ON THE DIFFERENCE EQUATION OF THE POINCARÉ TYPE (Part 2)

L.A. Gutnik

Dedicated to the memory of Professor A.O. Gelfond.

Table of contents

§0. Foreword.

Here I begin the presentation of the proof of Theorem 4, which was formulated in the Part 1 ([33]) and proved in [15]; this theorem plays important role in my work ([18] - [36]). With this aim I prove here the following auxiliary Theorem 6 , which is proved in [14] as Theorem 1.

Theorem 6. *Let us consider the following difference equation:*

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k)=0, \tag{1}$$

with $n \in \mathbb{N}$, $a_k(\nu) \in \mathbb{C}$ *for* $k = 0, \dots, n$ *and* $\nu \in \mathbb{N} - 1$. *Let*

$$a_k^{\sim} \in \mathbb{C},\ a_k(\nu) \in \mathbb{C},\ a_n(\nu)=1,\ a_k(\nu)-a_k^{\sim}=O(1/(\nu+1)), \tag{2}$$

where $k = 0, \dots, n$ *and* $\nu \in \mathbb{N} - 1$. *Let further*

$$q \in [1,n]\cap\mathbb{Z},\ p=n-q,\ a_q^{\sim}\neq 0, \tag{3}$$

$$T_1(z)=\sum_{k=0}^{p} a_{q+k}^{\sim} z^k \tag{4}$$

and suppose that the characteristic polynomial

$$T(z)=\sum_{k=0}^{n} a^{\sim}_{k} z^{k} \tag{5}$$

of the equation (1) satisfies the following equality:

$$T(z)=z^{q}T_1(z). \tag{6}$$

For $m\in\mathbb{N}-1$, *let* V_m *denote the* $\mathbb{C}$*-linear space of solutions* $y=y(\nu)$ *of the equation*

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k)=0, \tag{7}$$

where $\nu\in m-1+\mathbb{N}$, *related to equation (1). Then there exist* $C>0$ *and* $m\in\mathbb{N}$ *such that* V_m *splits into direct sum* $V^{\wedge}_{m}\oplus V^{\vee}_{m}$ *of two its subspaces* $V^{\wedge}_{m}$ *and* $V^{\vee}_{m}$, *which have the following propeties:*

a)

$$V^{\wedge}_{m}=\{y\in V_m\colon y(\nu)=O(1)(C/\nu)^{\nu/q})\}; \tag{8}$$

b) if $q=n$, *then*

$$V^{\vee}_{m}=\{0\}; \tag{9}$$

c) if $q<n$, *then* $V^{\vee}_{m}$ *coincides with the space of solutions of a difference equation of Poincaré type*

$$\sum_{k=0}^{p} b_k(\nu)y(\nu+k)=0, \tag{10}$$

where $p=n-q$, $b_k(\nu)\in\mathbb{C}$ *for* $k=0,\ \dots\ ,p$ *and* $\nu\in m-1+\mathbb{N}$,

$$b_0(\nu)\neq 0,\ b_p(\nu)=1, \tag{11}$$

for $\nu\in m-1+\mathbb{N}$,

$$b_k(\nu)-a^{\sim}_{q+k}=O(1/\nu), \tag{12}$$

where $k=0,\ \dots\ ,p$ *and* $\nu\in m-1+\mathbb{N}$.

Acknowledgment.I express my deepest thanks
to Professor B.Z.Moroz,
to Professor I.I. Piatetski-Shapiro,
to Professor A.G.Aleksandrov,
to Professor P.Bundshuh
and to Professor S.G.Gindikin for the support of my work.

§1. Begin of the proof of Theorem 6.

Lemma 1. *Let $C_0 \geq 1$, $m \in \mathbb{N} - 1$, $m_0 = [nC_0] + m$ and for the coefficients of the equation (7) the following inequality holds:*

$$|a_k(\nu)| \leq \frac{C_0}{\nu + 1}, \tag{13}$$

where $k = 0, \ldots, n-1$ and $\nu \in m - 1 + \mathbb{N}$. Let further $C > enC_0$. Then for any solution $y(\nu)$ of the equation (7) the following inequality holds:

$$y(\nu) = O(1)(C/\nu)^{\nu/n}, \tag{14}$$

where $\nu \in m - 1 + \mathbb{N}$.

Proof. If for some $\nu_0 \in m_0 - 1 + \mathbb{N}$ and all the $k = 0, \ldots, n-1$ the following inequality holds,

$$|y(\nu_0 + k)| \leq \gamma,$$

then the inequality $\nu_0 > nC_0$ implies that the inequality $|y(\nu)| \leq \gamma$ will be fulfilled for all the $\nu \in \nu_0 - 1 + \mathbb{N}$. Therefore, if

$$|y(m_0 + k)| \leq \gamma_0,$$

where $k = 0, \ldots, n-1$ then, in view of (13),

$$|y(\nu)| \leq \gamma_0$$

for $\nu \in m_0 - 1 + \mathbb{N}$,

$$|y(\nu)| \leq \gamma_0 C_0 n / m_0$$

for $\nu \in m_0 - 1 + n + \mathbb{N}$,

$$|y(\nu)| \leq \gamma_0 (C_0 n)^2 / (m_0(m_0 + n))$$

for $\nu \in m_0 - 1 + 2n + \mathbb{N}$,

$$|y(\nu)| \leq \gamma_0 (C_0)^\kappa / (m_0/n)_\kappa =$$

$$\gamma_0 (C_0)^\kappa \frac{\Gamma(m_0/n)}{\Gamma(m_0/n + \kappa)}$$

for $\nu \in m_0 - 1 + n\kappa + \mathbb{N}$. But $\kappa = \nu/n + O(1)$; therefore the equality (14) follows from the Stirling's formula. ■

Together with Lemma 1 we have proved the Theorem 6 for the case $q = n$. The following result was proved in [33]

Theorem 5. *Let the functin $\xi(x)$ is defined on $[0,+\infty)$, let $\xi(x)$ decreases together with increasing of the variable x in $[0,+\infty)$, let $\lim\limits_{x\to\infty}(\xi(x))=0$ and let $\xi(x)>0$ for $x\in[0,+\infty)$. Let*

$$\xi(x/2)=O(\xi(x)), \tag{15}$$

when $x\to\infty$,

$$\lim_{x\to 0}(\log(\xi(x)))/x=0. \tag{16}$$

Let $a_k(\nu)-a_k^{\sim}=O(\xi(\nu)), k=0,\,\dots\,,n,$ when $\nu\to\infty$. Let further the characteristical polynomial (5) of the equation (1) may be represented in the form

$$T(z)=T_1(z)T_2(z), \tag{17}$$

where

$$T_1(z)=\sum_{\alpha=0}^{p}b_\alpha^{\sim}z^\alpha,\, T_2(z)=\sum_{\beta=0}^{q}u_\beta^{\sim}z^\beta, b_p^{\sim}=u_q^{\sim}=a_n^{\sim}=1 \tag{18}$$

and absolute value of each root of $T_1(z)$ is greater than the absolute value of each root of $T_2(z)$.

Then there exist $m\in\mathbb{N}$,

$$b_\alpha(\nu)\in\mathbb{C}, \alpha=0,\,\dots\,,p, \nu\in\mathbb{N}+m-1,$$

and

$$u_\beta(\nu)\in\mathbb{C},\, \beta=0,\,\dots\,,q,\, \nu\in\mathbb{N}+m-1$$

such that

$$b_\alpha(\nu)-b_\alpha^{\sim}=O(\xi(\nu)),\, \alpha=0,\,\dots\,,p,\, b_p(\nu)=1,\, b_0(\nu)\neq 0, \tag{19}$$

$$u_\beta(\nu)-u_\beta^{\sim}=O(\xi(\nu)),\, \beta=0,\,\dots\,,q,\, u_q(\nu)=1, \tag{20}$$

where $\nu\in\mathbb{N}+m-1$, and, moreover, the connected with the equation (1) the equation (7) is equivqlent to the equation

$$\sum_{\alpha=0}^{p}b_\alpha(\nu)y(\nu+\alpha)=r(\nu), \tag{21}$$

where $\nu\in\mathbb{N}-1+m$ and $r(\nu)$ satisfies to the equation

$$\sum_{\beta=0}^{q}u_\beta(\nu)r(\nu+\beta)=0 \tag{22}$$

with $\nu \in \mathbb{N} - 1 + m$.

