Geometric Equimultiplicity

by

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Appendix to the book "EQUIMULTIPLICITY AND BLOWING-UP"
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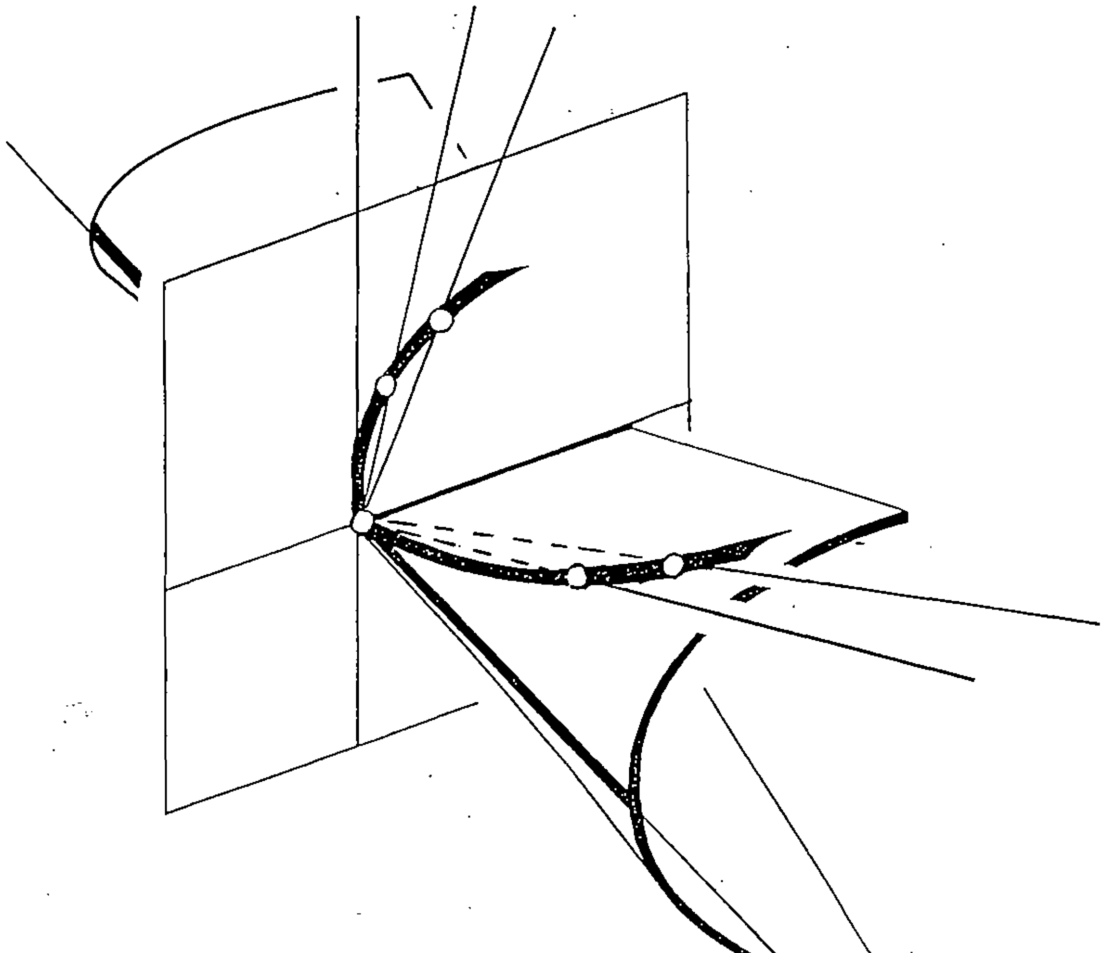
APPENDIX

GEOMETRIC EQUIMULTIPLICITY



CONTENTS

INTRODUCTION

The idea of a complex space emerged slowly over the decades as a natural generalization of the idea of a Riemann surface and its higher dimensional analogues, the complex manifolds. As in the classical theory of holomorphic functions of one variable, complex spaces arise in the attempt to understand holomorphic functions of several variables by constructing their natural home, "das analytische Gebilde", i.e. the maximal natural domain of definition. The nonuniformizable points, nowadays called singularities, caused great conceptual difficulties, so that a satisfactory definition had to wait until the 50's of this century when it was given by Behnke and Stein and, somewhat later in some greater generality, by Cartan and Serre. Subsequently it became clear that if one wants to gain a deeper understanding of complex manifolds, even of curves, complex spaces with nilpotents in their structure sheaf inevitably show up, be it in inductive proofs, or be it in the construction of such important geometric objects as moduli spaces of various, sometimes very classical, structures. This step was taken by Grauert and Grothendieck in the early 60's, who introduced the now generally accepted definition of, possibly nonreduced, complex spaces.

Aside from their intricate and important global properties, complex spaces possess a very rich and interesting local geometry, due to the presence of singularities. The algebraization of this local geometry was initiated by Weierstraß, who formulated his famous Preparation Theorem. Rückert, in a fundamental paper of 1931, was the first to use systematically algebraic tools in the local theory, and the consequent use of local algebra was further systematized in the Cartan Seminer of 1960/61, and Abhyankar's book of 1964 on local analytic geometry. It then became clear that the local geometry of complex spaces and the algebraic structure of the corresponding local rings are completely equivalent. In this way, then, algebraic statements within the category of local analytic algebras (i.e. quotients of convergent power series algebras) have an equivalent geometric interpretation which can be systematically exploited. Conversely, geometric considerations may provide particular insights and suggest natural algebraic statements which possibly would not have shown up easily within a pure algebraic context. It is this interplay between algebra and geometry which makes local analytic algebras a particularly intersting category, and a "testing ground" for conjectures and concepts in local algebra.

This Appendix sets out to give an introduction to Local Complex Analytic Geometry, to give the geometric interpretation of some fundamental algebraic concepts as dimension, system of parameters, multiplicity, and finally to explore to some extent the geometric meaning of the equimultiplicity results of [49], [77], and [78].

I now give a quick overview over the contents and intentions of the three parts;more details are provided in the introductory remarks of the various parts and their paragraphs.

In Part I, my intention was to give a rapid introduction to the local theory of complex spaces, but at the same time to maintain the contradictory principle of giving all main lines of thought, in order not to discourage the nonspecialist by refering constantly to a labyrinthic and sometimes extremely technical literature. The main results are the Equivalence Theorem 3.3.3, which establishes the equivalence of the algebraic and geometric viewpoint; and the Local Representation Theorem 6.3.1.This local description of a complex space as a branched cover, which was, in principle,known to Weierstraß,lies at the heart of algebraization of the analytic theory, expressing the fact that any complex spacegerm gives rise to a "relative algebraic situation" over a smooth germ. This geometric situation is the local analogue of the Noether Normalization and contains the notions of dimension, system of parameters, and multiplicity, in its geometry. Technically, I have tried to emphasize two points. Firstly, I have made constant use of the General Division Theorem of Grauert-Hironaka from the beginning. From my point of view, it is a natural and systematic procedure which classifies many technical points. Moreover, it is basic for Hironaka's resolution of complex space singularities (see III, 1.3.5) and its effective algorithmic character may someday point the way to an explicit resolution procedure. (Presently, at least, it provides an effective algorithm for computing standard bases, and so Hilbert functions and tangent cones, see I, 2.4.4) Secondly, following Grothendieck's treatment in [64], I have postponed the introduction of coherence to the point where it really becomes indispensible; since, in the complex analytic case, coherence is a deep and not at all obvious property, it should be used only for the proof of those results which depend crucially on it (in our case, the property that openness of a finite map at a point implies the map being open near that point). Large parts of the exposition are taken from [28], and I refer to it and [40],[64] for complete details.

In Part II, I expose the geometric theory of local multiplicity as a local mapping degree; for more historical and geometrical background I refer to the introductory remarks to that Part. The main technical concept, introduced in § 1, is that of a compact Stein neighbourhood. This concept allows to relate properties of nearby analytic local rings of a complex space to one unifying algebraic object, the coordinate ring of a compact Stein neighbourhood. This gives a systematic way of deducing local properties of complex spaces from results of local algebra, and vice versa. Here, coherence enters in a fundamental way, and it is via coherence and the Equivalence Theorem I 3.3.3 that local, not only punctual, properties of complex spaces can be deduced by doing local algebra. This technique seems to have originated in [33], and has been exploited by various authors to deduce results in Complex Analytic Geometry from corresponding results in Algebraic Geometry, starting with [4]; see [5], [29], [38], and [63]. Here, I have simplified the treatment by dropping the requirement of semianalyticity for the compact Stein neighbourhoods, thus avoiding the highly nontrivial stratification theory of semianalytic sets.

Part III, finally, deals with the geometric theory of equimultiplicity, and forms the central part of the Appendix. It also gives various instances of the method of compact Stein neighbourhoods. In § 1, we deduce properties of normal flatness in the complex analytic case from the algebraic case; in § 2 we give a geometric proof of the equivalence of the conditions $e(R) = e(R_{\mathfrak{p}})$ and $ht(\mathfrak{p}) = s(\mathfrak{p})$ of Chapter IV, Theorem (20.9)*); and in § 3, finally, we turn this principle around and establish the geometric contents of equimultiplicity via Theorem (20.5) of Chapter IV.**) Further, bearing in mind the title of a well-known paper by Lipman [49] I have made comments on the connections with, and the geometric significance of, the algebraic notions of reduction and integral dependence. The underlying fundamental geometric principle, which unifies equimultiplicity, reduction, and integral dependence, is the notion of transversality (this is a basic principle in the work of Teissier [69]); this becomes particularly clear from the geometric description of multiplicity as the mapping degree of a projection (see the introductory remarks to III, III § 2, and III § 3 below).

