j}(M),$$

where, letting ρ_B act on the vector space on the left via the first factor in the tensor product, we have

$$\rho_B(H^{i,j}(M)) = H^{j,i}(M).$$

The archetype of a motive arises from a smooth projective variety X defined over $\mathbb{Q}$; here we consider

$$H_B^m(X(\mathbb{C})), \quad H_{DR}^m(X), \quad H_l^m(X),$$

(the Betti cohomology with coefficients in $\mathbb{Q}$ of the complex manifold $X(\mathbb{C})$, the algebraic de Rham cohomology of the algebraic variety $X/\mathbb{Q}$, and for each prime number l, the l-adic cohomology of X viewed over $\overline{\mathbb{Q}}$). Another example is the motive $M(f)$ attached to a newform f by Deligne, Scholl and Jannsen.

Also, we have linear algebra operations on motives. In particular, one can consider $M^\vee$, the dual motive: its l-adic representation is contragredient to that of M. Given two motives M_1 and M_2 one can consider tensor product $M_1 \otimes M_2$: its l-adic representation is the tensor product of those for M_1 and M_2. We let $M(m) = M \otimes \mathbb{Q}(m)$, where $\mathbb{Q}(m)$ denotes the Tate motive.

The L-function $L(M,s)$ of M is defined for s with large $\mathrm{Re}\, s$ as the following Euler product

$$L(M,s) = \prod_p L_p(M, p^{-s}),$$

extended over all primes p and where

$$\begin{aligned} L_p(M,X)^{-1} &= \det\left(1 - X \cdot r_l(\mathrm{Fr}^{-1})|_{H_l(M)^{I_p}}\right) \\ &= (1-\alpha_p^{(1)}X)\ldots(1-\alpha_p^{(d)}X); \end{aligned}$$

here r_l is the representation giving $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module structure on $H_l(M)$, and $\mathrm{Fr}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the Frobenius element at p, defined modulo conjugation and modulo the inertia subgroup $I_p \subset G_p \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of the decomposition group G_p. We make the standard hypothesis that the coefficients of $L_p(M,X)^{-1}$ do not depend on $l \neq p$.

We next state the conjectural functional equation for $L(M,s)$. First let's recall the definition of the Euler factor $L_\infty(M,s)$ at infinity. We suppose that if $w(M)$ is even and $H^{\frac{w}{2},\frac{w}{2}}(M) \neq 0$, the involution ρ_B acts on it by a scalar ± 1. Put

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \quad h(i,j) = \dim_{\mathbb{C}} H^{i,j}(M),$$

and, if $w(M)$ is even and $H^{\frac{w}{2},\frac{w}{2}}(M) \neq 0$, we put

$$r^\alpha(M) = \dim_{\mathbb{C}}\{x \in H^{\frac{w}{2},\frac{w}{2}}(M) : \ \rho_B(x) = \alpha(-1)^{\frac{w}{2}}x\}, \quad \alpha = \pm 1.$$

We define

$$L_\infty(M,s) = L_{\infty,\mathbb{R}}(M,s)L_{\infty,\mathbb{C}}(M,s),$$

where

$$L_{\infty,\mathbb{R}}(M,s) = \Gamma_{\mathbb{R}}(s - \frac{w}{2})^{r^+(M)}\Gamma_{\mathbb{R}}(s+1-\frac{w}{2})^{r^-(M)},$$

and

$$L_{\infty,\mathbb{C}}(M,s) = \prod_{i<j}\Gamma_{\mathbb{C}}(s-i)^{h(i,j)}.$$

We put

$$\Lambda(M,s) = L_\infty(M,s)L(M,s). \tag{1.1}$$

We also assume the existence of a finite set of bad primes S such that, for each prime l and each $r \notin S \cup \{l\}$, the inertia group I_r acts trivially on $H_l(M)$. For each $r \in S$, and each $l \neq r$, we additionally assume that the conductor of $H_l(M)$ as a representation of the local Galois group D_r does not depend on l. The global conductor $c(M)$ of M is then defined as the product of local conductors at primes $r \in S$.

For each $p \notin S$, we assume that the reciprocal complex roots of $L_p(M,X)^{-1}$ have absolute value equal to $p^{\frac{w}{2}}$; according to the monodromy conjecture, a more refined version of the purity should hold also at bad primes. It implies, that $L(M,s)$ converges in the complex half-plane $\mathrm{Re}\, s > \frac{w}{2}+1$.

