A Pieri-type theorem for even Orthogonal Grassmannians

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MPI 96-83

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June 29, 1996

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1 Introduction

In the present paper, we complete our program of finding Pieri-type formulas for homogeneous spaces of the form G/P, where G is a classical semisimple algebraic group and P is a maximal parabolic subgroup. The case $G = SL(n)$ was known classically. In [P-R 0-2], we established Pieri-type formulas for the

Chow (or cohomology) rings of Lagrangian and odd orthogonal Grassmannians, i.e. in the cases $G = Sp(2m)$ and $G = SO(2m+1)$. In the present paper, we give a Pieri-type theorem in the case of even orthogonal Grassmannians, i.e. for $G = SO(2m)$.

For an outline of the whole theory, we refer the reader to Section 6 of [P2].

To formulate our Pieri-type formula, we define "special" Schubert cycles which generate multiplicatively the Chow rings of these Grassmannians and describe how to multiply an arbitrary Schubert cycle by a special one. It requires a different and more involved combinatorics than the one in the odd-orthogonal and symplectic cases.

The idea of the proof of the main theorem of this paper is like that in [P–R2] and many propositions and lemmas from [P–R2] can be directly applied here. Sections 1 – 5 contain basic information about permutations with even number of bars, shapes, reduced decompositions etc. In Section 6 we examine configurations of D- and $\sim D$-boxes which give "the vanishing" (that is, the corresponding operator acting on a certain generating function gives zero). In Section 7 we prove the main theorem by checking its validity in four separate cases. Subsection 7.5 contains examples illustrating the main theorem.

In the present paper, we treat the Grassmannians of non-maximal isotropic subspaces. For the case of maximal isotropic subspaces, we refer the reader to [P1, Section 6]. It is possible, however, to obtain a Pieri-type formula for these Grassmannians using the methods of this work. This will be implemented in the next version of the present paper.

It would be interesting (and valuable) to give a unified (group theoretic ?!) proof of the main theorems of [P-R 0-2] and the present paper (see also [P2, Section 6]), which does not depend on a particular root system chosen.

This research was carried out during the first author's stay at the Max-Planck Institut für Mathematik; he thanks the MPIfM for a generous hospitality. The second author was partially supported by the R. Bosch Foundation during the preparation of the present paper.

2 Preliminaries

We fix positive integers $m > n$. Suppose that $H = SO(2m)$ is the orthogonal group (of type D_m) over the field of complex numbers. Let us use the following notation:

B - a fixed Borel subgroup of H, $T \subset B$ - a fixed maximal torus.

$\mathcal{R}$ - the root system of H associated with T,

$\Sigma = \{\alpha_1, \dots, \alpha_m\}$ the set of simple roots of $\mathcal{R}$ associated with B,

W - the Weyl group of (H, T),

W_n - the subgroup of W generated by simple reflections associated with the simple roots: $\{\alpha_1, \dots, \alpha_m\} \setminus \{\alpha_n\}$, where $n < m$,

P_n - the maximal parabolic subgroup of H containing B and corresponding to the above subset of simple roots,

$F = H/B$ (an isotropic flag manifold),

$G = H/P_n$ (an isotropic Grassmannian).

In a standard Bourbaki [Bou] realization we have:

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \le i < j \le m\} \subset \mathbf{R}^m = \oplus_{i=1}^m \mathbf{R} e_i,$$

$$\Sigma = \{\alpha_1, \dots, \alpha_m\} = \{e_1 - e_2, \dots, e_{m-1} - e_m, e_{m-1} + e_m\},$$

$$W = S_m \ltimes \mathbf{Z}_2^{m-1},$$

$$W_n \simeq S_n \times (S_{m-n} \ltimes \mathbf{Z}_2^{m-n-1}).$$

A typical element of W can be written as a pair (τ, ϵ), where $\tau \in S_m$ and $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ is a sequence of elements of $\mathbf{Z}_2 = \{-1, 1\}$ such that $\#\{i : \epsilon_i = -1\}$ is even. Multiplication in W is given by

$$(\tau, \epsilon) \cdot (\tau', \epsilon') = (\tau \circ \tau', \delta),$$

where "$\circ$" denotes the composition of permutations and $\delta_i = \epsilon_{\tau'(i)} \cdot \epsilon'_i$. The length function on the group W, is defined by the equality:

$$l(w) = \sum_{i=1}^{m} a_i + \sum_{\epsilon_j = -1} 2b_j,$$

where

$$a_i = \#\{j \mid j > i \wedge w(j) < w(i)\}$$

and

$$b_j = \#\{i \mid j < i \wedge w(j) > w(i)\}.$$

The poset $W^{(n)}$ of the minimal length left coset representatives of W_n in W can be decomposed into two disjoint subsets:

$$W^{(n)} = W_1^{(n)} \,\dot\cup\, W_2^{(n)},$$

where:

$$W_1^{(n)} = \{(y_1, y_2, \dots, y_{n-k}; \overline{z}_k, \overline{z}_{k-1}, \dots, \overline{z}_1; v_1, \dots .v_{m-n}) \mid \quad k - even\}$$

and $y_1 < \dots < y_{n-k}$; $z_k > \dots > z_1$ and $v_1 < \dots < v_{m-n}$;

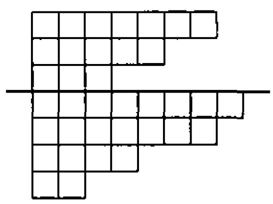
$$W_2^{(n)} = \{(y_1, y_2, \ldots, y_{n-k}; \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1; v_1, \ldots . v_{m-n-1}, \overline{v}_{m-n}) \mid \quad k - odd\}$$

$$\text{and} \quad y_1 < \ldots < y_{n-k};\ z_k > \ldots > z_1 \text{ and } v_1 < \ldots < v_{m-n-1}.$$

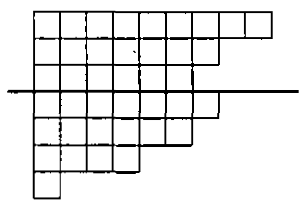
We refer to these sets as *permutations of type 1* and *of type 2* respectively.

Definition 2.1 *A pair* $\lambda = (\lambda^t//\lambda^b)$ *of strict partitions* λ^t *and* λ^b *is called a* shape *if* $\lambda^t \subset (m^{m-n})$, $\lambda^b \subset (m^n)$, $l(\lambda^b)$ *is even and* $\lambda^t_{m-n-1} \geq l(\lambda^b) + 1$, $\lambda^t_{m-n} \geq 1$.
If $\lambda^t_{m-n} \geq l(\lambda^b) + 1$, *then we say that* λ *is of type 1; if this is not true, then we say that* λ *is of type 2.*

Denote the set of shapes by $\mathcal{P}_n$. It would be useful to display shapes with the help of sets of boxes in the fourth quarter of the plane. Let D^t_λ and D^b_λ be the Ferrers' diagrams of λ^t and λ^b (see [M]; also the other terminology related to partitions, diagrams etc. is borrowed from loc. cit.). The diagram D_λ of shape $(\lambda^t//\lambda_b)$ is the juxtaposition of D^t_λ and D^b_λ with rows of successive lengths: $\lambda^t_1, \ldots, \lambda^t_{m-n}, \lambda^b_1, \ldots, \lambda^b_l$, $l = l(\lambda_b)$:



type 2



type 1

For a given element $w_\lambda \in W^{(n)}$, we define the corresponding shape $\lambda = (\lambda^t//\lambda^b)$ in the following way:

- If $w \in W_1^{(n)}$,

  $$w = \{(y_1, y_2, \ldots, y_{n-k}; \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n}) \mid \quad k - even\},$$

  then

  $\lambda^b_j = m + 1 - z_j$ for $j = 1, 2, \ldots, k$;
  $\lambda^t_r = m + 1 - v_r + d_r$ for $r = 1, 2, \ldots, m - n$; $d_r = \#\{j \mid z_j < v_r\}$.

- If $w \in W_2^{(n)}$,

  $$w = \{(y_1, y_2, \ldots, y_{n-k}; \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n-1}, \overline{v}_{m-n}) \mid \quad k{-}odd\},$$

then

$K := \{\lambda_1^b, \ldots, \lambda_{k+1}^b\} = \{m+1-z_1, \ldots, m+1-z_k, m+1-v_{m-n}\}$ and
$\lambda_1^b \geq \ldots \geq \lambda_{k+1}^b$;
$\lambda_r^t = m+1-v_r+\overline{d}_r; \quad r = 1, \ldots, m-n-1$ and $\lambda_{m-n}^t = 1+\overline{d}_{m-n}$,
$\overline{d}_r = \#\{u \in K \mid v_r > u\}$.

Lemma 2.2 *The correspondence described above is a bijection between $W^{(n)}$ and the poset of shapes $\mathcal{P}_n$. Moreover, if w_λ is of type 1 (resp. 2), then λ is of type 1 (resp. 2).*

Proof. Suppose first w is of type 1. The sequences (z_i) and (v_r) are increasing and (d_r) is nondecreasing. Thus λ^b and λ^t are strict; $\lambda^b \subset (m^n)$, $\lambda^t \subset (m^{m-n})$. Observe that

$$\lambda_{m-n}^t \geq l(\lambda^b) + 1 = k+1$$

because

$$m+1-v_{m-n}+d_{m-n} = 1+(m-v_{m-n}+d_{m-n}) \geq k+1.$$

(For the proof of the last inequality, note that $d_{m-n} = \#\{j \mid z_j < v_{m-n}\}$ and $m - v_{m-n} \geq \#\{j \mid z_j > v_{m-n}\}$). It follows that $\lambda_{m-n}^t \geq 1$ and $\lambda_{m-n-1}^t \geq l(\lambda^b)+1$. Moreover, $l(\lambda^b) = k$ is even.

Now, let w be of type 2. The same arguments as above show that $\lambda^b \subset (m^n)$ is strict, $l(\lambda^b)$ is even, $\lambda^t \subset (m^{m-n})$ and $\lambda_1^t > \ldots > \lambda_{m-n-1}^t$. Moreover, $\lambda_{m-n-1}^t \geq l(\lambda^b)+1$ (we use $\overline{d}$ instead of d). It remains to prove $\lambda_{m-n-1}^t > \lambda_{m-n}^t$. This inequality is equivalent to

$$m+1-v_{m-n-1}+\overline{d}_{m-n-1} > 1+\overline{d}_{m-n},$$

$$\text{or} \quad (m-v_{m-n-1}) + (\overline{d}_{m-n-1} - \overline{d}_{m-n}) > 0$$

Consider the following two cases:

- $v_{m-n-1} > v_{m-n}$; then $\overline{d}_{m-n-1} > \overline{d}_{m-n}$ ($v_{m-n} \in$) and the inequality holds;

- $v_{m-n} > v_{m-n-1}$; then $m - v_{m-n-1} > \overline{d}_{m-n} - \overline{d}_{m-n-1}$, because
$\overline{d}_{m-n} - \overline{d}_{m-n-1} = \#\{b \in K \mid v_{m-n-1} < b < v_{m-n}\} <$
$$< \#\{b \in K \mid v_{m-n-1} < b < v_{m-n}\} < m - v_{m-n-1}.$$

It follows that if w is of type 1, then $\lambda^t_{m-n} \geq l(\lambda_b) + 1$ and λ is of type 1; if w is of type 2, then $\lambda^t_{m-n} \leq l(\lambda_b)$ and λ is of type 2.

Suppose that a shape λ is given. Let us try to construct the permutation w_λ. There are two possibilities:

(1) $\lambda^t_{m-n} \geq l(\lambda^b) + 1$. We look for w_λ of type 1. First compute $z_j = m+1-\lambda^b_j$, $j = 1, 2, \ldots, k$. Define the numbers $p_r = \lambda^t_r - (m-n-r+l(\lambda^b)+1)$, $r = 1, 2, \ldots, m-n$. We have $p_r > 0$ because $\lambda^t_{m-n} \geq l(\lambda_b) + 1$ and $\lambda^t_r \geq (m-n-r+l(\lambda^b)+1)$, $r = 1, 2 \ldots, m-n$. The sequence (v_r) can be obtained in the following way : v_r is the p_r-th element (counting from right) in the sequence $(1, 2, \ldots, m)$ with removed $\{z_j \mid j = 1, 2, \ldots, l(\lambda_b)\}$ and $v_{m-n}, v_{m-n-1}, \ldots, v_{r+1}$. Note that such a v_r satisfies $v_r = m+1-\lambda^t_r + d_r$. Indeed,

$$
\begin{aligned}
m - v_r &= \#\{a \mid v_r < a \leq m\} = \\
&= (p_r - 1) + \#\{v_{m-n}, \ldots, v_{r+1}\} + \#\{j \mid z_j > v_r\} = \\
&= (\lambda^t_r - (m-n-r+l(\lambda^b))) + (m-n-r) + (k-d-r) = \\
&\hphantom{=}\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \lambda^t_r - d_r - 1.
\end{aligned}
$$

(2) $\lambda^t_{m-n} \leq l(\lambda^b)$. We look for w_λ of type 2. Since $K = \{\lambda^b_1, \ldots, \lambda^b_{k+1}\} = \{m+1-z_1, \ldots, m+1-z_k, m+1-v_{m-n}\}$ and $\lambda^t_{m-n} = 1 + \bar{d}_{m-n}$, we look for the numbers:

$$\{m+1-\lambda^b-1, \ldots, m+1-\lambda^b_{k+1}\} = \{a_1, \ldots, a_{k+1}\}$$

such that $a_{k+1} < a_k < \ldots < a_1$. It is clear that $v_{m-n} = a_{\lambda^t_{m-n}}$ and the set of remaining elements a_i is equal to $\{z_j\}$. Thus we can determine $z_1, \ldots, z_k$ and v_{m-n}. The elements v_j, $j < m-n$, can be obtained in the same way as in (1). □

Let

$$c : S^\bullet(X(T)) = \mathbf{Z}[x_1, \ldots, x_m] \longrightarrow A^\bullet(Sp(2m, \mathbf{C})/B) = A^\bullet(F)$$

be the Borel characteristic map (see [B–G–G, D2]). The induced map

$$c_G \ : \ R = A^\bullet(F)^{W_n} \longrightarrow A^\bullet(G)$$

after tensoring by $\mathbf{Z}[1/2]$ gives an isomorphism (see [D1, D2]). More explicitly, we have:

$$R = \frac{\mathcal{SP}(x_1, \ldots, x_n) \otimes \mathcal{SP}(x^2_{n+1}, \ldots, x^2_m)[U]}{(e_i(x^2_1, \ldots, x^2_m), \ \ i < m, \quad e_m(x_1, \ldots, x_m))}$$

where $U = x_{n+1} \ldots x_m$, $\mathcal{SP}(\ )$ denotes the ring of symmetric polynomials in the indicated indeterminates and $e_i(\ )$ is the i-th elementary symmetric polynomial in the indicated variables.

Proposition 2.3 *The Poincaré series of $A^*(G)$ is equal to:*

$$P(t) = \frac{(1-t^{2(m-n)})\dots(1-t^{2(m-1)})(1-t^m)}{(1-t)\dots(1-t^n)(1-t^{(m-n)})}.$$

Proof. The elements $e_i(x_1^2,\dots,x_m^2)$, $i=1,2,\dots,m-1$, and $e_m(x_1,\dots,x_m)$ are algebraically independent and have degrees $2,4,\dots,2(m-1),m$. The Poincaré series of $\mathcal{SP}(x_1,\dots,x_n)$ and $\mathcal{SP}(x_{n+1}^2,\dots,x_m^2)[U]$ are equal to:

$$\frac{1}{(1-t)\dots(1-t^n)} \quad \text{and} \quad \frac{1}{(1-t^2)\dots(1-t^{2(m-n-1)})(1-t^{m-n})}$$

respectively. Thus the Poincaré series of G is equal to:

$$\frac{(1-t^{2(m-n)})\dots(1-t^{2(m-1)})(1-t^m)}{(1-t)\dots(1-t^n)\cdot(1-t^2)(1-t^4)\dots(1-t^{2(m-n-1)})(1-t^{m-n})} =$$

$$= \frac{(1-t^{2(m-n)})\dots(1-t^{2(m-1)})(1-t^m)}{(1-t)\dots(1-t^n)(1-t^{(m-n)})}.$$

□

Proposition 2.4 *The elementary symmetric polynomials $e_i(x_1,\dots,x_n)$, $i = 1,2,\dots,n$ and the polynomial $e_{m-n}(x_{n+1},\dots,x_m) = x_{n+1}x_{n+2}\dots x_m$ generate multiplicatively the ring R.*

Proof. The assertion follows from the proof of [P–R2, Theorem 1.5]. □

3 Main Theorem

Let $G = H/P_n$ be the Grassmannian of n-dimensional isotropic subspaces of $\mathbf{C}^{2m}$ with respect to a nondegenerate orthogonal form on $\mathbf{C}^{2m}$ (recall that we assume $m > n$).

Given a shape λ, we denote by σ_λ the Schubert cycle in $A^*(G)$ corresponding to w_λ, i.e. the class of the closure of $B^- w_\lambda P_n/P_n$, where w_λ is the element of the poset $W^{(n)}$, associated with λ.

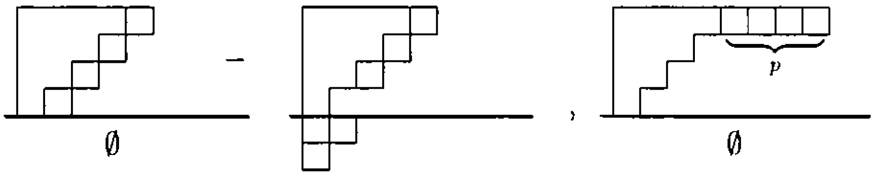
We denote by σ_i, $i = 1,2,\dots n$, and σ the Schubert cycles

$$\sigma(m-n+i, m-n-1, m-n-2, \dots, 2, 1//\emptyset)$$

and

$$(-1)^{m-n}\Big(\sigma(m-n+1, m-n, \dots, 3, 2//\emptyset) - \sigma(m-n+1, m-n, \dots, 4, 3, 1//2, 1)\Big)$$

respectively. In the following pictures, the cycles σ and σ_p, for $m = 9$, $n = 5$ and $p = 4$, are displayed:



If $c_G : R \to A^*(G) \otimes \mathbf{Z}[1/2]$ is the isomorphism induced by the Borel characteristic map (see Section 2), then $\sigma_1, \ldots, \sigma_n$ are the images of the elementary symmetric functions $e_i(x_1, \ldots, x_n) \in R$ and σ is the image of $x_{n+1}x_{n+2} \ldots x_m$. By Proposition 2.4, the elements $\sigma_1, \ldots, \sigma_n, \sigma$ generate $A^*(G)$ over $\mathbf{Z}$.