On one hand, this Appendix was intended to give an overview of the geometric significance of equimultiplicity and not to be a full detailed treatment. On the other hand, I felt that it would have been of little value just to state the results without providing some insights into the machinery producing them, especially as there seems to be some

*) this is Satz 2 of [77] **) this is basically Theorem 4 of §2 in [49]

interest on the side of algebraists to become more acquainted with complex-analytic methods. In connection with the confinements of space, time, and perseverance of the author, there results that the prsentation oscillates between rigour and loose writing, a dilemma I have been unable to solve. I can only offer my apologies and hope that those who approve of the one and disapprove of the other will appreciate seeing their approvals met instead of complaining about seeing their disapprovals aroused.

Concerning the notation, local rings are usually denoted R etc. instead of $(R,\mathfrak{m})$. The maximal ideal of R etc. is then denoted by $\mathfrak{m}_R$, and its nilradical by $\mathfrak{n}_R$. The notation $\mathfrak{m}_n$, $n \in \mathbb{N}$, refers specially to the maximal ideal of $\Bbbk\{z_1,\ldots,z_n\}$. If $(X,\mathcal{O}_X)$ is a complex space, $\mathfrak{m}_{X,x}$ or $\mathfrak{m}_x$, denotes the maximal ideal of $\mathcal{O}_{X,x}$, and $\mathcal{N}_{X,x}$, or $\mathcal{N}_x$, its nilradical. References within this Appendix usually are by full address; II 5.2.1 for instance refers to 5.2.1 of Part II. When they are made within one Part, the corresponding numbers I, II, III are suppressed. Numbers in brackets refer to formulas; I (2.3.1) for instance means the formula numbered (2.3.1) in Part I.

I wish to take the opportunity to express my profound indebtness to Professor Manfred Herrmann for the suggestion to include this work as a part of the book. I thank him, and O. Villamayor, for the interest they took in this work and for numerous hours of discussion, which saved me from error more than once. It goes without saying that all the remaining errors and misconceptions are entirely within the author's responsibility. I further express may gratitude to the Max-Planck-Institut für Mathematik and its director, Professor F. Hirzebruch, to be able to work in a stimulating atmosphere, and for financial support. Finally, I thank Mrs. Pearce for her skilful typing and for the patience with which she bore many hours of extra work and the ever-lasting threat of possible changes.

I. LOCAL COMPLEX ANALYTIC GEOMETRY

In this chapter I give an overview over the basic facts of the local theory of complex analytic spaces. The main references are the Cartan seminar [64], especially the exposés 9 - 11, 13 - 14 of Grothendieck and 18 - 21 of Houzel, and the excellent book [28]. For further information, one can also consult the book [40].

The main results are the Equivalence Theorem 3.3.3, which establishes the equivalence of the category of local analytic algebras and the category of complex space germs, the Integrality Theorem 4.4.1., which characterizes finiteness geometrically and algebraically, and, finally, the Local Representation Theorem 6.3.1., which is a local analogue of Noether normalization. It allows to represent a complex space germ locally as a branched cover of an affine space, and this gives the geometric interpretation of the dimension and of a system of parameters of the corresponding local ring. Moreover, this setup will be fundamental for the description of the multiplicity of this local ring in the next chapter.

§ 1. Local analytic algebras

In this section, I describe the category $\underline{\underline{la}}$ of local analytic algebras, which will be basic to all what follows. Its objects, the local analytic algebras, are the algebraic counterparts to the geometric objects formed by the germs of analytic spaces, or singularities, which will be introduced in § 3.

In what follows, $\mathbb{k}$ denotes any complete valued field. Proofs are mostly sketched, or omitted. For details I refer to [26], Kapitel 1, § 0 - 1; [40], and § 21.

1.1. Formal power series

I assume known the notion of a formal power series in n indeterminates $X_1,\dots,X_n$. They form a ring denoted $\mathbb{k}[[X_1,\dots,X_n]]$, or $\mathbb{k}[[X]]$ if n is understood. I use the multiindex notation; a monomial $X_1^{A^1}\dots X_n^{A^n}$ will be denoted X^A with $A = (A^1,\dots,A^n) \in \mathbb{N}^n$. Let $M(n) \subseteq \mathbb{k}[[X]]$ be the space of monomials; then

(1.1.1) $$\log : \mathcal{M}(n) \longrightarrow \mathbb{N}^n \qquad X^A \longrightarrow A$$

induces an isomorphism $(\mathcal{M}(n), \cdot, X^0) \longrightarrow (\mathbb{N}^n, +, 0)$ of monoids which I will freely use; in this way, one may view monomials as lattice points in $\mathbb{R}^n$, and divisibility properties of monomials turn into combinatorial properties of lattice points. This interplay between algebra and combinatorics will be quite crucial in establishing in § 2 fundamental properties of power series rings such as the Division Theorem, the noetherian property, or the Krull Intersection Theorem.

In the multiindex notation, $|A| := \sum_{j=1}^{n} A^j$, so that $|X^A| := |A|$ is the usual degree. Formal power series will be written as $f = \sum_{M\in\mathcal{M}(n)} f_M M = \sum_{A\in\mathbb{N}^n} f_A X^A$, with $f_M, f_A \in \mathbb{k}$. We define

(1.1.2) $$\operatorname{supp}(f) := \left\{M \in \mathcal{M}(n) \mid f_M \neq 0\right\},$$

the <u>support of</u> f, and

(1.1.3) $$\nu(f) := \min\left\{|M| \;\middle|\; M \in \operatorname{supp}(f)\right\}$$

the <u>order</u> or <u>subdegree</u> of f. We will make use of the following properties of $\mathbb{k}[[X_1,\dots,X_n]]$:

<u>Proposition 1.1.1</u>.

(i) $\mathbb{k}[[X_1,\dots,X_n]]$ <u>is a commutative ring with unit, and in fact a</u> $\mathbb{k}$-<u>algebra</u>.

(ii) $f \in \mathbb{k}[[X]]$ <u>is a unit if and only if</u> $f_0 \neq 0$.

(iii) $\mathbb{k}[[X_1,\dots,X_n]]$ <u>is a local ring with maximal ideal</u> $\hat{\mathfrak{m}}_n := \{f \mid \nu(f) \geq 1\} = (X_1,\dots,X_n)\cdot \mathbb{k}[[X_1,\dots,X_n]]$.

(iv) $\forall k \in \mathbb{N} : \hat{\mathfrak{m}}_n^k = \{f \mid \nu(f) \geq k\}$; <u>especially</u>

$$\bigcap_{k=0}^{\infty} \hat{\mathfrak{m}}_n^k = \{0\} \quad .$$

These properties are elementary. (i) is clear. For (ii), note that, when $f := 1 - u$ with $\nu(u) \geq 1$, $\sum_{j=0}^{\infty} u^j$ exists in $\mathbb{k}[[X]]$. Finally, if f has $\nu(f) \geq k$, it can be written as

$$(1.1.4) \qquad f = \sum_{\substack{M \in \mathcal{M}(n) \\ |M|=k}} M \cdot f^{(M)} \quad , \quad f^{(M)} \in \mathbb{k}[[X]] \quad ,$$

with the $\mathrm{supp}(M \cdot f^{(M)})$ pairwise disjoint (this will be systematized later on in the Division Algorithm 2.3.1.). This shows that $\{f \mid \nu(f) \geq k\} \subseteq (X_1, \ldots, X_n)^k$, which implies, together with (ii), the statements (iii) and (iv).

1.2. Convergent power series

Let $\mathbb{A}^n$ be the affine n-space over $\mathbb{k}$. A <u>polyradius</u> ρ is an element $\rho = (\rho^1, \ldots, \rho^n) \in (\mathbb{R}_{>0})^n$, and if $z_0 = (z_0^1, \ldots, z_0^n)$ is a point in $\mathbb{A}^n$, the set

$$(1.2.1) \qquad P(z_0;\rho) := \left\{ z \in \mathbb{A}^n \mid \forall\, 1 \leq i \leq n : |z^i - z_0^i| < \rho^i \right\}$$

is called <u>the polycylinder around</u> z_0 <u>of (poly-)radius</u> ρ.

<u>Proposition 1.2.1</u>. <u>For a formal power series</u> $f \in \mathbb{k}[[X]]$, <u>the following properties are equivalent</u>:

(i) $\exists$ <u>a polyradius</u> $\rho \in (\mathbb{R}_{>0})^n$ <u>such that the family</u> $(f_A z^A)_{A \in \mathbb{N}^n}$ <u>is summable in</u> $\mathbb{k}$ <u>for</u> $z \in P(0;\rho)$.

