K_1 BY GENERATORS AND RELATIONS

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Abstract. We give a presentation by generators and relations for K_1 of an arbitrary exact category.

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1. Review of the G-Construction

In [GG] Gillet and Grayson attached a simplicial set $G.\mathfrak{A}$ to any exact category $\mathfrak{A}$ and proved that $|G.\mathfrak{A}|$ is homotopy equivalent to $\Omega|Q\mathfrak{A}| \sim \Omega|S.\mathfrak{A}|$, the equivalence being natural in $\mathfrak{A}$. Thus one can take the formula

$$K_m(\mathfrak{A}) = \pi_m(G.\mathfrak{A}), \qquad m \geq 0,$$

for a definition of the higher K-groups of $\mathfrak{A}$.

An n-simplex in $G.\mathfrak{A}$ is a pair of triangular diagrams in $\mathfrak{A}$ of the form

$$\begin{array}{ccccccccc}
 & & & & P_{n/n-1} \\
 & & & & \uparrow \\
 & & & \cdots & \cdots \\
 & & P_{2/1} & \to \cdots \to & P_{n/1} \\
 & & \uparrow & & \uparrow \\
 & P_{1/0} \to & P_{2/0} & \to \cdots \to & P_{n/0} \\
 & \uparrow & \uparrow & & \uparrow \\
P_0 \to & P_1 \to & P_2 & \to \cdots \to & P_n
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & P_{n/n-1} \\
 & & & & \uparrow \\
 & & & \cdots & \cdots \\
 & & P_{2/1} & \to \cdots \to & P_{n/1} \\
 & & \uparrow & & \uparrow \\
 & P_{1/0} \to & P_{2/0} & \to \cdots \to & P_{n/0} \\
 & \uparrow & \uparrow & & \uparrow \\
P'_0 \to & P'_1 \to & P'_2 & \to \cdots \to & P'_n
\end{array}
\qquad (1.1)$$

subject to the conditions:

(i) the quotient index subtriangles in both diagrams coincide;
(ii) all the squares commute;
(iii) all the sequences of the form $P_j \to P_k \to P_{k/j}$, $P'_j \to P'_k \to P_{k/j}$, and $P_{j/i} \to P_{k/i} \to P_{k/j}$ with $i \leq j \leq k$ are short exact sequences in $\mathfrak{A}$.

In particular, a vertex in $G.\mathfrak{A}$ is a pair of objects (P, P'), and an edge connecting (P_0, P'_0) to (P_1, P'_1) is a pair of short exact sequences $(P_0 \to P_1 \to P_{1/0}, P'_0 \to P'_1 \to P_{1/0})$, with equal cokernels. The i-th face of (1.1) amounts to deleting all the objects whose indices contain i.

For instance, the faces of a generic 2-simplex

$$t = \left[\begin{array}{ccccccc} & & P_{2/1} & & & & P_{2/1} \\ & & \uparrow & & & & \uparrow \\ & P_{1/0} & \longrightarrow P_{2/0} & & & P_{1/0} & \longrightarrow P_{2/0} \\ & \uparrow & \uparrow & & & \uparrow & \uparrow \\ P_0 \longrightarrow & P_1 & \longrightarrow P_2 & & P'_0 \longrightarrow & P'_1 & \longrightarrow P'_2 \end{array}\right] \qquad (1.2)$$

are given by

$$\begin{aligned} d_0t &= (P_1 \to P_2 \to P_{2/1}, \quad P'_1 \to P'_2 \to P_{2/1}) \\ d_1t &= (P_0 \to P_2 \to P_{2/0}, \quad P'_0 \to P'_2 \to P_{2/0}) \\ d_2t &= (P_0 \to P_1 \to P_{1/0}, \quad P'_0 \to P'_1 \to P_{1/0}). \end{aligned}$$

Let 0 denote a distinguished zero object in $\mathfrak{A}$, then we let $(0,0)$ be the base point of $G.\mathfrak{A}$. Given $A \in \mathfrak{A}$, the standard edge $e(A)$ from $(0,0)$ to (A,A) is given by

$$e(A) = (0 \to A \xrightarrow{1} A, 0 \to A \xrightarrow{1} A).$$

2. The Main Result

Let $\mathfrak{A}$ be an exact category.

Definition. A *double short exact sequence* in $\mathfrak{A}$ (a dses. for short) is a pair of short exact sequences $0 \to A \xrightarrow{f_1} B \xrightarrow{g_1} C \to 0$, $0 \to A \xrightarrow{f_2} B \xrightarrow{g_2} C \to 0$ on the same objects. Given such data, we will write them in the form

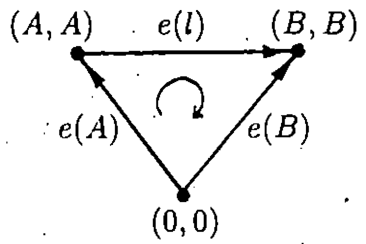
$$l = (A \underset{f_2}{\overset{f_1}{\rightrightarrows}} B \underset{g_2}{\overset{g_1}{\rightrightarrows}} C). \qquad (2.1)$$

If $A \in \mathfrak{A}$ and $\alpha \in \operatorname{Aut} A$, we will associate to α two dses.'s

$$l(\alpha) = (0 \rightrightarrows A \underset{\alpha}{\overset{1}{\rightrightarrows}} A), \quad \tilde{l}(\alpha) = (A \underset{1}{\overset{\alpha}{\rightrightarrows}} A \rightrightarrows 0), \qquad (2.2)$$

and in this way a dses. should be thought of as a generalization of an automorphism.

For any dses. l of the form (2.1), we denote by $e(l)$ the edge from (A,A) to $(B.B)$ in $G.\mathfrak{A}$ given by l. We associate to l a loop $\mu(l)$ in $G.\mathfrak{A}$ given by



$(2.2\frac{1}{2})$

and let $m(l)$ denote its class in $K_1(\mathfrak{A}) = \pi_1(G.\mathfrak{A})$.

Proposition 2.1. ([Ne2]) *The elements $m(l)$ are subject to the following two types of relations in $K_1(\mathfrak{A})$:*

(i) *If $f_1 = f_2$ and $g_1 = g_2$ in (2.1), then $m(l) = 0$ (in this case, we will say that l is diagonal).*

(ii) *Suppose we are given a diagram of the form*

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array} \tag{2.3}$$

which consists of six double short exact sequences and is subject to the condition: the first (the upper) arrows commute with the first (the left) ones and the second (the lower) arrows commute with the second (the right) ones. Let l_A, l_B, and l_C (respectively, l', l, and l'') denote the horizontal (respectively, the vertical) dses.'s in (2.3). Then we have

$$m(l_A) - m(l_B) + m(l_C) = m(l') - m(l) + m(l'').$$

Definition. We define $\mathcal{D}(\mathfrak{A})$ to be the abelian group with generators $\langle l \rangle$ for all dses.'s l subject to the above relations (i) and (ii) posed on the symbols $\langle l \rangle$ rather than on the elements $m(l)$ of $K_1(\mathfrak{A})$.