The main theorem of our paper gives a Pieri-type formula for the multiplication of an arbitrary Schubert cycle $\sigma(\lambda)$ by the special cycles σ_i, $i = 1, 2, \ldots, n$ and σ.

Let us define several notions which are necessary to state our theorem. We recall that all the terminology related to partitions, Ferrers' diagrams and shifted diagrams are borrowed from [M]. We use the following conventions for "strips":

A *1-strip* is (an ordinary) horizontal strip (that is a skew diagram with at most one box in a fixed column).

A *1/2-strip* is a horizontal strip with pairwise disconnected rows. [1]

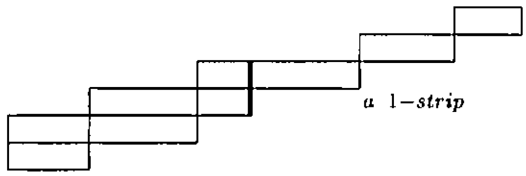
A *3/2-strip* is an almost horizontal strip in the terminology of [P–R2, Section 2]. That is, it is a (possibly) disconnected skew diagram with at most two boxes in each column such that the set of highest boxes in columns forms a 1-strip and the remaining boxes form a 1/2-strip. Every 3/2-strip has a decomposition into connected components. The set of non-highest boxes in columns of a component forms a set called *the excrescence* of the component. (Compare [P–R2, Section 2]).

A *2-strip* is a skew diagram with exactly two boxes in any (nonempty) column.

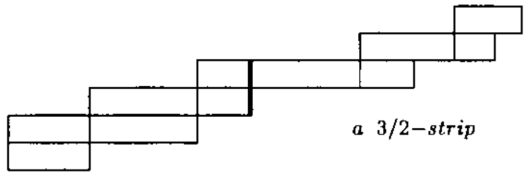
A *degenerate strip* is a 1/2-strip with at most one box in a fixed row.

For $i = 3/2, 1$, by an *extended i-strip* we understand a skew-diagram whose certain (nonempty) amount of initial columns is a 2-strip (we call this set of boxes *the 2-strip* of the extended i-strip) and, restricted to the remaining columns, it is an i-strip (we call this set of boxes *the i-strip* of the extended i-strip). In the pictures below, examples of an extended 3/2-strip and 1-strip are displayed:

[1] We say that a skew diagram D is connected if each of the sets $\{i \mid \exists_j \ (i,j) \in D\}$ and $\{j \mid \exists_i \ (i,j) \in D\}$ is an interval in the set of positive integers.



a 1−strip



a 3/2−strip

Suppose that $\mu^b = (\mu_1, \ldots, \mu_k)$, where k is even. We define $\tilde{l}$ - another function of length in the following way: if $\mu_k > 1$, then $\tilde{l}(D^b_\mu) = k$; if $\mu_k = 1$, then $\tilde{l}(D^b_\mu) = k - 1$. The same definition is valid for the diagram D^b_λ. (Here, the function $\tilde{l}(\ )$ counts the number of nontrivial rows of D^b_μ, that is, rows which have at least one box not marked with $\times$ - see Section 4.)

Now suppose that two shapes λ and μ are given. In what follows, by a *row* without further indications we will mean a row in the top part and by the λ- (resp. μ-part) of a row understand its restriction to D^t_λ (resp. D^t_μ).

A row will be called *exceptional* if its λ-part contains strictly its μ-part.

By a *component* we will understand a connected component of $D^b_\mu \setminus D^b_\lambda$.

A component will be called *extremal* if it meets the leftmost column.(Note that there exists at most one extremal component).

We will say that a box $\mathfrak{t} \in D^t_\lambda \cup D^t_\mu$ *lies over* a box $\mathfrak{b} \in D^b_\mu$, or that $\mathfrak{b}$ *lies under* $\mathfrak{t}$ if $\mathfrak{t}$ and the shifted $\mathfrak{b}$ lie in the same column. Similarly, a subset T of $D^t_\lambda \cup D^t_\mu$ *lies over* a subset B of D^b_λ if every box of T lies over some box of B. For a set T contained in one row, we will say that T *ends over* B if the rightmost box of T (called *the end of* T) lies over B.

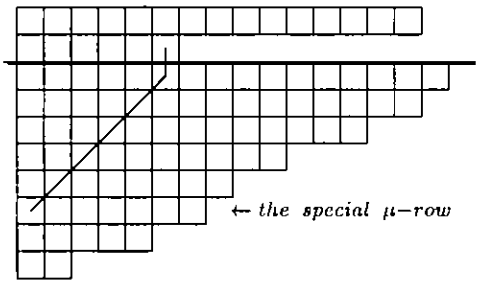
Also, for boxes $\mathfrak{b}_1$ and $\mathfrak{b}_2$ from D^b_μ, we say that $\mathfrak{b}_1$ *lies over* (resp. *lies under*) $\mathfrak{b}_2$ if the column of the shifted $\mathfrak{b}_1$ is equal to the column of the shifted $\mathfrak{b}_2$ and the row number of $\mathfrak{b}_1$ is smaller (resp. bigger) than the row number of $\mathfrak{b}_2$ in the increasing from top to bottom numbering of rows (see Section 4).

Suppose now that two subsets B_1 and B_2 of D^b_μ are given, appearing in disjoint sets of rows of D^b_μ. We will say that B_1 *appears/lies above* B_2 (resp. B_2 *appears/lies under* B_1) if all the row numbers of boxes of B_1 are smaller than all the row numbers of boxes of B_2. In particular, this definition applies to the components.

The boxes from the difference $D^t_\mu \setminus D^t_\lambda$ will be called $(\mu - \lambda)$-*boxes.*

If the μ-part (resp. λ-part) of the shortest (i.e. the $(m-n)$-th) row ends over the leftmost box of some row of D^b_μ (resp. D^b_λ), then the latter row will be called *the special μ-row* (resp. *special λ-row*).

Let us remark that if there exists the special μ-row (resp. special λ-row), then μ (resp. λ) is of type 2; if the special row does not exist, then μ (resp. λ) is of type 1.



In Definitions 3.1, 3.2, for technical reasons, by D^b_μ (resp. D^b_λ) we understand the diagram D^b_μ (resp. D^b_λ) with the last row removed provided it is of length 1.

The component meeting the special λ-row will be called the *special component.*

Finally, in the present paper, the word " diagram " may be used in a wider sense than usually; namely, by *a diagram* we will mean a subset of D_μ, which is the union of connected subsets of rows of D_μ, each starting from the leftmost column. Thus a diagram is uniquely determined by the specification of the lengths of its consecutive rows (using the row-coordinates explained in Section 4).

Definition 3.1 *μ is compatible with λ if the following conditions hold:*

(1) $D^b_\mu \supset D^b_\lambda$ and every component of $D^b_\mu \setminus D^b_\lambda$, which lies above the special one, is a 3/2-strip. Moreover, $D^t_\mu \setminus D^t_\lambda$ is a 1/2-strip.

(2) The λ-part of at most one row ends over a component. Such a pair will be called related (i.e. the row is related to the component and the component is related to the row).

(3) Each exceptional row is related to a component over which its μ-part ends.

(4) If a $(\mu-\lambda)$-box lies over the component, then this component is neither extremal nor related and this box lies over the leftmost box of the component.

(5) *An excrescence can appear only in a related component, under the λ-part of the related row; no box from the μ-part of the related row lies over the component.*

Moreover, for the special component the following conditions are satisfied:

(i) *If w_μ is of type 1 and w_λ is of type 2, then the special component is a 1-strip and all components below it form a 1/2-strip. Neither the special component nor the components below it are related.*

(ii) *If w_μ is of type 2 and w_λ is of type 1, then $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) = 2$; the special component is an extended 3/2-strip. The lower 1-strip of its 2-strip lies under the λ-part of a row, and the excrescence of its 3/2-strip is a degenerate strip appearing below the special μ-row.*

(iii) *If w_μ and w_λ are of type 2, then the special component is a 3/2-strip whose excrescence appears below the special μ-row and is a degenerate strip.*

Definition 3.2 *A shape μ is σ-compatible with λ if the following conditions hold:*

(1) *$D^b_\mu \supset D^b_\lambda$ and the components of $D^b_\mu \setminus D^b_\lambda$, which lie above the special one form a 1/2-strip. Moreover, $D^t_\mu \setminus D^t_\lambda$ is a degenerate strip, and $D^t_\lambda \setminus D^t_\mu$ is a 1/2-strip.*

(2) *The boxes of $D^t_\lambda \setminus D^t_\mu$ lie over the components of $D^b_\mu \setminus D^b_\lambda$. If a λ-row ends over the component of $D^b_\mu \setminus D^b_\lambda$, then its μ-part also ends over this component.*

(3) *The set of boxes of a component of $D^b_\mu \setminus D^b_\lambda$, over which no box of $D^t_\lambda \setminus D^t_\mu$ lies, is a set of pairwise disjoint boxes, and it contains the leftmost box of the component.*

(4) *If a $(\mu - \lambda)$-box lies over a component, then this component is special and this box lies over the leftmost box of the component.*

Moreover, for the special and extremal components the following conditions are satisfied:

(i) *If w_μ is of type 1 (and w_λ is of type 1 or 2), then the special and extremal components satisfy conditions (1)–(4).*

(ii) *If w_μ is of type 2 and w_λ is of type 1, then the special component is extremal and it forms an extended* 1*-strip. The λ-part of a row ends over the rightmost box of the lower* 1*-strip of the* 2*-strip, and its* 1*-strip satisfies (1)-(4).*

(iii) *If w_μ and w_λ are of type 2, then the extremal component is special and it forms a* 1*-strip.*

Our main theorem asserts the following:

Theorem 3.3 *For every $\lambda \in \mathcal{P}_n$ and $p = 1, \ldots, n$,*

(1)

$$\sigma(\lambda) \cdot \sigma_p = \sum 2^{e(\lambda,\mu)} \sigma(\mu),$$

where the sum is over all μ which are compatible with λ, $|\mu| = |\lambda| + l(\mu^b) - l(\lambda^b) + p$ and $e(\lambda, \mu)$ is the cardinality of the set of all components appearing above the special one, which are not related and have no $(\mu - \lambda)$-boxes over them.

(2) *For every $\lambda \in \mathcal{P}_n$,*

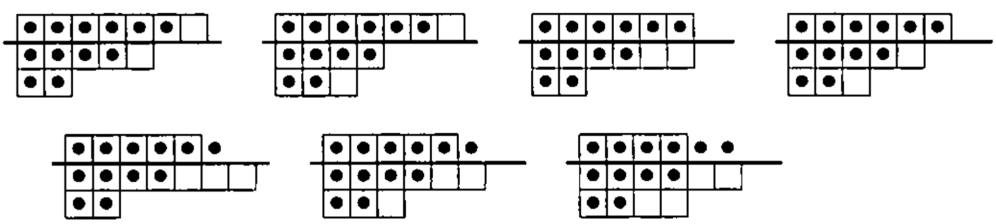
$$\sigma(\lambda) \cdot \sigma = \sum \sigma(\mu),$$

where the sum is over all μ which are σ-compatible with λ with $|\mu| = |\lambda| + l(\mu^b) - l(\lambda^b) + (m - n)$.

In Section 7 we prove part (1) of the theorem. A proof of part (2) is similar but much more easier than the proof of (1), and it is omitted.

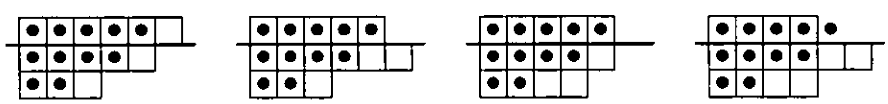
Example 3.4 $m = 7, n = 6$

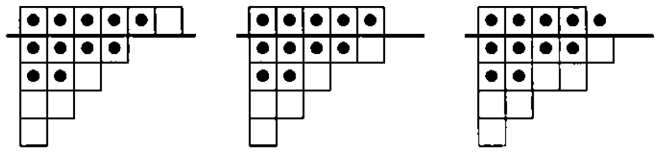
$\sigma(6//4,2) \cdot \sigma_2 = 2\sigma(7//5,2) + 2\sigma(7//4,3) + 2\sigma(6//6,2) + 2^2\sigma(6//5,3) + \sigma(5//7,2) + \sigma(5//6,3) + \sigma(4//6,4)$



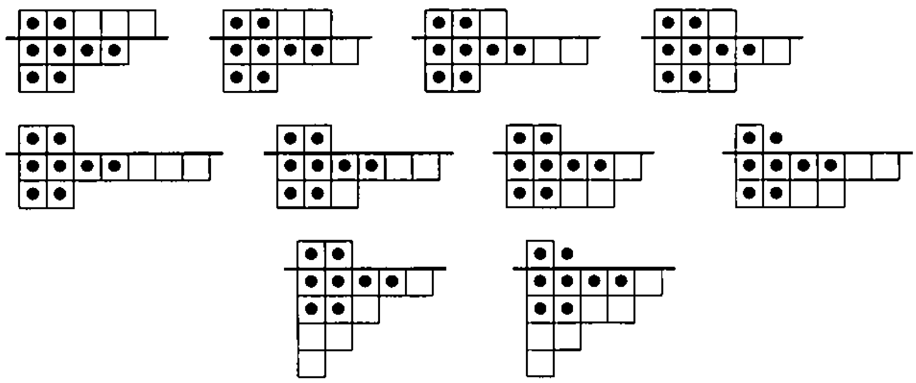
Example 3.5 $m = 7, n = 6$

$\sigma(5//4,2) \cdot \sigma_3 = 2\sigma(6//5,3) + 2\sigma(6//4,3,2,1) + 2\sigma(5//6,3) + \sigma(5//5,4) + 2\sigma(5//5,3,2,1) + \sigma(4//6,4) + \sigma(4//5,4,2,1)$

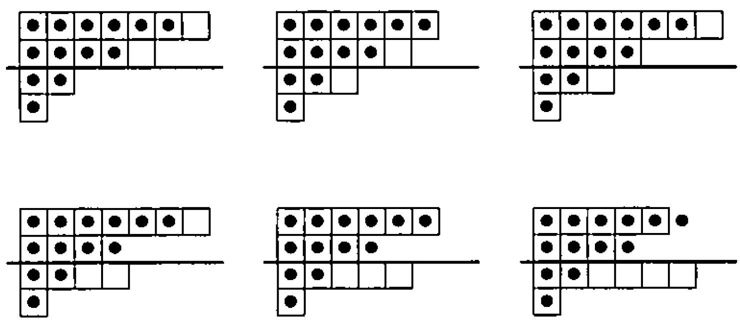




Example 3.6 $m = 7, n = 6$
$\sigma(2//4,2)\cdot\sigma_3 = \sigma(5//4,2)+2\sigma(4//5,2)+2\sigma(3//6,2)+2\sigma(3//5,3)+2\sigma(2//7,2)+2\sigma(2//6,3)+\sigma(2//5,4)+\sigma(1//6,4)+2\sigma(2//5,3,2,1)+2\sigma(1//5,4,2,1)$



Example 3.7 $m = 7, n = 5$
$\sigma_5 \cdot \sigma = 0$ *(follows from both the rules)*
$\sigma(6,4//2,1)\cdot\sigma = \sigma(7,5//2,1)+\sigma(6,5//3,1)+\sigma(7,4//3,1)+\sigma(7,3//4,1)+\sigma(6,3//5,1)+\sigma(5,3//6,1)$



4 Calculus of divided differences

Let us define "even orthogonal simple divided differences " which are operators $\partial_i : \mathbf{Z}[X] \to \mathbf{Z}[X]$, $i = 1, \ldots, m$, of degree -1 acting on the ring of polynomials $\mathbf{Z}[X]$ where X is a fixed set of indeterminates $X = (x_1, x_2, \ldots, x_m)$. We denote by s_i the transposition $(i, i+1) \in S_m \subset W$, $i = 1, 2, \ldots, m-1$, acting on X by interchanging x_i and x_{i+1}; and by s_m - the reflection which transposes x_{m-1} with x_m and changes the signs of both the variables; the remaining variables are invariant. This action is extended multiplicatively to the ring $\mathbf{Z}[X]$. Simple divided differences for the even orthogonal group are defined as follows:

$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}), \qquad i = 1, \ldots, m-1;$$

$$\partial_m(f) = (f - s_m f)/(x_{m-1} + x_m).$$

For every $f, g \in \mathbf{Z}[X]$, we have:

$$\partial_i(f \cdot g) = f \cdot (\partial_i g) + (\partial_i f) \cdot (s_i g) \tag{1}$$

(a Leibniz-type formula).

For a given $\mathbf{a} = (a_m, a_{m-1}, \ldots, a_2, a_1) \in \{-1, 0, 1\}^m$, we define the generating function:

$$E_{\mathbf{a}} = \prod_{i=1}^{m} (1 + a_i x_i).$$

In particular, for $\mathbf{a} = (0, \ldots, 0, 1, \ldots, 1)$, where 0 appears $(m-n)$-times and 1 occurs n-times, the resulting generating function, denoted by E, is the generating function for the elementary symmetric polynomials in $x_1, \ldots, x_n$.