(ii) $\exists$ <u>a polyradius</u> $\rho \in (\mathbb{R}_{>0})^n$ <u>such that</u>

$$(1.2.2) \qquad \|f\|_\rho := \sum_{A \in \mathbb{N}^n} |f_A| \rho^A < \infty$$

(iii) $\exists$ <u>constants</u> $C, N \in \mathbb{R}_{>0}$ <u>such that</u>

$$(1.2.3) \qquad |f_A| \leq C \cdot N^{|A|}$$

<u>for all</u> $A \in \mathbb{N}^n$.

Moreover, in these cases there is the "Cauchy estimate"

(1.2.4) $$|f_A| \leq \|f\|_\rho \cdot \rho^{-A} \quad \text{for all} \quad A \in \mathbb{N}^n \quad .$$

Definition 1.2.2. A formal power series $f \in \mathbb{k}[[X]]$ satisfying one of the properties of Proposition 1.2.1. is called a convergent power series. The convergent power series form a commutative, unitary ring and a $\mathbb{k}$-algebra, denoted $\mathbb{k}\{X_1,\ldots,X_n\}$, or $\mathbb{k}\{X\}$ for short.

The "norm" $\| \ \|_\rho$ defined in (1.2.2) is the main technical tool in manipulating convergent power series. Introduce the following subalgebras, for $\rho \in (\mathbb{R}_{>0})^n$, of $\mathbb{k}\{X\}$:

$$\mathbb{k}\{X\}_\rho := \{f \in \mathbb{k}[[X]] \ \Big| \ \|f\|_\rho < \infty\}$$

That these are in fact subalgebras follows from

Proposition 1.2.4. $\mathbb{k}\{X\}_\rho$ is a $\mathbb{k}$-Banach-algebra with norm $\| \ \|_\rho$, and has no zerodivisors.

The proof uses the Cauchy estimate (1.2.4).

We now find the units of $\mathbb{k}\{X\}$:

Lemma 1.2.5. For $f \in \mathbb{k}\{X\}$, $\lim_{\rho \to 0} \|f\|_\rho = |f_0|$.

Proof. Write, as in (1.1.4), $f = f_0 + \sum_{j=1}^{n} X_j f_j$ with the $\mathrm{supp}(X_j f_j)$ disjoint, then $\|f\|_\rho = |f_0| + \sum_{j=1}^{n} \rho^j \|f_j\|_\rho$, whence the claim.

Hence, if $f = 1 - u$ with $\nu(u) \geq 1$, $\|u\|_\rho < 1$ for suitable ρ, and so $\sum_{j=0}^{\infty} u^j$ in fact exists not only in $\mathbb{k}[[X]]$ but in $\mathbb{k}\{X\}$ because of Proposition 1.2.4. This proves

Proposition 1.2.6. $f \in \mathbb{k}\{X\}$ is a unit if and only if $f_0 \neq 0$.

Corollary 1.2.7. $\mathbb{k}\{X\}$ is a local ring with maximal ideal $\mathfrak{m}_n = (X_1,\ldots,X_n) \cdot \mathbb{k}\{X\}$.

Proof. $\mathbb{k}\{X\}$ is local by Proposition 1.2.6, with maximal ideal $m_n := \hat{m}_n \cap \mathbb{k}\{X\}$. By the proof of Lemma 1.2.5 we may write $f \in m_n$ as $f = \sum_{j=1}^{n} f_j X_j$ with $\|f\|_\rho = \sum_{j=1}^{n} \rho^j \|f_j\|_\rho$ which shows the f_j are in $\mathbb{k}\{X\}$.

Finally, reasonings analoguous to those above show the following lemma.

Lemma 1.2.8. $m_n^k = \{f \in \mathbb{k}\{X\} \mid \nu(f) \geq k\} = (X_1,\dots,X_n)^k \cdot \mathbb{k}\{X\}$.

Corollary 1.2.9. $\bigcap_{k=0}^{\infty} m_n^k = \{0\}$.

1.3. Local analytic algebras

We are now in a position to describe the category la/$\mathbb{k}$ of local analytic $\mathbb{k}$-algebras. The proofs are sketched, for more details see [26] or [40]; they are more or less straigthforward with the notations and results of 1.1. and 1.2.

The following definition makes sense because of Corollary 1.2.7.

Definition 1.3.1. Let R be a $\mathbb{k}$-algebra. R is called a local analytic $\mathbb{k}$-algebra if and only if R is isomorphic to a quotient algebra $\mathbb{k}\{X_1,\dots,X_n\}/I$, where $I \subseteq \mathbb{k}\{X_1,\dots,X_n\}$ is a finitely generated ideal.

The assumption on I being finitely generated is in fact superfluous due to the following famous theorem.

Theorem 1.3.2 (Rückert Basissatz). A local analytic $\mathbb{k}$-algebra is noetherian.

This is a nontrivial result. I will give a proof in 2.4. which makes it clear that this property comes from a combinatorial property of the monomials which puts the noetherianness of $\mathbb{k}[X]$, $\mathbb{k}[[X]]$ and $\mathbb{k}\{X\}$ on an equal footing. ("Dickson's Lemma"; see Proposition 2.2.1).

Here, we assume Theorem 1.3.2.

The local analytic $\mathbb{k}$-algebras with the local $\mathbb{k}$-algebra homomorphims form a category which I will call la/$\mathbb{k}$. The following remark is sometimes useful:

Remark 1.3.3 (Serre). Any $\mathbb{k}$-algebra homomorphism of local $\mathbb{k}$-algebras is local.

The proof is simple and left to the reader.

The following theorem is the main result of this section; it characterizes the convergent power series in $\underline{\underline{la/\mathbb{k}}}$.

Theorem 1.3.4. The algebras $\mathbb{k}\{X_1,\ldots,X_n\}$ are free objects in $\underline{\underline{la/\mathbb{k}}}$.

In other words, given a local analytic $\mathbb{k}$-algebra R and n elements $f_1,\ldots,f_n \in m_R$, there is a unique $\mathbb{k}$-algebra homomorphism $\varphi : \mathbb{k}\{X_1,\ldots,X_n\} \longrightarrow R$ with $\varphi(X_j) = f_j$ for $j = 1,\ldots,n$.
This property will be an essential step in the proof of the Equivalence Theorem 3.3.3; see Proposition 3.3.1.

Sketch of proof of 1.3.4.

For existence we may assume $R = \mathbb{k}\{U_1,\ldots,U_m\}$ is a convergent power series ring. Let $f_1,\ldots,f_n \in m_R$ be given. Write $g \in \mathbb{k}\{X_1,\ldots,X_n\}$ as

$$g = \sum_{k=0}^{\infty} g_k \qquad , \tag{1.3.1}$$

where the g_k are homogeneous polynomials of degree k. Then $g_k(f_1,\ldots,f_n)$ is a formal power series with $\nu(g_k(f_1,\ldots,f_n)) \geq k$, and so $g(f_1,\ldots,f_n) := \sum_{k=0}^{\infty} g_k(f_1,\ldots,f_n)$ is a well-defined formal power series. If then $\sigma \in (\mathbb{R}_{>0})^n$ is such that $\|g\|_\sigma < \infty$, there is a $\rho \in (\mathbb{R}_{>0})^m$ with $\|g(f_1,\ldots,f_n)\|_\rho \leq \|g\|_\sigma$; this follows from Lemma 1.2.5. So $g(f_1,\ldots,f_n) \in \mathbb{k}\{U\}_\rho \subseteq \mathbb{k}\{U\}$, and we put $\varphi(g) := g(f_1,\ldots,f_n)$.

For uniqueness assume $\varphi,\psi : \mathbb{k}\{X\} \longrightarrow R$ are such that $\varphi(X_j) = \psi(X_j)$, $1 \leq j \leq n$. Then, with the notation (1.3.1), $(\varphi-\psi)(g) = (\varphi-\psi)\left(\sum_{k=p}^{\infty} g_k\right)$ for all $g \in \mathbb{k}\{X\}$ and $p \in \mathbb{N}$. By Lemma 1.2.8., $\sum_{k=p}^{\infty} g_k \in m_n^p$, so $(\varphi-\psi)(g) \in \bigcap_{p=0}^{\infty} m_R^p$, but $\bigcap_{p=0}^{\infty} m_R^p = \{0\}$ because of Theorem 1.3.2. and the Krull Intersection Theorem (see Theorem 2.4.5, or [1], 10.19).

§ 2. Local Weierstrass Theory I: The Division Theorem.

The classical Weierstrass Preparation and Division Theorem lie at the foundation of local analytic geometry and are the most basic and important results of the theory. In their classical appearance, their use in proofs requires always induction on the dimension, which makes sometimes these proofs appear not very transparent. A more natural statement of the Division Theorem has been found independently by Grauert [23] and Hironaka [35], the main point being to divide a power series not by a single other one,but by several others at the same time. This is also related to the construction of standard bases, i.e. computing equations of tangent cones (for which by now an effective algorithm exists), and seems also to be of crucial importance in Hironaka's desingularization theory, since it allows to put generators of an ideal of power series into a canonical form. I will sketch a proof here which I think is the most simple one and clearly exhibits that it is based on a manifest division algorithm suggested by the usual euclidean algorithm for polynomials in one variable, the sole difference being that one divides with respect to ascending monomial degree instead of descending degree. See also [8],[18]-[21], and [62] .

In this section, $\mathbb{k}\langle X \rangle$ will stand either for $\mathbb{k}[[X]]$ or $\mathbb{k}\{X\}$.