Lemma 2. *Let the conditions of the Theorem 6 are fulfilled and* $q < n$. *Then there exist* $m \in \mathbb{N}$,

$$b_\alpha(\nu) \in \mathbb{C}, \alpha = 0, \dots, p, \nu \in \mathbb{N} + m - 1,$$

and

$$u_\beta(\nu) \in \mathbb{C}, \ \beta = 0, \dots, q, \ \nu \in \mathbb{N} + m - 1$$

such that

$$b_\alpha(\nu) - a^{\sim}_{q+\alpha} = O(1/\nu), \ \alpha = 0, \dots, p, \ b_p(\nu) = 1, b_0(\nu) \neq 0, \tag{23}$$

$$u_\beta(\nu) = O(1/\nu), \beta = 0, \dots, q, \ u_q(\nu) = 1, \tag{24}$$

where $\nu \in \mathbb{N} + m - 1$, *and, moreover, the connected with the equation (1) the equation (7) is equivalent to the equation (21) where* $\nu \in \mathbb{N} - 1 + m$ *and* $r(\nu)$ *satisfies to the equation (22) with* $\nu \in \mathbb{N} - 1 + m$.

Proof The Lemma is direct corollary of the Theorem 5. with $T_2(z) = z^q$ and $\xi(x) = 1/(x+1)$. ■

§2. The general plan of the construction of the spaces $V_m^\vee$ and $V_m^\wedge$.

First we take on the role of m in the Theorem 6 the m of the Lemma 2. Let R_m be the linear over $\mathbb{C}$ space of all the solutions of the equations (22). According the Lemma 1, there exists $C > 0$, such that

$$r(\nu) = O(1)(C/\nu)^{\nu/n} \tag{25}$$

for $r(\nu) \in R_m$ and $\nu \in m - 1 + \mathbb{N}$. The connected with the equation (21) map

$$y(\nu) \to r(\nu) = \sum_{\alpha=0}^{p} b_\alpha(\nu) y(\nu + \alpha) \tag{26}$$

is a $\mathbb{C}$-linear map of the space V_m onto R_m and the null-space $V_m^\vee$ of this map is a $\mathbb{C}$-linear subspace of V_m, which coincides with the space of solutions of the equation (10). The Theorem 6 will be proved, if after replacement of m in the Lemma 2 by the bigger $m \in \mathbb{N}$ we will constructed a splitting monomorphism ξ_m of the space R_m into V_m with the property:

$$y(\nu) = O(1)(C/\nu)^{\nu/n}, \tag{27}$$

for $y(\nu) \in V_m^\wedge = \xi_m(R_m)$ and $\nu \in m - 1 + \mathbb{N}$.

§3. On some linear normed spaces of sequenses of elements of a linear normed space.

Let K be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and L be a linear normed space over K with norm $p(x)$. In the case $L = K^n$ we fix as $p(x)$, wehre $x \in K^n$, the maximum of the absolute values of coordinates of x in the standard basis, i.e.

$$p(x) = h(x) = \sup(\{|x_1|, \dots, |x_n|\}), \tag{28}$$

where

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

If L is a Banach space with the norm p, then $K-$algebra of all the linear continuous operators acting in L will be denoted by $\mathfrak{M}^\wedge(L)$, and the norm on $\mathfrak{M}^\wedge(L)$, associated with the norm p will be denoted by $p^\sim$. So,

$$p^\sim(A) = sup(\{p(AX) \colon X \in L, \, p(X) \le 1\}).$$

It is well known that the associciated with h norm on $Mat_n(\mathbb{C})$ is defined as follows

$$h^\sim(A) = sup\left(\left\{\sum_{k=1}^{n} |a_{i,j}| \colon i = 1, \dots, n\right\}\right), \tag{29}$$

where $A = (a_{i,k}) \in Mat_n(\mathbb{C})$. The norms h and $h^\sim$ coincide respectiely with with the norms q_∞ and $q_\infty^\sim$ considered in section 6 of the paper [33].

Let $m \in \mathbb{N}$, and let $E_m(L)$ be the set $L^{m-1+\mathbb{N}}$ of all the maps of the set $m - 1 + \mathbb{N}$ into L. The set $E_m(L)$ is a linear space over K, where the muliplication of the elements by the number from K and addition of the elements is defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to L, and we identify this subspace with L.

We denote by $\mathfrak{M}^\vee(\mathfrak{L})$ the space of all the $K-$linear maps of the space L in L. If $\phi \in \mathfrak{M}^\vee(\mathfrak{L})$ and $\psi \in \mathfrak{M}^\vee(\mathfrak{L})$, then $\phi \circ \psi$ denotes the composition of operators ϕ and ψ, so that $(\phi \circ \psi) f = \phi((\psi f))$ for each $f \in L$. For $x \in E_m(L)$ let

$$p_{m,\infty}(x) = \sup(\{p(x(\nu)) \colon \nu \in m - 1 + \mathbb{N}\}.$$

Let further

$$E_{m,\infty}(L) = \{x \in E_m(L) \colon p_{m,\infty}(x) \ne \infty\},$$

$$E_{m,0}(L) = \{x \in E_m(L) \colon \lim_{\nu \to \infty} p(x(\nu)) = 0\},$$

$$E_m^{\rightarrow}(L) = L + E_{m,0}(L).$$

Clearly, the space $E_m^{\to}(L)$ consists of all the $y \in E_m(L)$, for which there exists

$$\lim(y) = \lim_{\nu\to\infty}(y(\nu)).$$

Let $m \in \mathbb{N}-1$, $\mu \in m-1+\mathbb{N}$ and ler $r_{m,\mu}$ be the operator of restriction of the elements $y \in E_m(L)$ on te set $m-1+\mathbb{N}$. Clearly, the map $r_{m,\mu}$ is an epimorphism of the space $E_m(L)$ onto the space $E_\mu(L)$. If L is a K-algebra, then $E_m(L)$ is a K-algebra, where the muliplication and addition of the elements is defined coordinate-wise; so, in this case $r_{m,\mu}$ is an epimorphism of K-algebra $E_m(L)$ onto K-algebra $E_\mu(L)$.

If L be an algebra with unity, let L^* denotes the group of all its invertible elements. Then

$$(L^*)^{m-1+\mathbb{N}} \subset L^{m-1+\mathbb{N}};$$

we denote below $(L^*)^{m-1+\mathbb{N}}$ by $E_m(L^*)$. Clearly,

$$E_m(L^*) = (E_m(L))^*.$$

Let $L = \mathbb{C}^n$, $y \in E_m(L)$, and let $y_i(\nu)$ denotes the i-th coordinate of the element $y(\nu)$, where $i = 1, \ldots, n$, $\nu \in m-1+\mathbb{N}$; then the space $(E_m(\mathbb{C}))^n$ contains an element $\omega(y)$, which has $y_i(\nu)$ as the value of its i-th coordinate at the point $\nu \in m-1+\mathbb{N}$. So we obtain the natural isomorphism ω of the algebra $E_m(\mathbb{C}^n)$ onto $(E_m(\mathbb{C}))^n$. This map ω induces an isomorphism of the algebra $E_m(Mat_n(\mathbb{C}))$ onto $Mat_n(E_m(\mathbb{C}))$.

Clearly, if L is a $K-algebra$, then each $a \in E_m(L)$ determines an acting on $E_m(L)$ K-linear operator $\mu_a \in \mathfrak{M}^\vee(\mathfrak{E}_\mathfrak{m}(\mathfrak{L}))$, which turns any $y \in E_m(L)$ into $\mu_a y = ay$. On $E_m(L)$ acts also K-linear operator $\bigtriangledown\mathfrak{M}^\vee(\mathfrak{L})$, which turns any $y \in E_m(L)$ in the $\bigtriangledown y \in E_m(L)$ such that

$$(\bigtriangledown y)(\nu) = y(\nu+1)$$

for any $\nu \in m-1+\mathbb{N}$. Let us consider the subring $\mathfrak{A}_m(L)$ of the ring $\mathfrak{M}^\vee(\mathfrak{L})$ generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m(L)$. Clearly,

$$\mu_a \circ \bigtriangledown^r \circ \mu_b \circ \bigtriangledown^s = \mu_{a\bigtriangledown^r b_k} \circ \bigtriangledown^{r+s}, \tag{30}$$

where $\{r, s\} \subset \mathbb{N}-1$, $\{a, b\} \subset E_m(L)$. For each $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$ are uniquelly defined the number $\deg(\alpha)$ and representation of α in the form

$$\alpha = \sum_{k=0}^{\deg(\alpha)} \mu_{a_k} \circ \bigtriangledown^k, \tag{31}$$

where $a_k \in E_m(L)$ for $k = 0, \ldots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$. Clearly, (31) may be rewritn in the form

$$\alpha = \sum_{k=0}^{\infty} \mu_{a_k} \circ \bigtriangledown^k, \tag{32}$$

where $a_k = 0_{E_m(L)}$ for $k \in \deg(\alpha) + \mathbb{N}$. It follows from (30) that $\mathfrak{A}_m(L)$ is a graduated algebra, and if

(33) $$\beta = \sum_{r=0}^{p} \mu_{b_r} \circ \bigtriangledown^r \in \mathfrak{A}_m(L),$$