By Proposition 2.1, we have a well-defined homomorphism

$$m \colon \mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A}), \qquad \langle l \rangle \mapsto m(l).$$

Consider the category DSES($\mathfrak{A}$) of all dses.'s in $\mathfrak{A}$. We can make it an exact category, a short exact sequence of dses.'s being a diagram of the form

$$\begin{array}{ccccc} A' & \rightrightarrows & A & \rightrightarrows & A'' \\ \downarrow & & \downarrow & & \downarrow \\ B' & \rightrightarrows & B & \rightrightarrows & B'' \\ \downarrow & & \downarrow & & \downarrow \\ C' & \rightrightarrows & C & \rightrightarrows & C'' \end{array} \tag{2.4}$$

where the columns are short exact sequences in $\mathfrak{A}$ and the upper (the lower) horizontal arrows commute with the upper (the lower) ones. We can regard (2.4) as a particular case of (2.3) in which the vertical dses.'s are diagonal. Thus in the same notation, we have in $\mathcal{D}(\mathfrak{A})$

$$\langle l_A \rangle - \langle l_B \rangle + \langle l_C \rangle = 0, \tag{2.5}$$

for any diagram of the type (2.4), i.e., the symbol $\langle l \rangle$ is additive and we have a well-defined map

$$K_0(\mathrm{DSES}(\mathfrak{A})) \to \mathcal{D}(\mathfrak{A}).$$

Consider the exact category $\mathrm{Aut}(\mathfrak{A})$ of pairs $(A;\alpha)$ with $A \in \mathfrak{A}$ and $\alpha \in \mathrm{Aut}\, A$. We have an exact functor $\mathrm{Aut}(\mathfrak{A}) \to \mathrm{DSES}(\mathfrak{A})$ given by $\alpha \mapsto l(\alpha)$ (see (2.2)) which yields a homomorphism $K_0(\mathrm{Aut}(\mathfrak{A})) \to K_0(\mathrm{DSES}(\mathfrak{A}))$. Given two automorphisms $\alpha, \beta \in \mathrm{Aut}(A)$, consider the diagram

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ 0 & \rightrightarrows & A & \overset{1}{\underset{\alpha\beta}{\rightrightarrows}} & A \\ \downdownarrows & & {\scriptstyle 1}\downdownarrows{\scriptstyle \beta} & & {\scriptstyle 1}\downdownarrows{\scriptstyle 1} \\ 0 & \rightrightarrows & A & \overset{1}{\underset{\alpha}{\rightrightarrows}} & A \end{array}$$

It is a particular case of (2.3), hence we get in $\mathcal{D}(\mathfrak{A})$

$$\langle l(\alpha\beta) \rangle = \langle l(\alpha) \rangle + \langle l(\beta) \rangle. \tag{2.6}$$

It follows that we have a well-defined homomorphism

$$K_0(\mathrm{Aut}(\mathfrak{A}))/\sim \;\longrightarrow\; \mathcal{D}(\mathfrak{A}), \qquad \alpha \mapsto \langle l(\alpha) \rangle,$$

where the equivalence relation is generated by $(A;\alpha\beta) \sim (A;\alpha) + (A;\beta)$ for all $A \in \mathfrak{A}$ and $\alpha, \beta \in \mathrm{Aut}\, A$. The left hand side group is known as the group $K_1^{\det}(\mathfrak{A})$ of H. Bass (cf. [Ba1], [Ba2], and [Ge]), and the composite map $K_1^{\det}(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$ ia also well-known (for instance, see [Ge]). If every short exact sequence in $\mathfrak{A}$ splits, then $K_1^{\det}(\mathfrak{A}) \to K_1(\mathfrak{A})$ is an isomorphism ([We], [Sh1]). However, in general this map need not be either surjective or injective (see [Ge]), i.e., $K_1^{\det}$ does not provide a good algebraic substitute for K_1 in the general case. But $\mathcal{D}(\mathfrak{A})$ does the job.

Theorem. *For any exact category $\mathfrak{A}$, the map $m\colon \mathcal{D}(\mathfrak{A}) \to K_1(\mathfrak{A})$ is an isomorphism.*

In [Ne1] we have shown that for any element $x \in K_1(\mathfrak{A})$, there exists a dses. l such that $x = m(l)$ (we use the results of [Sh2] and [Sh3] in our proof of this fact). Thus m is surjective. In the present paper, we construct a homomorphism $b\colon K_1(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A})$ and show that $b \circ m = \mathrm{id}_{\mathcal{D}(\mathfrak{A})}$, which implies injectivity of m.

3. Some Lemmas about $K_1^{\det}(\mathfrak{A})$ and $\mathcal{D}(\mathfrak{A})$

Lemma 3.1. *Let $A \in \mathfrak{A}$. In the notation (2.2), we have in $\mathcal{D}(\mathfrak{A})$*

(i) $\langle l(\alpha) \rangle = \langle \tilde{l}(\alpha) \rangle$ *for any* $\alpha \in \mathrm{Aut}\, A$;

(ii) $\langle 0 \rightrightarrows A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \rangle = \langle l(\alpha^{-1}\beta) \rangle = \langle l(\beta\alpha^{-1}) \rangle = \langle l(\beta) \rangle - \langle l(\alpha) \rangle,$

$\langle A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \rightrightarrows 0 \rangle = \langle \tilde{l}(\alpha\beta^{-1}) \rangle = \langle \tilde{l}(\beta^{-1}\alpha) \rangle = \langle \tilde{l}(\alpha) \rangle - \langle \tilde{l}(\beta) \rangle,$

for any $\alpha, \beta \in \mathrm{Aut}\, A$.

Proof. (i)

$$\begin{array}{ccccc}
0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
A & \overset{\alpha}{\underset{1}{\rightrightarrows}} & A & \rightrightarrows & 0 \\
{\scriptstyle 1}\Downarrow{\scriptstyle 1} & & {\scriptstyle 1}\Downarrow{\scriptstyle \alpha} & & \Downarrow \\
A & \overset{\alpha}{\underset{\alpha}{\rightrightarrows}} & A & \rightrightarrows & 0
\end{array}$$

(ii) use (2.6) and the diagrams like

$$\begin{array}{ccccc}
0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
0 & \rightrightarrows & A & \overset{\alpha}{\underset{\beta}{\rightrightarrows}} & A \\
\Downarrow & & {\scriptstyle \alpha}\Downarrow{\scriptstyle \alpha} & & {\scriptstyle 1}\Downarrow{\scriptstyle 1} \\
0 & \rightrightarrows & A & \overset{1}{\underset{\beta\alpha^{-1}}{\rightrightarrows}} & A
\end{array} \qquad \square$$

Lemma 3.2. *Let $A \in \mathfrak{A}$.*

(i) *The class of the automorphism*

$$\alpha_A = \begin{pmatrix} 0 & 1_A \\ -1_A & 0 \end{pmatrix} \in \mathrm{Aut}(A \oplus A) \tag{3.1}$$

vanishs in $K_1^{\det}(\mathfrak{A})$.