2.1. Ordering the monomials.

Usually, in order to prove noetherianness of power series rings, or the Weierstrass theorems, one uses the valuation on power series given by the subdegree $\nu \in \mathbb{N}$ (1.1.3). The crucial idea of getting a refined division theorem is to manipulate power series by using the finer valuation given by the monomial degree $\log(M) = A \in \mathbb{N}^n$ for $M = X^A$. For this, one has to choose an ordering on the monomials, or, equivalently (because of (1.1.1)) on the monoid $\mathbb{N}^n$. The idea of putting an order on the monomials appears for the first time in a famous paper of Macaulay ([52], p. 533). We require that this order is compatible with the monoid structure. Nevertheless, there are quite a lot of orders fulfilling these requirements; they have been classified by Robbiano [58] and, in fact, there are infinitely many. We will temporarily work with the following one.

<u>Definition 2.1.1.</u> <u>The lexicographic degree order on</u> $M(n)$ <u>is defined as follows:</u>

(2.1.0) $X^A < X^B$ if and only if
either $|A| < |B|$,
or $|A| = |B|$, and the last nonzero coordinate of
of $A - B$ is negative.

It has the properties

(2.1.1) (i) $1 < M$ for all $M \in (n)$;

(ii) $M < M' \Rightarrow MN < M'N$ for all N ;

(iii) $<$ is a well-ordering.

Definition 2.1.2. For $f \in \mathbb{k}[[X]]$,

$$LM(f) := \min(\mathrm{supp}(f)) \in \mathcal{M}(n) \cup \{\infty\}$$

is called the leitmonomial of f , with the convention $LM(0) = \infty$.

Recall that $f \in \mathbb{k}[[X]]$ has a unique decomposition $f = \sum_{k=\nu(f)}^{\infty} f_k$, with $f_k \in \mathbb{k}[[X]]$ homogeneous of degree k ; $f_{\nu(f)} =: L(f) =: in(f)$ is called the leitform or initial form of f . The following properties are immediate from the definitions:

(2.1.2) (i) $LM(f) = LM(L(f))$, and so $|LM(f)| = \nu(f)$;

(ii) $LM(f+g) \geq \mathrm{Min}(LM(f), LM(g))$, with equality holding when $LM(f) \neq LM(g)$;

(iii) $LM(f \cdot g) = LM(f) \cdot LM(g)$.

2.2. Monomial ideals and leitideals

A monomial ideal $I \subseteq \mathbb{k}\langle X \rangle$ is defined to be an ideal generated by monomials. The following lies at the heart of the noetherian property of $\mathbb{k}[X]$, $\mathbb{k}[[X]]$, and $\mathbb{k}\{X\}$:

Proposition 2.2.1 ("Dickson's Lemma"). A monomial ideal is finitely generated. A canonical basis consists of those monomials which are minimal with respect to the divisibility relation.

For this, introduce the "stairs of I", $E(I)$, for a monomial ideal I:

$$E(I) := \left\{ A \in \mathbb{N}^n \;\middle|\; X^A \in I \right\} \quad \text{(see Fig. 1)} \quad .$$

$E(I)$ is translation invariant: $E(I) + \mathbb{N}^n \subseteq E(I)$. In 1913, Dickson studied numbers with only finitely many given prime factors and proved ([11]):

<u>2.2.2.</u> $E \subseteq \mathbb{N}^n$ <u>is translation invariant if and only if</u> E <u>can be written as</u>

$$E = \bigcup_{j=1}^{k} \left(A_k + \mathbb{N}^n \right)$$

<u>for some</u> $A_j \in \mathbb{N}^n$, $j = 1,\ldots,k$. <u>These</u> A_j <u>are unique up to permutation when they are taken as the minimal elements of</u> E <u>with respect to the partial orders</u> $A < B :\Longleftrightarrow \forall 1 \leq j \leq n : A^j \leq B^j$ <u>on</u> $\mathbb{N}^n$.

Looking at Figure 1, this is intuitively clear, since when approaching the coordinate hyperplanes the steps of the stairs decrease by integral amounts in the coordinate directions, which can happen only finitely many times. The precise proof is left to the reader. This result proves Proposition 2.2.1.

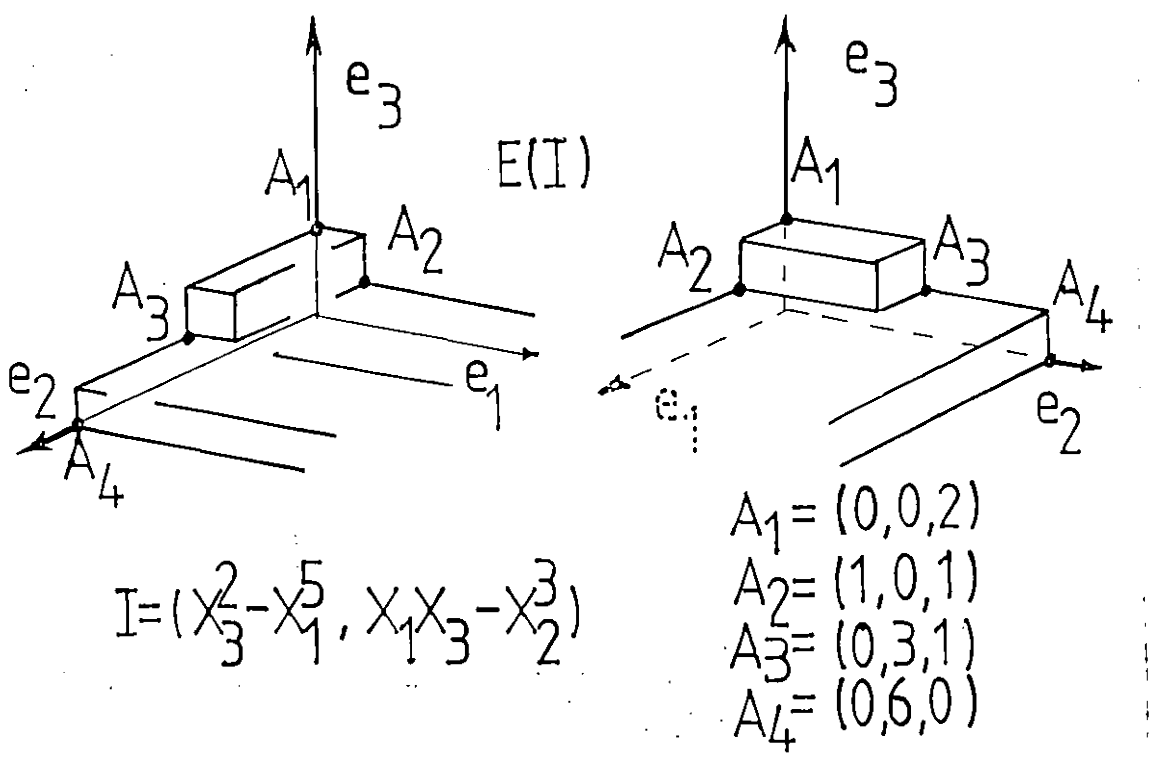


Fig. 1

If I is monomial, $f \in \mathbb{k}\langle X \rangle$ belongs to I if and only if all $M \in \mathrm{supp}(f)$ belong to I; this is analogous to the fact that a polynomial belongs to a homogeneous ideal if and only if all its homogeneous components belong to it. The crucial property of monomial ideals is now that membership of a monomial is effectively decidable if generators are known, since a monomial belongs to it if and only if it is divisible by the generators. But testing the divisibility of monomials is a simple effective operation; this operation will be put to work in the Division Algorithm 2.3.1. below. One therefore associates to any ideal I a monomial ideal $LM(I)$:

Definition 2.2.3. Let $I \subseteq \mathbb{k}\langle X \rangle$ be an ideal. The monomial ideal

$$LM(I) := \text{ideal generated by the } LM(f) \text{ for } f \in I$$

is called the leitideal of I.

$LM(I)$ reflects many properties of I. For instance, a famous result of [52] is that, if I is homogeneous, the Hilbert function $H(I,t)$ of I equals $H(M(I),t)$, and we will see in Section 2.4 that a base of an ideal I whose leitmonomials generate $M(I)$ has special pleasant properties and allows to deduce in an elegant way various facts about ideals in the rings $\mathbb{k}\langle X \rangle$; see 2.4.3, 2.4.4. , and 2.5.2.

2.3. The Division Theorem

In order to give some motivation for the Division Theorem, consider the problem of finding a finite basis for an ideal I. The idea of how to obtain a finite basis is as follows: By Dickson's Lemma there are finitely many $f^1,\dots,f^k \in I$ such that the $LM(f^1),\dots,LM(f^k)$ generate $LM(I)$. Given $f \in I$, we then may write

$$(2.3.1) \qquad LM(f) = g_1^{(0)} LM(f^1) + \dots + g_k^{(0)} LM(f^k)$$

for some $g_1^{(0)},\dots,g_k^{(0)}$; note this step is constructive. We regard this as the 0-th approximation to a wanted equation

$$(2.3.2) \qquad f = g_1 f^1 + \dots + g_k f^k \ .$$

For the first approximation, we form

(2.3.3) $$f^{(1)} := f - \left(g_1^{(0)} f^1 + \dots + g_k^{(0)} f^k\right) .$$

and iterate the step (2.3.1) with f replaced by $f^{(1)}$. Continuing this way, we get formal solutions $g_j = \sum_{p=0}^{\infty} g_j^{(p)}$ to (2.3.2)(which actually converge). This process is constructive when the f^j are given, and so it is feasible to call it an algorithm. The development of this idea leads to the Division Algorithm 2.3.1, which technically proceeds a little differently. Of course, this is only one aspect of the Division Algorithm, and its full power can only be seen from the consequences to which it will lead.