(34) $$\gamma = \sum_{s=0}^{q} \mu_{c_s} \circ \bigtriangledown^s \in \mathfrak{A}_m(L),$$

then

(35) $$\beta\gamma = \sum_{k=0}^{p+q} \sum_{\substack{\leq r \leq p \\ 0 \leq s \leq q \\ r+s=k}} \mu_{b_r \bigtriangledown^r c_s} \circ \bigtriangledown^{r+s};$$

clearly, $\deg(\beta\gamma) = \deg(\beta) + \deg(\gamma)$, if $b_p(\nu)^r c_q(\nu+p)$ is different from 0 at least for one $\nu \in m-1+\mathbb{N}$. Let $\mathfrak{A}_m^{\rightarrow}(L)$ be the ring generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m^{\rightarrow}(L)$. Since $\bigtriangledown a \in E_m^{\rightarrow}(L)$, if $a \in E_m^{\rightarrow}(L)$, it follows, in view of (30), that $\mathfrak{A}_m^{\rightarrow}(L)$ is a graduated subalgebra $\mathfrak{A}_m^{\rightarrow}(L)$ of the algebra $\mathfrak{A}_m(L)$, each $\alpha \in \mathfrak{A}_m(L) \setminus \{0_{\mathfrak{A}_m(L)}\}$ admits a representation in the form (31) with $a_k \in E_m^{\rightarrow}(L)$ for $k = 0, \ldots, \deg(\alpha)$ and $a_{\deg(\alpha)} \neq 0_{E_m(L)}$; to each such α corresponds the limit operator

(36) $$\lim(\alpha) = \sum_{k=0}^{\deg(\alpha)} \mu_{\lim(a_k)} \circ \bigtriangledown^k,$$

and polynomial

(37) $$P(\alpha, z) = \sum_{k=0}^{\deg(\alpha)} \lim(a_k) z^k \in L[z].$$

If $\alpha = 0_{\mathfrak{A}_m(L)}$, then we put

$$\lim(\alpha) = 0_{\mathfrak{A}_m(L)},\ P(\alpha, z) = 0_{L[z]}.$$

The equality (30) shows that the map

(38) $$\alpha \to P(\alpha, z)$$

is an epimorphism of the algebra $\mathfrak{A}_m^{\rightarrow}(L)$ on on the algebra $L[z]$. We note that, if $\alpha \in \mathfrak{A}_m(\mathbb{C})$, then $Ker(\alpha)$ coincides with the linear space of all the solutions of the equation (7); moreover if $\alpha \in \mathfrak{A}_m^{\rightarrow}(\mathbb{C})$, then the corresponding to α equation (7) is an equation of the Poincar'e type and $P(\alpha, z)$ is its characterictical polynomial.

Let L be an algebra with unity. The set of all the $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (31) with $a_{\deg(\alpha)} \in E_m(L^*)$ will be denoted further by $\mathfrak{A}_m(L)^\circ$. The set of all the $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (31) with $a_{\deg(\alpha)} = 1_{E_m(L)}$ will be denoted further by $\mathfrak{A}_m(L)^\vee$. The set of all the $\alpha \in \mathfrak{A}_m(L)\diagdown\{0_{\mathfrak{A}_m(L)}\}$, which have the representation (31) with $a_0 \in E_m(L^*)$, will be denoted further by $\mathfrak{A}_m(L)^\wedge$. Let further

$$\mathfrak{A}_m(L)^{\circ\wedge} = \mathfrak{A}_m(L)^\circ \bigcap \mathfrak{A}_m(L)^\wedge,\ \mathfrak{A}_m(L)^{\vee\wedge} = \mathfrak{A}_m(L)^\vee \bigcap \mathfrak{A}_m(L)^\wedge,$$

$$\mathfrak{A}_m(L)^{\circ\to} = \mathfrak{A}_m(L)^\circ \bigcap \mathfrak{A}_m(L)^\to,\ \mathfrak{A}_m(L)^{\vee\to} = \mathfrak{A}_m(L)^\vee \bigcap \mathfrak{A}_m(L)^\to,$$

$$\mathfrak{A}_m(L)^{\circ\wedge\to} = \mathfrak{A}_m(L)^{\circ\wedge} \bigcap \mathfrak{A}_m(L)^\to,\ \mathfrak{A}_m(L)^{\vee\wedge\to} = \mathfrak{A}_m(L)^{\vee\wedge} \bigcap \mathfrak{A}_m(L)^\to,$$

Clearly, $\mathfrak{A}_m(L)^\circ$. consists of epimorphisms of the space $E_m(L)$ onto $E_m(L)$. The above map $r_{m,\mu}$ induces epimorphism $r^{\triangleright}_{m,\mu}$ of the algebra $\mathfrak{A}_m(L)$ on the algebra $\mathfrak{A}_\mu(L)$ defined as follows:

if $\alpha \in \mathfrak{A}_m(L)$,

$$\alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \bigtriangledown^k, \tag{39}$$

then

$$r^{\triangleright}_{m,\mu}(\alpha) = \sum_{k=0}^{n} \mu_{r_{m,\mu}(a_k)} \circ \bigtriangledown^k, \tag{40}$$

where the operator $\bigtriangledown$ in (39) acts in $E_m(L)$ and the operator $\bigtriangledown$ in (40) acts in $E_\mu(L)$. Clearly, $r^{\triangleright}_{m,\mu}$ surjectively maps
$\mathfrak{A}_m(L)^\circ$ onto $\mathfrak{A}_\mu(L)^\circ$, $\mathfrak{A}_m(L)^\vee$ onto $\mathfrak{A}_\mu(L)^\vee$,
$\mathfrak{A}_m(L)^\wedge$ onto $\mathfrak{A}_m u(L)^\wedge$, $\mathfrak{A}_m(L)^{\circ\wedge}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge}$,
$\mathfrak{A}_m(L)^{\vee\wedge}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge}$, $\mathfrak{A}_m(L)^{\circ\to}$ onto $\mathfrak{A}_\mu(L)^{\circ\to}$,
$\mathfrak{A}_m(L)^{\vee\to}$ onto $\mathfrak{A}_\mu(L)^{\vee\to}$, $\mathfrak{A}_m(L)^{\circ\wedge\to}$ onto $\mathfrak{A}_\mu(L)^{\circ\wedge\to}$,
$\mathfrak{A}_m(L)^{\vee\wedge\to}$ onto $\mathfrak{A}_\mu(L)^{\vee\wedge\to}$. Since the diagram

$$\begin{array}{ccc} E_m(L) & \xrightarrow{r_{m,\mu}} & E_\mu(L) \\ \alpha\Big\downarrow & & \Big\downarrow r^{\triangleright}_{m,\mu}(\alpha) \\ E_m(L) & \xrightarrow[r_{m,\mu}]{} & E_\mu(L) \end{array}$$

is commuative and therefore

$$r_{m,\mu}\alpha = r^{\triangleright}_{m,\mu}(\alpha) r_{m,\mu}, \tag{41}$$

it follows that $r_{m,\mu}$ surjectively maps $Ker(r_{m,\mu}\alpha)$ onto

$$Ker(r^{\triangleright}_{m,\mu}(\alpha)) \supset r_{m,\mu}Ker(\alpha).$$

Lemma 3. *If $\mu \in m-1+\mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$, then the operator α bijectively maps $Ker(r_{m,\mu})$ onto $Ker(r_{m,\mu})$.*

Proof. For $m=\mu$ we have equality $Ker(r_{m,\mu}) = 0_{E_m(L)}$ and assertion of the Lemma is obvious. Let $\mu > m$. Clearly,

$$Ker(r_{m,\mu}) = \{x \in E_m(L) \colon x(\nu) = 0_L,\ \nu \in \mu - \mathbb{N} - 1\}.$$

Let $\alpha \in \mathfrak{A}_m(L)^\wedge$ has the form (39) with $a_0(\nu) \in L^*$. If $x \in Ker(r_{m,\mu})$, and

$$y = \alpha(x), \tag{42}$$

then

$$y(\nu) = \sum_{k=0}^{n} a_k(\nu)x(\nu+k) \tag{43}$$

and therefore $y(\nu) = 0_L$ for $\nu \in \mu-1+\mathbb{N}$, if $x(\nu) = 0_L$ for $\nu \in \mu-1+\mathbb{N}$. On the other hand for given $y \in Ker(r_{m,\mu})$ the coordinates $x(\nu)$ in (42) must be equal to 0_L, if $\nu \in \mu-1+\mathbb{N}$, and the equalities

(44)

$$x(\mu-j) = (a_0)(\mu-j))^{-1}\left(y(\mu-j) - \sum_{k=1}^{n} a_k(\mu-j)x(\mu-j+k)\right).$$

determined successesvely and in the unique way the coordinatess $x(\mu-j)$ for $j \in [1, \mu-m]\bigcap\mathbb{Z}$. ■