Lemma 4.1 a) *We have $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$, where*

$$\mathbf{a}' = \begin{cases} (a_m, \ldots, a_{i+2}, a_i, a_{i+1}, a_{i-1}, \ldots, a_1) & i < m, \\ (-a_{m-1}, -a_m, \ldots, a_1) & i = m. \end{cases}$$

b) *For $i = 1, 2, \ldots, m-1$,*

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \qquad \text{if } a_i = a_{i+1} + d \qquad (d = -2, -1, 0, 1, 2),$$

where $\mathbf{a}' = (a_m, \ldots, 0, 0, \ldots, a_1)$ is the sequence $\mathbf{a}$ with a_{i+1}, a_i replaced by zeros. In particular if Δ is a composition of some s- and ∂-operations, then for every $\mathbf{a}$, $\Delta(E_{\mathbf{a}}) = (scalar) \cdot E_{\mathbf{a}'}$, where $\mathbf{a}'$ is uniquely determined if this scalar is not zero.

c) $\partial_m(E_{\mathbf{a}}) = (a_m + a_{m-1}) \cdot E_{(0,0,a_{m-2},\ldots,a_1)}$. $\square$

Let w be an element of the Weyl group W and let $s_{i_1} \cdot \ldots \cdot s_{i_l}$ be its reduced decomposition. There exists an operator $\partial_w := \partial_{i_1} \circ \ldots \circ \partial_{i_l}$ on $\mathbf{Z}[X]$ of degree $-\, l(w)$ which does not depend on the reduced decomposition chosen and allows us to give an explicit description of the characteristic map

$$c : \mathbf{Z}[X] \to A^*(F)$$

in terms of Schubert cycles:

$$c(f) = \sum_{l(w)=\deg f} \partial_w(f) X_w,$$

where $f \in \mathbf{Z}[X]$ is homogeneous and $X_w = [\overline{B^- wB/B}]$ is the Schubert cycle in $A^*(F)$ corresponding to $w \in W$ (see [B–G–G, D2] for details). The following lemma holds:

Lemma 4.2 **(1)** *For* $p = 1, 2, \ldots, n$,

$$c(e_p(x_1, \ldots, x_n)) = X_{s_{n-p+1}\cdot\ldots\cdot s_{n-1}\cdot s_n} = \sigma_p \in A^p(G).$$

(2)

$$c(x_{n+1}\cdot \ldots \cdot x_m) = (-1)^{m-n}(X_{s_n\cdot\ldots\cdot s_{m-2}\cdot s_{m-1}} - X_{s_n\cdot\ldots\cdot s_{m-3}\cdot s_{m-2}\cdot s_m}) = \sigma.$$

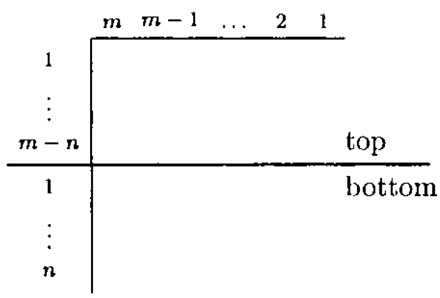
Note that

$$\sigma_p = X_{(1,\ldots,n-p,n-p+2,\ldots,n+1;\emptyset;n-p+1,n+2,\ldots,m)}$$

and

$$\sigma = (-1)^{m-n}(X_{(1,2,\ldots,n-1,m;\emptyset;n,n+1,\ldots,m-1)} - X_{(1,2,\ldots,n-1,\overline{m};\emptyset;n,n+1,\ldots,m-2,\overline{m-1})}).$$

Suppose a shape μ is given, which corresponds to the permutation w_μ. Let us use the following coordinates for boxes in D^t_μ and D^b_μ:



We associate with μ a certain distinguished reduced decomposition of $w_\mu \in W$. First, let us modify the diagram D_μ in the following way:

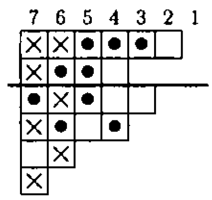
- Remove from D^t_μ the set of boxes with coordinates (a, b) satisfying the inequality $b - a \leq n$. (this is the same modification as in "B&C–case" – see [P–R2, Section 3]).
- Remove one box from each row of D^b_μ : from rows with even number remove the box in the m-th column, and from rows with odd number remove the box in the $(m-1)$-th column.

We display the removed boxes in the picture using the symbol $\times$ and denote the so obtained diagram by $\overset{\circ}{D}_\mu$.

Assume now, that a subset $D \subset \overset{\circ}{D}_\mu$ is given. A box belonging to D will be called *a D-box* and a box from the difference $D_\mu \setminus D$ will be called *a $\sim$$D$-box.*

Definition 4.3 *Read $\overset{\circ}{D}_\mu$ row by row from left to right and from top to bottom. Every D-box (resp. $\sim$ D-box) in the i-th column gives us s_i (resp. ∂_i). Then ∂_μ^D is the composition of the resulting s_i's and ∂_i's (the composition written from right to left).*

Definition 4.4 *Read $\overset{\circ}{D}_\mu$. Every D-box in the i-th column gives us s_i. $\sim$$D$-boxes give no contribution. Then, r_D is the word obtained by writing the resulting s_i's from right to left.(In other words, one obtains r_D by erasing all the ∂_i's from ∂_μ^D).*



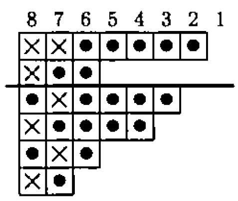
$m = 7 \;\; n = 5$

$\mu = ((6,4)//(5,4,2,1))$

$$r_D = s_4 \cdot s_6 \cdot s_5 \cdot s_7 \cdot s_5 \cdot s_6 \cdot s_3 \cdot s_4 \cdot s_5$$

$$\partial_\mu^D = \partial_7 \circ s_4 \circ \partial_5 \circ s_6 \circ \partial_3 \circ \partial_4 \circ s_5 \circ s_7 \circ \partial_4 \circ s_5 \circ s_6 \circ \partial_2 \circ s_3 \circ s_4 \circ s_5$$

One can easily prove that if $D = \overset{\circ}{D}_\mu$ then $r_D \in R(w_\mu)$ [2] – this is our distinguished reduced decomposition of w_μ.



$m = 8 \;\; n = 6$

$\mu = ((7,3)//(6,5,3,2))$

$w_\mu = (1,5,8;\overline{7},\overline{4},\overline{3};2,\overline{6})$

$$w_\mu = r_{D_\mu} = s_7 \cdot s_6 \cdot s_8 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_8 \cdot s_6 \cdot s_7 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6$$

Now we follow the same strategy as in [P–R2]; we choose a homogeneous $f_\lambda \in \mathbf{Z}[1/2][X]$ such that $c(f_\lambda) = \sigma(\lambda)$. Then, for $w \in W$, $l(w) = |\overset{\circ}{D}_\lambda|$, one

[2] For a given $w \in W$, we denote by $R(w)$ the set of its reduced decompositions.

has $\partial_w(f_\lambda) \neq 0$ iff $w = w_\lambda$ and $\partial_{w_\lambda}(f_\lambda) = 1$. We want to find the coefficients m_μ in the expansion

$$c(f_\lambda \cdot e_p(x_1, \ldots, x_n)) = \sum m_\mu \sigma(\mu).$$

Proposition 4.5 *In the above notation,*

$$m_\mu = \sum \partial_\mu^D (e_p(x_1, \ldots, x_n)),$$

where the sum is over all $D \subset \overset{\circ}{D}_\mu$ *such that* $r_D \in R(w_\lambda)$.

Proof. See [P–R2, Section 3]. □

5 Ribbons

Let us fix an element $w_\lambda \in W^{(n)}$. In this section we treat a given reduced decomposition $w_\lambda = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_l}$ as a sequence of simple transposition operations, which produces w_λ from the identity permutation:

$$w_\lambda = (\ldots((1, 2, \ldots, m) \cdot s_{i_1}) \ldots) \cdot s_{i_l}.$$

In the following, the simple transpositions involved will be called "the s_{i_h}-operations" $(h = 1, \ldots, l)$.

For a given $w_\lambda \in W^{(n)}$, the following proposition holds: (compare [P–R2, Proposition 4.1])

Proposition 5.1 **(1)** *Suppose* $w_\lambda = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n})$ *is of type 1 (k is even). Every* s_{i_h}*-operation appearing in a reduced decomposition of* w_λ *belongs to one of the following types:*

(i) $(\ldots, z_i, x, \ldots) \longrightarrow (\ldots, x, z_i, \ldots)$ $\quad x \neq z_j, i_h < m;$

(ii) $(\ldots, v_i, x, \ldots) \longrightarrow (\ldots, x, v_i, \ldots)$ $\quad x \neq z_j, v_j$ *and* $i_h < m;$

(iii) $(\ldots, a, b) \longrightarrow (\ldots, \overline{b}, \overline{a})$
$(a, b) = (z_i, z_{i+1}), (z_i, v_{m-n}), (v_{m-n}, z_i), (\overline{v}_{m-n}, z_i), (z_i, \overline{v}_{m-n});$

(iv) $(\ldots, \overline{v}_{m-n}, x, \ldots) \longrightarrow (\ldots, x, \overline{v}_{m-n}, \ldots)$
$x \neq v_j, z_i$ *and* $i_h < m$.

(2) *Suppose* $w_\lambda = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n-1}, \overline{v}_{m-n}) \in W^{(n)}$ *is of type 2 (k is odd). Every* s_{i_h}*-operation appearing in a reduced decomposition of* w_λ *belongs to one of the following types:*

(i) $(\ldots, z_i, x, \ldots) \longrightarrow (\ldots, x, z_i, \ldots)$ $\quad x \neq z_j, i_h < m;$

(ii) $(\ldots, v_{m-n}, x, \ldots) \longrightarrow (\ldots, x, v_{m-n}, \ldots)$ $\qquad$ $x \neq z_j$ *and* $i_h < m$;

(iii) $(\ldots, v_i, x, \ldots) \longrightarrow (\ldots, x, v_i, \ldots)$
$x \neq z_j, v_k, \overline{v}_{m-n}$ *and* $i_h < m$;

(iv) $(\ldots, a, b) \longrightarrow (\ldots, \overline{b}, \overline{a})$
$(a,b) = (z_i, v_{m-n}), (v_{m-n}, z_i)$ *if* $z_i < v_{m-n}$
or
$(a,b) = (z'_i, z'_{i+1})$ *where* $\{z'_1 < \ldots < z'_{k+1}\} = \{z_1, \ldots, z_k, v_{m-n}\}$
or
$$(a,b) = (z_i, v_{m-n-1}), (v_{m-n-1}, z_i), (\overline{v}_{m-n-1}, z_i), (z_i, \overline{v}_{m-n-1}), (z_i, v_{m-n}) \text{ if } v_{m-n-1} > v_{m-n};$$

(v) $(\ldots, \overline{v}_{m-n-1}, x, \ldots) \longrightarrow (\ldots, x, \overline{v}_{m-n-1}, \ldots)$
$x \neq z_j$, $v_{m-n-1} > v_{m-n}$ *and* $i_h < m$.

Proof. (1) Acting from right on the identity permutation $(1, 2, \ldots, m)$ we want to obtain $w_\lambda = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n})$ (k is even). Let us try first to compute the number of simple transposition operations which are necessary for this purpose. Remembering that some elements receive bars, we omit, for the moment, writing them for brevity. It is clear that we must transpose each pair (z_i, v_j) where $z_i < v_j$ at least twice (z_i must receive a bar and z_i precedes v_j in w_λ); each pair (z_i, v_j) where $z_i > v_j$ at least once (v_j is preceded by z_i in w_λ) and each pair (z_i, z_j) where $i < j$ at least once (in w_λ we have the ordering $z_k, \ldots, z_1$). Moreover, we must perform at least one transposition (y_i, z_j) if $z_j < y_i$ and (y_j, v_i) if $y_j > v_i$.

In sum we need at least:

$$\begin{aligned} 2\sum \#\{(z_i, v_j) \mid z_i < v_j\} \; &+ \; \sum \#\{(z_i, v_j) \mid z_i > v_j\} \; + \\ &+ \; \sum \#\{(z_i, z_j) \mid i < j\} + \sum \#\{(y_i, z_j) \mid z_j < y_i\} \; + \\ &+ \; \sum \#\{(y_j, v_i) \mid y_j > v_i\} \; = \; 2 \sum_{\epsilon_j = -1} a_j \; + \; \sum_{j=1}^{m} b_j \; = \; l(w_\lambda) \end{aligned}$$

operations. Here:

$a_j = \#\{w_i \mid i > j \wedge w_i > w_j\}$ $\qquad$ $b_j = \{w_i \mid i > j \wedge w_i < w_j\}$

$w_\lambda = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n}) = (\overset{\epsilon_1}{w_1}, \ldots, \overset{\epsilon_m}{w_m})$,
where $(w_1, \ldots, w_m) \in S_m$ $\epsilon_i = \pm 1$ and $\epsilon_i = -1$ means that w_i has a bar.

Therefore these transpositions exhaust all s_{i_h}-operations. It follows that only the following pairs can be transposed by s_{i_h}-operations (recall that we do not write bars for the moment):

(i) (z_i, v_j) twice if $z_i < v_j$: firstly z_i moves forward (that is, toward the m-th place), then v_j moves forward; and once if $z_i > v_j$: v_j moves forward;

(ii) (y_i, z_j) once if $y_i > z_j$: z_j moves forward;

(iii) (y_j, v_i) once if $y_j > v_i$: v_i moves forward;

(iv) (z_i, z_j) once if $z_i < z_j$.

Now let us take into account bars. There is no possibility to make a transposition (v_i, v_j); hence only v_{m-n} can receive a bar (and lose it before the end of the process). There is no transposition (y_i, y_j) and every s_{i_h}-operation moves y_i backward. It follows that y_i cannot receive a bar. Hence every $(s_{i_h} = s_m)$-operation is of the form described in (1)(iii). From (iv) we know that the transposition (z_i, z_j) can be performed no more than once. It follows that there is no transposition (z_i, z_j) except $(\ldots z_i, z_{i+1}) \to (\ldots \overline{z}_{i+1}, \overline{z}_i)$ or $(\ldots z_i, \overline{z}_j, \ldots) \ldots) \to (\ldots \overline{z}_j, z_i \ldots)$. Thus, if z_i is moved forward, then s_{i_h}-operation, $i_h < m$, acts as in (1)(i).

If v_j is moved forward, then the corresponding s_{i_h}-operation, $i_h < m$, is of the form given in (1)(ii).

If v_{m-n} receives a bar, then it must lose it. The element $\overline{v}_{m-n}$ must be moved toward the m-th place; for the same reasons as in (1)(ii), we have $x \neq z_i, v_j$.

(2) Let us specify, as in (1), all transpositions which must be performed:

- each pair (z_i, v_j), $z_i < v_j$, $j = 1, 2, \ldots, m - n - 1$, must be transposed at least twice;
- each pair (z_i, v_{m-n}) must be transposed at least twice if $z_i < v_{m-n}$ and once if $z_i > v_{m-n}$;
- each pair (z_i, z_j) - at least once if $z_i < z_j$;
- each pair (y_i, z_j) - at least once if $z_j < y_i$;
- each pair (y_i, v_j) - at least once if $v_j < y_i$;
- each pair (v_i, v_{m-n}) - at least once if $v_i > v_{m-n}$.

In sum we need at least

$$2\sum \#\{(z_i, v_j) \mid z_i < v_j\} + \sum \#\{(z_i, z_j) \mid z_i < z_j\} +$$
$$+ \sum \#\{(y_i, z_j) \mid y_i > z_j\} + \sum \#\{(y_i, v_j) \mid y_i > v_j\} +$$

$$+ \sum \#\{(v_i, v_{m-n}) \mid v_i > v_{m-n}\} = 2\sum_{\epsilon_j=-1} a_j + \sum_{j=1}^{m} b_j = l(w_\lambda)$$

operations. Here:

$a_j = \#\{w_i \mid i > j \wedge w_i > w_j\}$ $\qquad$ $b_j = \{w_i \mid i > j \wedge w_i < w_j\}$

$w_\lambda = (y_1, \ldots, y_{n-k}; \overline{z}_k, \ldots, \overline{z}_1; v_1, \ldots, v_{m-n-1}, \overline{v}_{m-n}) = (\overset{\epsilon_1}{w_1}, \ldots, \overset{\epsilon_m}{w_m})$,

where $(w_1, \ldots, w_m) \in S_m$, $\epsilon_i = \pm 1$ and $\epsilon_i = -1$ means that w_i has a bar.

Therefore these transpositions exhaust all s_{i_h}-operations.

As in the proof of (1), we observe:

1° There are no transpositions (v_i, v_j), $i, j \neq m-n$, (z_i, z_j), (y_i, y_j).

2° Transpositions between z_i and $\overline{z}_j$ are $(\ldots z_i, \overline{z}_j \ldots) \to (\ldots \overline{z}_j, z_i \ldots)$.

3° If z_i moves forward and x goes backward, then $x \neq z_j$. (This proves (2)(i).)

4° If v_i, $i < m-n$, moves forward and x goes backward, then $x \neq z_i, v_{m-n}$.

5° If $v_{m-n} > v_{m-n-1}$, then for $i = 1, 2, \ldots, m-n-1$ we have no transpositions (v_i, v_{m-n}) so no v_i, $i < m-n$, can receive a bar. If $v_{m-n} < v_{m-n-1}$, then v_{m-n-1} can receive a bar. (This gives the operations in (2)(iv).)

6° If v_{m-n-1} receives a bar, then it must lose it - this gives the operations from (2)(v).

Corollary 5.2 **(1)** *Let w_λ be of type 1. The following changes can be caused by s_{i_h}-operations:*

- **(i)** *move $z_\bullet$ forward and $v_\star$ backward;*
- **(ii)** *move $z_\bullet$ forward without moving any $v_\star$;*
- **(iii)** *transpose, with changing the sign, a pair (a, b), where:*
  $(a, b) = (z_i, z_{i+1}), (z_i, v_{m-n}), (\overline{v}_{m-n}, z_i), (z_i, \overline{v}_{m-n})$
  a and b are on the $(m-1)$-th and m-th place respectively;
- **(iv)** *move $\overline{v}_{m-n}$ forward without moving any $z_\bullet$ or $v_\star$.*

(2) *Let w_λ be of type 2. The following changes can be caused by s_{i_h}-operations:*

- **(i)** *move $z_\bullet$ forward and $v_\star$ backward;*
- **(ii)** *move $z_\bullet$ forward without moving any $v_\star$;*
- **(iii)** *move $\overline{v}_i$, $i < m-n$, forward without moving any $z_\bullet$ or v_{m-n};*

(iv) *transpose with changing the sign a pair* (a, b), *where:*
$(a, b) = (z'_i, z'_{i+1})$, $\{z'_1 < z'_2 < \ldots < z'_k < z'_{k+1}\} = \{z_1, \ldots, z_k, v_{m-n}\}$
or
$(a, b) = (z_i, v_{m-n})$, (v_{m-n}, z_i) *if* $z_i < v_{m-n}$
or
$(a, b) = (z_i, v_{m-n-1}), (v_{m-n-1}, z_i), (z_i, \overline{v}_{m-n}), (\overline{v}_{m-n-1}, z_i)$
if $v_{m-n-1} > v_{m-n}$.

Proof. : This is just a list of necessary operations appearing in the proof of Proposition 5.1. □

Assume that a subset $D \subset D_\mu$ encodes a reduced decomposition of the permutation w_λ ($r_D \in R(w_\lambda)$). Boxes of D (D-boxes) correspond to s_{i_h}-operations appearing in r_D. Corollary 5.2 allows us to define the notion of a *mark*. We will say that a D-box $\mathfrak{a}$ has *z-mark* (resp. *v-mark*) j if z_j (resp. v_j) is nontrivially involved in the s_{i_h}-operation associated with $\mathfrak{a}$. Also, we say that a box $\mathfrak{a}$ has $\overline{v}$*-mark* j ($j = m - n,\ m - n - 1$) if $\overline{v}_j$ is nontrivially involved in the s_{i_h}-operation corresponding to $\mathfrak{a}$.

We say that a box $\mathfrak{a}$ is a *pure v-box* (resp. a *pure* $\overline{v}$*-box* with v-mark j) if the associated operation moves v_j (resp. $\overline{v}_j$) forward (toward the m-th place). In a similar way, $\mathfrak{a}$ is called a *z-box* if the corresponding s_{i_h}-operation moves forward some z_j.