(ii) *The class of the d.s.e.s.*

$$A \overset{\binom{1}{0}}{\underset{\binom{0}{1}}{\rightrightarrows}} A \oplus A \overset{(0,1)}{\underset{(-1,0)}{\rightrightarrows}} A \tag{3.2}$$

vanishs in $\mathcal{D}(\mathfrak{A})$.

Proof. (i) Observe that

$$\alpha_A = \begin{pmatrix} 1_A & 0 \\ -1_A & 1_A \end{pmatrix} \begin{pmatrix} 1_A & 1_A \\ 0 & 1_A \end{pmatrix} \begin{pmatrix} 1_A & 0 \\ -1_A & 1_A \end{pmatrix}.$$

The classes of $\begin{pmatrix} 1_A & 0 \\ -1_A & 1_A \end{pmatrix}$ and $\begin{pmatrix} 1_A & 1_A \\ 0 & 1_A \end{pmatrix}$ vanish in $K_1^{\det}(\mathfrak{A})$. For consider the short exact sequences in the category $\mathrm{Aut}(\mathfrak{A})$

$$0 \longrightarrow 1_A \xrightarrow{\binom{1}{0}} \begin{pmatrix} 1_A & 1_A \\ 0 & 1_A \end{pmatrix} \xrightarrow{(0,1)} 1_A \longrightarrow 0$$

$$0 \longrightarrow 1_A \xrightarrow{\binom{0}{1}} \begin{pmatrix} 1_A & 0 \\ -1_A & 1_A \end{pmatrix} \xrightarrow{(1,0)} 1_A \longrightarrow 0.$$

(ii) It follows from the diagram

$$\begin{array}{ccccc} 0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\ \downdownarrows & & \downdownarrows & & \downdownarrows \\ A & \underset{\binom{0}{1}}{\overset{\binom{1}{0}}{\rightrightarrows}} & A\oplus A & \underset{(-1,0)}{\overset{(0,1)}{\rightrightarrows}} & A \\ 1\downdownarrows 1 & & 1\downdownarrows \alpha_A & & 1\downdownarrows 1 \\ A & \underset{\binom{1}{0}}{\overset{\binom{1}{0}}{\rightrightarrows}} & A\oplus A & \underset{(0,1)}{\overset{(0,1)}{\rightrightarrows}} & A \end{array} \qquad (3.3)$$

that the class of (3.2) in $\mathcal{D}(\mathfrak{A})$ equals $\langle l(\alpha_A)\rangle$, the latter vanishs by (i). □

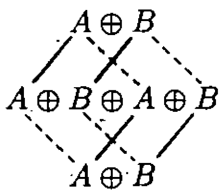
Lemma 3.3. *Let (A, B) be a vertex in the base point component of $G.\mathfrak{A}$ (i.e $[A] = [B]$ in $K_0(\mathfrak{A})$).*

(i) *The class of the automorphism*

$$\alpha_{A,B} = \begin{pmatrix} 0 & 1_{A\oplus B} \\ 1_{A\oplus B} & 0 \end{pmatrix} \in \operatorname{Aut}(A \oplus B \oplus A \oplus B) \qquad (3.4)$$

vanishs in $K_1^{\det}(\mathfrak{A})$.

(ii) *The class of the d.s.e.s.*



$\left(\text{\textit{In this notation, the regular lines yield the first arrows and the dashed lines yield the second arrows of the d.s.e.s.,}}\right)$

vanishs in $\mathcal{D}(\mathfrak{A})$.

Proof. (i) Let $(A \to A' \to N, B \to B' \to N)$ be an edge in $G.\mathfrak{A}$. Then we have a short exact sequence in the category $\operatorname{Aut}(\mathfrak{A})$,

$$0 \to \alpha_{A,B} \to \alpha_{A',B'} \to \alpha_{N,N} \to 0.$$

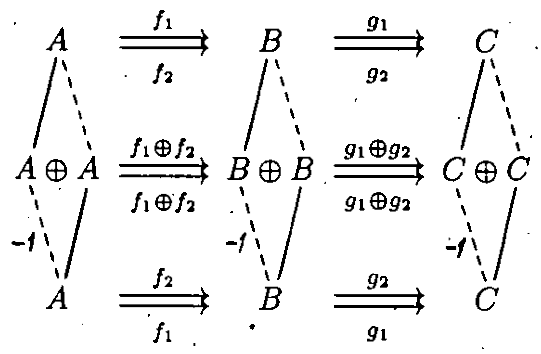
Consider the matrix $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ over $\mathbb{Z}$. Since its determinant equals 1, it can be represented as a product of elementary matrices. Replacing the integer 1 by $1_N \in \operatorname{Aut} N$, we obtain the same representation for $\alpha_{N,N}$. Now the short exact sequences as in the proof of Lemma 3.2(i) show that the class of $\alpha_{N,N}$ vanishs in $K_1^{\det}(\mathfrak{A})$. Thus the classes of $\alpha_{A,B}$ and $\alpha_{A',B'}$ are equal. Since we can connect (A, B) to $(0, 0)$ by a sequence of edges, assertion (i) is proved.

(ii) follows from (i) by a diagram similar to (3.3). □

In the notation (2.1), we put $l^{\mathrm{op}} = (A \underset{f_1}{\overset{f_2}{\rightrightarrows}} B \underset{g_1}{\overset{g_2}{\rightrightarrows}} C)$.

Lemma 3.4. *$\langle l^{\mathrm{op}}\rangle = -\langle l\rangle$ in $\mathcal{D}(\mathfrak{A})$ for any d.s.e.s. l.*

Proof. Consider the diagram



The vertical d.s.e.s.'s are of the type (3.2), hence we are done. □

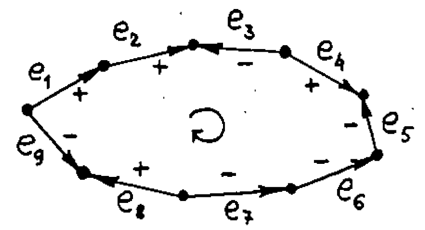
Lemma 3.5. *Isomorphic d.s.e.s.'s give rise to the same element of $\mathcal{D}(\mathfrak{A})$.*

Proof. It is an obvious particular case of the additivity (2.5). □

4. The Inverse Map $b : K_1(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A})$

Let $G.\mathfrak{A}^\circ$ denote the base point component of $G.\mathfrak{A}$. We will assign a d.s.e.s. to any combinatorial loop in $G.\mathfrak{A}^\circ$ and show that this leads to a well-defined homomorphism $K_1(\mathfrak{A}) = \pi_1(G.\mathfrak{A}^\circ) \to \mathcal{D}(\mathfrak{A})$.