Corollary 1. *Let $\mu \in m-1+\mathbb{N}$ and let $\alpha \in \mathfrak{A}_m(L)^\wedge$. If*

$$g \in E_m(L),\ x \in E_\mu(L),\ m \le \mu,\ \alpha \in \mathfrak{A}_m(L)^\wedge,$$

$$r_{m,\mu}(g) = (r^{\triangleright}_{m,\mu}(\alpha))(x),$$

then there exists a unique $y \in E_m(L)$ such that

$$\alpha(y) = g,\ r_{m,\mu}(y) = x;$$

Proof. Since $r_{m,\mu}$ is an epimorphism of $E_m(L)$ onto $E_\mu(L)$, it follows that there exists $z \in E_m(L)$ such that $r_{m,\mu}(z) = x$ In view of (41),

$$r_{m,\mu}(\alpha(z)) = (r^{\triangleright}_{m,\mu}(\alpha))(r_{m,\mu}(z)) = (r^{\triangleright}_{m,\mu}(\alpha))(x) = r_{m,\mu}(g).$$

Then $g - \alpha(z) \in Ker(r_{m,\mu})$. According to the Lemma 3, $Ker(r_{m,\mu})$ contains an element u such that $g - \alpha(z) = \alpha(u)$. Let $y = z + u$. Then

$$\alpha(y) = g,\ r_{m,\mu}(y) = r_{m,\mu}(z) = x.$$

If $\alpha(y) = r_{m,\mu}(y) = 0_{E_m(L)}$, then $y \in Ker(r_{m,\mu})$, and the Lemma 3 implies the equality $y = 0_{E_m(L)}$. ■

Corollary 2. *Let $\mu \in m - 1 + \mathbb{N}$ and $\alpha \in \mathfrak{A}_m(L)^\wedge$. Then and $r_{m,\mu}$ bijectively maps $Ker(\alpha)$ onto $Ker(r^{\triangleright}_{m,\mu}(\alpha)) = r_{m,\mu}(Ker(\alpha))$.*

Proof. Let $x \in Ker(r^{\triangleright}_{m,\mu}(\alpha))$. Clearly, the conditions of the Corollary 1 are fulfilled for $g = 0_{E_m(L)}$ and x. Therefore there exist a unique $y \in Ker(\alpha)$ such that $r_{m,\mu}(y) = x$. ■

If for the equation (1) are fulfilled the conditions (2) then

$$a_k = (a_k(1),\, a_k(2),\, \dots,\, a_k(\nu),\, \dots) \in E_1^{\rightarrow}(\mathbb{C}), \tag{45}$$

where $k = 0, \dots, n$. Moreover $a_n = 1_{E_1(\mathbb{C})}$, for α in (39) $Ker(\alpha)$ coincides with the linear over $\mathbb{C}$ space of all the solutions of the equation (1), polynomial (5) is equal to $P(\alpha, z) = P(r^{\triangleright}_{0,m}(\alpha), z)$, where $m \in \mathbb{N}$, and $Ker(r^{\triangleright}_{1,m}(\alpha))$ coincides with linear over $\mathbb{C}$ space V_m of all the solutions of the equation (7).

Let $\mathfrak{v}$ be the element in $E_{0,0}$, for which

$$\mathfrak{v}(\nu) = \frac{1}{\nu + 1},$$

where $\nu \in \mathbb{N} - 1$. Clearly, $r_{0,m}(\mathfrak{v})E_{m,\infty}(\mathbb{C}) \subset E_{m,0}(\mathbb{C})$ for any $m \in \mathbb{N} - 1$. Let

$$E^{\succ}_{m,0}(L) = r_{0,m}(\mathfrak{v})E_{m,\infty}(L),\; E^{\succ}_m(L) = L + E^{\succ}_{m,0}(L).$$

Let us consider the ring $\mathfrak{A}^{\succ}_m(L)$ generated by the operator $\bigtriangledown$ and by all the operators μ_a, where $a \in E_m(L)^{\succ}$ and let

$$\mathfrak{I}^{\succ}_m(L) = \{\alpha \in \mathfrak{A}^{\succ}_m(L) \colon P(\alpha, z) = 0\}.$$

The Lemma 2 may be reformulated now as follows:

Lemma 4. *Let $\alpha \in \mathfrak{A}^{\succ}_0(\mathbb{C}) \bigcap \mathfrak{A}^{\vee}_0(\mathbb{C})$, and $P(\alpha, z)$ coincides with the polynomial $T(z)$ in (5) and (6).*

Then there exist $m \in \mathbb{N}$ and representation of the oprator $r^{\triangleright}_{0,m}(\alpha)$ in the form

$$r^{\triangleright}_{0,m}(\alpha) = \psi\beta \tag{46}$$

such that

$$\psi \in \mathfrak{A}^{\succ}_m(\mathbb{C}) \bigcap \mathfrak{A}^{\vee}_m(\mathbb{C}),\; \psi - \bigtriangledown^q \in \mathfrak{I}^{\succ}_m(\mathbb{C}), \tag{47}$$

$$\beta \in \mathfrak{A}^{\succ}_m(\mathbb{C}) \bigcap \mathfrak{A}^{\vee\wedge}_m(\mathbb{C}),\; \deg(\beta) = p = n - q, \tag{48}$$

and $P(\beta, z)$ coincides with the polynomial $T_1(z)$ in (6).

Let $C > 0$, $n\mathbb{N}$. *Let* $w_{C,n}$ *denotes the element in* $E_{0,0}(\mathbb{C})$, *for which*

(49) $$w_{C,n}(\nu) = \left(\frac{C}{\nu+1}\right)^{\nu/n},$$

where $\nu \in \mathbb{N} - 1$. *The Lemma 1 admits the following reformulation:*

Lemma 5. *Let* $m \in \mathbb{N}$ *and* α *from (31) belongs to* $\mathfrak{A}_m^{\vee}(\mathbb{C})$ *and*

$$\deg(\alpha) = n,\ a_k \in r_m(\mathfrak{v})E_{m,\infty},$$

where $k = 0, \ldots, n-1$. *Let further*

$$C_0 \geq 1,\ q_\infty((r_m(\mathfrak{v}))^{-1})a_{n-k} \leq C_0,$$

where $k = 0, \ldots, n-1$. *Let*

(50) $$C > enC_0.$$

Then $Ker(\alpha) \subset r_{0,m}(w_{C,n})E_{m,\infty}(\mathbb{C}) \subset E_{m,0}(\mathbb{C})$.

Each $A \in E_m(\mathfrak{M}^{\vee}(L))$ *defines a linear operator* $A^{\vee} \in \mathfrak{M}^{\vee}(E_m(L))$, *such that*

$$(A^{\vee}y)(\nu) = (A(\nu))(y(\nu)),$$

where $\nu \in m-1+\mathbb{N}$, $y \in E_m(L)$. *The operator* $A^{\vee}$ *is invertible if and only if, when the operator* $A(\nu)$ *is invertible for any* $\nu \in m-1+\mathbb{N}$; *in this case the map*

$$\nu \mapsto (A(\nu))^{-1},$$

where $\nu \in m-1+\mathbb{N}$, *will be denoted by* A^{-1}. *So,*

$$A^{-1}(\nu) = (A^{-1})(\nu),$$

where $\nu \in m-1+\mathbb{N}$. *Clearly,* $(A^{\vee})^{-1} = (A^{-1})^{\vee}$. *For* $\lambda > 0$ *let* $T_{m,\lambda}(L)$ *denotes the element of* $E_m(\mathfrak{M}^{\vee}(L))$, *for which* $((T_{m,\lambda}(L)(\nu))y)(\nu) = \lambda^{\nu}y(\nu)$, *where* $\nu \in m-1+\mathbb{N}$. *Clearly,*

(51) $$(T_{m,\lambda}(\mathfrak{M}^{\vee}(L))A)^{\vee} = (T_{m,\lambda}(L))^{\vee}A^{\vee} = A^{\vee}(T_{m,\lambda}(L))^{\vee},$$

$$T_{m,1}(L))^{\vee} = 1_{\mathfrak{M}^{\vee}(E_m(L))}, T_{m,\lambda_1\lambda_2}(L))^{\vee} = (T_{m,\lambda_1}(L))^{\vee})(T_{m,\lambda_2}(L))^{\vee}),$$

where $A \in E_m(\mathfrak{M}^{\vee}(L)), \lambda > 0$, $\lambda_1 > 0$ *and* $\lambda_2 > 0$. *Let*

$$E_{m,\lambda}(L) = \{y \in E_m(L)\colon p_{m,\lambda}(y) = p_{m,\infty}((T_{m,\lambda}(L))^{-1}y) < +\infty\}.$$

Then $(E_{m,\lambda}(L), p_{m,\lambda})$ *is a Banach space, and*

$$E_{m,1}(L) = E_{m,\infty}(L),\ p_{m,1} = p_{m,\infty}.$$

Clearly, the map $T_{m,\lambda}(L)$ is an isometry of $E_{m,1}(L) = E_{m,\infty}(L)$ onto $E_{m,\lambda}(L)$, and the map $T_{m,\lambda}(L)\bigtriangledown^m$ is an isometry of $(E_{0,1}(L), p_{0,1}) = (E_{0,\infty}(L), p_{0,\infty})$ onto $(E_{m,\lambda}(L), p_{m,\lambda})$.