Definition 5.3 *A z-ribbon (resp. v-ribbon,* $\overline{v}$ *-ribbon) with mark* j *is the set of all boxes of* $D \subset D_\mu$ *whose z-marks (resp. v-marks,* $\overline{v}$*-marks) are equal to* j*. The sum of v-ribbon and* $\overline{v}$*-ribbon with mark* j ($j = m-n, m-n-1$) *will be called a* $\overline{v}/v$*-ribbon with mark* j.

Proposition 5.4 **(1)** *(Connectedness) The z-boxes, pure v-boxes and pure* $\overline{v}$*-boxes with a fixed mark form connected sets in each row.*

(2) *(Separation) In a fixed row, any two sets of D-boxes are disconnected (i.e. there is at least one* $\sim$*D-box between them) provided:*

- *they are equipped with two different z-marks;*
- *they are pure v-boxes (or* $\overline{v}$*-boxes) equipped with two different marks;*
- *one of them consists of z-boxes with a fixed mark and the second of pure v- or* $\overline{v}$*-boxes with a fixed mark.*

(3) *The z-ribbon with a fixed mark j is contained entirely in the bottom part of D_μ and is of the form:*

$$(t_m, m)(t_{m-2}, m-2)(t_{m-3}, m-3)\dots(t_{z_j}, z_j)$$

or

$$(\bar{t}_{m-1}, m-1)(\bar{t}_{m-2}, m-2)(\bar{t}_{m-3}, m-3)\dots(\bar{t}_{z_j}, z_j),$$

where $t_m \le t_{m-2} \le \dots \le t_{z_j}$ *and* $\bar{t}_{m-1} \le \bar{t}_{m-2} \le \dots \le \bar{t}_{z_j}$.

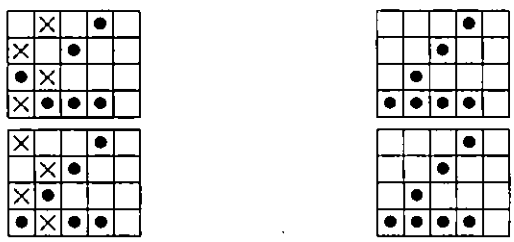
(4) *The z-marks in a given column whose number is smaller than m, strictly increase from top to bottom.*

(5) *In D^t_μ only pure v-boxes or $\bar{v}$-boxes appear, and in a fixed column their marks strictly increase from top to bottom.*

(6) *If a D-box $\mathfrak{a}$ appears in the m-th column, then it has z-marks $i, i+1$ or z-mark i and v-mark j (resp. $\bar{v}$-mark j), where $j = m-n$, $m-n-1$. If $\mathfrak{b}$ is a D-box in the m-th column and the row of $\mathfrak{a}$ is above the row of $\mathfrak{b}$, then the z-mark of $\mathfrak{a}$ is smaller than the z-mark of $\mathfrak{b}$.*

Proof. Mimic the proof of [P–R2, Proposition 4.4] and use Corollary 5.2.
□

Proposition 5.4 describes the behavior of z-ribbons in D^b_μ. It remains to analyze the picture of v- and $\bar{v}$-ribbons in D^b_μ. Since boxes marked with $\times$ are removed from D^b_μ , the $\bar{v}$-ribbons are slightly irregular. For the convenience, we simplify the pictures of ribbons in D^b and display them in the following way:



a real ribbon   the ribbon after " simplifying"

(We treat $\times$-boxes as "real" boxes in D^b_μ and "simplify" the ribbons.)

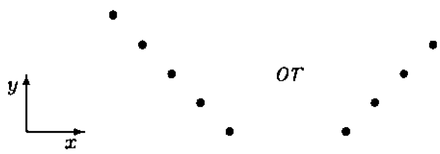
Using this convention we can state a proposition which describes v- and $\bar{v}$-ribbons:

Proposition 5.5 *Read the bottom part of $\bar{v}/v$-ribbon with mark r. The graph of the function:*

x = *the number of a box in the bottom part of the ribbon*	$\mapsto$	y = *the column number of the box*

for all x such that $y < m$, has the following properties:

1° *It is the union of sets of points of the form (which we will call a decreasing and increasing part of the graph, respectively):*



each of cardinality ≥ 1. (Note that the set consisting of a single point only can be both an increasing or decreasing part of the graph).

2° *No two decreasing (resp. increasing) parts of the graph can appear successively.*

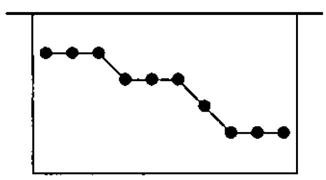
3° *The end and the beginning of two successive parts have the same y-coordinate.*

If $y = m$ for some x, then, for $x + 1$, we have $y = m - 1$.

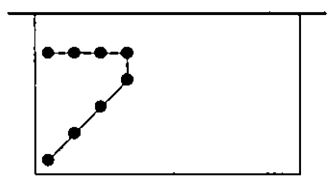
(Under this identification the function $y(x)$ is decreasing on pure v-boxes and increasing on non pure v-boxes).

(Compare [P–R2] , note that we do not draw the $\times$-boxes and use the "simplifying" convention for ribbons explained above.)

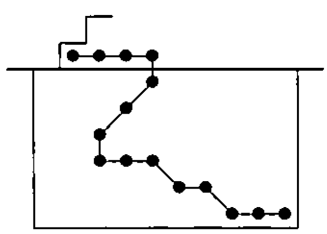
Let us display typical pictures of ribbons of different kind:



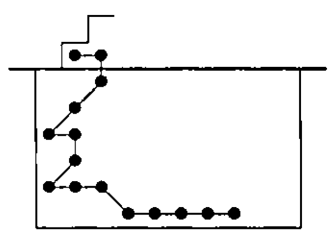
a z-ribbon



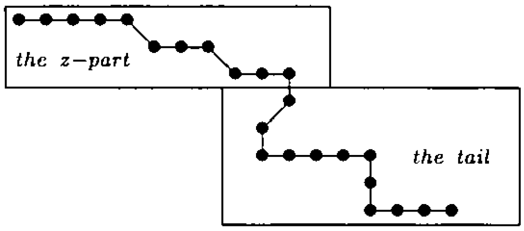
a $\overline{v}$-ribbon



a v-ribbon



a $\overline{v}/v$-ribbon

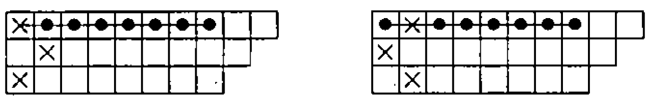


The last picture shows the v_{m-n}-ribbon when w_λ is of type 2. In that case, the v_{m-n}-ribbon can be decomposed into two parts: its *z-part* which looks like an ordinary z-ribbon and the remaining part called *the tail.*

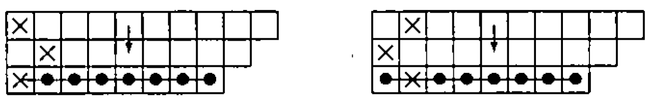
Propositions 5.4 and 5.5 allow us to control all diagrams $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$.

At the end of this section we will show some important, for this paper, operations on boxes and ribbons of D. These operations transform $D \subset D_\mu$ into a new diagram $D' \subset D_\mu$ such that if $r_D \in R(w_\lambda)$ then $r_{D'} \in R(w_\lambda)$.

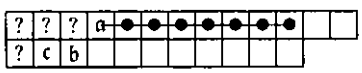
- ("Push down") Assume that we have the following configuration: the i-th row is a z-ribbon with a fixed mark.

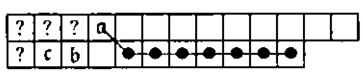
  Then we can "push down" the i-th row into the $(i+2)$-th row and obtain a new set of boxes D':

- ("Breaking a ribbon") Assume that the following configuration of D-boxes appears: $\mathfrak{a}$ cannot be in the m-th or $(m-1)$-th column, $\mathfrak{b}$ is a $\sim D$-box) or $\mathfrak{b}$ is a $\times$-box and $\mathfrak{c}$ is a $\sim D$-box:

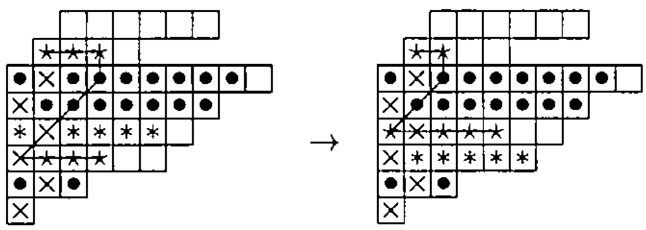
  Such an $\mathfrak{a}$ will be called *a breaking box.* Replace this configuration by:

Using the Coxeter relations in W one easily shows that if $r_D \in R(w_\lambda)$, then, after breaking a ribbon, we get $r_{D'} \in R(w_\lambda)$. In the case of the push down operation, it is clear that $r_D = r_{D'}$.

The third transformation is "an exchanging operation" and can be applied only in case if w_λ is of type 2. In the definition below, we treat the $\times$-boxes as "real" boxes in D_μ - i.e. we use the simplifying convention for ribbons. Suppose that w_λ is of type 2, all z-ribbons and the z-part of the v-ribbon with mark $(m-n)$ are the consecutive rows in D^b_λ starting from the first row of the bottom part and the row of D^b_λ corresponding to the v-ribbon with mark $(m-n)$ is not the first row of the bottom part. The following transformation of a diagram $D \subset D_\mu$ will be called *the exchanging operation* provided the resulting diagram D' is contained in D_μ:

- Transpose the z-part of the v-ribbon with mark $(m-n)$ and the row of D_λ appearing immediately above it.
- Add one D-box at the end of the z-part of the v-ribbon with mark $(m-n)$.
- Remove the rightmost D-box from the $(m-n)$-th row of D^t_μ to get D'.



Lemma 5.6 *If D' is obtained from D via the exchanging operation then r_D and $r_{D'}$ are reduced decompositions of the same element of W.*

Proof. Observe that r_D and $r_{D'}$ applied to the identity permutation give the same barred permutation. Moreover, the cardinalities of D and D' are equal. Thus the assertion follows. □

Definition 5.7 *Assume that one is given a family of ribbons (or parts of ribbons) which form consecutive rows of a diagram $D \subset D_\mu$. The maximal deformation of this family is the diagram obtained in the following way:*

- *Take the last ribbon of the family. Push it down as many times as possible. Then choose the leftmost breaking box and break the ribbon. Choose the next breaking box in the ribbon and continue this as long as there exists a breaking box.*

- *Apply these operations to the next ribbon.*

6 Lemmas about vanishing

In this section, we describe configurations of D- and $\sim D$-boxes in D_μ for which $\partial^D_\mu(E) = 0$. We will say about a configuration with this property that "it causes/gives the vanishing". Let us start with a simple but useful lemma.

Lemma 6.1 *Assume that $r_D \in R(w_\lambda)$. The following configuration:*

$$\begin{array}{c|c|c|c}
 & & \vdots & \\
\hline
? & \mathfrak{c} & \mathfrak{b} & ? \\
\hline
? & \mathfrak{a} & \multicolumn{2}{c}{?} \\
\hline
 & (h) & \vdots &
\end{array}$$

cannot appear in D^b_μ, where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ have the column number h smaller than $m-1$ and:

1. *$\mathfrak{a}$ is a pure v-box with mark smaller than $(m-n)$, $\mathfrak{b}$ is a $\sim D$-box and $\mathfrak{c}$ is a D-box, or*

2. *$\mathfrak{a}$ is a pure v-box with mark $(m-n)$, $\mathfrak{b}$ is a $\sim D$-box, $\mathfrak{c}$ is a D-box with a nontrivial z-mark.*

Proof.

1. Since $\mathfrak{a}$ is a pure v-box with mark different from $(m-n)$, the operator of $\mathfrak{a}$ acts as:
$$(\ldots v_*, x \ldots) \to (\ldots x, v_* \ldots)$$
and $x \neq v_\bullet$. But the operator of $\mathfrak{c}$ moves x forward and x must be $v_\bullet$ or $z_\bullet$ ($\mathfrak{b}$ is a $\sim D$-box, see Proposition 5.1); we get a contradiction.

2. The operator of $\mathfrak{a}$ moves v_{m-n} forward and a certain x goes backward:
$$(\ldots v_{m-n}, x \ldots) \to (\ldots x, v_{m-n} \ldots),$$
so $x \neq z_*$. But $\mathfrak{c}$ has a nontrivial z-mark and its operator moves x forward so x must be z_*, and we get a contradiction again. $\square$

In almost all proofs in this section, we must apply compositions of the operators of boxes of D_μ to the generating functions $E_{\mathbf{a}}$. The following example shows how such operators act.

Example 6.2 *We apply the operators of boxes from left to right to $E_{\mathbf{a}}$ and obtain $E_{\mathbf{a}'}$; $\bullet$ denotes a D-box and empty boxes are $\sim$D-boxes.*

$$m = 9 \qquad\qquad \mathbf{a} = (a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1)$$

1. *Action of the operators associated with a row in D^t_μ:*

9	8	7	6	5	4	3	2	1
×	×	×	•	•	•	•		

$$\mathbf{a}' = (a_9, a_8, a_6, a_5, a_4, a_3, 0, 0, 0)$$

2. *Action of the operators associated with a row in D^b_μ:*

9	8	7	6	5	4	3	2	1
×	•	•	•	•	•			

$$\mathbf{a}' = (a_8, a_7, a_6, a_5, a_4, 0, 0, 0, 0)$$

9	8	7	6	5	4	3	2	1
•	×	•	•	•	•	•	•	

$$\mathbf{a}' = (-a_8, a_7, a_6, a_5, a_4, a_3, a_2, 0, 0)$$

9	8	7	6	5	4	3	2	1
•	×	•	•	•	•	•	•	•

$$\mathbf{a}' = (-a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, -a_9)$$

9	8	7	6	5	4	3	2	1
×	•	•	•		•	•		•

$$\mathbf{a}' = (a_8, a_7, a_6, 0, a_4, a_3, 0, a_1, 0)$$

3. *Action of the operators associated with rows of D-boxes in D^b_μ:*

$$\mathbf{a} = (*, *, *, *, *, *, b, *, *)$$

9	8	7	6	5	4	3	2	1
×	•	•	•	•	•	$\boxdot$	•	•
•	×	•	•	•	$\boxdot$	•	•	
×	•	•	•	$\boxdot$	•	•		
•	×	•	$\boxdot$	•	•			

$$\mathbf{a}' = (*, *, b, *, *, *, *, *, *)$$

$$\mathbf{a} = (*, *, *, b, *, *, *, *, *)$$

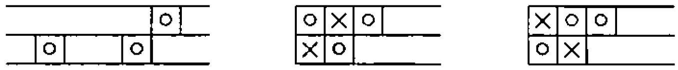
9	8	7	6	5	4	3	2	1
•	×	•	$\boxdot$	•	•	•	•	•
×	•	$\boxdot$	•	•	•	•	•	
$\boxdot$	×	•	•	•	•	•		
×	$\boxdot$	$\boxdot$	$\boxdot$	$\boxdot$	$\mathfrak{a}$			

$$\mathbf{a}' = (*, *, *, *, b, *, *, *, *)$$

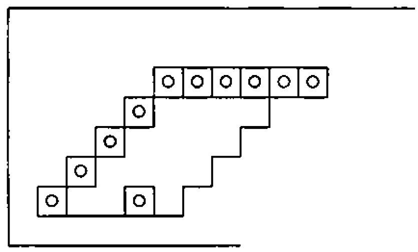
The symbols $\boxdot$ denote the operators which move b; in the second picture, $\mathbf{a}'$ is obtained after applying all operators of boxes preceding $\mathfrak{a}$.

Lemma 6.3 *The following configurations of $\sim$D-boxes in D^b_μ give the vanishing:*

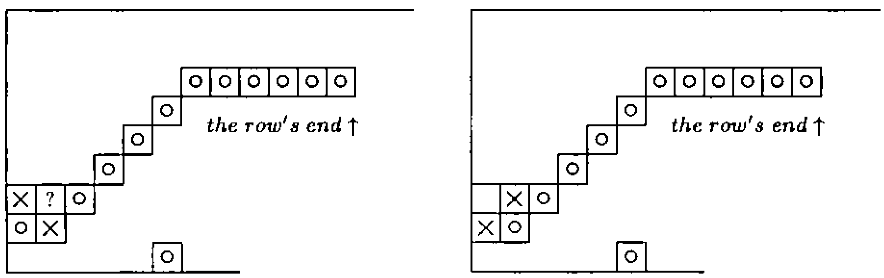
1.



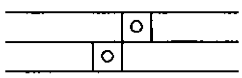
2.



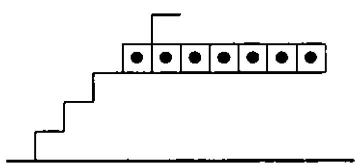
3.



Lemma 6.4 *The following configuration of* $\sim D$*-boxes in* D^t_μ *gives the vanishing:*



Lemma 6.5 *If* $\partial^D_\mu(E) \neq 0$*, then the top segment of a* v*-ribbon (resp. the* $\overline{v}_{m-n}$*-ribbon) is of the form:*



Proof. See the proof of [P–R2, Lemma 5.5]. □

Corollary 6.6 *If* $r_D \in R(w_\lambda)$ *and* $\partial^D_\mu(E) \neq 0$*, then*

1. $D^t \subset D^t_\mu$ *is the diagram of a strict partition.*

2. $D^t_\mu \setminus D^t$ *is a* $1/2$*-strip.*

Proof. 1. If D^t is not strict, then there exists an s_{i_h}-operation which interchanges the pair (v_i, v_j); but this is impossible (see Proposition 5.1).