A (combinatorial oriented) loop in a simplicial set is a circular sequence of edges like



We attach plus or minus to an edge if its orientation inherited from the simplicial set structure coincides with (respectively, is opposite to) the orientation of the loop. Thus a loop is a sequence of edges $e_1, \dots, e_n$ and signs $\varepsilon_1, \dots, \varepsilon_n \in \{+,-\}$ such that for each $i = 1, \dots, n$

$$\begin{aligned}
d_0 e_i &= d_1 e_{i+1} && \text{if } \varepsilon_i = \varepsilon_{i+1} = + \\
d_0 e_i &= d_0 e_{i+1} && \text{if } \varepsilon_i = +,\ \varepsilon_{i+1} = - \\
d_1 e_i &= d_1 e_{i+1} && \text{if } \varepsilon_i = -,\ \varepsilon_{i+1} = + \\
d_1 e_i &= d_0 e_{i+1} && \text{if } \varepsilon_i = \varepsilon_{i+1} = -
\end{aligned} \tag{4.1}$$

under the convention $e_{n+1} = e_1$, $\varepsilon_{n+1} = \varepsilon_1$. Condition (4.1) simply means that the target of each edge coincides with the source of the next edge, where we use the words "source" and "target" in the sense of the loop orientation, and the four cases in (4.1) mean that this sense might be opposite to the sense inherited from the simplicial set structure. We consider the source of e_1 as the base point of such a loop.

A free (combinatorial oriented) loop is a loop up to a cyclic permutation of indices, i.e. without base point.

If $e = (P \xrightarrow{\alpha} P' \xrightarrow{\gamma} N, Q \xrightarrow{\beta} Q' \xrightarrow{\delta} N)$ is an edge in $G.\mathfrak{A}$, we put $e^+ = e$ and $e^- = (Q \xrightarrow{\beta} Q' \xrightarrow{\delta} N, P \xrightarrow{\alpha} P' \xrightarrow{-\gamma} N)$. (NB: we change the sign of γ.) Let $\mu = (e_1, \ldots, e_n; \varepsilon_1, \ldots, \varepsilon_n)$ be a loop in $G.\mathfrak{A}$ with $\varepsilon_i \in \{+, -\}$ and

$$e_i = (P_i \xrightarrow{\alpha_i} P_i' \xrightarrow{\gamma_i} N_i, Q_i \xrightarrow{\beta_i} Q_i' \xrightarrow{\delta_i} N_i).$$

Let

$$e(\mu) = \bigoplus_{i=1}^{n} e_i^{\varepsilon_i}, \tag{4.2}$$

i.e., we take the term-wise direct sums of the first and the second short exact sequences that appear in e_i or e_i^- accordingly to $\varepsilon_i = +$ or $-$, over all i. We introduce an explicit notation for the edge $e(\mu)$,

$$e(\mu) = (P(\mu) \to P'(\mu) \to N(\mu), Q(\mu) \to Q'(\mu) \to N(\mu)). \tag{4.3}$$

Thus for instance, $P(\mu)$ is the direct sum of those P_i for which $\varepsilon_i = +$ and those Q_i for which $\varepsilon_i = -$.

We claim that the objects $P(\mu) \oplus Q'(\mu)$ and $P'(\mu) \oplus Q(\mu)$ are isomorphic. For let $1 \leq i \leq n$ and (P, Q) denote the vertex between e_i and e_{i+1}. Then in each of the four cases in (4.1), there is a copy of P and a copy of Q as a direct summand in both expressions under question. (These copies correspond to the pair of indices $(i, i+1)$. There might be other copies of P and Q in $P(\mu) \oplus Q'(\mu)$ and $P'(\mu) \oplus Q(\mu)$ if the loop passes through the vertex (P, Q) several times). For instance, in the third case

$(\ldots \xleftarrow{e_i} \underset{(P,Q)}{\bullet} \xrightarrow{e_{i+1}} \ldots)$ we have $P_i = P_{i+1} = P$, $Q_i = Q_{i+1} = Q$,

$$\begin{aligned} e_i^- &= (Q \to Q_i' \to N_i, P \to P_i' \to N_i) \\ e_{i+1}^+ &= (P \to P_{i+1}' \to N_{i+1}, Q \to Q_{i+1}' \to N_{i+1}), \end{aligned}$$

and the claim is obvious. Let

$$\eta(\mu) \colon P(\mu) \oplus Q'(\mu) \xrightarrow{\sim} P'(\mu) \oplus Q(\mu)$$

denote the isomorphism that takes those copies of P (respectively Q) to each other, for every vertex.

Given an edge $e = (P \xrightarrow{\alpha} P' \xrightarrow{\gamma} N, Q \xrightarrow{\beta} Q' \xrightarrow{\delta} N)$ endowed with an isomorphism $\eta \colon P \oplus Q' \xrightarrow{\sim} P' \oplus Q$, consider the two short exact sequences

$$\begin{array}{ccccc} P \oplus Q & \xrightarrow{\alpha \oplus 1_Q} & P' \oplus Q & \xrightarrow{(\gamma, 0)} & N \\ & & \wr \uparrow \eta & & \\ P \oplus Q & \xrightarrow{1_P \oplus \beta} & P \oplus Q' & \xrightarrow{(0, \delta)} & N \end{array} \tag{4.4}$$

Replacing the object $P \oplus Q'$ in the second short exact sequence by $P' \oplus Q$ by means of η we obtain a double short exact sequence,

$$l(e;\eta) = (P \oplus Q \underset{\eta \circ (1_P \oplus \beta)}{\overset{\alpha \oplus 1_Q}{\rightrightarrows}} P' \oplus Q \underset{(0,\delta)\circ\eta^{-1}}{\overset{(\gamma,0)}{\rightrightarrows}} N).$$

We put $l(\mu) \doteq l(e(\mu); \eta(\mu))$, and let $b(\mu) = \langle l(\mu) \rangle$ be its class in $\mathcal{D}(\mathfrak{A})$, for any loop μ in $G.\mathfrak{A}$. A cyclic permutation of indices gives rise to a permutation of the direct summands in the whole procedure, therefore it leads to an isomorphic dses.. Thus by Lemma 3.5, $b(\mu)$ is well-defined for a free loop.

Let μ_1 and μ_2 be free loops, and suppose that they have a common vertex (P, Q).

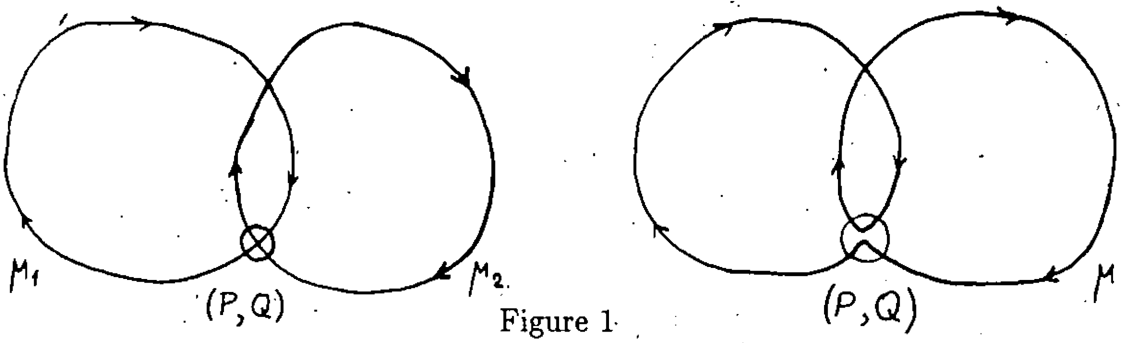


Figure 1

We define the product of μ_1 and μ_2 at (P, Q) by the right hand side of the above figure, i.e., we take the disjoint union of the sets of indices for μ_1 and μ_2 and put them in the obvious cyclic order. Let μ denote the resulting loop. It might happen that μ_1 or μ_2 passes through (P, Q) several times. In order to make the definition correct, we fix a pair of indices $(i, i+1)$ (respectively $(j, j+1)$) such that (P, Q) is the vertex between the i-th and $(i+1)$-th edges of μ_1 and between the j-th and $(j+1)$-th edges of μ_2.