We begin with elements $f^1,\dots,f^k \in \mathbb{k}\langle X\rangle$. Let $LM(f^j) = X^{A_j}$, $j = 1,\dots,k$, and fix the ordering $(A_1, A_2,\dots,A_k)$ of the A_j.

Definition 2.3.1. Let $f^1,\dots,f^h \in \mathbb{k}\langle X\rangle$. The Division Algorithm with respect to $(f^1,\dots,f^k)$ is defined by the following recursion scheme:

Start: For $f \in \mathbb{k}\langle X\rangle$ put

(2.3.4) (i) $\forall 1 \le j \le k : g_j^{(-1)} := 0 , \quad h^{(-1)} := 0 ;$

(ii) $f^{(0)} := f .$

Recursion: Let $g_1^{(0-1)},\dots,g_k^{(p-1)}, h^{(p-1)}, f^{(p)}$ be defined for $0 \le p \le q$. Then put

(2.3.5) $$f^{(q)} =: g_1^{(q)} \cdot X^{A_1} + \dots + g_k^{(q)} \cdot X^{A_k} + h^{(q)} ,$$

where the $g_j^{(q)}$, $j = 1,\dots,k$ and $h^{(q)}$ are defined uniquely by the requirements

(2.3.6) (i) $\mathrm{supp}(g_j^{(q)} \cdot X^{A_j})$ and $\mathrm{supp}(h^{(q)})$ are pairwise disjoint for $1 \le j \le n$;

(ii) if $X^B \in \mathrm{supp}(g_j^{(q)} X^{A_j})$, B is in no $A_i + \mathbb{N}^n$ where A_i precedes A_j in the given order. (In other words, one first collects all $M \in \mathrm{supp}(f^{(q)})$ divisible by X^{A_1} to obtain $g_1^{(q)}$, then those divisible by X^{A_2} to obtain $g_2^{(q)}$, and so on.)

Finally, put

(2.3.7) $$f^{(q+1)} := f^{(q)} - g_1^{(q)} f^1 - \ldots - g_k^{(q)} f^k - h^{(q)}$$
$$= g_1^{(q)} (X^{A_1} - f^1) + \ldots + g_1^{(q)} (X^{A_k} - f^k) \quad .$$

From (2.3.7), it is easy to see that, because of (2.1.2), one gets a strictly increasing sequence

$$LM(f^{(0)}) < LM(f^{(1)}) < LM(f^{(2)}) < \ldots \quad ,$$

which implies that, for $1 \leq j \leq n$,

(2.3.8) $$g_j := \sum_{q=0}^{\infty} g_j^{(q)}$$

and

(2.3.9) $$h := \sum_{q=0}^{\infty} h^{(q)}$$

exist in $\mathbb{k}[[X]]$, and so

(2.3.10) $$f = g_1 f^1 + \ldots + g_k f^k + h$$

holds in $\mathbb{k}[[X]]$, with the g_j and h uniquely determined by (2.3.6).

The miracle which now happens is that, if $f \in \mathbb{k}\{X\}$, the g_j and h are also in $\mathbb{k}\{X\}$, and (2.3.10) holds in $\mathbb{k}\{X\}$. I will just collect together the necessary estimates and leave the details, which are elementary after all, to the reader.

<u>(i)</u>. The conditions (2.3.6) guarantee , because of (2.3.5):

(2.3.11) $$\|g_1^{(q)}\|_\rho \rho^{A_1} + \ldots + \|g_k^{(q)}\|_\rho \rho^{A_k} + \|h^{(q)}\|_\rho = \|f^{(q)}\|_\rho$$

for all q and ρ , and so, fixing some ρ :

(2.3.12) $$\forall 1 \leq j \leq k : \; \|g_j^{(q)}\|_\rho \leq \|f^{(q)}\|_\rho \rho^{-A_k}$$

and $$\|h^{(q)}\|_\rho \leq \|f^{(q)}\|_\rho$$

for all q .

(ii). Because of (2.3.7), (2.3.12) implies

$$(2.3.13) \quad \| f^{(q+1)} \|_\rho \le \left(\| X^{A_1} - f^1 \|_\rho \rho^{-A_1} + \ldots + \| X^{A_k} - f^k \|_\rho \rho^{-A_k} \right) \| f^{(q)} \|_\rho$$

for all q .

The crucial point is now that the expression in brackets can be made smaller then a given ε for $\rho = \lambda \rho_0$, where $\rho_0 \in (\mathbb{R}_{>0})^n$ is suitable, and $\lambda \in (0,1)$ arbitrary. This is a tedious, but elementary point which the reader should try to convince himself of; trouble is caused by those monomials of f^j which are different from X^{A_j} but have the same degree. It depends on the fact that all monomials in $\mathrm{supp}(X^{A_j} - f^j)$ are strictly greater then X^{A_j} , and on the lexicographic order (see [23], Satz 2.). Hence, choosing $\varepsilon \in (0,1)$ and $\rho_0 \in (\mathbb{R}_{>0})^n$ suitably we get from (2.3.11), by summing over q and using (2.3.13):

$$(2.3.14) \quad \| g_1 \|_\rho \rho^{A_1} + \ldots + \| g_k \|_\rho \rho^{A_k} + \| h \|_\rho \le \frac{1}{1-\varepsilon} \| f \|_\rho$$

for all $\rho = \lambda \rho_0$, $\lambda \in (0,1) \subseteq \mathbb{R}$, which gives the desired estimates on the $\| g_j \|_\rho$ and $\| h \|_\rho$ to ensure $g_j , h \in \Bbbk\{X\}$, $1 \le j \le n$.

This establishes the Division Algorithm with respect to the lexicographic degree order. There are, however, further important orders which arise, more generally, from a strictly positive linear form

$$(2.3.15) \quad \begin{aligned} \Lambda : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x^1, \ldots, x^n) &\longrightarrow \sum_{i=1}^{n} \lambda_i x^j \end{aligned} ,$$

with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}_{>0}$, by defining

$X^A <_\Lambda X^B$ if and only if either $\Lambda(A) < \Lambda(B)$ or $\Lambda(A) = \Lambda(B)$ and the last nonzero coordinate of $A - B$ is negative.

Call such an order a linear order. It again defines, for $f \in \Bbbk\langle X \rangle$, a leitmonomial which I denote by $LM_\Lambda(f)$, or $LM(f)$ if Λ is understood. With these leitmonomials, one can again set up the Division Algorithm 2.3.1. To arrive at the estimates (2.3.12) and (2.3.13), one changes the definition of $\| \; \|_\rho$ to

$$\|f\|_{\rho,\Lambda} := \sum_{A\in\mathbb{N}^n} |f_A|\cdot\rho^{\Lambda(A)}$$

with $\rho^{\Lambda(A)} := (\rho^1)^{\lambda_1 A^1}\dots(\rho^n)^{\lambda_n A^n}$; this norm clearly is equivalent to the former norm $\|\ \|_\rho$ defined by (1.2.2). One gets again the estimates (2.3.12) and (2.3.13), with $\rho^{\pm A_j}$ replaced by $\rho^{\pm\Lambda(A_j)}$, and the conclusion that the bracket in (2.3.13) can be made arbitrarily small still holds.

Finally, one may even allow positive linear forms, i.e. with $\lambda_i \in \mathbb{R}_{\geq 0}$, since a generic small perturbation of the λ_i defines a strictly positive linear form with the <u>same</u> division algorithm. Summing up one gets

<u>Theorem 2.3.2</u> (The Division Theorem, or "Weierstrass préparé à la Grauert-Hironaka"). <u>Let</u> $\Lambda : \mathbb{R}^n \longrightarrow \mathbb{R}$ <u>be a positive linear form. Let</u> $f^1,\dots,f^k \in \mathbb{k}\langle X\rangle$, <u>and</u> $LM_\Lambda(f^j) = X^{A_j}$, $1 \leq j \leq k$, <u>be the leitmonomials with respect to the linear order on</u> $\mathcal{M}(n)$ <u>induced by</u> Λ. <u>Fix the order</u> $(f^1,\dots,f^k)$ <u>of the</u> f^j, <u>and put recursively</u>

$$\Delta_0 := \emptyset\ ,$$

$$\Delta_j := (A_j + \mathbb{N}^n) - \coprod_{i<j} \Delta_j\ ,\quad j = 1,\dots,k\ ,$$

$$\bar{\Delta} := \mathbb{N}^n - \coprod_{j\leq k} \Delta_j\ .$$

<u>Finally, let</u> $f \in \mathbb{k}\langle X\rangle$. <u>Then the following statements hold</u>:

(i) <u>The Division Algorithm 2.3.1. gives a unique representation</u>

$$f = g_1 f^1 + \dots + g_k f^k + h$$

<u>with</u> $g_j = \sum_{A\in\Delta_j} g_{jA} X^{A-A_j}$, $1 \leq j \leq k$, <u>and</u> $h = \sum_{A\in\bar{\Delta}} h_A X^A$ <u>power series</u> in $\mathbb{k}[[X]]$.