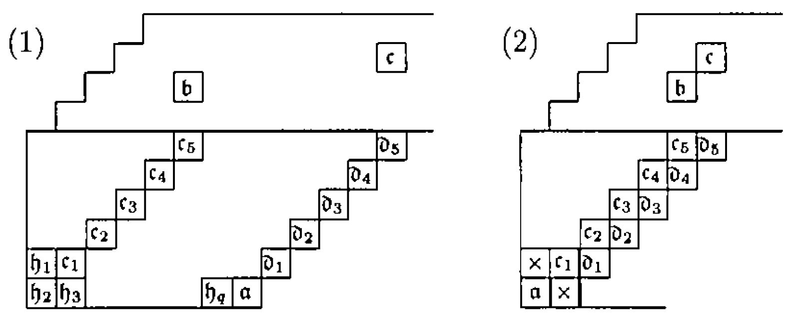
2. This follows from Lemmas 6.4 and 6.5. □

Lemma 6.7 *Let $\partial_\mu^D(E) \neq 0$ and let Δ be the operator corresponding to the top part of the diagram $D \subset D_\mu$. Then $\Delta(E) = 1 \cdot E_{\mathbf{a}}$, where the sequence* **a** *is defined as follows:*

$$a_h = \begin{cases} 0 & \text{if } h \text{ is the column number of the end of a row of } D^t, \\ 0 & \text{if } h \text{ is the column number of a box in } D_\mu \setminus D, \\ 1 & \text{in the remaining case.} \end{cases}$$

Proof. We know from Corollary 6.6 that D^t is the diagram of a strict partition and $D^t_\mu \setminus D^t$ is a 1/2-strip. The calculation from Example 6.2 applied to the consecutive rows of D^t gives the formula for **a**. □

For a given box $\mathfrak{a}$, let $\Delta_{\mathfrak{a}}$ be the composition of operators of boxes preceding $\mathfrak{a}$ in D_μ. It is clear that if $\partial_\mu^D(E) = 0$, then there exists a $\sim D$-box $\mathfrak{a}$ such that $\Delta_{\mathfrak{a}}(E) = c \cdot E_{\mathbf{a}} \neq 0$ and $\partial_j(E_{\mathbf{a}}) = 0$, where j is the column number of $\mathfrak{a}$. Such a box will be called *bad*. It follows that $\partial_\mu^D(E) \neq 0$ iff there are no bad boxes in $D_\mu \setminus D$. Corollary 6.6 gives a necessary and sufficient condition for the absence of bad boxes in D^t_μ. The next proposition allows us to decide whether a given $\sim D$-box is bad or not. Suppose that the following configurations of boxes are given:



Proposition 6.8 **(1)** *Suppose that the column number of $\mathfrak{a}$ is equal to $h < m$. Then $\mathfrak{a}$ is bad if and only if at least one of the following conditions holds:*

- *$\mathfrak{c}$ and $\mathfrak{b}$ are $\sim D$-boxes or the rightmost D-boxes in their rows;*
- *$\mathfrak{c}$ is a $\sim D$-box or the rightmost D-box in the row , and, for some i, $\mathfrak{c}_i$ is a $\sim D$-box or $\mathfrak{h}_i$ is a $\sim D$-box;*

- $\mathfrak{b}$ *is a* $\sim$*D-box or the rightmost D-box in the row , and, for some* j, $\mathfrak{d}_j$ *is a* $\sim$*D-box;*
- *There exist numbers* i, j *such that* $\mathfrak{c}_i$ *and* $\mathfrak{d}_j$, *or* $\mathfrak{h}_i$ *and* $\mathfrak{d}_j$, *are* $\sim$*D-boxes.*

(2) *Suppose that the column number of* $\mathfrak{a}$ *is equal to* m. *Then* $\mathfrak{a}$ *is bad if and only if at least one of the following conditions holds:*

- $\mathfrak{c}$ *and* $\mathfrak{b}$ *are* $\sim$*D-boxes or the rightmost boxes in their rows;*
- $\mathfrak{c}$ *is a* $\sim$*D-box or the rightmost box in the row, and, for some* i, $\mathfrak{d}_i$ *is a* $\sim$*D-box;*
- $\mathfrak{b}$ *is a* $\sim$*D-box or the rightmost box in the row, and, for some* j, $\mathfrak{c}_j$ *is a* $\sim$*D-box.*

Proof. (1) Let $E_{\mathbf{a}}$ be the function obtained from E after applying the operators of boxes of D^t_μ and let $\Delta^b_{\mathfrak{a}}$ be the composit of all operators of boxes preceding $\mathfrak{a}$ in D^b_μ. We want to calculate the components a'_h, a'_{h+1} in the sequence $\mathbf{a}'$ defined by $\Delta^b_{\mathfrak{a}}(E_{\mathbf{a}}) = c \cdot E_{\mathbf{a}'}$. Clearly, $\mathfrak{a}$ is bad iff $a'_h = a'_{h+1}$. Assume that $\mathbf{a} = (\ldots b \ldots c \ldots)$, where $b = a_s$ and $c = a_t$ (s and t are equal to the column numbers of $\mathfrak{b}$ and $\mathfrak{c}$, respectively). We know that b and c are equal to 1 or 0 (Lemma 6.7). Note that in $\Delta_{\mathfrak{a}}$, only the operators of the $\mathfrak{d}$'s have an influence on c , and only the operators of the $\mathfrak{h}$'s and $\mathfrak{c}$'s have an influence on b. One has:

$b = 0$ iff $\mathfrak{b}$ is $\sim$$D$-box or the rightmost box in a row in D^t_μ;
$c = 0$ iff $\mathfrak{c}$ is $\sim$$D$-box or the rightmost box in a row in D^t_μ.
Clearly, $a'_{h+1} = a'_h = 0$ iff:

1. $b = c = 0$;
2. $b = 0$ and $\exists_i \quad \mathfrak{d}_i$ is a $\sim$$D$-box;
3. $c = 0$ and $\exists_j \quad \mathfrak{c}_j$ is a $\sim$$D$-box or $\mathfrak{h}_j$ is a $\sim$$D$-box;
4. $\exists_{i,j} \quad \mathfrak{d}_i$ is a $\sim$$D$-box and $\mathfrak{h}_j$ is a $\sim$$D$-box, or $\mathfrak{d}_i$ is a $\sim$$D$-box and $\mathfrak{c}_j$ is a $\sim$$D$-box.

Observe that the operation associated with $\mathfrak{h}_1$ or $\mathfrak{h}_2$ changes the sign of b so $a'_h = -1$ or 0 and $a'_{h+1} = 0$ or 1 (see Lemma 6.7 and Example 6.2). Thus $a'_{h+1} = a'_h$ iff $a'_{h+1} = a'_h = 0$. Thus (1) is proved. The proof of (2) is almost the same and we omit it. $\square$

It is clear that if $\partial^D_\mu(E) = c \cdot E_{\mathbf{a}}$, then

1. c is a power of 2 (see Section 4);

2. if there exists at most one $D \subset D_\mu$ such that $\partial_\mu^D(E) \neq 0$ and $r_D \in R(w_\lambda)$, then c is equal to the multiplicity m_μ.

Suppose that $\Delta_{\mathfrak{a}}(E) = E_{\mathbf{a}}$ for a some $\sim D$-box $\mathfrak{a}$. We will say that $\mathfrak{a}$ is *essential* if $\partial_h(E_{\mathbf{a}}) = 2 \cdot E_{\mathbf{a}'} \neq 0$, where ∂_h is the operator of $\mathfrak{a}$. As a consequence of the proof of Proposition 6.8, we have the following corollary (in the situation displayed in pictures before Proposition 6.8).

Corollary 6.9 *A box $\mathfrak{a}$ is essential if and only if:*

1. *The boxes $\mathfrak{c}$ and $\mathfrak{b}$ are D-boxes but not the rightmost boxes in their rows; the boxes $\mathfrak{c}_i, \mathfrak{h}_j$ are D-boxes or $\times$-boxes; and $\mathfrak{d}_i$ are D-boxes (see the picture before Proposition 6.8);*

2. *The boxes $\mathfrak{c}$ and $\mathfrak{b}$ are D-boxes but not the rightmost boxes in their rows; and the boxes $\mathfrak{d}_i$ and $\mathfrak{h}_i$ are D-boxes.*

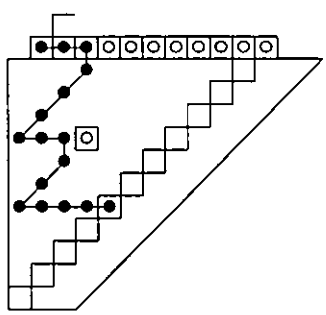
7 Proof of the main theorem

In this section, we assume tacitly that $r_D \in R(w_\lambda)$ and freely use the notions associated with such a D in the earlier sections.

7.1 Case 1: w_μ and w_λ are of type 1

Lemma 7.1 *Suppose w_μ and w_λ are of type 1. If $\partial_\mu^D(E) \neq 0$, then there is no s_{i_h}-operation supplying v_* with a bar.*

Proof. Suppose that some v_* receives a bar. Then it must lose it (there is no v_* with a bar in w_λ). Then, the v-ribbon of v_* looks like:

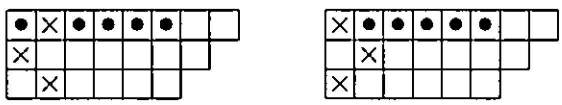


We thus obtain the vanishing by Proposition 6.8. $\square$

Lemma 7.2 *If $\partial_\mu^D(E) \neq 0$, then the set of z-boxes $D_Z \subset D_\mu^b$ is the maximal deformation of D_λ^b in D_μ^b.*

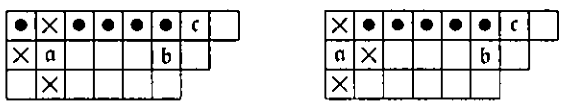
Proof. First observe that the inclusion $D_\lambda^b \subset D_\mu^b$ follows from the properties of ribbons. Suppose that D_Z is not the maximal deformation. So D_Z is some deformation of D_λ^b but not performed in the maximal way. The following cases must be examined:

- There is a possibility of pushing down a row but we do not do it:

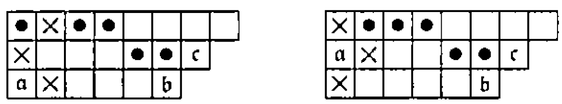


We have the following cases:

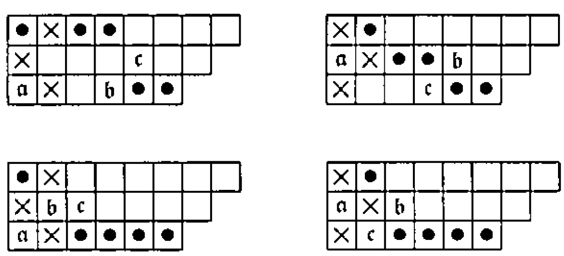
– We do nothing:



– We break the ribbon from the first to the second row:



– We deform the ribbon from the first to the third row, but not in the maximal way:



In each case, $\sim\!D$ boxes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ give the vanishing (see Lemma 6.3). The existence of these $\sim\!D$-boxes follows from Lemma 6.1 and the separation property for ribbons.

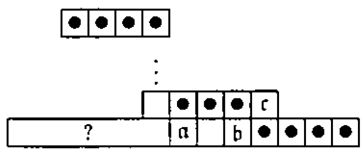
- There is a possibility of breaking a row (or a ribbon) but we do not do it in the maximal way:

  – We do nothing:

  – We break the row but not in the maximal way:

  As before, using Lemma 6.1, the separation property and the fact that no v_* can receive a bar, we get the vanishing caused by $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$. It follows that to avoid the vanishing we must perform the maximal deformation of D_λ^b.

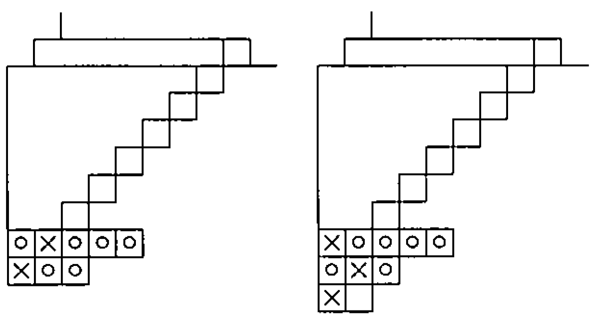
  (If we do not break, in the maximal way, the ribbon which has been already deformed, then the situation is almost the same:



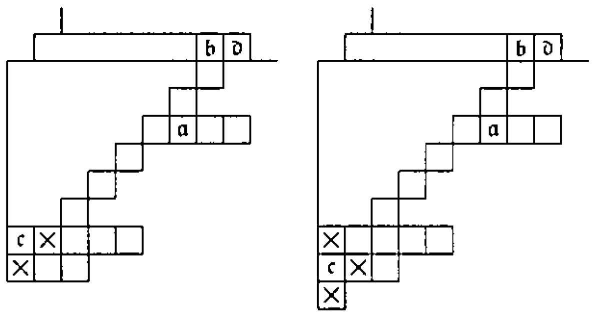
  Here, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ cause the vanishing. $\square$

Lemma 7.3 *If $\partial_\mu^D(E) \neq 0$, then $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

Proof. Suppose that $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \geq 2$. The following pictures, where only the boxes in $D_\mu \setminus D_\lambda$ are marked, will help to end the proof:



After the maximal deformation, we must obtain:



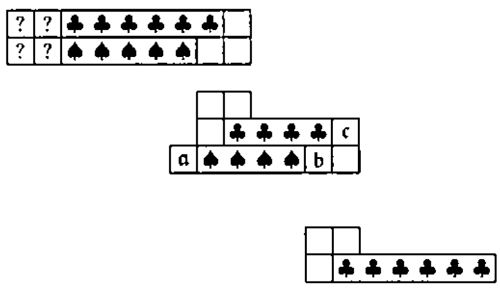
Here, $\mathfrak{c}$ is always a $\sim D$-box. If $\mathfrak{a}$ is a $\sim D$-box, then Proposition 6.8 gives the vanishing. If $\mathfrak{a}$ is a D-box, then it is a v-box and $\mathfrak{b}$ is a v-box too. So $\mathfrak{d}$ is a $\sim D$-box and Proposition 6.8 gives the vanishing again. □

Corollary 7.4 *There is no push down operation in the process of the maximal deformation.*

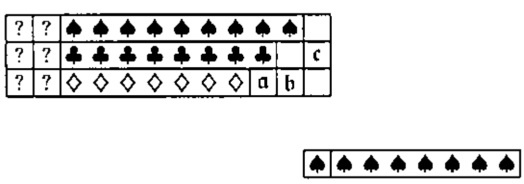
Proof. The assertion follows from Lemma 7.3: the push down requires the inequality $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \geq 2$. □

Proposition 7.5 *If $\partial^D_\mu(E) \neq 0$, then the z-boxes with the same mark can appear in at most two successive rows.*

Proof. If some z-boxes with the same mark appear in three different rows, then we have the following situation (compare [P-R2, Proposition 6.2]):



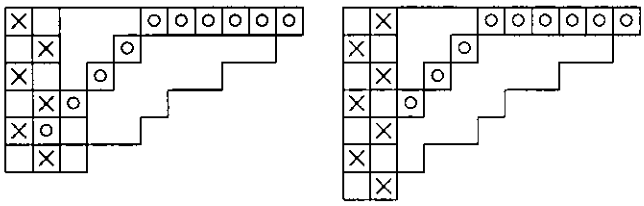
Then some $\sim D$-boxes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ cause the vanishing. If some z-boxes appear in the rows which are not successive, then we have:



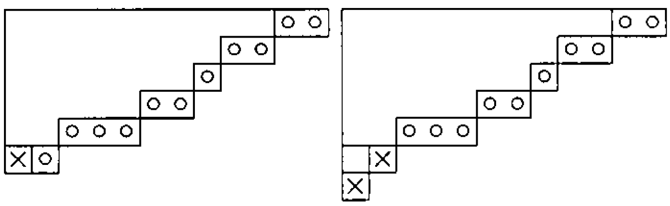
and the $\sim D$-boxes $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ cause the vanishing (see Lemma 6.3). In fact, this proof is almost the same as the proof of Proposition 6.2 in [P–R2]. A possible difference can appear only if the first breaking box of the z-ribbon is situated in the m-th or $(m-1)$-th column. But $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$, and, if the z-ribbon meets (at least) three rows, or if it meets two rows which are non-consecutive, then the first breaking box can lie neither in the m-th nor $(m-1)$-th column. Thus the pictures above show all situations which can actually happen. $\square$

Proposition 7.6 *If $\partial^D_\mu(E) \neq 0$, then $D^b_\mu \setminus D^b_\lambda$ is a $3/2$-strip and its extremal component is a 1-strip.*

Proof. The latter assertion follows easily from Lemma 7.2, the proof of Lemma 7.3 and Lemma 6.3. The extremal deformed component looks like:

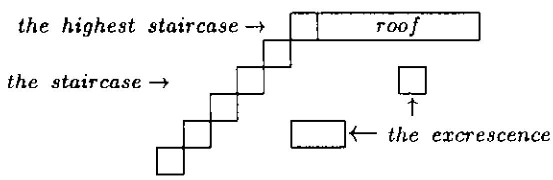


Therefore, before the deformation, it must be a 1-strip:



The former assertion can be proved directly in the same way as the first part of [P–R2, Proposition 6.7]. $\square$

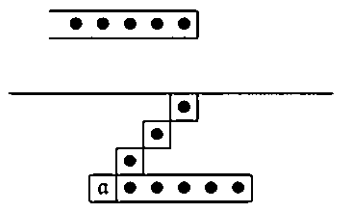
It follows from Proposition 7.6 that a typical nonextremal component looks like (cf. [P-R2, the end of Section 3]):



A typical extremal component looks like in pictures (1) and (2) of the proof of Proposition 7.6.

Lemma 7.7 *An excrescence can appear only under the roof of a deformed component and there are no two boxes of the excrescence lying one over the other. Moreover, the segment of a row between the staircase box and the excrescence must contain a z-box. (Compare [P–R2, Lemma 6.8] and the picture above.)*

We have proved that if $\partial_\mu^D(E) \neq 0$, then the positions of z-boxes are uniquely determined. Now, we must determine the positions of v-boxes. We will show that the condition $\partial_\mu^D(E) \neq 0$ can be satisfied for at most one D (that is, the positions of v-boxes are determined in a unique way too). We will use Proposition 6.8 in the following situation:



Here, $\mathfrak{a}$ is a $\sim D$-box and its column number is h. D-boxes form a part of a v-ribbon, and it is clear that $\Delta_{\mathfrak{a}}(E) = E_{\mathfrak{a}}$ where $a_h = 0$. We need to determine a_{h+1} if we want to know whether $\mathfrak{a}$ is bad or not.