Conjecture 1. [D, CP-R] *The function $\Lambda(M,s)$ has a meromorphic continuation over the whole complex plane, and satisfies the functional equation*

$$\Lambda(M,s) = \epsilon(M)c(M)^{\frac{w+1}{2}-s}\Lambda(M^\vee(-w), w+1-s),$$

where $\epsilon(M)$ is a complex number of modulus 1 which is independent of s. Moreover, $L(M,s)$ is holomorphic unless both $w(M)$ is even and $\mathbb{Q}\left(-\frac{w}{2}\right)$ is a direct summand of M, in which case $L(M,s)$ may possibly have a pole at the point $s = \frac{w}{2}+1$.

We also conjecture that $\Lambda(M,s)$ minus the singular part is bounded in every vertical strip:

$$-\infty < \sigma_0 \leq \mathrm{Re}(s) \leq \sigma_1 < \infty.$$

By Stirling's formula there exist an absolute constant $c > 0$ such that for $\sigma_0 \leq \mathrm{Re}\, s \leq \sigma_1$ we have

$$L_\infty\big(M^\vee(-w), w+1-s\big)\big(L_\infty(M,s)\big)^{-1} = O\big((\mathrm{Im}\, s)^c\big),$$

Using the above functional equation and Phragmen-Lindelöf principle we obtain for s in the above strip

$$L(M,s) = O\big((\mathrm{Im}\, s)^{c_0})\big), \tag{1.2}$$

where c_0 is some positive constant.

Let χ be a quadratic character of conductor $c(\chi)$. To such χ one can associate a motive $[\chi]$ over $\mathbb{Q}$ of weight 0 and rank 1 so that $L([\chi],s) = L(s,\chi)$. Also given any (pure) motive M, we can define $M_\chi = M \otimes [\chi]$; it is again a motive over $\mathbb{Q}$ of the same weight and rank so that we have

$$L(M_\chi, s) = \prod_p L_p\big(M, \chi(p)p^{-s}\big).$$

Moreover we have

Lemma 1. ([CP-R]; lemma 3.1 (3)) *Assume that $(c(\chi), c(M)) = 1$. Then*

$$c(M_\chi) = c(M)c(\chi)^{d(M)}.$$

The above conjectures will be assumed in the following sections.

2. THE MEAN VALUE ESTIMATE.

We slightly modify Heath-Brown's arguments ([H-B], section 10).
The main tool is the following large sieve type estimate ([H-B], Corollary 3)

$$\sum_{\chi\in\mathcal{C}(D)}\Big|\sum_{n\le N}a_n\chi(n)\Big|^2 \ll (DN)^\varepsilon N(D+N)\max_{n\le N}|a_n|^2. \tag{2.1}$$

Define A to be the number of $\Gamma_\mathbb{R}'s$ plus twice number of $\Gamma_\mathbb{C}'s$ factors in $L_\infty(M,s)$. We will prove (cf. [P-P])

Theorem 2. *Let $\varepsilon>0$ and assume that $d=d(M)\le 2$. Then for any $s=\sigma+it$ with $\sigma\in[\frac{w+1}{2},\frac{w+2}{2}]$* we have

$$\sum_{\chi\in\mathcal{C}(D)}|L(M_\chi,\sigma+it)|^2 \ll_\varepsilon \Big(D+\big(D(|t|+1)^A\big)^{w+2-2\sigma}\Big)\big(D(|t|+1)\big)^\varepsilon.$$

Proof. Let

$$S^+(D,s)=\sum_{\chi\in\mathcal{C}(D)}|L(M_\chi,s)|^2,$$

$$S^-(D,s)=\sum_{\chi\in\mathcal{C}(D)}\big|L(M_\chi^\vee(-w),s)\big|^2$$

and $\nu^\pm(\sigma)$ be the infimum of those exponents $\nu^\pm$ respectively, for which

$$S^\pm(D,\sigma+it)\ll\Big(D+(DT^A)^{w+2-2\sigma}\Big)(DT^A)^{\nu^\pm}$$

uniformly in D and t, where $T=|t|+1$. We wil show that $\nu^\pm\le 0$. Similarly as in [H-B] p.268 we have for any $X>0$ and $\frac{w+1}{2}\le\alpha\le\sigma\le\frac{w+2}{2}$

$$S^+(D,s)\ll\sum_{\chi\in\mathcal{C}(D)}\Big|\sum_n a_n\chi(n)n^{-s}e^{-\frac{n}{X}}\Big|^2+X^{2(\alpha-\sigma)}\int_\mathbb{R}S^+(D,\alpha+iu)e^{-|u-t|}du. \tag{2.2}$$