Lemma 6. *Let $m \in \mathbb{N} - 1$ and α from (31) belongs to $\mathfrak{A}_m^{\rightarrow}(\mathbb{C})$ Then α is bounded linear operator on the space $E_{m,\lambda}(\mathbb{C})$ and*

(52) $$p^{\sim}_{m,\lambda}(\alpha) \leq \sum_{k=0}^{\deg(\alpha)} p_{m,1}(a_k)\lambda^k$$

Proof. Since $\alpha \in \mathfrak{A}_m^{\rightarrow}(\mathbb{C})$, it follows that $a_k \in E_{m,1}(\mathbb{C})$ for $k = 0, \ldots, n$ In view of (31), if $y \in E_{m,\lambda}(\mathbb{C})$, then

$$(\alpha(y))(\nu) = \sum_{k=0}^{\deg(\alpha)} a_k(\nu)\lambda^{\nu+k}\lambda^{-\nu-k}y(\nu+k) =$$

$$\lambda^\nu \sum_{k=0}^{\deg(\alpha)} a_k(\nu)\lambda^k\lambda^{-\nu-k}y(\nu+k)$$

and

(53) $$\left|\sum_{k=0}^{\deg(\alpha)} a_k(\nu)\lambda^k\lambda^{-\nu-k}y(\nu+k)\right| \leq$$

$$p_{m,\lambda}(y)\sum_{k=0}^{\deg(\alpha)} p_{m,1}(a_k)\lambda^k.$$

The inequality (52) follows from (53). ■

Lemma 7. *If $\lambda > 0$, $\theta > 0$,*

(54) $$A \in E_{m,\theta/\lambda}(\mathfrak{M}(L)),$$

then $A^\vee$ turns $E_{m,\lambda}(L)$ in $E_{m,\theta}(L)$, and

$$(p^{\sim})_{m,\theta/\lambda}(A) = \sup(\{p_{m,\theta}(A^\vee y)\colon y \in E_{m,\lambda}(L),\, p_{m,\lambda}(y) \leq 1\}).$$

Proof. Let $y \in E_{m,\lambda}(L)$ and $z = (T_{m,\lambda}(L))^{-1}y$. Then

$$z \in E_{m,1}(L),\, p_{m,\lambda}(y) = p_{m,1}(z).$$

Let $B = (T_{m,\theta/\lambda}(\mathfrak{M}(L)))^{-1}A$. Then

$$B \in E_{m,1}(\mathfrak{M}(L)),\, (p^{\sim})_{m,\theta/\lambda}(A) = (p^{\sim})_{m,1}(B).$$

Therefore, in view of (51),

$$B^\vee z \in E_{m,1}(L),\, T_{m,\theta}(L)(B^\vee z) \in E_{m,\theta}(L),$$

$$A^\vee y = A^\vee T_{m,\lambda} z = A^\vee T_{m,\theta}(L))T_{m,\lambda/\theta}(L)z =$$
$$T_{m,\theta}(L))A^\vee T_{m,\lambda/\theta}(L)z =$$
$$T_{m,\theta}(L)(T_{m,\lambda/\theta}(\mathfrak{M}(L))A)^\vee z =$$
$$T_{m,\theta}(L))((T_{m,\theta/\lambda}(\mathfrak{M}(L)))^{-1}A)^\vee z \in E_{m,\theta}(L).$$

Further we have

$$(p^\sim)_{m,\theta/\lambda}(A) = \sup(\{p^\sim((\theta/\lambda)^{-\nu}A(\nu))) \colon \nu \in m-1+\mathbb{N}\}) =$$

$$\sup(\{p((\theta/\lambda)^{-\nu}A(\nu)z(\nu))) \colon z(\nu) \in L,\, p(z(\nu)) \le 1,\, \nu \in m-1+\mathbb{N}\}) =$$
$$\sup(\{p((\theta^{-\nu}A(\nu)y(\nu))) \colon y(\nu) \in L,\, p(y(\nu)) \le \lambda^\nu,\, \nu \in m-1+\mathbb{N}\}) =$$
$$\sup(\{p((\theta^{-\nu}((A^\vee y)(\nu))) \colon y(\nu) \in L,\, \lambda^{-\nu}p(y(\nu)) \le 1,\, \nu \in m-1+\mathbb{N}\}) =$$
$$\sup(\{p_{m,\theta}(A^\vee y) \colon y(\nu) \in E_{m,\lambda}(L),\, p_{m,\lambda}(y)) \le 1\}).$$

■

Corollary. *If $A \in E_{m,1}(\mathfrak{M}(L))$, then $A^\vee$ turns $E_{m,\lambda}(L)$ in $E_{m,\lambda}(L)$, and*

(55) $$(p^\sim)_{m,1}(A) =$$

$$\sup(\{p_{m,\lambda}(A^\vee y) \colon y \in E_{m,\lambda}(L),\, p_{m,\lambda}(y)) \le 1\}) = (p_{m,\lambda})^\sim(A^\vee) = (p^\sim_{m,1})(A^\vee).$$

Proof. The assertion of the Corollary follows directly from the assertion of the Lemma for $\theta = \lambda$. ■

Clearly, if $\lambda > 0$, $\theta > 0$, $A \in E_{m,\lambda}(\mathfrak{M}(L)), B \in E_{m,\theta}(\mathfrak{M}(L))$, then AB is contained in $E_{m,\lambda\theta}(\mathfrak{M}(L))$. Clearly, $\bigtriangledown$ maps $E_{m,\lambda}(L)$ in $E_{m,\lambda}(L)$ and

(56) $$(p_{m,\lambda})^\sim(\bigtriangledown) = \lambda.$$

Clearly, for any $k \in \mathbb{N}-1$, $A \in E_m(\mathfrak{M}(L))$

(57) $$(\bigtriangledown \circ A^\vee)^k = (\prod_{\kappa=1}^{k}(\bigtriangledown^\kappa A)) \circ \bigtriangledown^\kappa.$$

Let L is a Banach space over th field K and $A \in E_{m,1}(\mathfrak{M}^\wedge(L))$. Let further there exists $A^{-1} \in E_{m,1}(\mathfrak{M}^\wedge(L))$, and

(58) $$(p^\sim)_{m,1}(A^{-1}) = \rho < 1/\lambda.$$

Then, clearly, $\mathfrak{M}^\wedge(E_{m,\lambda}(L))$ contains the linear operator

(59) $$-(A^{-1})^\vee \sum_{k=0}^{\infty}(\bigtriangledown \circ (A^{-1})^\vee)^k =$$

$$-(A^{-1})^\vee(1_{\mathfrak{M}^\wedge(E_{m,\lambda}(L))} - \bigtriangledown \circ A^{-1})^\vee)^{-1} = (\bigtriangledown - A^\vee)^{-1},$$

and in view of (56) and (59),

(60) $$(p_{m,1})((\bigtriangledown - A^{\vee})^{-1}) \le \rho/(1-\rho\lambda).$$

According to (57), the equality (59) may be rewritten in the form

(61) $$(\bigtriangledown - A^{\vee})^{-1} = -(A^{-1})^{\vee}\sum_{k=0}^{\infty}\left(\prod_{\kappa=1}^{k}(\bigtriangledown^{\kappa}(A^{-1}))^{\vee}\right)\circ\bigtriangledown^{k}.$$

Lemma 8. *([21], Lemma 2, [15], Lemma 2) Let $A \in Mat_n(\mathbb{C})$ an let k is a maximal order of its Jordan blocks. Then there exists a constante $\gamma^*(A) > 0$ with the following properties:*

for any $\varepsilon > 0$ there exists a norm $p_{A,\varepsilon}$ on $\mathbb{C}^n$ such that

(62) $$p_{A,\varepsilon} \le \gamma^*(A)(\max(1, 1/\varepsilon)^{k-1}h,$$

(63) $$h \le \gamma^*(A)(\max(1, \varepsilon)^{k-1}p_{A,\varepsilon},$$

(64) $$(p_{A,\varepsilon})^{\sim} \le (\gamma^*(A))^2(\max(\varepsilon, 1/\varepsilon)^{k-1}h^{\sim},$$

(65) $$h^{\sim} \le (\gamma^*(A))^2(\max(\varepsilon, 1/\varepsilon)^{k-1}(p_{A,\varepsilon})^{\sim},$$

(66) $$\|A\|_{sp} \le (p_{A,\varepsilon})^{\sim} \le \|A\|_{sp} + (sign(k-1))\varepsilon,$$

where $\|A\|_{sp}$ denotes the maximum of the absolute values of eigenvalues of the matrix A. If, moreover,

(67) $$\det(A) \ne 0,\ \left\|A^{-1}\right\|_{sp}^{-1} > (sign(k-1))\varepsilon,$$

then

(68) $$\left\|A^{-1}\right\|_{sp} \le (p_{A,\varepsilon})^{\sim}(A^{-1}) \le \left(\left\|A^{-1}\right\|_{sp}^{-1} - (sign(k-1))\varepsilon\right)^{-1}$$