Proposition 4.1. $b(\mu) = b(\mu_1) + b(\mu_2)$ *provided all the three loops are in* $G.\mathfrak{A}^\circ$.

Lemma 4.2. *Let* $e = (P \xrightarrow{\alpha} P' \xrightarrow{\gamma} N; Q \xrightarrow{\beta} Q' \xrightarrow{\delta} N)$ *be an edge and* $\eta, \eta' : P \oplus Q' \xrightarrow{\sim} P' \oplus Q$ *be isomorphisms. Let* $\phi = \eta' \oplus \eta^{-1} \in \mathrm{Aut}(P' \oplus Q)$. *Then in* $\mathcal{D}(\mathfrak{A})$ *we have*

$$\langle l(e;\eta) \rangle = \langle l(e;\eta') \rangle + \langle l(\phi) \rangle.$$

Proof.

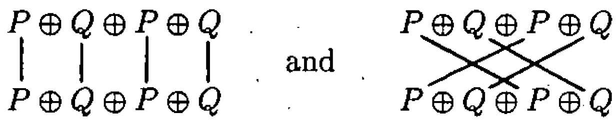
$$\begin{array}{ccccc}
0 & \rightrightarrows & 0 & \rightrightarrows & 0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
P \oplus Q & \underset{\eta\circ(1_P\oplus\beta)}{\overset{\alpha\oplus 1_Q}{\rightrightarrows}} & P' \oplus Q & \underset{(0,\delta)\circ\eta^{-1}}{\overset{(\gamma,0)}{\rightrightarrows}} & N \\
1 \Downarrow 1 & & 1 \Downarrow \phi & & 1 \Downarrow 1 \\
P \oplus Q & \underset{\eta'\circ(1_P\oplus\beta)}{\overset{\alpha\oplus 1_Q}{\rightrightarrows}} & P' \oplus Q & \underset{(0,\delta)\circ\eta'^{-1}}{\overset{(\gamma,0)}{\rightrightarrows}} & N
\end{array}$$

□

Proof of the proposition. There is an obvious isomorphism

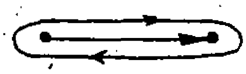
$$e(\mu_1) \oplus e(\mu_2) \cong e(\mu)$$

since both expressions consist of the same summands. The difference between $\eta(\mu_1) \oplus \eta(\mu_2)$ and $\eta(\mu)$ amounts essentially to the difference between the isomorphisms



accordingly to what we change at the point (P, Q) in Figure 1. By the lemma, $\langle l(\mu_1)\rangle + \langle l(\mu_2)\rangle = \langle l(\mu_1) \oplus l(\mu_2)\rangle = \langle l(\mu)\rangle + \langle l(\alpha_{P,Q})\rangle$, where $\alpha_{P,Q}$ is the matrix of (3.4). Thus by Lemma 3.3 we are done. $\square$

For any edge $e = (P \xrightarrow{\alpha} P' \xrightarrow{\gamma} N, Q \xrightarrow{\beta} Q' \xrightarrow{\delta} N)$, let $\mu(e)$ denote the loop $(e, e; +, -)$ which goes along e to and back,



Lemma 4.3. *If e is an edge in $G.\mathfrak{A}^\circ$, then $b(\mu(e)) = 0$.*

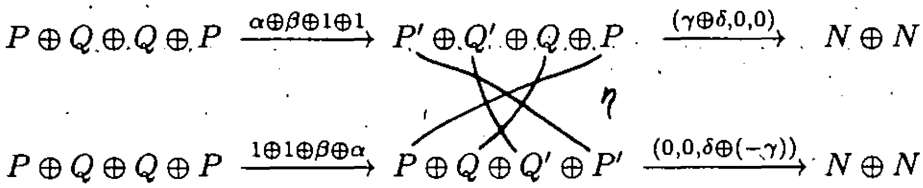
Proof. According to (4.2), we have $e(\mu(e)) = e \oplus e^- =$

$$= (P \oplus Q \xrightarrow{\alpha\oplus\beta} P' \oplus Q' \xrightarrow{\gamma\oplus\delta} N \oplus N, \quad Q \oplus P \xrightarrow{\beta\oplus\alpha} Q' \oplus P' \xrightarrow{\delta\oplus(-\gamma)} N \oplus N).$$

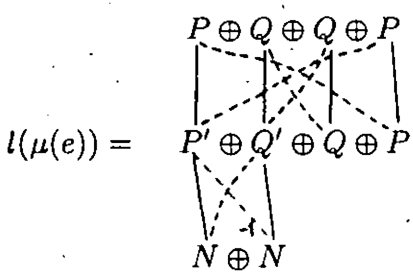
In the notation (4.3)

$$P(\mu(e)) \oplus Q'(\mu(e)) = P \oplus Q \oplus Q' \oplus P', \qquad P'(\mu(e)) \oplus Q(\mu(e)) = P' \oplus Q' \oplus Q \oplus P,$$

and the isomorphism η is the obvious permutation of summands. According to (4.4), we form two short exact sequences

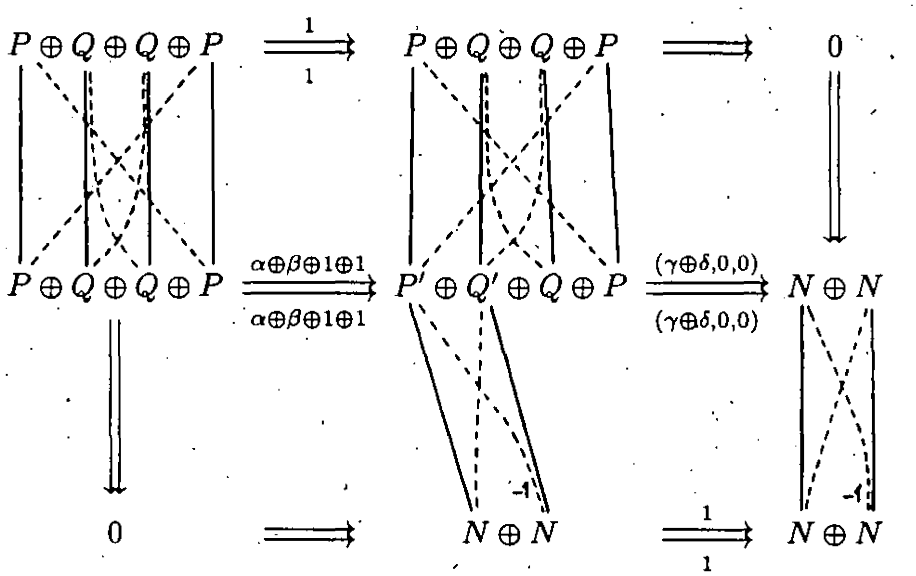


and proceed to the dses.