Using the functional equation we obtain

$$S^+(D,\alpha+iu)=\sum_{\chi\in\mathcal{C}(D)}|U(D,\alpha+iu)|^2|L(M_\chi^\vee(-w),w+1-\alpha-iu)|^2$$

where

$$U=U(D,\alpha+iu)=\epsilon(M_\chi)c(M_\chi)^{\frac{w+1}{2}-(\alpha+iu)}L_\infty(M_\chi^\vee(-w),w+1-s)\big(L_\infty(M_\chi,s)\big)^{-1} \tag{2.3}$$

The last quotient contains the Γ-factors of type

$$\Gamma\Big(\frac{1}{2}(w+1-s+a)\Big)\Big(\Gamma\big(\frac{1}{2}(s+a)\big)\Big)^{-1}$$

or the similar ones with double arguments. In the second case we use the duplication formula

$$\Gamma(s)=\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})2^{s-1}\pi^{-\frac{1}{2}}$$

to have $\frac{s}{2}$ in all arguments of Γ-factors. Since A is the total number of them, then using the equality

$$\Gamma(\sigma+it)=\sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}e^{-\frac{\pi}{2}|t|}\big(1+O(|t|^{-1})\big)$$

valid for $\sigma_0\le\sigma\le\sigma_1$ and $t>t_0(\sigma_1,\sigma_2)$, we deduce that the ratio of the L_∞-terms in (2.3) is $\ll_\sigma |t|^{A(\frac{w+1}{2}-\sigma)}$. Hence by Lemma 1 we have

$$\begin{aligned}|U|^2 &\ll \big(c(\chi)^d\big)^{w+1-2\alpha}|t|^{A(w+1-2\alpha)}\\ &\ll \big(D^d|t|^A\big)^{w+1-2\alpha}.\end{aligned}$$

The summation over $n\ge X(\log DT)^2$ in (2.2) is negligible due to the decay of $\exp(-\frac{n}{X})$. We split the remaining part of the sum into the intervals of type $(N,2N]$ and apply (2.1) to obtain

$$\sum_{\chi\in\mathcal{C}(D)}\Bigg|\sum_{N<n\le 2N}a_n\chi(n)n^{-s}e^{-\frac{n}{X}}\Bigg|^2\ll(D+X^{w+2-2\sigma})(DX)^\varepsilon$$

for $\frac{w+1}{2}\le\sigma\le\frac{w+2}{2}$. Let $\Delta=\sigma-\frac{w+1}{2}$ and $\eta>0$ be sufficiently small and $X=(DT^A)^{1+\eta}$. By the definition of $\nu^\pm$ and the above estimate we obtain

$$\begin{aligned}S^+(D,s)\ll{}&\Big(\big(DT^A\big)^{w+2-2\sigma}+D\Big)(DT^A)^{\eta+3\varepsilon}\\ &+(DT^A)^{2(\alpha-\sigma)(1+\eta)}(D^dT^A)^{w+1-2\alpha}\big(D+(DT^A)^{2\alpha-w}\big)(DT^A)^{\nu^-(w+1-\alpha)+\varepsilon}.\end{aligned}$$

If $\frac{w+1}{2}+\eta\le\sigma\le\frac{w+2}{2}$ then we choose $\alpha=w+1-\sigma$ getting

$$\begin{aligned}S^+(D,s)\ll{}&\Big(\big(DT^A\big)^{w+2-2\sigma}+D\Big)(DT^A)^{\eta+3\varepsilon}\\ &+\big(DT^A\big)^{-4\Delta(1+\eta)}(D^dT^A)^{2\Delta}D\big(DT^A\big)^{\nu^-(w+1-\alpha)+\varepsilon}\end{aligned}\tag{2.4}$$

and by the inequality $d\le 2$ we obtain, by comparing the exponents that $\nu^+(\sigma)\le\max(\eta+3\varepsilon,-4\eta^2+\varepsilon+\nu^-(\sigma))$. Writting the same for the dual motive we conclude by the symmetry that $\nu^-(\sigma)\le\max(\eta+3\varepsilon,-4\eta^2+\varepsilon+\nu^+(\sigma))$. Choosing $2\eta>\sqrt{\varepsilon}$ we conclude that $\nu^+(\sigma)\le 0$ in this case.