Lemma 7.8 *No v-box can appear in an excrescence.*

Proof. Use the separation property, the remark above and Proposition 6.8. □

Lemma 7.9 *A family of v-boxes can appear only in the roof of a deformed component and it forms a segment starting from the leftmost box of the roof. No two pure v-boxes with different marks can appear in the same roof.*

Proof. Use Proposition 6.8 again. □

Proposition 7.10 *No two different roofs can contain pure v-boxes with the same mark.*

Proof. Analogous to that of [P–R2, Proposition 7.4]. □

Corollary 7.11 *The marks of segments of pure v-boxes in the roofs of consecutive deformed components increase from top to bottom.*

(Compare [P–R2, Proposition 7.4])

Theorem 7.12 *For given two shapes λ, μ, there exists at most one $D \subset D_\mu$ such that $\partial_\mu^D(E) \neq 0$.*

Proof. Analogous to that of [P–R2, Theorem 8.1]. □

Recipe 7.13 *(Compare [P–R2, Recipe 8.4].) Let λ, μ be two shapes satisfying the conditions:*

(1) *$D_\mu^b \supset D_\lambda^b$, $D_\mu^b \setminus D_\lambda^b$ is a 3/2-strip and $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

(2) *The λ-part of at most one row ends over a component.*

The recipe is:

(i) *Perform the maximal deformation of D_λ^b in D_μ^b.*

(ii) *Shift the bottom part of D_μ together with the deformed D_λ^b. For every shifted component of D_μ^b choose a row of D_λ^t which ends over the component. Subtract the segment of the row, which ends over the roof and push it down to the roof.*

Lemma 7.14 *If $\partial_\mu^D(E) \neq 0$, then:*

(1) *$D_\mu^t \setminus D_\lambda^t$ is a 1/2-strip.*

(2) *No $(\mu - \lambda)$-box lies over the staircase of a related component.*

Proof. Analogous to the one of [P–R2, Lemma 8.5]. □

Definition 7.15 *Let w_λ and w_μ be of type 1. Then μ is compatible with λ if*

(1) *$D_\mu^b \supset D_\lambda^b$ and $D_\mu^b \setminus D_\lambda^b$ is a 3/2-strip; $D_\mu^t \setminus D_\lambda^t$ is a 1/2-strip.*

(2) *The λ-part of at most one row ends over a component. If a row ends over a component we say that they are related. A component which is related to some row is called related. Similarly, a row which is related to some component is called related.*

(3) *Each exceptional row is related to a component over which the μ-part of this row ends.*

(4) *If a $(\mu-\lambda)$-box lies over a component, then this component is not related and this box lies over the leftmost box of the component.*

(5) *An excrescence can appear only in a related component under the λ-part of the related row; no box from the μ-part of the related row lies over the excrescence.*

Proposition 7.16 *There exists (a unique) $D \subset D_\mu$ such that $\partial_\mu^D(E) \neq 0$ if and only if μ is compatible with λ.*

Proof. Suppose first that there exists D such that $\partial_\mu^D(E) \neq 0$. We will show that μ is compatible with λ. Note that:

(1) holds because of Proposition 7.6;

(2) is a consequence of condition (2) of Recipe 7.13;

(3) follows from Recipe 7.13 and $D \subset D_\mu$;

(4) follows from Recipe 7.13 and Proposition 6.8;

(5) is a consequence of Proposition 6.8.

Assume conversely, that (1) — (5) hold. We will prove that the set of bad boxes is empty. First, observe that the extremal component is a 1-strip — this follows from (1) and (5). Thus, the deformed component looks like in the pictures preceding Lemma 7.7. It follows from Proposition 6.8 that no box from the staircase can be bad. Suppose that a component is not related. This means that no λ-row ends over a component and if a $\sim D$-box lies over a component, then this box lies over the highest staircase. Thus no box from the roof can be bad. If a component is related, then no box in the roof can be bad (use (4) and Proposition 6.8), and no box from the excrescence can be bad (use (5) and Proposition 6.8). There are no bad boxes in the extremal component: this follows from (4) and Proposition 6.8. □

Let, for compatible μ and λ, $D^{\lambda,\mu}$ denote the unique D from the proposition. Clearly, m_μ is the number defined by $\partial_\mu^{D^{\lambda,\mu}}(E) = m_\mu \cdot E_{\mathfrak{a}}$. It follows that m_μ is equal to $2^{m(\lambda,\mu)}$ where $m(\lambda, \mu)$ is the number of essential boxes.

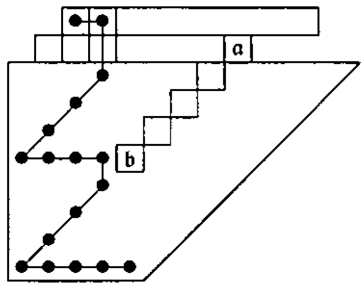
Proposition 7.17 *A $\sim D$-box $\mathfrak{a}$ is essential if and only if it is the highest staircase box in a non-related component.*

Proof. Use Corollary 6.9. □

7.2 Case 2: w_μ is of type 1 and w_λ is of type 2.

Lemma 7.18 *If $\partial_\mu^D(E) \neq 0$, then no s_{i_k}-operation supplies v_{m-n-1} with a bar.*

Proof. Suppose that v_{m-n-1} receives a bar. It follows that the $\overline{v}/v$-ribbon with mark $(m-n-1)$ must meet (at least) twice the m-th column. Thus we have the following picture of the $\overline{v}/v$-ribbon:



Here, $\mathfrak{a}$ is a $\sim D$-box (w_μ is of type 1 so $\mathfrak{a}$ cannot belong to the v-ribbon with mark $(m-n)$) and $\mathfrak{b}$ is a $\sim D$-box (use the separation property). Then Proposition 6.8 applied to this configuration gives the vanishing. □

Assume that $D \subset D_\mu$ is such that $\partial_\mu^D(E) \neq 0$. Let $D_Z \subset D_\mu^b$ be the set of all z-boxes together with the z-part of the v-ribbon with mark $(m-n)$ (see the picture after Proposition 5.5).

We will show that D_Z is determined in a unique way provided D does not cause the vanishing.

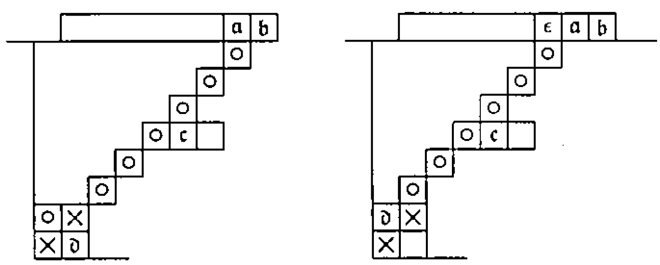
It is clear from Section 5 that by inverting the operations of breaking a row and pushing down a row, we obtain the diagram, denoted by D', in which all z-ribbons and the z-part of the v-ribbon with mark $(m-n)$ appear as consecutive rows. After applying the maximal deformation to D', we get a certain new subset of D_μ^b; denote it by D''.

Proposition 7.19 *If $D_Z \neq D''$, then $\partial_\mu^D(E) = 0$.*

Proof. Analogous to that of Lemma 7.2. □

Proposition 7.20 *If $\partial_\mu^D(E) \neq 0$, then $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

Proof. Suppose that this is not true. Then the extremal component must contain (after the deformation) one of the following two configurations:



It is clear that $\mathfrak{a}$ and $\mathfrak{b}$ belong to D_μ^t (w_μ is of type 1).

(1) Consider first the picture on the left-hand side. If $\mathfrak{c}$ is a $\sim D$-box, then $\mathfrak{d}$ gives the vanishing. Thus $\mathfrak{c}$ must be a D-box and $\mathfrak{c}$ is a v-box. Moreover, $\mathfrak{a}$ belong to the same v-ribbon and $\mathfrak{b}$ is a $\sim D$-box which gives the vanishing because of $\mathfrak{d}$ (see Proposition 6.8).

(2) In the case of the picture on the right-hand side, the argument is almost the same: if $\mathfrak{c}$ is a $\sim D$-box, then $\mathfrak{d}$ gives the vanishing. If $\mathfrak{c}$ is a D-box, it is a v-box and ϵ is the rightmost v-box in the row: $\mathfrak{a}$ and $\mathfrak{b}$ are $\sim D$-boxes. Therefore $\mathfrak{d}$ gives the vanishing by Proposition 6.8. □

Before stating the next proposition, suppose that:

$$z_1 < z_2 < \ldots < z_p < v_{m-n} < z_{p+1} < \ldots < z_k$$

and let u be the length of the z-part of the v_{m-n}-ribbon in D. The element v_{m-n} must receive a bar before or together with z_{p+1}. It follows that the rows of the diagram D' which is defined before Proposition 7.19, have lengths

$$m+1-z_1 < m+1-z_2 < \ldots < m+1-z_q,\ u,\ m+1-z_{q+1} < \ldots < m+1-z_k$$

where $q \leq p$.

We have $D' \subset D^b_\mu$ because $r_D \in R(w_\lambda)$. Let D'_λ be the diagram (contained in D^b_μ) with the row-lengths:

$$m+1-z_1 < m+1-z_2 < \ldots < m+1-z_p,\ u,\ m+1-z_{p+1} < \ldots < m+1-z_k.$$

(Note that by inverting the exchanging operations, D'_λ is gotten from D'.) Conversely, applying exchanging operations to D'_λ (in fact, it will be shown in the next proposition, that we must perform all possible exchanging operations to avoid the vanishing), we obtain the diagram D'. Then we deform the diagram D' (in the maximal way) and get D. We know (see Proposition 7.19) that D must be the maximal deformation of D' (in the opposite case we get the vanishing).

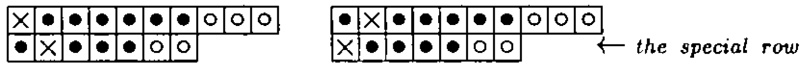
Proposition 7.21 *If $\partial^D_\mu(E) \neq 0$, then D is the result of following operations applied to D'_λ:*

(1) Apply the exchanging operation to D'_λ as many times as possible to get D'.

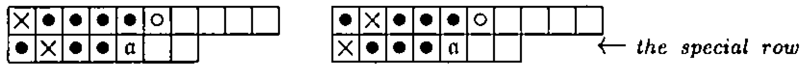
(2) Deform D' in the maximal way to obtain D.

Proof. Part (2) has been proved in Proposition 7.19.

For the proof of (1), observe that no push down operation can be applied here because $\tilde{l}(D_\mu)-\tilde{l}(D_\lambda) \leq 1$. Suppose that there is a possibility to perform the exchanging operation and we do not do it:



Then, after the maximal deformation, we get D in which the following configuration appears (this is guaranteed by the absence of the push down operation):



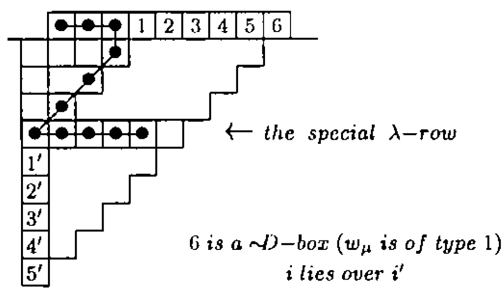
and the $\sim D$-box a causes the vanishing (see Proposition 6.8). □

Proposition 7.22 *The boxes of D_Z can appear in at most two successive rows.*

Proof. Analogous to that of Proposition 7.5. □

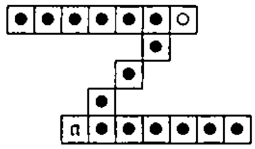
Proposition 7.23 *$D^b_\mu \setminus D^b_\lambda$ is a $3/2$-strip, the special component is a 1-strip and all components below the special one form a $1/2$-strip.*

Proof. For all components above the special one the proof is the same as that of [P–R2, Proposition 6.7]. Since w_μ is of type 1, the length of $(m-n)$-th row of D^t_μ is bigger than the number of rows in D^b_μ. Hence, there exists a $\sim D$-box (or the rightmost D-box) in the $(m-n)$-row of D^t_μ over every box appearing in the m-th column of D^b_μ, in the special row or below it. In the picture, $1,2,3,4,5,6$ are $\sim D$-boxes and $1,2,3,4,5$ lie over $1',2',3',4',5'$.

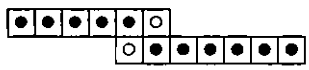


For the special row and the rows below we can apply Proposition 6.8 and we obtain:

(1) The v_{m-n}-ribbon is equal to its z-part (if this is not true, then the $\sim D$-box $\mathfrak{a}$ causes the vanishing):



(2) The deformed component below the special one is contained in a single row (if this is not true, then the following configuration of boxes, in the deformed component, causes the vanishing):

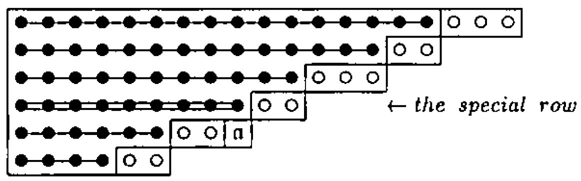


It remains to prove that the special component is a 1-strip. It follows from (1) that $D'_\lambda = D^b_\lambda$. After applying the exchanging operations to D_λ, we obtain $D' \subset D^b_\mu$ and then we deform D' to D_Z. $\square$

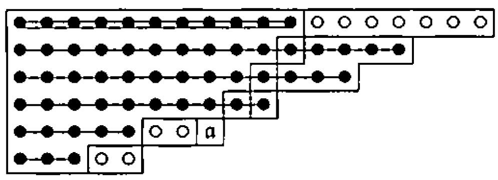
Corollary 7.24 *If $\partial^D_\mu(E) \neq 0$, then $D^b_\lambda \subset D^b_\mu$.*

Proof. Recall that $r_D \in R(w_\lambda)$, and hence $D_Z \subset D^b_\mu$. Thus $D' \subset D^b_\mu$. Since D' is obtained from $D'_\lambda = D^b_\lambda$ by the exchanging operations, $D^b_\lambda \subset D^b_\mu$. (In fact, it is true that if $r_D \in R(w_\lambda)$, then $D^b_\lambda \subset D^b_\mu$ without the assumption $\partial^D_\mu(E) \neq 0$ but we do not need this result.)

Let us return to the special component. Look at the picture:



The component must contain at least a horizontal strip; $\mathfrak{a}$ is a $\sim D$-box and $\mathfrak{a}$ must exist (C is connected). After the exchanging operations we get:



It is clear that there is no deformation of rows (see (2) before Corollary 7.24). Hence $\mathfrak{a}$ causes the vanishing. It follows that the component C is a 1-strip and the special row is the lowest row in C. □

We have a complete description of the connected components of $D^b_\mu \setminus D^b_\lambda$. For the components appearing above the special one, Lemmas 7.7, 7.8, 7.9, Proposition 7.10 and Corollary 7.11 hold true.

Theorem 7.25 *For given two shapes λ, μ such that w_λ is of type 2 and w_μ is of type 1, there exists at most one $D \subset D_\mu$, denoted $D^{\lambda,\mu}$, such that $\partial^D_\mu(E) \neq 0$ (and $r_D \in R(w_\lambda)$).*

Proof. It follows from Propositions 7.21 and 7.23 that D_Z is uniquely determined if $\partial^D_\mu(E) \neq 0$ and $r_D \in R(w_\lambda)$. Moreover, there are no v-boxes in the special component and below it. Thus, arguing as in the proof of [P–R2, Theorem 8.1], we infer that at most one D has the needed properties. □

Recipe 7.26 *(Compare [P–R2, Recipe 8.4].) Let λ, μ be two shapes satisfying the conditions:*

(1) *$D^b_\mu \supset D^b_\lambda$, $D^b_\mu \setminus D^b_\lambda$ is a 3/2-strip and $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

(2) *The λ-part of at most one row ends over a component.*

The recipe is:

(i) *Perform the exchanging operation to D_λ as many times as possible to obtain the diagram D'.*

(ii) *Apply the maximal deformation to D' to obtain the set D''.*

(iii) *Shift the bottom part of the diagram D^b_μ together with D''. For every (shifted deformed) component of D^b_μ choose a row of D^t_μ, which ends over the component. Subtract the segment of the row which ends over the roof of the component and push it down to the roof.*

Lemma 7.27 *(Compare Lemma 7.14.) If $\partial^D_\mu(E) \neq 0$, then:*

(1) *$D^t_\mu \setminus D^t_\lambda$ is a 1/2 strip.*

(2) *No $(\mu - \lambda)$-box lies over the staircase of a related component.*

Proof. Analogous to that of [P–R2, Lemma 8.5]. □

Definition 7.28 *Let w_λ be of type 2 and w_μ be of type 1. Then, μ is compatible with λ if*

(1) $D^b_\mu \supset D^b_\lambda$ *and* $D^b_\mu \setminus D^b_\lambda$ *is a* $3/2$*-strip; the special component is a* 1*-strip; the components appearing below the special one form a* $1/2$*-strip. Moreover,* $D^t_\mu \setminus D^t_\lambda$ *is a* $1/2$*-strip.*

(2) *The* λ *part of at most one row ends over a component, the special component and the components lying below it are not related.*

(3) *Each exceptional row is related to the component over which the* μ*-part of this row ends.*

(4) *If a* $(\mu-\lambda)$*-box lies over a component, then this component is not related and this box lies over the leftmost box of the component.*

(5) *An excrescence can appear only in a related component under the* λ*-part of the related row; no box from the* μ*-part of the related row lies over the excrescence.*

Proposition 7.29 $\partial_\mu^{D^{\lambda,\mu}}(E) \neq 0$ *if and only if* μ *is compatible with* λ*.*

Proof. We can almost repeat the proof of Proposition 7.16. The main difference between Definitions 7.15 and 7.28 is the addition to (1) and (2) the conditions for the special component and for the components lying below it. These modifications are necessary by Proposition 7.23. The only thing which must be proved is that (1)–(5) imply the non-existence of bad boxes in the special component and in the components below it. But this is clear by Proposition 6.8. □

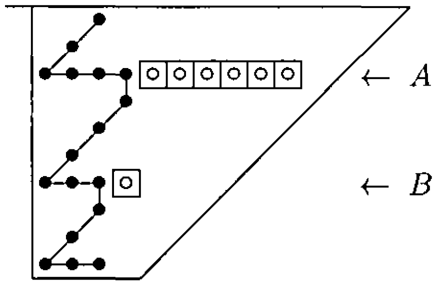
Proposition 7.30 *One has* $m_\mu = 2^{m(\lambda,\mu)}$*, where* $m(\lambda,\mu)$ *is the cardinality of the set of non-related components above the special component, with no* $(\mu-\lambda)$*-boxes over them.*

Proof. Each essential box gives the multiplicity 2. Essential boxes are the highest staircase boxes in non-related components, with no $(\mu-\lambda)$-box over them. No essential boxes can appear in the special component and below it (see Proposition 6.8). □

7.3 Case 3: w_μ is of type 2 and w_λ is of type 1.

Lemma 7.31 *If* $\partial_\mu^D(E) \neq 0$*, then the* $\overline{v}/v$*-ribbon with mark* $(m-n)$ *meets the* m*-th column no more than twice.*

Proof. Suppose that this is not true. Then the $\overline{v}/v$-ribbon meets the m-th column at least four times (w_λ is of type 1 and w_{m-n} has no bar).