We keep using the way of displaying dses.'s as in Lemma 3.3. Each line here means an obvious map, i.e., α, β, 1_P, etc.; -1 near a line means that we change the sign of the corresponding map; the collection of dotted lines yields the second arrows in the dses., and the regular lines yield the first arrows.

Consider the diagram

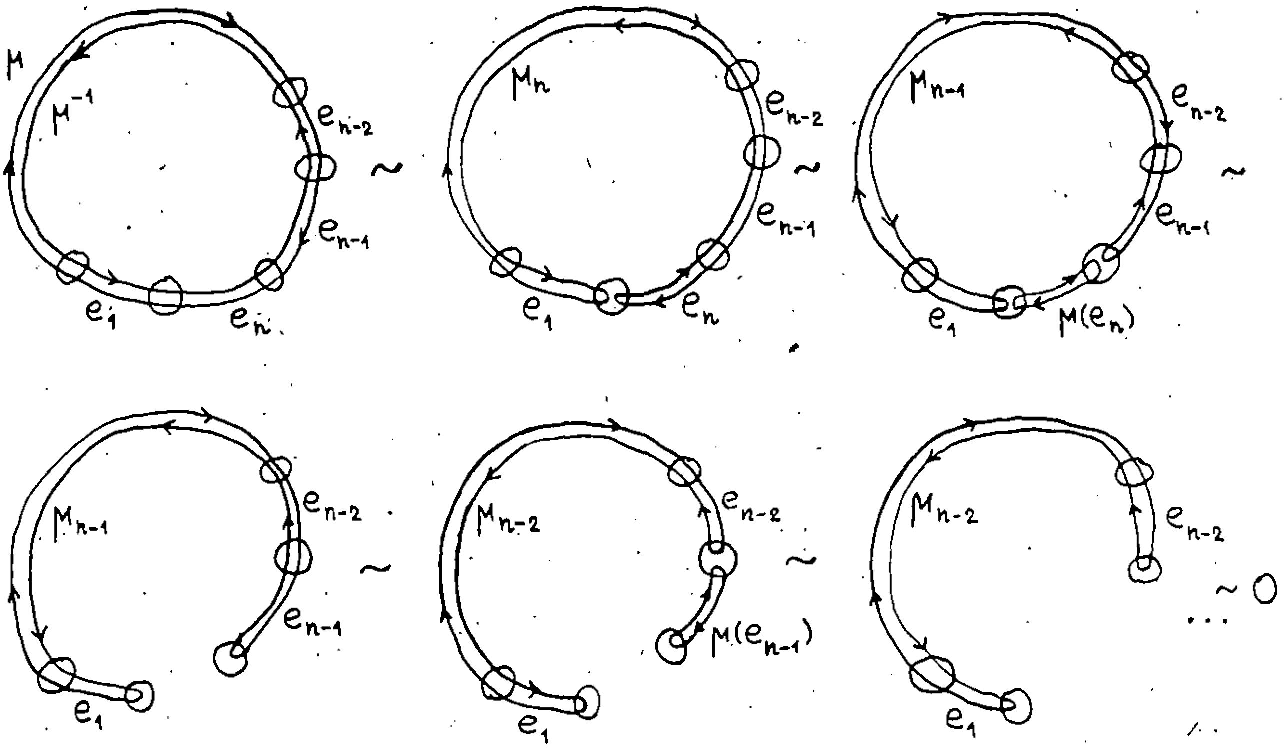


The horizontal dses.'s here are diagonal. The middle vertical dses. is $l(\mu(e))$. The right vertical dses. is $l(\alpha_N)$, it vanishs in $\mathcal{D}(\mathfrak{A})$ by Lemma 3.2(i). The left vertical dses. is isomorphic to $\bar{l}(\alpha_{P,Q})^{\mathrm{op}}$ (transpose the last copies of P and Q). By Lemmas 3.5 and 3.3(i), its class vanishs in $\mathcal{D}(\mathfrak{A})$, hence $\langle l(\mu(e))\rangle = 0$. □

For any loop μ let μ^{-1} denote the inverse loop (obtained from μ by changing orientation).

Corollary 4.4. *If μ is in $G.\mathfrak{A}^{\circ}$, then $b(\mu^{-1}) = -b(\mu)$.*

Proof. Let $\mu = (e_1, \ldots, e_n; \varepsilon_1, \ldots, \varepsilon_n)$ and define the loops $\mu_n, \mu_{n-1}, \mu_{n-2}, \ldots$ by the pictures



The sign $\sim$ above means that we apply Proposition 4.1 and Lemma 4.3 and get

$$\begin{aligned} b(\mu)+b(\mu^{-1}) &= b(\mu_n) = b(\mu_{n-1}) + b(\mu(e_n)) \\ &= b(\mu_{n-1}) = b(\mu_{n-2}) + b(\mu(e_{n-1})) \\ &= b(\mu_{n-2}) = b(\mu_{n-3}) + b(\mu(e_{n-2})) \\ &\ \vdots \\ &= 0 \quad \square \end{aligned}$$

Next we want to show that $b(\mu)$ vanishs provided μ is the contour of a triangle. We will prove a more general assertion concerning admissible triples of edges.

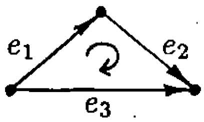
Definition. An *admissible triple of edges* in $G.\mathfrak{A}$ is a triple $\tau = (e_1, e_2, e_3)$ of the form

$$\begin{aligned} e_1 &= (P_0 \xrightarrow{\alpha_{0,1}} P_1 \xrightarrow{\alpha_{1,1/0}} P_{1/0}, P_0' \xrightarrow{\alpha_{0,1}'} P_1' \xrightarrow{\alpha_{1,1/0}'} P_{1/0}) \\ e_2 &= (P_1 \xrightarrow{\alpha_{1,2}} P_2 \xrightarrow{\alpha_{2,2/1}} P_{2/1}, P_1' \xrightarrow{\alpha_{1,2}'} P_2' \xrightarrow{\alpha_{2,2/1}'} P_{2/1}) \\ e_3 &= (P_0 \xrightarrow{\alpha_{0,2}} P_2 \xrightarrow{\alpha_{2,2/0}} P_{2/0}, P_0' \xrightarrow{\alpha_{0,2}'} P_2' \xrightarrow{\alpha_{2,2/0}'} P_{2/0}) \end{aligned}$$

subject to the condition

$$\alpha_{1,2} \circ \alpha_{0,1} = \alpha_{0,2} \text{ and } \alpha_{1,2}' \circ \alpha_{0,1}' = \alpha_{0,2}'.$$

It looks like



and we let $\mu(\tau)$ denote the loop formed by e_1, e_2, and e_3 with the orientation shown in the figure. Thus in the notation discussed above, $\varepsilon_1 = \varepsilon_2 = +$, $\varepsilon_3 = -$.