If $\frac{w+1}{2} \leq \sigma \leq \frac{w+1}{2} + \eta$ then choosing $\alpha = \frac{w+1}{2} - \eta$ we have by the above that $\nu^+(w+1-\alpha) \leq 0$ and by (2.4) we obtain

$$\begin{aligned} S^+(D,s) &\ll \Big(\big(DT^A\big)^{w+2-2\sigma} + D\Big)(DT^A)^{\eta+3\varepsilon} + (D^dT^A)^{w+1-2\alpha}\big(D + (DT^A)^{2\alpha-w}\big) \\ &\ll \Big(\big(DT^A\big)^{w+2-2\sigma} + D\Big)(DT^A)^{\eta+3\varepsilon} + D(DT^A)^{4\eta+\varepsilon}, \end{aligned}$$

hence in this case also $\nu^+(\sigma) \leq 0$. Analogously we prove that $\nu^-(\sigma) \leq 0$. This completes the proof of Theorem 2.

3. APPLICATION OF THE ZERO DETECTION METHOD.

In this section we show how to deduce from (0.1) and Theorem 2, the zero density estimate (see Theorem 3 below). We apply the technique familiar from the Montgomery's book [Mo] (see Chapter 12). Since we are not interested in obtaining the best exponent here, the corresponding steps are essentially simpler. Let

$$N(\sigma,T,\chi) = \sharp\{\rho : L(M_\chi,\rho) = 0,\ \operatorname{Re}\rho \geq \sigma, |\gamma| \leq T\}.$$

The result is the following
Theorem 3. *Let $0 \leq \delta \leq 1/2$ be fixed. Then for any $\varepsilon > 0$ and $D > D_0(M,\varepsilon)$ we have*

$$\sum_{\chi\in\mathcal{C}(D)} N\left(\frac{w+1}{2} + \delta, T, \chi\right) \ll_\varepsilon (DT^{A+3})^{1+\varepsilon-\delta}.$$

Proof. We start from the Euler product representation

$$L(M_\chi,s) = \prod_p \big(1-\alpha_p^{(1)}\chi(p)p^{-s}\big)^{-1} \ldots \big(1-\alpha_p^{(d)}\chi(p)p^{-s}\big)^{-1},$$

in the corresponding half-plane, where $|\alpha_p^{(i)}| \leq p^{\frac{w}{2}} \quad (1 \leq i \leq d)$. In this half-plane the inverse

$$\frac{1}{L(M_\chi,s)} = \sum_{n\geq1} b_n\chi(n)n^{-s},$$

is determined by the coefficients b_n satisfying the bound

$$|b_n| \leq \tau_d(n)n^{\frac{w}{2}},$$

where $\tau_d(n)$ is the number of representations of n as a product of d-factors.

Let $B_X(M_\chi,s) = \sum_{n\leq X} b_n\chi(n)n^{-s}$. Then

$$L(M_\chi,s)B_X(M_\chi,s) = \sum_{n\geq1} e_n\chi(n)n^{-s},$$

where

$$e_n = e_n(X) = \begin{cases} 1 & n = 1 \\ 0 & 1 < n \leq X \\ \ll \tau_{2d}(n) n^{\frac{w}{2}}, & n > X. \end{cases}$$

Therefore applying the Mellin transform we obtain (for sufficiently large $Y \leq \frac{X}{\log^2 X}$)

$$\frac{1}{2\pi i} \int_{(\alpha)} L(M_\chi, s+v) B_X(M_\chi, s+v) Y^v \Gamma(v) dv = \sum_{n \geq 1} e_n \chi(n) n^{-s} e^{-\frac{n}{Y}}$$
$$= e^{\frac{-1}{Y}} + O(X^{-1}).$$

where the integration above is taken along the vertical line $\operatorname{Re} v = \alpha$ with $\alpha > \frac{w}{2} + 1$. Moving the line of integration $\operatorname{Re} v = \alpha$ to $\operatorname{Re} v = \frac{w+1}{2} - \operatorname{Re} s$, we pass a simple pole of gamma function at $v = 0$ with residuum $L(M_\chi, s) B_X(M_\chi, s)$, and possibly a pole of $L(M_\chi, s+v)$ at $v = \frac{w}{2} + 1 - s$. Since the last condition may happen at most for one character $\chi \in \mathcal{C}$, the corresponding contribution does not depend on D. Therefore