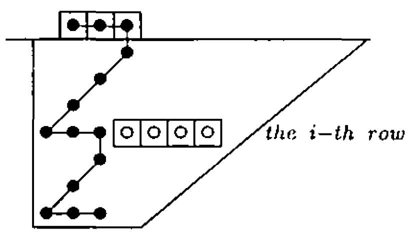


The empty places in row A can be occupied neither by v-boxes nor z-boxes (in the opposite case we get the vanishing by Proposition 6.8). A $\sim D$-box in row B must exist because D^b_μ is the diagram of a strict partition. But this configuration gives the vanishing by Proposition 6.8 applied again. □

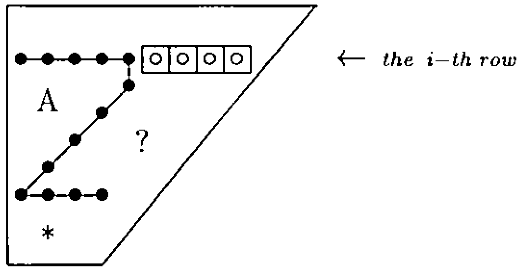
Now, let us consider separately the following two situations. Firstly, assume that v_{m-n} can receive a bar in the process of transforming the identity permutation into the permutation w_λ.

Proposition 7.32 *If the $\overline{v}/v$-ribbon with mark $(m-n)$ meets the m-th column twice and $\partial^D_\mu(E) \neq 0$, then the positions of z-boxes in D^b_μ are uniquely determined.*

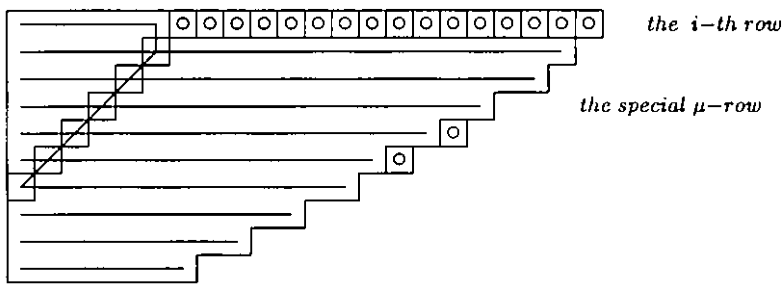
Proof. Assume that the $\overline{v}/v$-ribbon meets the m-th column twice. This means that v_{m-n} receives and then loses a bar. A typical situation is shown in the picture:



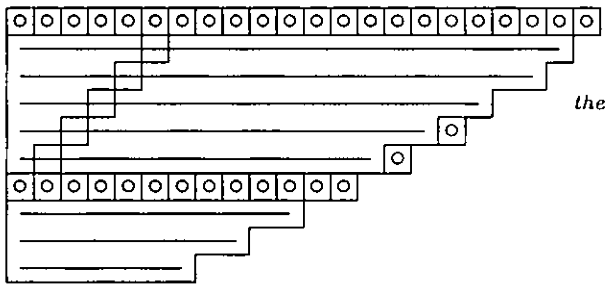
In the i-th row, only D-boxes from the $\overline{v}/v$-ribbon can appear; so if $\partial_\mu^D(E) \neq 0$, then our configuration looks like:



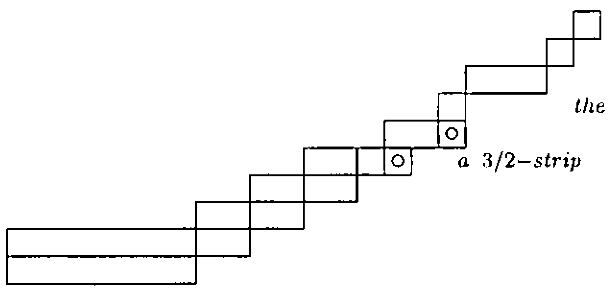
There are no $\sim D$-boxes in the area marked with "$*$" (this follows from the properties of ribbons).
In area A, only D-boxes can appear.
In the area marked with "?", there is no more than one $\sim D$-box in a fixed row (see Proposition 6.8).
If a $\sim D$-box $\mathfrak{a}$ appears in "?" , then there is no $\sim D$-box over the leftmost box of the row of $\mathfrak{a}$.
Therefore, if $\partial_\mu^D(E) \neq 0$, then the part of the diagram D_μ below the i-th row looks like:



(The z-ribbons are rows, the pure v-boxes with mark $(m - n)$ appear in precisely two rows (in the i-th row and some lower one), single $\sim D$-boxes can occupy rows below the i-th one and the special μ-row. We want to translate these conditions into the initial shape-data. Let us remove from the diagram all D-boxes with (pure) v-mark $(m - n)$. The lengths of the z-ribbons are equal to the lengths of the rows of D_λ:



Therefore, the extremal component of $D_\mu^b \setminus D_\lambda^b$ is an extended 3/2-strip and looks like:



Let us fix i as above.

Corollary 7.33 *1. The i-th row is the highest row of the extremal component.*

2. In the diagram $D \subset D_\mu$, the positions of boxes of the v_{m-n}-ribbon are uniquely determined.

Arguing as in case 1, one can prove that for the remaining components of $D_\mu^b \setminus D_\lambda^b$, conditions (1)–(5) of Definition 3.1 hold true.

Proposition 7.34 *If the $\overline{v}/v$-ribbon with mark $(m-n)$ meets the m-th column twice and, for some c, $\partial_\mu^D(E) = c \cdot E_{\mathbf{a}} \neq 0$, then:*

(1) *D is uniquely determined.*

(2) *The extremal component of $D_\mu^b \setminus D_\lambda^b$ looks like the one in the picture above, that is, it is an extended 3/2-strip such that the excrescence of the 3/2-strip appears below the special μ-row and is a degenerate strip.*

(3) *The $(m-n)$-th row ends over the extremal component and the rightmost D-box of this row ends over the rightmost D-box of the lower 1-strip of the 2-strip (of the extremal component).*

(4) *The remaining components satisfy conditions (1)-(5) of Definition 3.1.*

It is easy to see that conditions (2) – (5) of this proposition are sufficient for the existence of a subset $D \subset D_\mu$ which gives a non-zero multiplicity in our formula.

Now we must examine the case when the $\overline{v}/v$-ribbon with mark $(m-n)$ does not meet the m-th column. Observe that this is a situation similar to case 1 when w_μ and w_λ are of type 1. The main difference is that the length of the $(m-n)$-th row of D^t_μ is smaller than $l(D^b_\mu)+1$.

Proposition 7.35 *If $\partial^D_\mu(E) \neq 0$, then the set of z-boxes $D_Z \subset D^b_\mu$ is the maximal deformation of D^b_λ in D^b_μ.*

Proof. The v-ribbon with mark $(m-n)$ does not meet the m-th column and all arguments of the proof of Lemma 7.2 can be repeated. □

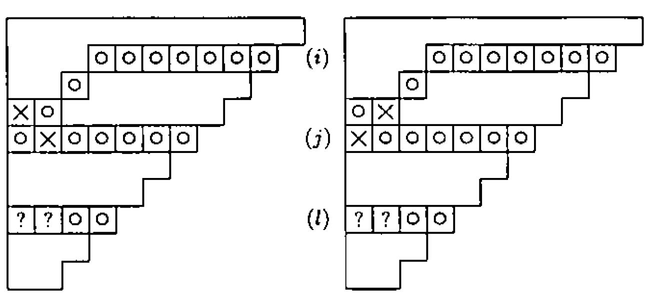
Proposition 7.36 *If $\partial^D_\mu(E) \neq 0$, then $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) = 2$.*

Proof. The inequality $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) > 2$ means that there exist at least three rows with $\sim D$-boxes in $D^b_\mu \setminus D^b_\lambda$. We know from Proposition 7.35 that, to avoid the vanishing, we must perform the maximal deformation of D^b_λ in D^b_μ. Moreover, the following result holds:

Proposition 7.37 *If $\partial^D_\mu(E) \neq 0$, then the z-boxes with the same mark can appear in at most two successive rows.*

Proof. Analogous to that of Proposition 7.5. □

Therefore, after the maximal deformation, we have the following configuration of boxes:



It follows from the properties of ribbons that the v-boxes with mark $(m-n)$ can appear only in the i-th row. But then the j-th and l-th rows consist of $\sim D$-boxes (no v-boxes can appear there) and we get the vanishing. $\square$

From Proposition 7.37, we infer:

Proposition 7.38 *If $\partial_\mu^D(E) \neq 0$, then every component of $D_\mu^b \backslash D_\lambda^b$ appearing above the special one is a $3/2$-strip satisfying the conditions of Definition 3.1.*

Proof. Analogous to that of Proposition 7.5 (use Propositions 7.35 and 7.37). $\square$

It follows that $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 2$. Suppose that $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) < 2$. In this case, there is no push down operation in the process of the maximal deformation. Since w_μ is of type 2 and w_λ is of type 1, the $(m-n)$-th row of D_μ^t is exceptional. But the partitions μ^b and λ^b have equal lengths and there is no component of D_μ^b over which this row ends. Hence it is impossible to obtain $D \subset D_\mu$ satisfying $r_D \in R(w_\lambda)$ if $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 2$. Thus $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) = 2$. $\square$

Proposition 7.38 gives necessary conditions for the non-vanishing. Note that an excrescence can appear also in the extremal component.

Proposition 7.39 *If $\partial_\mu^D(E) \neq 0$ and the v-ribbon with mark $(m-n)$ does not meet the m-th column then D is determined uniquely.*

The proof of this proposition is the same as the one of [P–R2, Theorem 8.1]. See also Lemmas 7.7 – 7.9, Proposition 7.10 and Corollary 7.11. $\square$

It is clear that the unique D satisfying the condition: the v-ribbon with mark $(m-n)$ does not meet the m-th column, can be obtained by performing the operations of Recipe 7.13 to shapes λ, μ for which:

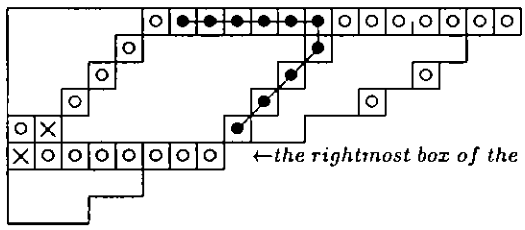
1. $D_\mu^b \supset D_\lambda^b$; the extremal component is special and forms a 2-strip; the remaining components of $D_\mu^b \setminus D_\lambda^b$ are $3/2$-strips.

2. At most one row from D_λ^t ends over a component.

(Compare with (1),(2) in Recipe 7.13.)

Theorem 7.40 *There exists at most one D for which $\partial_\mu^D(E) \neq 0$ (without any assumptions on the v-ribbon with mark $(m-n)$).*

Proof. Suppose that the extremal component of D_λ and D_μ looks like that in the picture before Corollary 7.33. Apply Recipe 7.13. We get the following configuration of boxes:



This configuration gives the vanishing. Hence, it is impossible that the pair D_λ, D_μ satisfies the assumptions of Proposition 7.34 and, applying Recipe 7.13, we can find $D' \neq D$ such that $\partial_\mu^{D'}(E) \neq 0$. □

Definition 7.41 *A shape μ is compatible with λ if*

(1) *We have $D_\mu^b \supset D_\lambda^b$ and every non-special component of $D_\mu^b \setminus D_\lambda^b$ is a 3/2-strip; moreover, $D_\mu^t \setminus D_\lambda^t$ is a 1/2-strip.*

(2) *The λ-part of at most one row ends over a component. If the λ-part of a row ends over a component, then we say that the row and the component are related. A component which is related to some row is called related. Similarly, a row which is related to some component is called related.*

(3) *Each exceptional row is related to a component over which the μ-part of this row ends.*

(4) *If a $(\mu-\lambda)$-box lies over a component, then this component is not related and this box lies over the leftmost box of the component.*

(5) *An excrescence can appear only in a related component under the λ-part of the related row; no box from the μ-part of the related row lies over the excrescence.*

(6) *The extremal component is special and is an extended 3/2-strip such that the excrescence of the 3/2-strip is a degenerate strip. The lower 1-strip of the 2-strip lies under the λ-part of a row.*

It is easy to see that the extremal component has no influence on the multiplicity m_μ.

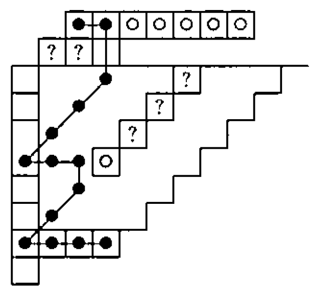
Theorem 7.42 *If w_λ is of type 1 and w_μ is of type 2, then $m_\mu \neq 0$ iff μ is compatible with λ. In case (a) of Definition 7.41, $m_\mu = 2^{m(\lambda,\mu)}$ where $m(\lambda, \mu)$ is the number of non-extremal components which are above the special component and have no $(\mu-\lambda)$-boxes over them.*

Proof. Observe that the only difference between case 1 where w_λ and w_μ are of type 1 and the present situation, is that the length of the $(m-n)$-th row of D^t_μ is smaller then $l(D^b_\mu)+1$ and the extremal component does not need to be a 1-strip. Hence (1) — (5) are necessary – the arguments are the same as in case 1. In the deformed diagram, the $(m-n)$-th row of D^t_λ ends over the leftmost box of the top row of the special component and the $(m-n)$-th row of D^t_μ ends over the special μ-row. We conclude the proof using Proposition 6.8. □

7.4 Case 4 : w_μ and w_λ are of type 2

Lemma 7.43 *If $\partial^D_\mu(E) \neq 0$, then no s_{i_h}-operation can supply v_{m-n-1} with a bar.*

Proof. Suppose that some s_{i_h}-operation supplies v_{m-n-1} with a bar. Then the v-ribbon with mark $(m-n-1)$ meets the m-th column at least twice (in the barred permutation w_λ, v_{m-n-1} has no bar):



We get the vanishing by Proposition 6.8. □

Lemma 7.44 *If $\partial^D_\mu(E) \neq 0$, then the v_{m-n}-ribbon meets the m-th column no more than once.*

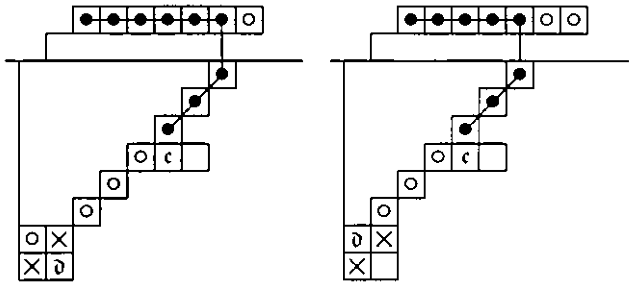
Proof. Analogous to that of Lemma 7.31. □

Now, let us assume that $D \subset D_\mu$ is such that $\partial^D_\mu(E) \neq 0$. Let D_Z be the set of all z-boxes together with the z-part of the v-ribbon with mark $(m-n)$. Let D' and D'' be as in Subsection 7.2, between Lemma 7.18 and Proposition 7.19. We state the following proposition (whose proof is analogous to Proposition 7.19).

Proposition 7.45 *If $D_Z \neq D''$, then $\partial^D_\mu(E) = 0$.*

Proposition 7.46 *If $\partial_\mu^D(E) \neq 0$, then $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

Proof. Suppose that $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \geq 2$:



If $\mathfrak{c}$ is a $\sim D$-box, then we get the vanishing. If $\mathfrak{c}$ is a D-box, then it is a v-box with mark $(m - n - 1)$ but in this case $\sim D$-boxes from the $(m - n - 1)$-th row give the vanishing. □

This present case and case 2 are quite similar; essentially the same arguments show:

Proposition 7.47 *If $\partial_\mu^D(E) \neq 0$, then D_Z is the result of the following operations applied to D'_λ:*

(1) *Apply the exchanging operation as many times as possible and denote the so obtained diagram by D'.*

(2) *Deform D' in the maximal way to obtain D_Z.*

(Compare Proposition 7.21)

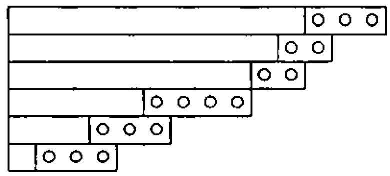
Proposition 7.48 *D_Z-boxes can appear in at most two successive rows.*

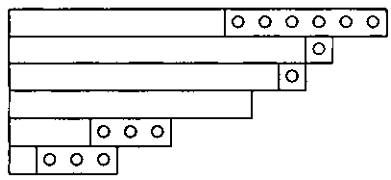
Proof. Analogous to that of Proposition 7.5. □

Proposition 7.49 *(1) An ordinary component of $D_\mu^b \setminus D_\lambda^b$ is a 3/2-strip.*
(2) The special component is a 3/2-strip whose excrescence appears between the special μ-row and the special λ-row and is a degenerate strip.

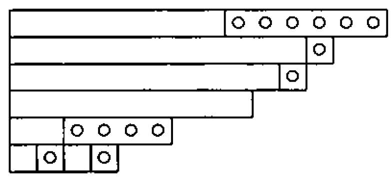
Proof. The proof of (1) is the same as that of [P–R2, Proposition 6.7]. For (2) let us divide the special component into two parts: the "upper part" consisting of boxes in the special λ-row and in rows above it; and the "lower part" consisting of boxes below the special λ-row. Perform the exchanging operations (this is necessary to avoid the vanishing - see Proposition 7.47).

The set of boxes of the special component can break up (or not) into some number of disconnected sets:

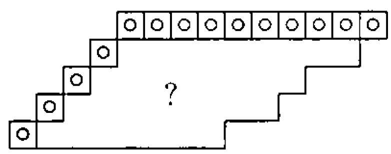




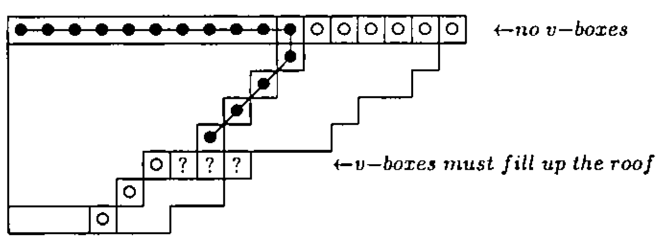
After the maximal deformation we get:



In general, from the lower part of the special component, we get the set of the form:



and the special λ-row ends over its roof. It follows that the tail of the v-ribbon with mark $(m-n)$ must appear in the roof:

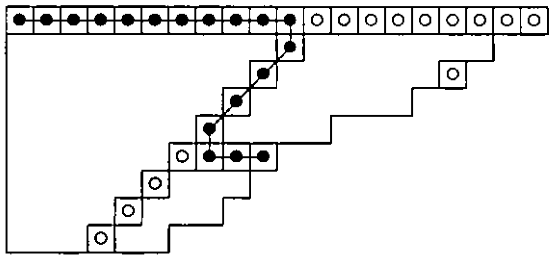


Note the following two facts which are consequences of Proposition 6.8:

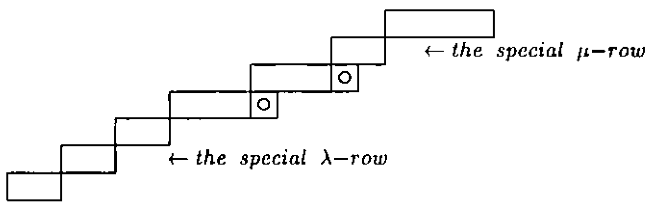
1. No $\sim D$-box can appear in the part of row marked with "?" as well no $\sim D$-box lies under a box marked with "?".