Proof. Let $C \in Mat_n(\mathbb{C})$, $\det(C) \ne 0$ and

(69) $$J = C^{-1}AC$$

is a Jordan form of A. Let J is composed by s Jordan $k_i \times k_i$-blocks J_i, where $i = 1, \dots, s$ and $\sum\limits_{i=1}^{s} k_i = n$. Let $\varepsilon > 0$, and let $T^{\wedge}_{m,\varepsilon}$ denotes the diagonal $m \times m-$matrix, which $i-th$ diagonal element is equal to ε^{i-1}, where $i = 1, \dots, m$. Let further $T_\varepsilon^\vee$ denotes the $n \times n-$diagonal matrix composed by the blocks $T_{k_i,\varepsilon}^\wedge$, where $i = 1, \dots, s$. Let

$$\gamma^*(A) = \max(h^\sim(C^{-1}), h(C)), \tag{70}$$

$$p_{A,\varepsilon}(X) = h((CT_\varepsilon^\vee)^{-1}X), \tag{71}$$

where $X \in \mathbb{C}^n$. Then

(72)
$$p_{A,\varepsilon}(X) \le h^\sim(C)h^\sim((T_\varepsilon^\vee)^{-1})h(X) \le \gamma^*(A)\max(1, (1/\varepsilon)^{k-1})h(X)$$

for $X \in \mathbb{C}^n$; therefore (62) holds. Clearly,

(73)
$$h(X) = h(CT_\varepsilon^\vee(CT_\varepsilon^\vee)^{-1}X) \le h(C)h(T_\varepsilon^\vee)h(CT_\varepsilon^\vee)^{-1}X) \le$$
$$\gamma^*(A)\max(1, \varepsilon^{k-1})p_{A,\varepsilon}(X)$$

for $X \in \mathbb{C}^n$; therefore (63) holds. In view of 71,

(74)
$$(p_{A,\varepsilon})^\sim(B) = \sup(\{p_{A,\varepsilon}(BX)\colon X \in \mathbb{C}^n,\, p_{A,\varepsilon}(X) \le 1\}) =$$
$$\sup(\{h((CT_\varepsilon^\vee)^{-1}BX)\colon X \in \mathbb{C}^n,\, h((CT_\varepsilon^\vee)^{-1}X) \le 1\}) =$$
$$\sup(\{h((CT_\varepsilon^\vee)^{-1}BCT_\varepsilon^\vee Y)\colon Y \in \mathbb{C}^n,\, h(Y) \le 1\}) =$$
$$h^\sim(CT_\varepsilon^\vee)^{-1}BCT_\varepsilon^\vee),$$

where $B \in Mat_n(\mathbb{C})$. The equalities (74) imply (64) and (66). It follows from the equalities (74) that

$$h^\sim(B) = (p_{A,\varepsilon})^\sim(CT_\varepsilon^\vee B(CT_\varepsilon^\vee)^{-1}), \tag{75}$$

where $B \in Mat_n(\mathbb{C})$. The equality (75) implies (65). Let $\det(A) \ne 0$, and let Λ is the diagonal $n \times n-$matrix, which diagonal elements are equal to the corresponding diagonal elements of the matrix J. If (67) holds, then

$$(T_\varepsilon^\vee)^{-1})JT_\varepsilon^\vee = \Lambda(E - N), \tag{76}$$

where E is the unit $n \times n$-matrix, N is a nilpotent $n \times n$-matrix and

$$h^\sim(N) \le \|A^{-1}\|_{sp}(sign(k-1))\varepsilon, h^\sim(\Lambda^{-1}) = \|A^{-1}\|_{sp},$$

$$(T_\varepsilon^\vee)^{-1})J^{-1}T_\varepsilon^\vee = (E - N)^{-1}\Lambda^{-1}, (p_{A,\varepsilon})^\sim(A^{-1}) =$$
$$h^\sim(CT_\varepsilon^\vee)^{-1}A^{-1}CT_\varepsilon^\vee) = h^\sim((T_\varepsilon^\vee)^{-1}J^{-1}T_\varepsilon^\vee) \le$$
$$\|A^{-1}\|_{sp}\sum_{\kappa=0}^{\infty}(\|A^{-1}\|_{sp}sign(k-1))\varepsilon)^\kappa =$$

$\|A^{-1}\|_{sp}(1-\|A^{-1}\|_{sp}(sign(k-1))\varepsilon)^{-1} = ((\|A^{-1}\|_{sp})^{-1} - (sign(k-1))\varepsilon)^{-1}$.
■

Corollary. *If all the eigenvalues of the matrix A are symple, then*

(77) $$(p_{A,\varepsilon})^{\sim} = \|A\|_{sp}.$$

If, moreover,

(78) $$\det(A) \neq 0,$$

then

(79) $$(p_{A,\varepsilon})^{\sim}(A^{-1}) = \left(\|A^{-1}\|_{sp}\right)^{-1}.$$

Proof. Since in this case $k = 1$, and, consequently, (78) implies (67), it follows that the assertion of the Lemma follows directly from (66) - (68). ■

Lemma 9. *([15], Lemma 2). Let are fulfilled all the conditions of the Lemma 8 and let $B \in Mat_n(\mathbb{C})$, $\varepsilon_1 > 0$,*

(80) $$(p_{A,\varepsilon})^{\sim}(B-A) \leq \varepsilon_1,$$

then

(81) $$(p_{A,\varepsilon})^{\sim}(B) \leq \|A\|_{sp} + (sign(k-1))\varepsilon + \varepsilon_1.$$

If, moreover, the inequalities (67) hold and

(82) $$\left\|A^{-1}\right\|_{sp}^{-1} > (sign(k-1))\varepsilon + \varepsilon_1,$$

then

(83)
$$\det(B) \neq 0, (p_{A,\varepsilon})^{\sim}(B^{-1}) \leq \left(\left\|A^{-1}\right\|_{sp}^{-1} - (sign(k-1))\varepsilon - \varepsilon_1\right)^{-1}$$

Proof. The inequality (81) follows directly from (66) and (80). If, moreover, all the inequalities (67) and (82) hold, then let us to represent B in the form

(84) $$B = A(E - A^{-1}(A-B));$$

in view of (68), (80) and (82),

$$(p_{A,\varepsilon})^{\sim}(A^{-1}(A-B)) \leq \left(\left\|A^{-1}\right\|_{sp}^{-1} - (sign(k-1))\varepsilon\right)^{-1}\varepsilon_1 < 1;$$

therefore the matrices $(E - A^{-1}(A-B))^{-1}$,

(85) $$B^{-1} = (E - A^{-1}(A-B))^{-1}A^{-1}$$

exist and

$$(p_{A,\varepsilon})^{\sim}(B^{-1}) = (p_{A,\varepsilon})^{\sim}((E - A^{-1}(A - B))^{-1}A^{-1}) \leq$$
$$(p_{A,\varepsilon})^{\sim}((E - A^{-1}(A - B))^{-1})(p_{A,\varepsilon})^{\sim}(A^{-1}) \leq$$
$$\left(1 - \frac{\varepsilon_1}{\left(\|A^{-1}\|_{sp}\right)^{-1} - (sign(k-1))\varepsilon}\right)^{-1} \times$$
$$\left(\left(\|A^{-1}\|_{sp}\right)^{-1} - (sign(k-1))\varepsilon\right)^{-1} =$$
$$\left(\left(\|A^{-1}\|_{sp}\right)^{-1} - (sign(k-1))\varepsilon - \varepsilon_1\right)^{-1}.$$

■

Corollary 1. *Let are fulfilled all the conditions of the Lemma 9, and all the eigenvalues of the matrix A are symple. Then*

(86) $$(p_{A,\varepsilon})^{\sim}(B) \leq \|A\|_{sp} + \varepsilon_1.$$

If, moreover,

(87) $$det(A) \neq 0,\ \left\|A^{-1}\right\|_{sp}^{-1} > \varepsilon_1,$$

then

(88) $$\det(B) \neq 0, (p_{A,\varepsilon})^{\sim}(B^{-1}) \leq \left(\left\|A^{-1}\right\|_{sp}^{-1} - \varepsilon_1\right)^{-1}.$$

Proof. Since in this case $k = 1$, and, consequently, (87) implies (82), it follows that the assertions of the Lemma follows directly from (81) - (83). ■

Corollary 2. *([14], Lemma 3). Let are fulfilled all the conditions of the Lemma 8,* $\det(A) \neq 0$,

(89) $$0 < \varepsilon < \left(\left\|A^{-1}\right\|_{sp}\right)^{-1}/2$$

and let $B \in Mat_n(\mathbb{C})$,

(90) $$(p_{A,\varepsilon})^{\sim}(B - A) \leq \varepsilon,$$

then

(91) $$(p_{A,\varepsilon})^{\sim}(B) \leq \|A\|_{sp} + 2\varepsilon,$$

the matrix B^{-1} *exists and*

(92) $$(p_{A,\varepsilon})^{\sim}(B^{-1}) \leq \left(\left\|A^{-1}\right\|_{sp}^{-1} - 2\varepsilon\right)^{-1}.$$

Proof. Let us take $\varepsilon_1 = \varepsilon$. Then (82) follows from (88) and (89). ■

§4. End of the proof of Theorem 6.