(ii) <u>If</u> $f \in \mathbb{k}\{X\}$, <u>then for any</u> ε <u>with</u> $0 < \varepsilon < 1$ <u>there exists</u> a <u>neighbourhood basis</u> $\mathcal{B}$ <u>of</u> $0 \in \mathbb{A}^n$ <u>consisting of polycylinders</u> $P(0;\rho)$ such that for any $P(0;\rho) \in \mathcal{B}$ the estimate

$$\|g_1\|_{\rho,\Lambda}\,\rho^{\Lambda(A_1)} + \dots + \|g_k\|_{\rho,\Lambda}\,\rho^{\Lambda(A_k)} + \|h\|_{\rho,\Lambda} \leq \frac{1}{1-\varepsilon}\,\|f\|_{\rho,\Lambda}$$

holds. In particular, the g_j and h are in $\mathbb{k}\{X\}$.

2.4. Division with respect to an ideal; standard bases

We are now in a position to carry out the suggestions at the beginning of 2.3. and prove the Rückert Basissatz, Theorem 1.3.2. We also give a proof of the Krull Intersection Theorem.

Let $I \subseteq \mathbb{k}\langle X\rangle$ be an ideal. Fix some linear order and choose $f^1,\dots,f^k \in I$ such that the leitmonomials $LM(f^1),\dots,LM(f^k)$ generate the leitideal $LM(I)$, which is possible by Dickson's Lemma 2.2.1.

Proposition 2.4.1 (Division with respect to I). Let $f \in \mathbb{k}\langle X\rangle$, and let I and $f^1,\dots,f^k \in I$ be as above.

(i) In the representation

$$f = g_1 f^1 + \dots + g_k f^k + h$$

given by the Division Theorem 2.3.2, h does not depend on the choice of $f^1,\dots,f^k$, hence depends only on I , and is called $red_I f$.

(ii) $f \in I$ if and only if $red_I f = 0$.

(iii) $\{f^1,\dots,f^k\}$ is an ideal base of I .

The proof is left to the reader; (i) depends on the fact that $\mathbb{N}^n = \bigsqcup_j \Delta_j \sqcup \overline{\Delta}$ is a disjoint decomposition, and (ii), (iii) are simple consequences.

Because of (iii), the following definiton makes sense:

Definition 2.4.2. Let $I \subseteq \mathbb{k}\langle X\rangle$. Then $\{f^1,\dots,f^k\} \subseteq I$ is called a standard base of I (with respect to a given linear ordering of the monomials) if and only if $\{LM(f^1),\dots,LM(f^k)\}$ is a base of $LM(I)$.

Since standard bases exist, we get

Corollary 2.4.3. The rings $\mathbb{k}\langle X\rangle$ are noetherian.

This clearly implies Theorem 1.3.2.

Remark 2.4.4. a) Not every ideal base is a standard base.

b) If $\{f^1,\ldots,f^k\}$ is a standard base, the initial forms $L(f^1),\ldots,L(f^k)$ generate the initial ideal $L(I)$, i.e. the ideal $L(I)$ having the property that $gr_{m_n}(\mathbb{k}<X>/I) \cong \mathbb{k}<X>/L(I)$. In particular, $L(f^1),\ldots,L(f^k)$ define the tangent cone at 0 of $\mathrm{Spec}(\mathbb{k}<X>/I)$.

c) If $f^1,\ldots,f^k$ are polynomials defining an ideal $I \subseteq \mathbb{k}<X>$, there is a constructive procedure for deriving a standard base from them using the division algorithm, which is based on the fact that $\{f^1,\ldots,f^k\}$ is a standard base if and only if each monomial syzygy of the leitmonomials lifts to a syzygy of $f^1,\ldots,f^k$. Dividing the $f := \sum g_i f^i$, where the $g_1,\ldots,g_k$ run through a generating system of the monomial syzygies of the leitmonomials, by $f^1,\ldots,f^k$ and adding the nonzero remainders leads eventually to a standard base. See [44], [55] and [62]. An implementation of the algorithm of [44] is available on the computer algebra system Macaulay [4]. This allows the computation of the Hilbert function of a homogeneous ideal (based on III, (1.3.4)), [53].

d) For $\mathbb{k}[X]$ one obtains, using maximal monomials instead of minimal ones, a proof of the Hilbert Basissatz along identical lines. In this case, a standard base is known as a Gröbner base ([3],[44], and [46]).

2.5. Applications of standard bases: The General Weierstrass Preparation Theorem and the Krull Intersection theorem.

Any ideal $I \subseteq \mathbb{k}<X>$ has a canonical standard base with respect to a given linear order in the following way: By Dickson's Lemma, $LM(I)$ has a unique base of monomials minimal with respect to divisibility, $\{X^{A_1},\ldots,X^{A_k}\}$ say. By Proposition 2.4.1, we have well-defined remainders $red_I X^{A_j}$. We thus obtain

Theorem 2.5.1 (The General Weierstrass Preparation Theorem). *Let $I \subseteq \mathbb{k}<X>$ be an ideal, Λ a linear order on $\mathcal{M}(n)$, and $\mathcal{B} \subseteq LM_\Lambda(I)$ the canonical base consisting of the minimal elements with respect to the divisibility relation. Then $\{\omega_M \mid \omega_M := M - red_I M \text{ for } M \in \mathcal{B}\}$ is a standard base of I.*

We refer to this base as the Weierstrass base of I (with respect to the given linear order).

As a further application of the Division Theorem we prove:

Theorem 2.5.2 (Krull Intersection Theorem). If $R \in \mathrm{la}/\mathbb{k}$,

$$\bigcap_{p=0}^{\infty} m_R^p = \{0\} \quad .$$

Proof. One has to show that, if $I \subseteq \mathbb{k}\{X_1,\dots,X_n\}$ is any ideal, $\bigcap_{p=0}^{\infty} (I + m_n^p) = I$.

Choose a standard base $\{f^1,\dots,f^k\}$ of I . Let $f \in \bigcap_{p=0}^{\infty} (I+m_n^p)$, and let p_0 be so large that all the $X^{A_j} := LM(f^j)$, $1 \leq j \leq k$, have degree less that p_0 . Let $p \geq p_0$. Fix the order $(X^{A_j},\dots,X^{A_k})$ and then all X^A with degree $X^A = p$ in some order, and apply the Division Algorithm with respect to $f^1,\dots,f^k$ and the X^A to f ; so f can be written

$$f = g_1^{(p)} f^1 + \dots + g_k^{(p)} f^k + \sum_{\deg X^A = p} g_A^{(p)} \cdot X^A \quad .$$

But the Division Algorithm 2.3.1 shows that the $g_j^{(p)}$, $1 \leq j \leq k$, do not depend on p as soon as $p \geq p_0$. Hence the remainder $\sum_{\deg X^A = p} g_A^{(p)} \cdot X^A$ does not depend on p and so is in $\bigcap_{p \geq p_0} m_n^p$, which is zero by Corollary 1.2.9.

2.6. The classical Weierstrass Theorems.

These are the classical cornerstones of Local Complex Analysis and direct consequences of the Division Theorem 2.3.2.

We introduce the notation $X' := (X_1,\dots,X_{n-1})$, and so $X = (X',X_n)$.

The Weierstrass Division Theorem 2.6.1 (Stickelberger-Späth; see the discussion in [26], p. 36). Let $f \in \mathbb{k}<X>$ be such that $f(0,X_n) = X_n^b \cdot u$ for some integer $b \geq 1$ and $u \in \mathbb{k}<X_n>$ a unit (we then say f is regular in X_n of order b.) Then any $e \in \mathbb{k}<X>$ can be uniquely written as

$$e = g \cdot f + h$$

<u>with</u> $g \in \mathbb{k}\langle X \rangle$ <u>and</u> $h \in \mathbb{k}\langle X' \rangle [X_n]$ <u>of</u> X_n-<u>degree strictly less than</u> b.

For this, just note that the condition on f ensures, after reversing the numbering of the coordinates, the existence of a positive linear form Λ on $\mathbb{R}^n$ making $LM_\Lambda(f) = X_n^b$; then apply the Division Theorem 2.3.2. Uniqueness, i.e. independence of g and h of the order, comes from the fact that $\Delta_1 = (0,b) + \mathbb{N}^n$, $\overline{\Delta} = \mathbb{N}^{n-1} \times [0,b-1]$ do not depend on the choice of order.

<u>Corollary 2.6.2.</u> $\mathbb{k}\langle X \rangle/(f) \cong \mathbb{k}\langle X' \rangle^b$ <u>as a</u> $\mathbb{k}\langle X' \rangle$-<u>module</u>.

Hence $\mathbb{k}\langle X \rangle/(f)$ is a <u>finite</u> $\mathbb{k}\langle X' \rangle$-<u>module</u>. This fact is the main reason why Local Complex Analysis is accessible to algebraic methods. It will be considerably generalized in the sequel to the extent that any local analytic algebra is finite over a convergent power series ring (see 6.2.4), leading in geometric terms to the Local Representation Theorem 6.3.1, which realizes any analytic space germ as a finite branched cover of a domain in some number space.