We get two short exact sequences

$$P_{1/0} \xrightarrow{\alpha_{1/0,2/0}} P_{2/0} \xrightarrow{\alpha_{2/0,2/1}} P_{2/1} \quad \text{and} \quad P_{1/0} \xrightarrow{\alpha_{1/0,2/0}'} P_{2/0} \xrightarrow{\alpha_{2/0,2/1}'} P_{2/1}$$

from the diagrams

$$\begin{array}{ccccc} P_0 & \xrightarrow{\alpha_{0,1}} & P_1 & \xrightarrow{\alpha_{1,1/0}} & P_{1/0} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha_{1,2}} & & \downarrow{\scriptstyle \alpha_{1/0,2/0}} \\ P_0 & \xrightarrow{\alpha_{0,2}} & P_2 & \xrightarrow{\alpha_{2,2/0}} & P_{2/0} \\ \downarrow & & \downarrow{\scriptstyle \alpha_{2,2/1}} & & \downarrow{\scriptstyle \alpha_{2/0,2/1}} \\ 0 & \longrightarrow & P_{2/1} & \xrightarrow{1} & P_{2/1} \end{array} \qquad \begin{array}{ccccc} P_0' & \xrightarrow{\alpha_{0,1}'} & P_1' & \xrightarrow{\alpha_{1,1/0}'} & P_{1/0} \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \alpha_{1,2}'} & & \downarrow{\scriptstyle \alpha_{1/0,2/0}'} \\ P_0' & \xrightarrow{\alpha_{0,2}'} & P_2' & \xrightarrow{\alpha_{2,2/0}'} & P_{2/0} \\ \downarrow & & \downarrow{\scriptstyle \alpha_{2,2/1}'} & & \downarrow{\scriptstyle \alpha_{2/0,2/1}'} \\ 0 & \longrightarrow & P_{2/1} & \xrightarrow{1} & P_{2/1} \end{array}$$

and we call

$$l(\tau) = (P_{1/0} \underset{\alpha'_{1/0,2/0}}{\overset{\alpha_{1/0,2/0}}{\rightrightarrows}} P_{2/0} \underset{\alpha'_{2/0,2/1}}{\overset{\alpha_{2/0,2/1}}{\rightrightarrows}} P_{2/1})$$

the dses. associated to the admissible triple τ.

Note that τ provides the contour of a (uniquely determined) 2-simplex (see (1.2)) if and only if the associated dses. $l(\tau)$ is diagonal, i.e., if $\alpha_{1/0,2/0} = \alpha'_{1/0,2/0}$ and $\alpha_{2/0,2/1} = \alpha'_{2/0,2/1}$. Admissible triples played an essential role in the proof of relation (ii) for the elements $m(l)$ in [Ne2].

Proposition 4.5. *Let $\tau = (e_1, e_2, e_3)$ be an admissible triple in $G.\mathfrak{A}^\circ$. Then $b(\mu(\tau)) = \langle l(\tau) \rangle$.*

Corollary 4.6. *If τ is the contour of a 2-simplex in $G.\mathfrak{A}^\circ$, then $b(\mu(\tau)) = 0$.*

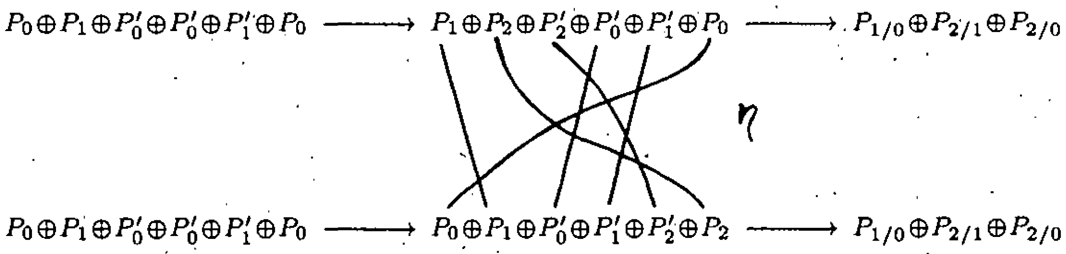
Proof of the proposition. According to (4.2), we have $e(\mu(\tau)) = e_1 \oplus e_2 \oplus e_3^- =$

$$= (P_0 \oplus P_1 \oplus P'_0 \xrightarrow{\alpha_{0,1} \oplus \alpha_{1,2} \oplus \alpha'_{0,2}} P_1 \oplus P_2 \oplus P'_2 \xrightarrow{\alpha_{1,1/0} \oplus \alpha_{2,2/1} \oplus \alpha'_{2,2/0}} P_{1/0} \oplus P_{2/1} \oplus P_{2/0},$$
$$P'_0 \oplus P'_1 \oplus P_0 \xrightarrow{\alpha'_{0,1} \oplus \alpha'_{1,2} \oplus \alpha_{0,2}} P'_1 \oplus P'_2 \oplus P_2 \xrightarrow{\alpha'_{1,1/0} \oplus \alpha'_{2,2/1} \oplus (-\alpha_{2,2/0})} P_{1/0} \oplus P_{2/1} \oplus P_{2/0}).$$

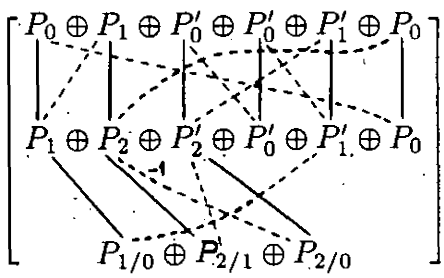
Thus in notation (4.3),

$$P(\mu(\tau)) \oplus Q'(\mu(\tau)) = P_0 \oplus P_1 \oplus P'_0 \oplus P'_1 \oplus P'_2 \oplus P_2,$$
$$P'(\mu(\tau)) \oplus Q(\mu(\tau)) = P_1 \oplus P_2 \oplus P'_2 \oplus P'_0 \oplus P'_1 \oplus P_0,$$

and the isomorphism η is the obvious permutation of summands. We form two short exact sequences as in (4.4),

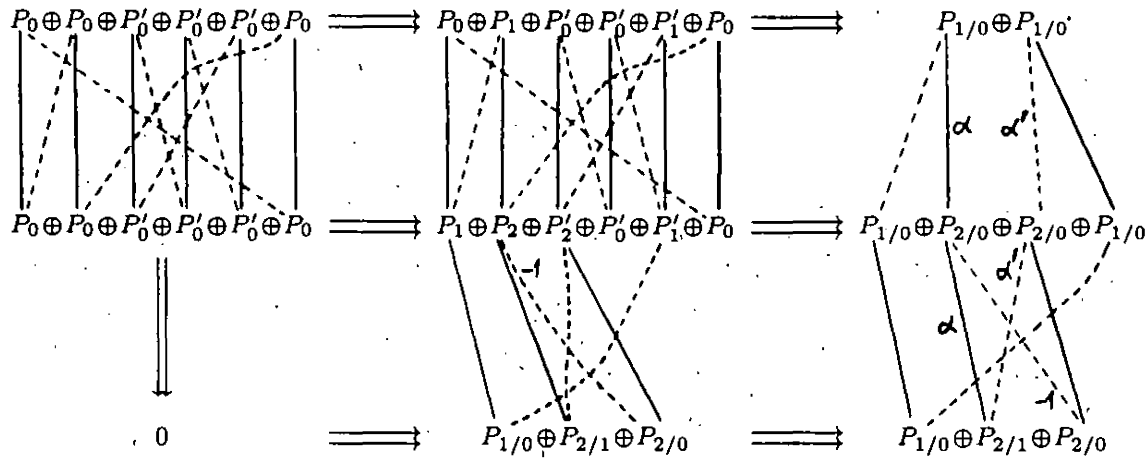


and get the resulting dses.