$$1 \ll L(M_\chi, s) B_X(M_\chi, s)| + \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| (LB_X)(M_\chi, \frac{w+1}{2} + iu) Y^{\frac{w+1}{2} - \sigma + i(u-t)} \, \Gamma\left(\frac{w+1}{2} - \sigma + i(u-t)) du \right) \right| \tag{3.1}$$

where $s = \sigma + it$. We apply the above inequality with $s = \rho$ such that $\operatorname{Re} \rho \geq \frac{w+1}{2} + \delta$. Moreover we assume that $\delta \geq (\log D)^{-1}$ since otherwise Theorem 3 is trivial. Let $|t| + 1 \leq T$. Then for the gamma factor we have

$$\Gamma\big(\frac{w+1}{2} - \sigma + i(u-t)\big) \ll \log D \exp\left(-c|u|/T\right)$$

with some $c > 0$. Summing both sides of (3.1) over the zeros ρ and then over $\chi \in \mathcal{C}$ we see that the RHS of (3.1) contributes at most to

$$\sum_{\chi \in \mathcal{C}} Y^{-\delta} N(\frac{w+1}{2} + \delta, T, \chi) \int_{-\infty}^{\infty} \left| (LB_X)(M_\chi, \frac{w+1}{2} + iu) e^{-c|u|/T} \right| du.$$

Hence by the the trivial bound

$$N(\frac{w+1}{2} + \delta, T, \chi) \ll T \log DT$$

and the Cauchy's inequality we obtain that

$$\sum_{\chi \in \mathcal{C}(D)} N\left(\frac{w+1}{2} + \delta, T, \chi\right) \ll Y^{-\delta} T \log(DT) \int_{-\infty}^{+\infty} \left(\sum_{\chi \in \mathcal{C}(D)} \left| L(M_\chi, \frac{w+1}{2} + iu) \right|^2 \right)^{\frac{1}{2}} \\ \times \left(\sum_{\chi} \left| B_X(M_\chi, \frac{w+1}{2} + iu) \right|^2 \right)^{\frac{1}{2}} e^{-c|u|/T} du \tag{3.2}$$

Now taking $X = DT^A$ and $Y = \frac{X}{\log^2 X}$, Theorem 3 follows directly from (0.1) and Theorem 2.

4. THE EXPLICIT FORMULA.

Here we follow the approach by Mestre [Me]. Given an integrable function $f : \mathbb{R} \to \mathbb{C}$ with compact support, define by the capital letter

$$F(s) = \int_{-\infty}^{+\infty} f(x) e^{sx} dx$$

the suitable Laplace transform. For a function f of real or complex variable and for $\lambda > 1$ we define $f_\lambda(z) = f(\frac{z}{\lambda})$.
Let

$$\begin{aligned} \Lambda_1(s,\chi) &= \Lambda(M_\chi, s)\Big(c(M)c(\chi)^{d(M)}\Big)^{\frac{s}{2}}, \\ \Lambda_2(s,\chi) &= \Lambda(M^\vee(-w)_\chi, s)\Big(c(M)c(\chi)^{d(M)}\Big)^{\frac{s}{2}}, \end{aligned} \tag{4.1}$$

Applying Cauchy's integration formula we have (for a suitable weight function Φ) the equality

$$\sum_\rho m_\chi(\rho)\Phi(\rho) = \frac{1}{2\pi i}\int_{(\frac{w+1}{2})} \left(\frac{\Lambda_1'}{\Lambda_1}(s,\chi)\Phi(s) + \frac{\Lambda_2'}{\Lambda_2}(s,\chi)\Phi(w+1-s)\right) ds$$

where the above sum is understood as a symmetric limit of $|\operatorname{Im}\rho| \le T$ as T tends to infinity. Mestre's explicit formula is derived when spliting the logarithmic derivative of Λ_1 and Λ_2 into the conductor term, term containing the gamma-factors (archimedean) and term containing the L-function (non-archimedean). For $\Phi(s) = F(s - \frac{w+1}{2})$ it has the following form (see [Me] p.213)

$$\begin{aligned} \sum_\rho m(\rho)F(\rho - \frac{w+1}{2}) &= \log\Big(c(M)c(\chi)^{d(M)/2}\Big) - \sum_i I(a_i,b_i) - \sum_i I(a_i',b_i') \\ &\quad - \sum_{l\ge 1, p-prime, i=1,2,\ldots d(M)} (\alpha_p^{(i)l} + \beta_p^{(i)l}) f(l\log p)\frac{\log p}{p^{(w+1)l/2}} \end{aligned} \tag{4.2}$$