<u>The Weierstrass Preparation Theorem 2.6.3.</u> <u>Let</u> $f \in \mathbb{k}\langle X \rangle$ <u>be regular in</u> X_n <u>of order</u> b . <u>Then</u> f <u>can be uniquely written as</u>

$$f = e \cdot \omega ,$$

<u>where</u> e <u>is unit in</u> $\mathbb{k}\langle X \rangle$ <u>and</u> $\omega \in \mathbb{k}\langle X' \rangle [X_n]$ <u>a Weierstrass polynomial, i.e. it is monic with coefficients in</u> $\mathfrak{m}_{n-1}$.

Just apply Theorem 2.5.1. to $I = (f)$, and with linear order as above. The fact that the coefficients of ω are in $\mathfrak{m}_{n-1}$ follows from comparison of coefficients in the relation $X_n^b \cdot u = e(0,X_n) \cdot \omega(0,X_n)$.

<u>§ 3. Complex spaces and the Equivalence Theorem.</u>

<u>From now on,</u> $\mathbb{k} = \mathbb{C}$, <u>and</u> <u>la</u> := <u>la/$\mathbb{C}$</u> . The standard coordinates on $\mathbb{C}^n$ are denoted $z_1, \dots, z_n$.

The main result of this section will be Grothendieck's Equivalence Theorem which states the equivalence of the "algebraic" category of local analytic $\mathbb{C}$-algebras and the "geometric" category of complex analytic spacegerms (or "singularities"), or rather its dual. This is a local analogue to the equivalence in Algebraic Geometry between the categories of rings and of affine schemes. Although well-known, proofs are not readily accessible; one is in [64], Exposé 13, and one in [40], the latter one, however, makes use of the machinery of coherence, which we will, following the viewpoint of Grothendieck ([64], p. 9 - 10) make no unnecessary use of.

3.1. Complex spaces.

Higher dimensional complex manifolds and complex spaces with singular points arise naturally in the deformation and classification of varying complex structures on smooth complex curves. The systematic construction of these spaces by means of his philosophy of representable functors led Grothendieck to consider nilpotents in the structure sheaf (see his exposés 7 - 17 in [64]), and it is only when allowing arbitrary nonreduced spaces that phenomena as, for instance subspaces which have plenty of infinitesimal deformations but no actual one within the ambient space (corresponding to nonreduced isolated points of the Hilbert scheme), can be satisfactorily understood. At the same time Grauert [22], also led by the consideration of moduli problems, introduced nonreduced complex spaces.

I will assume that the notion of a ringed space is known and just fix some notation concerning them; full discussions are available in [28], [31], [40] and [64], Exposé 9.

As usual, a <u>ringed space</u> consists of a topological space X and a sheaf of (commutative, unitary) rings $\mathcal{O}_X$ on it and is denoted $(X,\mathcal{O}_X)$, or $\underline{X}$, if $\mathcal{O}_X$ is understood. We denote the stalk of $\mathcal{O}_X$ at $x \in X$ by $\mathcal{O}_{X,x}$, and, if it is a local ring, its maximal ideal by m_x. A <u>morphism</u> between ringed spaces is a pair $(f,f^0) : (X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$, where $f : X \longrightarrow Y$ is continuous and f^0 a sheaf morphism $\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$; if no confusion is possible, we also denote the canonical adjoint by $f^0 : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ because $\mathrm{Hom}(\mathcal{O}_Y, f_*\mathcal{O}_X) = \mathrm{Hom}(f^{-1}\mathcal{O}_Y,\mathcal{O}_X)$ naturally. Again, we abbreviate by writing $\underline{f} : \underline{X} \longrightarrow \underline{Y}$.

I further assume the notions of an <u>open subspace</u> and an <u>closed subspace</u> defined by an ideal $\mathcal{J} \subseteq \mathcal{O}_X$ which we always will assume to be locally finitely generated or, as I will say, locally finite. A subspace will always mean a locally closed subspace, i.e. a closed subspace of an open subspace. Corresponding to these notions there are the notions of an open immersion, closed immersion and immersion.

For later use, we note the following simple Lemma:

Lemma 3.1.1. Let $(X,\mathcal{O}_X)$ be a ringed space, $\mathcal{I},\mathcal{J} \subseteq \mathcal{O}_X$ ideals, and $\mathcal{I}$ locally finite. Then any $x \in X$ such that $\mathcal{I}_x \subseteq \mathcal{J}_x$ has a neighbourhood U such that $\mathcal{I}|U \subseteq \mathcal{J}|U$.

The proof is left to the reader.

We make $\mathbb{C}^n$ into a ringed space by defining the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ to be the sheaf of germs of holomorphic functions, in other words, $\mathcal{O}_{\mathbb{C}^n}(U) := \{f \mid f : U \longrightarrow \mathbb{C} \text{ holomorphic}\}$ for any open $U \subseteq \mathbb{C}^n$. For any $a = (a_1,\ldots,a_n) \in \mathbb{C}^n$ the stalk $\mathcal{O}_{\mathbb{C}^n,a}$ is then canonically isomorphic to the convergent power series ring $\mathbb{C}\{X_1-a_1,\ldots,X_n-a_n\}$, and we will identify these two rings: in particular, $\mathcal{O}_{\mathbb{C}^n,0} = \mathbb{C}\{X_1,\ldots,X_n\}$. Moreover, we will identify the indeterminates X_j with the standard coordinate functions z_j on $\mathbb{C}^n$. We can now define complex (analytic) spaces.

Definition 3.1.2.

(i) (Local model spaces). A local model space is a ringed space $(M,\mathcal{O}_M)$ given by the following data:

1) an open set $U \subseteq \mathbb{C}^n$,

2) elements $f^1,\ldots,f^k \in \mathcal{O}_{\mathbb{C}^n}(U)$ ("equations"),

in the following way: If $\mathcal{I} := (f^1,\ldots,f^k)\cdot\mathcal{O}_U$, then

$$
\begin{aligned}
M &:= \operatorname{supp}(\mathcal{O}_U|\mathcal{I}) \\
&= \left\{x \in U \mid \forall 1 \leq j \leq k : \; f^j_x \in \mathfrak{m}_x \subseteq \mathcal{O}_{\mathbb{C}^n,x}\right\} \\
&= \left\{x \in U \mid \forall 1 \leq j \leq k : \; f^j(x) = 0\right\} ,
\end{aligned}
$$

and $\mathcal{O}_M := (\mathcal{O}_X/I)|M$. We then write $\underline{M} = \underline{N}(n,U,(f^1,\dots,f^k))$ or $\underline{M} = \underline{N}(n,U,I)$; if $U \subseteq \mathbb{C}^n$ is understood we simply write $\underline{M} = \underline{N}(f^1,\dots,f^k)$ or $\underline{M} = \underline{N}(I)$, and call it the null space of I .

(ii) (Morphisms of local models). A morphism between local models $\underline{M} = N(m,U,(f^1,\dots,f^k))$ and $\underline{N} = N(n,V,(g^1,\dots,g^\ell))$ is a morphism $(f,f^0) : (M;\mathcal{O}_M) \longrightarrow (N,\mathcal{O}_N)$ induced by a holomorphic map $F : U \longrightarrow V$ with the property $\forall\, 1 \le j \le \ell : g^j \circ F \in (f^1,\dots,f^k)\cdot\mathcal{O}_U$ in the following way:

1) $f := F|M$;

2) $f^0 : \mathcal{O}_N \longrightarrow f_*\mathcal{O}_M$ is induced by the mapping
$F^0_W : \mathcal{O}_V(W) \longrightarrow \mathcal{O}_U(F^{-1}W)$, $g \longmapsto g \circ F$, for all open $W \subseteq V$.

(iii) (The Category of complex spaces). A complex space is a ringed space which is locally isomorphic to a local model. A morphism of complex spaces, or holomorphic map, is morphism $(f,f^0) : (X,\mathcal{O}_X) \longrightarrow (Y,\mathcal{O}_Y)$ of the complex spaces $(X,\mathcal{O}_X),(Y,\mathcal{O}_Y)$ within the category of ringed spaces which locally is isomorphic to a morphism of local models. This defines the category $\underline{\underline{\text{cpl}}}$ of complex spaces.

In fact, any morphism between complex spaces within the category of ringed spaces turns out to be a holomorphic map; see Corollary 3.3.4.

If X is a complex space, an open or closed subspace in the category of ringed spaces, as defined before, is itself a complex space, and we can talk about open, closed, or arbitrary subspaces, and of open, closed, and arbitrary, immersions.

Example 3.1.3. Let $X = \{x\}$ be a one point space and $A \in \underline{\underline{\text{la}}}$ be artinian. Then $(\{x\},A)$ is a complex space. In fact the converse is true: any one point complex space arises in this way. This is astonishingly difficult to prove; it is a special case of the Rückert Nullstellensatz, and essentially equivalent to it; see § 5.