2. No $\sim D$-boxes can appear in the special row and in the rows above it.

Hence, the following picture shows a special (deformed) component without bad boxes.



It follows that the nondeformed special component looks like:



□

Corollary 7.50 *If $\partial_\mu^D(E) \neq 0$ (and $r_D \in R(w_\lambda)$), then $D_\lambda^b \subset D_\mu^b$.*

Proof. This follows from Proposition 7.49 and the properties of z-ribbons. □

Theorem 7.51 *For given shapes λ, μ such that w_λ and w_μ are of type 2, there exists at most one $D =: D^{\lambda,\mu}$ such that $\partial_\mu^D(E) \neq 0$ (and $r_D \in R(w_\lambda)$).*

Proof. It follows from Propositions 7.47 and 7.49 that the positions of z-boxes and boxes of the v-ribbon with mark $(m-n)$ are uniquely determined. Using the arguments as in the proof of [P–R2, Theorem 8.1], one proves that the positions of v-boxes with mark different than $(m-n)$ are also uniquely determined. □

Recipe 7.52 *Let λ, μ be two shapes satisfying the conditions:*

(1) *$D_\mu^b \supset D_\lambda^b$, $D_\mu^b \setminus D_\lambda^b$ is a 3/2-strip and $\tilde{l}(D_\mu) - \tilde{l}(D_\lambda) \leq 1$.*

(2) *At most one row from D_λ^t ends over a component.*

The recipe is:

(i) *Perform the exchanging operation as many times as possible; call the so obtained set D'.*

(ii) *Perform the maximal deformation of D' in D^b_μ but do not change the special λ-row; denote by D'' the result of this deformation.*

(iii) *Shift the bottom part of the diagram $D^b_\mu \supset D''$. For every component, choose a row of D^t_μ which ends over the component. Subtract the segment of the row which ends over the roof and push it down to the roof.*

(iv) *Repeat (iii) with the special λ-row and the bottom part of the special component in D^b_μ.*

Lemma 7.53 *1. If $\partial^D_\mu(E) \neq 0$, then $D^t_\mu \setminus D^t_\lambda$ is a 1/2-strip.*

2. No $(\mu - \lambda)$-box lies over the staircase of a related component.

Proof. The assertions follow from Corollary 6.6, Lemma 6.7 and Proposition 6.8. □

Definition 7.54 *Let w_λ and w_μ be of type 2. Then μ is compatible with λ if:*

(1) *$D^b_\mu \supset D^b_\lambda$ and $D^b_\mu \setminus D^b_\lambda$ is a 3/2-strip; the special component is a 3/2-strip whose excrescence appear below the special μ-row and is a degenerate strip; moreover, $D^t_\mu \setminus D^t_\lambda$ is a 1/2-strip.*

(2) *The λ-part of at most one row ends over a component. The special component is not related.*

(3) *Each exceptional row is related to a component over which the μ-part of this row ends.*

(4) *If a $(\mu - \lambda)$-box lies over a component, then this component is not related and this box lies over the leftmost box of the component.*

(5) *An excrescence in a non-special component can appear only in a related component under the λ-part of the related row; no box from the μ-part of the related row lies over the excrescence.*

Proof. Observe that the only difference between this case and case 2 is the condition about the absence of bad boxes in a deformed component. □

Theorem 7.55 *One has $\partial_\mu^{D^{\mu,\lambda}}(E) \neq 0$ if and only if μ is compatible with λ. In this case, the multiplicity m_μ is equal to the number of non-related components above the special component, with no $(\mu - \lambda)$-boxes over them.*

End of the proof
Observe that the differences between results for cases 1–4 occur only for the special component and components lying below the special one. The equivalence of Definition 3.1 and the definitions of compatibility for cases 1–4 is obvious.

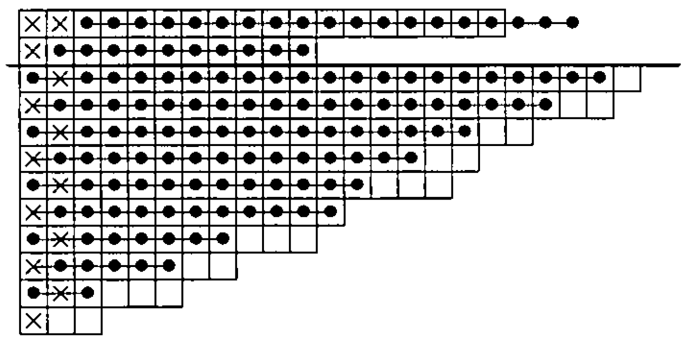
Remark 7.56 In [S], the author, using the linear algebra methods, proves some partial result about the intersection theory of G: the so called "triple intersection formula". His theorem gives necessary (but not sufficient) conditions for a nontrivial intersection of two arbitrary Schubert cycles with the special one (special cycles in [S] are equal to the Chern classes of the universal *quotient* boundle on G). This result, however, gives no information about the multiplicities occuring in the intersection and does not imply a Pieri-type formula.

7.5 Examples

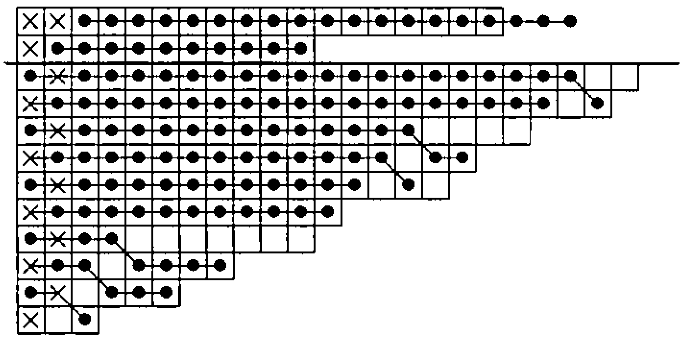
In this subsection we will show the examples of pairs of compatible diagrams D^b_μ and D^b_λ in cases 1 – 4. We also display the resulting deformed diagrams $D \subset D_\mu$ such that $\partial^D_\mu(E) \neq 0$ and $r_D \in R(w_\lambda)$.

(1) Case 1: w_μ and w_λ are of type 1.

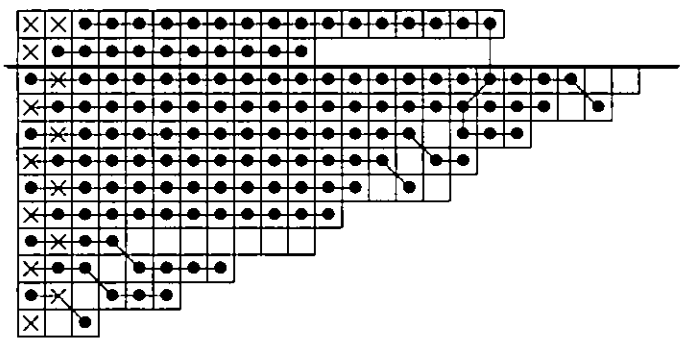
$\lambda = ((21, 11)//(22, 20, 17, 15, 13, 12, 8, 6, 3))$,
$\mu = ((18, 11)//(23, 22, 19, 17, 16, 12, 11, 8, 6, 3))$,



We perform the maximal deformation of D^b_λ in D^b_μ:

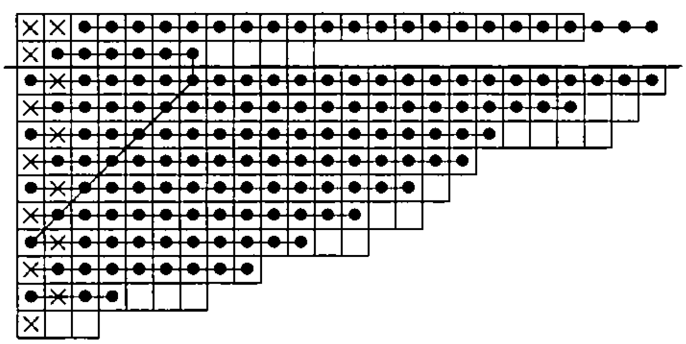


The final deformation of the v-ribbons looks like:

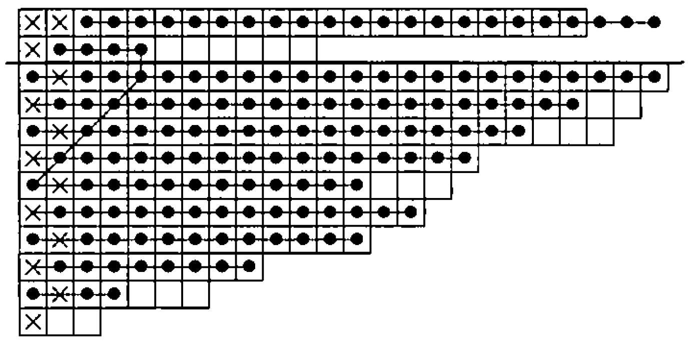


(2) Case 2: w_μ is of type 1 and w_λ is of type 2.

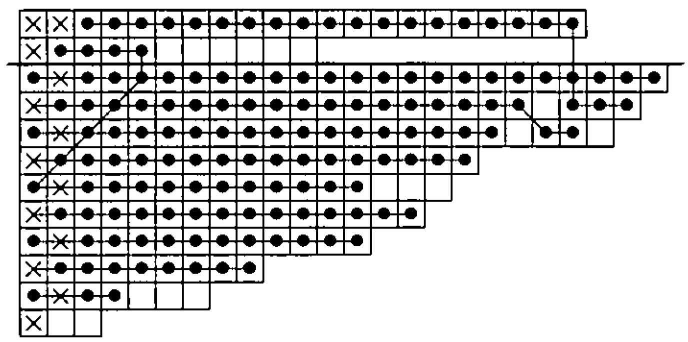
$\lambda = ((24,7)//(24,21,18,17,15,13,11,9,4))$,
$\mu = ((21,11)//(24,23,22,17,16,15,13,9,7,3))$,



We perform the exchanging operation:



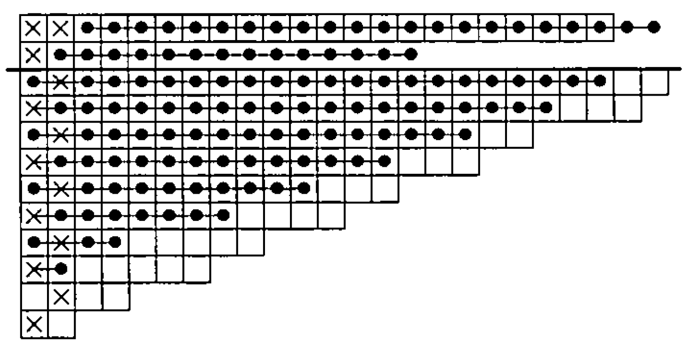
We deform the z-ribbons and the v-ribbon from D^t:



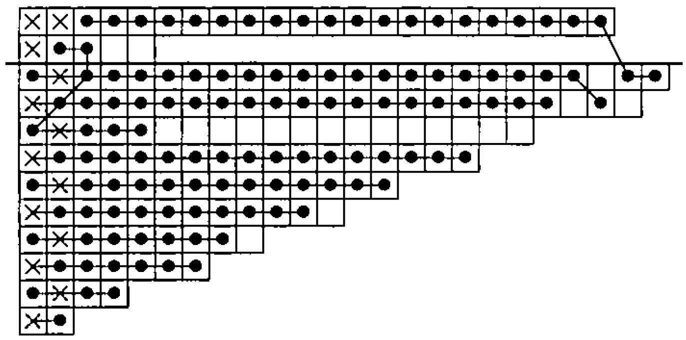
(3) Case 3: w_μ is of type 2 and w_λ is of type 1.

First part of case 3 : $\overline{v}/v$-ribbon meets the m-th column twice.

$\lambda = ((24, 5)//(22, 20, 17, 14, 11, 8, 4, 2))$,
$\mu = ((22, 15)//(24, 23, 19, 17, 14, 13, 10, 8, 4, 2))$,

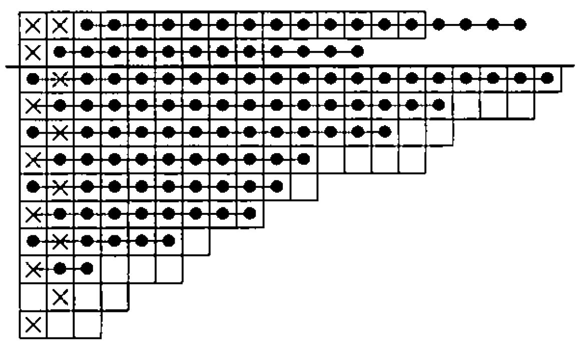


The resulting deformed diagram looks like:

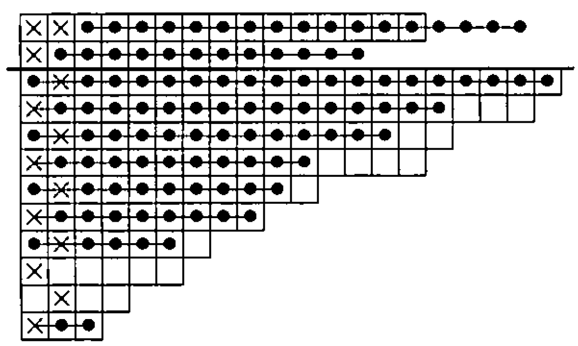


The second part of case 3: v_{m-n} does not receive a bar.

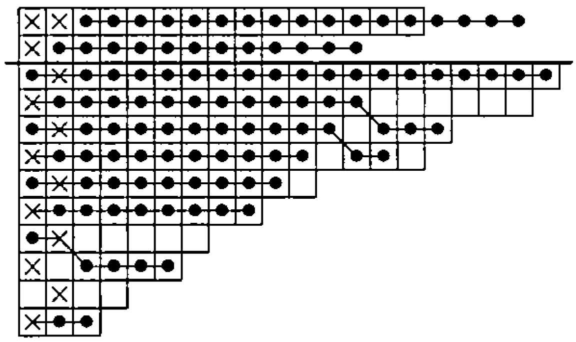
$\lambda = ((19, 13)//(20, 16, 14, 11, 10, 9, 6, 3))$,
$\mu = ((23, 9)//(20, 19, 16, 15, 11, 9, 7, 9, 6, 4, 3))$,



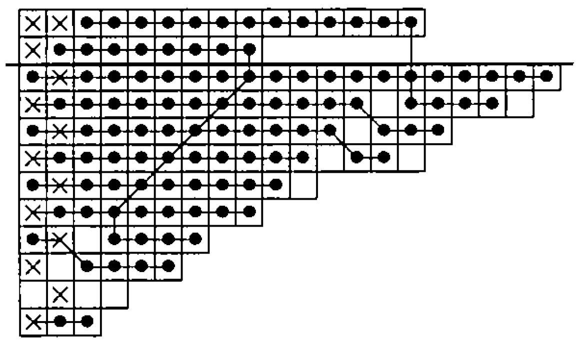
We perform the push down operation:



We break the ribbons:

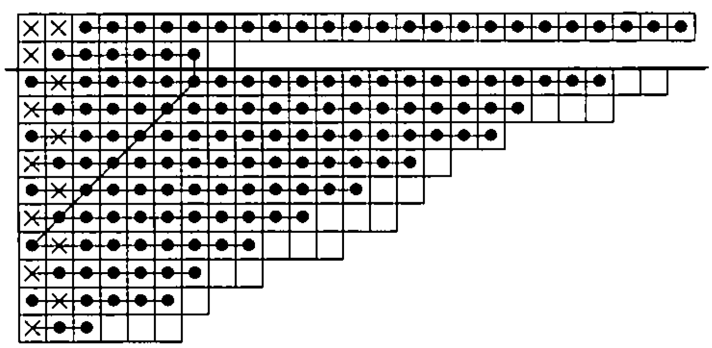


We deform the v-ribbon from D^t:

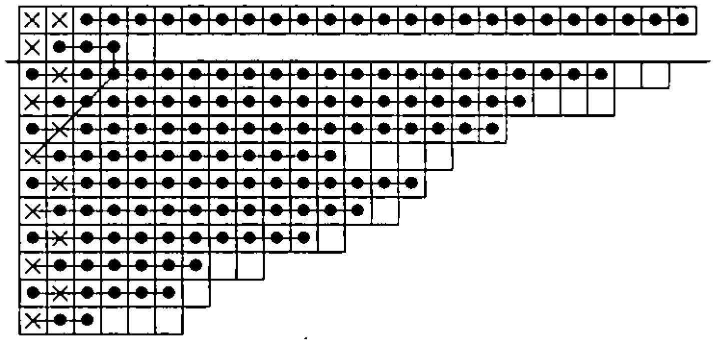


(4) Case 4: w_μ and w_λ are of type 2.

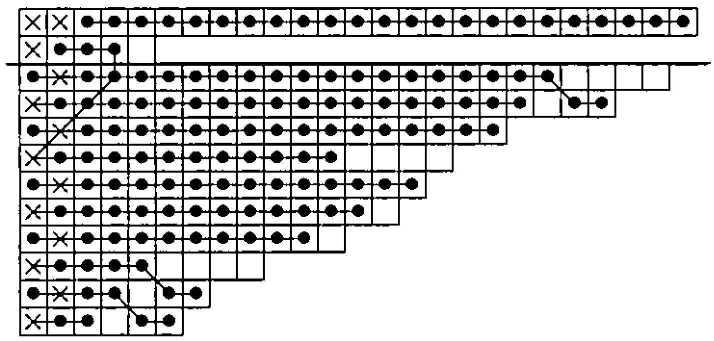
$\lambda = ((25, 7)//(22, 19, 18, 15, 13, 11, 9, 7, 6, 3))$,
$\mu = ((25, 8)//(24, 22, 18, 16, 15, 14, 12, 9, 7, 6))$,



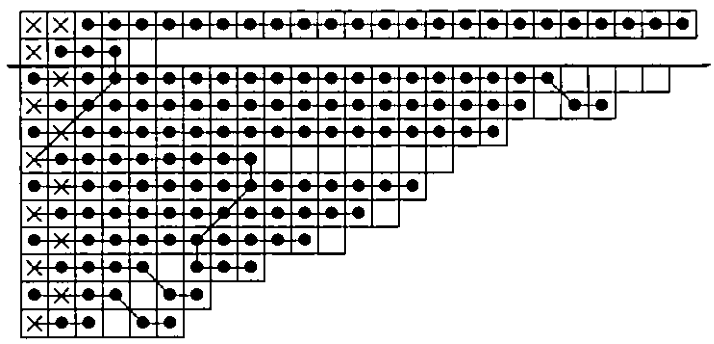
We apply the exchanging operation:



We apply the maximal deformation:



We deform the v_{m-n}-ribbon:



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