Here both, the contribution of the gamma factors $I(a_i', b_i')$ and $\beta_p^{(i)}$ (the inverse p-roots of the dual motive) correspond to the dual part $\Lambda_2(s,\chi)$ (see (1.1) and (4.1)). We apply (4.2) with $f(x)$ multiplied by an additive character e^{it_0x}, where $f(x) \in C_0^k(\mathbb{R})$ $(k \ge 2)$ is a real, even function such that $f(0) = 1$. Regarding $f(x)e^{it_0x}$ as a new test function (which is not real as in Mestre's formulation) the formula (4.2) still holds if we replace $F(s - \frac{w+1}{2})$ by $F(s - s_0)$ and multiply the coefficients $\alpha_p^{(i)}, \beta_p^{(i)}$ by the corresponding additive characters (cf.[P]). To see this let us consider only the archimedean part. The integral to compute is the following

$$\begin{aligned}I_{s_0}(a,b) &= \int_{(\mathfrak{c})} \psi(as+b)F(s-s_0)ds \\ &= \int_{-\infty}^{+\infty} \psi\Big((a\mathfrak{c}+b)+ia(t+t_0)\Big)\hat{f}(t)dt\end{aligned}$$

where $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$, $a = 1/2$ or 1, b is a half-integer and $\mathfrak{c} = \frac{w+1}{2}$. To evaluate it we apply the equality due to Gauss

$$\psi(\sigma+iat)-\psi(\sigma) = \int_{-\infty}^{+\infty} k(x/a)\frac{1-e^{itx}}{x}dx,$$

where $\sigma = a\frac{w+1}{2}+b$, $k(x) = xe^{-\sigma x}/(1-e^{-x})$ for $x>0$ and 0 otherwise. Therefore changing the order of integration we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(t)\Big(\psi(\sigma+ia(t+t_0)-\psi(\sigma)\Big)dt$$

$$\begin{aligned}&= \int_{-\infty}^{+\infty} \frac{k(x/a)}{x}\int_{-\infty}^{+\infty} \hat{f}(t)\big(1-e^{i(t+t_0)x}\big)dtdx \\ &= \int_{-\infty}^{+\infty} \left(\frac{f(ax)e^{-\sigma x}}{1-e^{-x}}e^{iat_0x}-\frac{f(0)e^{-x}}{x}\right)dx\end{aligned}$$

Hence we get the equality

$$\frac{1}{2\pi i}\int_{(\mathfrak{c})} \psi(as+b)G(s-\frac{w+1}{2})ds = -\int_{-\infty}^{+\infty} \left(\frac{g(ax)e^{-\sigma x}}{1-e^{-x}}-g(0)\frac{e^{-x}}{x}\right)dx,$$

where $g(x) = f(x)e^{it_0x}$. Replacing (f,F) by $(f_\lambda, \lambda F_{\frac{1}{\lambda}})$ and summing both sides of (4.2) over $\chi \in \mathcal{C}(D)$ we obtain by Lemma 2 of [P-P] the equality

$$\lambda \sum_{\chi\in\mathcal{C}(D)}\sum_{\rho} m_\chi(\rho)F_{\frac{1}{\lambda}}(\rho-s_0) = d(M)f_\lambda(0)\sum_{\chi\in\mathcal{C}(D)}\log c(\chi) + O_{s_0}(D\lambda+\sqrt{D}e^{2\lambda}). \tag{4.3}$$