Let in accordance with (10)

$$\nabla^p+\sum_{k=0}^{p-1}\mu_{b_k}\circ\nabla^k, \tag{93}$$

where $b_k \in \mathbb{C}+(r_{0,m}\mathfrak{v})E_{m,1}\mathbb{C}$ for $k=0, \dots, p-1$. In view of (23),

$$\lim(b_k)=a_{q+k} \tag{94}$$

where $k=0, \dots, p-1$. Let

$$B_1=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ -b_0 & -b_1 & -b_2 & \dots & -b_{p-1}\end{pmatrix},$$

$B=\omega^{-1}(B_1)$, where ω is the above isomorphism of the algebra $E_m(Mat_p(\mathbb{C}))$ onto $Mat_p(E_m(\mathbb{C}))$, and let

$$B^{\sim}=\begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ -a^{\sim}_q & -a^{\sim}_{q+1} & -a^{\sim}_{q+2} & \dots & -a^{\sim}_{q+p-1}\end{pmatrix}$$

We take now on the role of the matrix A in the Lemma 8 and Lemma 9 the matrix $B^{\sim}$. Since, in view of (3) $a^{\sim}_q\neq 0$, it follows that $(B^{\sim})^{-1}$ exists. We take now on the role ε in the Lemmata 8 and 9 and their corollaries the number

$$\varepsilon_0=\left(\left\|(B^{\sim})^{-1}\right\|_{sp}\right)^{-1}/3, \tag{95}$$

and we take

$$\mathfrak{q}=p_{B,\varepsilon_0}.$$

Since $\lim(B)=B^{\sim}$, it follows that we can (making use the operator $r_{m,\mu}$) replace the number m on some bigger m, such that for C from (49) and (50) the inequality

$$m\geq C\max\left(1,\left(6\left\|(B^{\sim})^{-1}\right\|_{sp}\right)^q\right), \tag{96}$$

holds and

(97) $$\mathfrak{q}(B(\nu) - B^{\sim}) \le \varepsilon_0,$$

where $\nu \in m-1+\mathbb{N}$. It follows from (95) and (97) that for $B^{\sim}$ and $B(\nu)$ with $\nu \in m-1+\mathbb{N}$ are fulfilled all the conditions of the Corrollary 2 of the Lemma 9; therefore there exists $(B(\nu))^{-1}$ for $\nu \in m-1+\mathbb{N}$ and

$$\mathfrak{q}^{\sim}((B(\nu))^{-1}) \le (3\varepsilon_0 - 2\varepsilon_0)^{-1} = 3\left\|(B^{\sim})^{-1}\right\|_{sp}.$$

Consequenty, there exists $B^{-1} \in E_{m,1}(Mat_p(\mathbb{C}))$ and

(98) $$(\mathfrak{q}^{\sim})_{m,1}(B^{-1}) \le 3\left\|(B^{\sim})^{-1}\right\|_{sp}.$$

In view of (96) and (49),

(99) $$(w_{C,q}(\nu))^{1/\nu} = \left(\frac{C}{\nu+1}\right)^{1/q} <$$

$$\min\left(1, \left(\left\|(B^{\sim})^{-1}\right\|_{sp}\right)^{-1}/6\right) = \min\left(1, \varepsilon_0/2\right),$$

where $\nu \in m-1+\mathbb{N}$. In accordance with (58)-(61), (96),(98) and (99), if

(100) $$3\left\|(B^{\sim})^{-1}\right\|_{sp} \le \frac{1}{2\lambda},$$

then

(101) $$\rho = (q^{\sim})_{m,1}(B^{-1}) \le 3\left\|(B^{\sim})^{-1}\right\|_{sp} \le \frac{1}{2\lambda} < \frac{1}{\lambda},$$

the algebra $\mathfrak{M}^{\wedge}(E_{m,\lambda}(\mathbb{C}^p))$ contains the linear operator

(102) $$-((B^{-1}))^{\vee}\sum_{k=0}^{\infty}(\nabla\circ(B^{-1})^{\vee})^k =$$

$$-(B^{-1})^{\vee}(1_{\mathfrak{M}(E_{m,\lambda}(\mathbb{C}))} - \nabla\circ B^{-1})^{\vee})^{-1} = (\nabla - B^{\vee})^{-1},$$

and, in view of the Lemma 7, its corollary, Lemma 6, (98),(60)

(103) $$(\mathfrak{q}_{m,\lambda})^{\sim}((\nabla - B^{\vee})^{-1}) \le \rho/(1-\rho\lambda) \le$$

$$3\left\|(B^{\sim})^{-1}\right\|_{sp}/(1 - 3\left\|(B^{\sim})^{-1}\right\|_{sp}\lambda) \le 6\left\|(B^{\sim})^{-1}\right\|_{sp}.$$

For any $y \in E_m(\mathbb{C})$ and $n \in \mathbb{N}$ let $Y_{n,y}$ and $Y^{\#}_{n,y}$ denote the elements in the space $E_m(\mathbb{C}^n)$, wich are determined respectively by means the following equalities:

(104) $$Y_{n,y}(\nu) = \begin{pmatrix} y(\nu) \\ \vdots \\ y(\nu+n-1) \end{pmatrix},$$

(105) $$Y^{\#}_{n,y}(\nu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y(\nu) \end{pmatrix},$$

where $\nu \in m - 1 + \mathbb{N}$. Clearly,

(106) $$p_\lambda(y) = h_\lambda(y) = p_\lambda(Y^{\#}_{n,y}) = h_\lambda(Y^{\#}_{n,y}) \leq h_\lambda(Y_{n,y}) \leq$$
$$\max(1, |\lambda|^{n-1}) h_\lambda(Y^{\#}_{n,y}),$$

where $y \in E_m(\mathbb{C})$. If

(107) $$|\lambda| \leq 1,$$

then all the inequalities (106) turn into equalities. Let

$$\lambda(\nu) = (w_{C,q}(\nu))^{1/\nu},$$

where $\nu \in m - 1 + \mathbb{N}$. In view of (99, for $\lambda = \lambda(\mu)$ with $\mu \in m - 1 + \mathbb{N}$ are fulfilled all the conditions (100) and (107). Let $z \in (r_{o,m}(w_{C,q}) E_{m,\infty}(\mathbb{C})$ and

$$p_{m,1}(r_{0,m}(w_{C,q})^{-1} z) = h_{m,1}(r_{0,m}(w_{C,q})^{-1} z) = \gamma$$

Then

(108) $$p_{\mu,\lambda(\mu)}(r_{m,\mu} z) = h_{\mu,\lambda(\mu)}(r_{m,\mu} z) =$$
$$\sup\{(\lambda(\mu))^{-\nu} |(r_{m,\mu} z)(\nu)| \colon \nu \in \mu - 1 + \mathbb{N}\} =$$
$$\sup\left\{\left(\frac{\lambda(\nu)}{\lambda(\mu)}\right)^{\nu} (\lambda(\nu))^{-\nu} |(r_{m,\mu} z)(\nu)| \colon \nu \in \mu - 1 + \mathbb{N}\right\} =$$
$$\sup\left\{\left(\frac{1+\mu)}{1+\nu)}\right)^{\nu} (\lambda(\nu))^{-\nu} |(r_{m,\mu} z)(\nu)| \colon \nu \in \mu - 1 + \mathbb{N}\right\} \leq$$
$$\sup\left\{(\lambda(\nu))^{-\nu} |(r_{m,\mu} z)(\nu)| \colon \nu \in \mu - 1 + \mathbb{N}\right\} \leq$$
$$\sup\left\{(\lambda(\nu))^{-\nu} |(r_{m,\mu} z)(\nu)| \colon \nu \in m - 1 + \mathbb{N}\right\} =$$
$$h_{m,1}((r_{0,m}(w_{C,q})^{-1} z) = p_{m,1}((r_{0,m}(w_{C,q})^{-1} z) = \gamma,$$

where $\mu \in m - 1 + \mathbb{N}$; consequently $r_{m,\mu} z \in E_{\mu,\lambda(\mu)}$, where $\mu \in m - 1 + \mathbb{N}$. In view of (108),

$$p_{\mu,\lambda(\mu)}\left(Y^{\#}_{q,r_{m,\mu} z}\right) = h_{\mu,\lambda(\mu)}\left(Y^{\#}_{q,r_{m,\mu} z}\right) \leq \gamma,$$