$l(\mu(\tau)) =$ 

There are many ways to show that $\langle l(\mu(\tau)) \rangle = \langle l(\tau) \rangle$ in $\mathcal{D}(\mathfrak{A})$ by constructing diagrams of the form (2.3). Here is one of them.

Consider the diagram



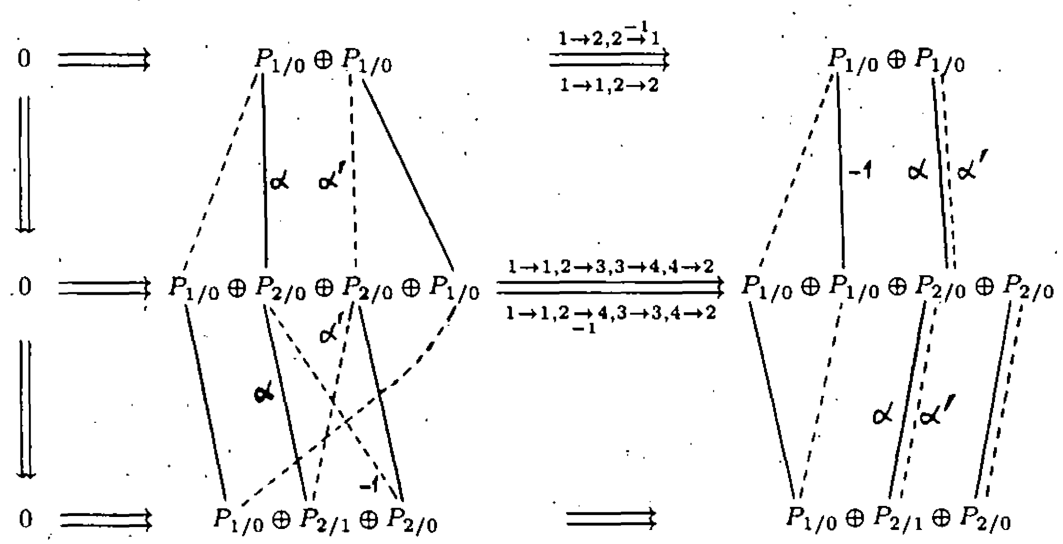
Here the middle vertical dses. is $l(\mu(\tau))$ and the horizontal dses.'s are diagonal, with the obvious arrows that do not change the order of summands. Let l' and l'' denote the left and the right vertical dses. respectively.

Lemma 4.7. $\langle l' \rangle = 0$.

Proof. This dses. amounts to the direct sum of the matrices $\begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$ and $\begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$ (the P_0- and P_0'-part, respectively), under the map $K_1^{\det}(\mathfrak{A}) \to \mathcal{D}(\mathfrak{A})$. Both automorphisms vanish in $K_1^{\det}(\mathfrak{A})$ by the argument as in the proof of Lemmas 3.2 and 3.3. $\square$

Lemma 4.8. $\langle l'' \rangle = \langle l(\tau) \rangle$.

Proof. Consider the diagram



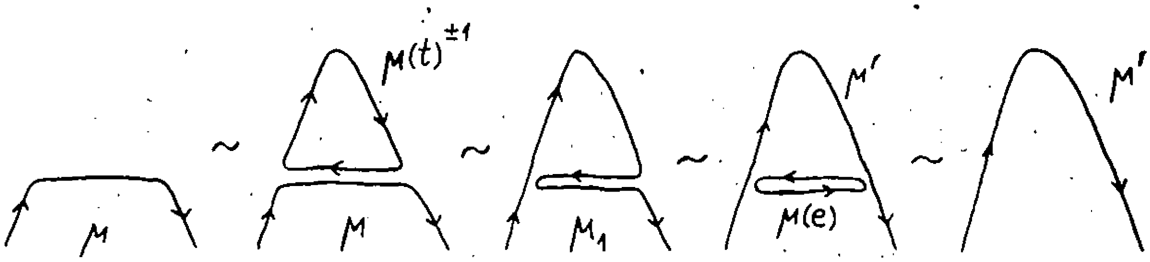
The notation at the horizontal arrows shows the permutation of the direct summands, -1 means the change of the sign of the corresponding identity map. The upper horizontal dses. amounts to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and vanishs in $\mathcal{D}(\mathfrak{A})$ by Lemma 3.2.

The horizontal dses. in the middle is isomorphic to the direct sum of a diagonal dses. (the $P_{1/0}$-part) and a dses. which amount to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ (the $P_{2/0}$-part). Thus $\langle l'' \rangle$ equals to the class of the right vertical dses.. The latter is the direct sum of a dses. of the form (3.2) (the $P_{1/0}$-part), a diagonal one (the $P_{2/0}$-part), and $l(\tau)$. The lemma and the proposition are proved. □ □

Let t be a 2-simplex in $G.\mathfrak{A}^\circ$, and let e be one of its edges. Suppose that a loop μ contains e. Let μ' be the loop obtained from μ by replacing e by the other two edges of t in the obvious way.

Lemma 4.9. $b(\mu) = b(\mu')$.

Proof.



By the above lemmas, $b(\mu) = b(\mu) + b(\mu(t)^{\pm 1}) = b(\mu_1) = b(\mu') + b(\mu(e)) = b(\mu')$. □

According to the well-known combinatorial description for the fundamental group of a simplicial set, it follows from the above lemmas that we have a well-defined homomorphism

$$b\colon K_1(\mathfrak{A}) = \pi_1(G.\mathfrak{A}^\circ) \to \mathcal{D}(\mathfrak{A}).$$

Proposition 4.10. $b \circ m = \mathrm{id}_{\mathcal{D}(\mathfrak{A})}$.

Proof. Let l be a dses., then $b(m(\langle l \rangle)) = b(\mu(l))$, where $\mu(l)$ is the loop defined by $(2.2\frac{1}{2})$. We can regard $\mu(l)$ as an admissible triple of edges, the associated dses. being l. By Proposition 4.5, $b(\mu(l)) = \langle l \rangle$ and we are done. □

This completes the proof of the theorem.

Remark. The diagrams in the proof of Proposition 4.5 are less complicated in the case of $\mu(l)$ than for an arbitrary admissible triple. We leave it to reader to show directly that $b(\mu(l)) = \langle l \rangle$.

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