5. COMPLETION OF THE PROOF.

In order to use the density estimate we first split the RHS of (4.3) into sums over $|\operatorname{Re}(\rho-s_0)| \le 1/\lambda$ (main term) and $|\operatorname{Re}(\rho-s_0)| > 1/\lambda$ (error term). From the classical arguments (see e.g. [Pe]) we know that $m(\rho) < \log(D|\operatorname{Im}\rho|)$, hence the error term restricted additionally by the condition $|\operatorname{Im}\rho - t_0| \le 1$ contributes at most to

$$\log(DT_0)\sum_{\chi}\sum_{|\operatorname{Re}\rho-\mathfrak{c}|>\frac{1}{\lambda}, |\operatorname{Im}\rho-t_0|\le 1} e^{\lambda(\operatorname{Re}\rho-\mathfrak{c})} \ll \log(DT_0)\sum_{\chi}\int_{\mathfrak{c}+\lambda^{-1}}^{\frac{w+2}{2}} e^{\lambda(\sigma-\mathfrak{c})}dN(\sigma,T_0,\chi)$$

$$\ll \log(DT_0)\sum_{\chi}\Big(N(\mathfrak{c}+\lambda^{-1},T_0,\chi)+\lambda\int_{\mathfrak{c}+\lambda^{-1}}^{\frac{w+2}{2}} e^{\lambda(\sigma-\mathfrak{c})}N(\sigma,T_0,\chi)\Big)d\sigma$$

where $T_0 = |t_0|+1$ and χ varies in the set $\mathcal{C}(D)$. Now using the density estimate we conclude that the corresponding contribution is

$$O_\varepsilon\Big(\lambda T_0(DT_0^{A+3})^{1+\varepsilon-\frac{1}{2\lambda}}\Big).$$

Choosing $f\in C_0^k(\mathbb{R})$ with sufficiently large $k=k(A)$, the decay of $F(s)$ over the imaginary lines shows that the above estimate holds without the additional restriction $|\operatorname{Im}\rho-t_0|\le 1$. To complete the proof of Theorem 1 we choose F_1 and F_2 such that both $\operatorname{Re}F_1(s)$ and $\operatorname{Re}F_2(s)$ are $\gg 1$ for $|s|\le 1$, both are positive for $|\operatorname{Re}s|\le 1$ and $|\operatorname{Im}s|\le 1$, and in the case when $|\operatorname{Re}s|\le 1$ and $|\operatorname{Im}s|>1$ we have $\operatorname{Re}F_1(s)>0$ and $\operatorname{Re}F_2(s)<0$. In the first case we obtain by (4.3)

$$\lambda\sum_{\chi}\sum_{|\operatorname{Re}\rho-s_0|\le 1/\lambda} m_\chi(\rho)\le c_1D\log D+O_{\varepsilon,s_0}\Big(D\lambda+\sqrt{D}e^{2\lambda}+\lambda D^{1+\varepsilon-\frac{1}{2\lambda}}\Big)$$

while in the second

$$\lambda\sum_{\chi}\sum_{|\operatorname{Re}(\rho-s_0|\le 1/\lambda,|\operatorname{Im}(\rho-s_0)|\le 1/\lambda} m_\chi(\rho)\ge c_2D\log D+O_{\varepsilon,s_0}\Big(D\lambda+\sqrt{D}e^{2\lambda}+\lambda D^{1+\varepsilon-\frac{1}{2\lambda}}\Big)$$

where c_1 and c_2 are some absolute, positive constants. Then the conclusions follow if we take $\lambda=\frac{1}{\delta}$ in the first case, $\lambda=\frac{2}{\delta}$ in the second one, ε small enough and D tending to infinity. It remains to prove the existence of the functions f_1 and f_2 with Laplace transforms F_1 and F_2 satisfying the required conditions. The construction of f_1 was given in [P-P]. To construct f_2 assume that $h\in C_0^\infty(\mathbb{R})$ satisfies the condition $\hat{h}\ge 0$ and define $h_*=h+ch''$ with a suitable, positive constant c. Then we have $\hat{h}_*(\xi)=\hat{h}(\xi)-c\xi^2\hat{h}(\xi)<0$ for $|\xi|\gg 1$ and taking $f_2(x)=h_*(x)\big(\operatorname{ch}(2x)\big)^{-1}$ we obtain, by Proposition 3, p.59 [S-Z] that

$$\begin{aligned} F_2(\sigma+it) &= \pi^2\int_{-\infty}^{+\infty}\frac{\hat{h}_*(t-x)}{\operatorname{ch}(\pi(x-i\sigma)/4)}dx\\ &= \pi^2\cos(\pi\sigma/4)\int_{-\infty}^{+\infty}\hat{h}_*(t-x)\frac{\operatorname{ch}(\pi x/4)}{|\operatorname{ch}(\pi(x-i\sigma)/4)|^2}dx\end{aligned}$$

hence the required properties follow.

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