where $\mu \in m - 1 + \mathbb{N}$. Therefore, in view of (62),

$$\mathfrak{q}_{\mu,\lambda(\mu)}\left(Y^{\#}_{q,r_{m,\mu} z}\right) \leq$$

$$\gamma^*(B^\sim)(\max(1,1/\varepsilon_0))^{p-1}h_{\mu,\lambda(\mu)}\left(Y^{\#}_{q,r_{m,\mu}z}\right) \leq$$
$$\gamma^*(B^\sim)(\max(1,1/\varepsilon_0))^{p-1}\gamma,$$

where $\mu \in m-1+\mathbb{N}$. Consequently, in view of (103),

(109)
$$\mathfrak{q}_{\mu,\lambda(\mu)}\left((\nabla - B^\vee)^{-1})Y^{\#}_{q,r_{m,\mu}z}\right) \leq$$
$$6\gamma\gamma^*(B^\sim)(\max(1,1/\varepsilon_0))^{p-1}\left\|(B^\sim)^{-1}\right\|_{sp},$$

where $\mu \in m-1+\mathbb{N}$. In view of (109) and (63),

(110)
$$h\left((\lambda(\mu))^{-\nu}\left(((\nabla - B^\vee)^{-1})Y^{\#}_{q,r_{m,\mu}z}\right)(\nu)\right) \leq$$
$$6\gamma(\gamma^*(B^\sim))^2(\max(\varepsilon_0,1/\varepsilon_0))^{p-1}\left\|(B^\sim)^{-1}\right\|_{sp},$$

where $\mu \in m-1+\mathbb{N}$ and $\nu \in \mu-1+\mathbb{N}$. It follows from the inequality (110) for $\nu = \mu \in m-1+\mathbb{N}$ that

(111)
$$h\left((\lambda(\mu))^{-\nu}\left(((\nabla - B^\vee)^{-1})Y^{\#}_{q,r_{m,\mu}z}\right)(\nu)\right) \leq$$
$$6\gamma(\gamma^*(B^\sim))^2(\max(\varepsilon_0,1/\varepsilon_0))^{p-1}\left\|(B^\sim)^{-1}\right\|_{sp}.$$

For any $X \in \mathbb{C}^q$ let $\pi(X)$ denotes the first coordiate of the column X, and let π be the map of $\mathbb{C}^q$ on $\mathbb{C}$, which turns each $X \in \mathbb{C}^q$ into $\pi(X)$. In view of (111),

(112)
$$h\left((\lambda(\nu))^{-\nu}\left(\pi\left(((\nabla - B^\vee)^{-1})Y^{\#}_{q,r_{m,\nu}z}\right)(\nu)\right)\right) \leq$$
$$6\gamma(\gamma^*(B^\sim))^2(\max(\varepsilon_0,1/\varepsilon_0))^{p-1}\left\|(B^\sim)^{-1}\right\|_{sp},$$

where $\nu \in m-1+\mathbb{N}$. Let $\mathfrak{p}$ denotes the norm on $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$, defined by means the equality $\mathfrak{p}(z) = p_{m,\infty}((r_{0,m}(w_{\mathbb{C},q}))^{-1}z)$, and let ϕ be the map of the space $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$ in $E_{m,\infty}(\mathbb{C})$, such that

$$(\phi(z))(\nu) = \pi\left(\left((\nabla - B^\vee)^{-1})Y^{\#}_{q,r_{m,\nu}z}\right)(\nu)\right)$$

for any $z \in (r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$ and any $\nu \in m-1+\mathbb{N}$. It follows now from (112) that ϕ maps $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$ into $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$, is a bounded linear operator on $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$, and

$$\mathfrak{p}^\sim(\phi) \leq 6\gamma(\gamma^*(B^\sim))^2(\max(\varepsilon_0,1/\varepsilon_0))^{p-1}\left\|(B^\sim)^{-1}\right\|_{sp},$$

So, we can take now on the role of the mentioned in the section 2 the splitting homomorphism ξ_m the restriction of the map ϕ on the subspace $Ker(\psi)$ of the space $(r_{0,m}(w_{\mathbb{C},q}))E_{m,\infty}(\mathbb{C})$, where ψ is a homomorpism in (46). ■

References.

[1] O.Perron, Über die Poincaresche Differenzengleichumg,
Journal für die reine und angewandte mathematik,
1910, 137, 6 – 64;
[2] A.O.Gelfond, Differenzenrechnung (in Russian),
1967, Nauka, Moscow.
[3] A.O.Gelfond and I.M.Kubenskaya, On the theorem of Perron
in the theory of differrence equations (in Russian),
IAN USSR, math. ser., 1953, 17, 2, 83 – 86.
[4] M.A.Evgrafov, New proof of the theorem of Perron
(in Russian),IAN USSR, math. ser., 1953, 17, 2, 77 – 82.
[5] G.A.Frejman, On theorems of of Poincaré and Perron
(in Russian), UMN, 1957, 12, 3 (75), 243 – 245.
[6] N.E.Nörlund, Differenzenrechnung,
Berlin: Springer Verlag, 1924.
[7] I.M.Vinogradov, Foundtions of the Number Theory
(in Russian), 1952, GIT-TL.
[8] CH.-J. de la Vallée Poussin, Course d'analyse infinitésimale,
Russian translation by G.M.Fikhtengolts,
GT-TI, 1933.
[9] L.A.Gutnik, On the decomposition of the difference operators
of Poincaré type (in Russian),
VINITI, Moscow, 1992, 2468 - 92, 1 – 55.
[10] __________, On the decomposition of the difference operators
of Poincaré type in Banach algebras (in Russian),
VINITI, Moscow, 1992, 3443 - 92, 1 – 36.
[11] __________, On the difference equations of Poincaré type (in Russian),
VINITI, Moscow 1993, 443 -B93, 1 – 41.
[12] __________, On the difference equations
of Poincaré type in normed algebras (in Russian),
VINITI, Moscow, 1994, 668 -B94, 1 – 44.
[13] __________, On the decomposition of the difference equations
of Poincaré type (in Russian),
VINITI, Moscow, 1997, 2062 -B97, 1 – 41.
[14] __________, The difference equations with of Poincaré type
characteristic polynomial having roots equal to zero
(in Russian), VINITI, Moscow, 1997, 2418 - 97, 1 – 20.
[15] __________, On the behavior of solutions of difference equations
of Poincaré type (in Russian),
VINITI, Moscow, 1997, 3384-B97, 1 - 41.
[16] __________, On the variability of solutions of difference equations
of Poincaré type (in Russian),
VINITI, Moscow, 1999, 361-B99, 1 – 9.
[17] __________, To the question of the variability of solutions
of difference equations of Poincaré type (in Russian),
VINITI, Moscow, 2000, 2416-B00, 1 – 22.
[18] __________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$,
Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2000, 3, 1 – 13.

[19] ___________, On the Irrationality of Some Quantities Containing $\zeta(3)$, Eleven papers translated from the Russian, American Mathematical Society, 1988, 140, 45 - 56.

[20] ___________, Linear independence over $\mathbb{Q}$ of dilogarithms at rational points (in Russian), UMN, 37 (1982), 179-180; english transl. in Russ. Math. surveys 37 (1982), 176-177.

[21] ___________, On a measure of the irrationality of dilogarithms at rational points (in Russian), VINITI, 1984, 4345-84, $1-74$.

[22] ___________, To the question of the smallness of some linear forms (in Russian), VINITI, 1993, 2413-B93, $1-94$.

[23] ___________, About linear forms, whose coefficients are logarithms of algebraic numbers (in Russian), VINITI, 1995, 135-B95, $1-149$.

[24] ___________, About systems of vectors, whose coordinates are linear combinations of logarithms of algebraic numbers with algebraic coefficients (in Russian), VINITI, 1994, 3122-B94, $1-158$.

[25] ___________, About linear forms, whose coefficients are $\mathbb{Q}$-proportional to the number $\log 2$, and the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1996, 3258-B96, $1-70$.

[26] ___________, The lower estimate for some linear forms, coefficients of which are proportional to the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1997, 3072-B97, $1-77$.

[27] ___________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 1), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2001, 15, $1-20$.

[28] ___________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 2), Max-Plank-Institut für Mathematik, Bonn, Preprint Series,2001, 104, $1-36$.

[29] ___________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 57, $1-33$.

[30] ___________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 4), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 142, $1-27$.

[31] ___________, On the dimension of some linear spaces over finite extension of $\mathbb{Q}$ (part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 107, $1-37$.

[32] ___________, On the dimension of some linear spaces over $\mathbb{Q}$ (part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 16, $1-45$.

[33] ___________,On the difference equation of Poincaré type (Part 1). Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 52, $1-43$.

[34] ____________, On the dimension of some linear spaces over $\mathbb{Q}$, (part 4)
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 73, 1 – 37.

[35] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$
(the detailed version, part 5),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 83, 1 – 13.

[36] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$
(the detailed version, part 6),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 99, 1 – 33.

Moscow State Institute
of Electronics and Mathematics
Bolshoi Trekhsvyatitelskij pereulok, 1-3/12, str.8
Moscow 109028
Russian Federation
E-mail: gutnik@gutnik.mccme.ru