3.2. Constructions in $\underline{\underline{\text{cpl}}}$. It should be kept in mind that the following constructions are categorical; that means that the spaces and morphisms whose existence is asserted do not exist only settheoretically, but also the sheaves and sheaf maps have to be considered, and I urge the reader to convince himself of the details.

a) Glueing. Glueing data for a complex space consist of

(i) a family $(M_i, \mathcal{O}_{M_i})_{i \in I}$ of local models,

(ii) open subsets $U_{ij} \subseteq M_i$, $U_{ji} \subseteq M_j$ and isomorphisms

$$\underline{f}_{ij} : (U_{ij}, \mathcal{O}_{M_i}) | U_{ij} \xrightarrow{\cong} (U_{ji}, \mathcal{O}_{M_j} | U_{ji})$$

for all $i,j \in I$ such that the cocycle identity

$$\underline{f}_{jk} \circ \underline{f}_{ij} = \underline{f}_{ik}$$

holds for all $i,j,k \in I$.

Given glueing data, there is, up to isomorphism, a unique complex space $(X, \mathcal{O}_X)$ which has local models $(M_i, \mathcal{O}_{M_i})$.

In a similar way, a morphism $(f, f^0) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ can be given by glueing data which I will not write down explicitely.

b) Intersections. Let $\underline{X}, \underline{X}' \hookrightarrow \underline{Y}$ be closed complex subspaces of the complex space $\underline{Y}$, defined by the locally finite ideals $I, I' \subseteq \mathcal{O}_Y$. The intersection $\underline{X} \cap \underline{X}'$ is defined to be the largest complex subspace $\underline{X}'' \hookrightarrow \underline{Y}$ such that any morphism $\underline{Z} \longrightarrow \underline{Y}$ which factors through $\underline{X}$ and $\underline{X}'$ also factors through $\underline{X}''$; it is given by the locally finite ideal $I + I'$.

c) Inverse images. Let $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ be a morphism in cpl . If $\underline{Z} \hookrightarrow \underline{Y}$ is a complex subspace, the inverse image $\underline{f}^{-1}(\underline{Z}) \hookrightarrow \underline{X}$ is the complex subspace with the universal property that if $\underline{f}' : \underline{X}' \longrightarrow \underline{X}$ is in cpl and $f \circ f'$ factors through $\underline{Z}$, $\underline{f}'$ factors through $\underline{f}^{-1}\underline{Z}$. If $\underline{Z} \hookrightarrow \underline{Y}$ is a closed complex subspace defined by the locally finite ideal I, $\underline{f}^{-1}(\underline{Z}) \hookrightarrow \underline{X}$ is defined by $\underline{f}^{-1} I := I \cdot \mathcal{O}_X$, the ideal generated in $\mathcal{O}_X$ by I under $f^0 : f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. A special case of this construction are the fibres $\underline{f}^{-1}(\underline{y}) \hookrightarrow \underline{X}$, $y \in Y$, of the morphism $\underline{f}$.

d) Products. In cpl , the categorical product

(3.2.1) $$\underline{X} \times \underline{Y} \begin{array}{l} \xrightarrow{\underline{pr}_X} \underline{X} \\ \xrightarrow[\underline{pr}_Y]{} \underline{Y} \end{array}$$

exists for $\underline{X}, \underline{Y} \in \underline{\underline{cpl}}$. Locally, it is given as follows:
If U, V are open subsets of number spaces,

(3.2.2) $$\underline{U} \times \underline{V} \begin{array}{l} \xrightarrow{\underline{pr}_U} \underline{U} \\ \xrightarrow[\underline{pr}_V]{} \underline{V} \end{array}$$

is given by the usual product $U \times V$ with the canonical complex structure, and $\underline{pr}_U$, $\underline{pr}_V$ by the usual projections and, on the sheaf level, by lifting holomorphic functions via these. If $\underline{X} = (m, U, (f^1, \dots, f^k))$ and $\underline{V} = (n, V, (g^1, \dots, g^\ell))$ are local models, (3.2.1) is given by b) and c) as $\underline{X} \times \underline{Y} := \underline{pr}_U^{-1}(\underline{X}) \cap \underline{pr}_V^{-1}(\underline{Y})$ and $\underline{pr}_X := \underline{pr}_U | \underline{X}$, $\underline{pr}_Y := pr_V | \underline{Y}$; this means that $\underline{X} \times \underline{Y}$ is the local model $(m+n, U \times V, f^1 \circ pr_U, \dots, f^k \circ pr_U, g^1 \circ pr_V, \dots, g^\ell \circ pr_V)$. In the general case, cover $\underline{X}$ and $\underline{Y}$ by local models, form their products, and use the universal property of the product to obtain glueing data for (3.2.1) according to a).

e) <u>Diagonals</u>. If $\underline{X} \in \underline{\underline{cpl}}$, the diagonal $\underline{\Delta}_X \hookrightarrow \underline{X}$ is the complex subspace with the property that for any morphism $\underline{f} : \underline{Z} \longrightarrow \underline{X}$ in $\underline{\underline{cpl}}$, $\underline{f} \times \underline{f} : \underline{Z} \longrightarrow \underline{X} \times \underline{X}$ factors uniquely through $\underline{\Delta}_X$. For a local model $\underline{X} \subseteq \underline{U}$, where U is open in some $\mathbb{C}^n$, $\underline{\Delta}_X := (\underline{X} \times \underline{X}) \cap \underline{\Delta}_U$, and $\underline{\Delta}_U$ is the obvious diagonal of $\underline{U}$; for the general case, glue according to a).

f) <u>Fibre products</u>. In $\underline{\underline{cpl}}$, categorical fibre products exist. Given $\underline{f} : \underline{X} \longrightarrow \underline{Y}$, $g : \underline{Y}' \longrightarrow \underline{Y}$, the cartesian square

(3.2.3) $$\begin{array}{ccc} X' & \xrightarrow{\underline{g}'} & \underline{X} \\ {\scriptstyle \underline{f}'}\downarrow & & \downarrow{\scriptstyle \underline{f}} \\ \underline{Y}' & \xrightarrow[\underline{g}]{} & \underline{Y} \end{array}$$

is defined by putting $\underline{X}' := \underline{X} \times_{\underline{Y}} \underline{Y}' := ((\underline{f} \circ \underline{pr}_X) \times (\underline{g} \circ \underline{pr}_{Y'}))^{-1}(\underline{\Delta}_Y)$, and f', g' defined by the projections $\underline{pr}_X, \underline{pr}_Y : \underline{X} \times \underline{Y}' \longrightarrow \underline{X}, \underline{Y}'$. The universal property of the fibre product is implied by the universal properties of the inverse image and the diagonal.

<u>g) Graph spaces</u>. A special case of f) is the graph space $\underline{\Gamma}_f$ of a morphism $\underline{f} : \underline{X} \longrightarrow \underline{Y}$; it is defined by the cartesian square

(3.2.4)
$$\begin{array}{ccc} \underline{\Gamma}_f & \xrightarrow{\ \underline{p}\ } & \underline{X} \\ {\scriptstyle \underline{g}}\downarrow & & \downarrow{\scriptstyle \underline{f}} \\ \underline{Y} & \xrightarrow[\ \underline{id}_Y\ ]{} & \underline{Y} \end{array} ,$$

and is a complex subspace of $\underline{X} \times \underline{Y}$. By the universal property of the fibre product the morphisms $\underline{id}_X : \underline{X} \longrightarrow \underline{X}$ and $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ define $\underline{i} := \underline{id}_X \times_Y \underline{f} : \underline{X} \longrightarrow \underline{\Gamma}_f$, and one gets the commutative diagram

(3.2.5)
$$\begin{array}{ccccc} \underline{X} & \xrightarrow{\ \underline{i}\ } & \underline{\Gamma}_f & \hookrightarrow & \underline{X} \times \underline{Y} \\ & {\scriptstyle \underline{f}}\searrow & \downarrow{\scriptstyle \underline{g}} & \swarrow{\scriptstyle \underline{pr}_Y} & \\ & & \underline{Y} & & \end{array} ,$$

where $\underline{i}$ is an isomorphism, inverse to $\underline{p}$. Hence, we have:

<u>Proposition 3.2.1</u>. <u>Any morphism $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ is isomorphic to the restriction of a projection to a complex subspace</u>.

If X and Y are Hausdorff, $\underline{\Gamma}_f$ is a closed complex subspace, and so $\underline{id} \times f : \underline{X} \longrightarrow \underline{X} \times \underline{Y}$ is a closed immersion with image $\underline{\Gamma}_f$.

The proposition will be important in the study of finite morphisms in the following paragraphs, since it allows to reduce locally to the situation of linear projections of number spaces restricted to closed complex subspaces.

h) Supports of modules.

Definition 3.2.2. Let $X \in \underline{cpl}$, and $\mathcal{M}$ be an $\mathcal{O}_X$-module. $\mathcal{M}$ is called admissible if and only if it is locally of finite presentation, i.e. if and only if every $x \in X$ has an open neighbourhood such that there is a short exact sequence

(3.2.6) $$\mathcal{O}_X^q \mid U \xrightarrow{\varphi} \mathcal{O}_X^p \mid U \longrightarrow \mathcal{M} \mid U \longrightarrow 0 \quad .$$

If $\mathcal{M}$ is admissible, the Fitting ideals $F_n(\mathcal{M})$ are defined as

(3.2.7) $F_n(\mathcal{M}) \mid U :=$ ideal generated in $\mathcal{O}_X \mid U$ by the $(p-n) \times (p-n)$ - minors of the $p \times q$-matrix given by φ in (3.2.6).

A theorem of Fitting [15] implies that the $F_n(\mathcal{M})$ are globally well-defined. By construction, they are locally finite. We then define the support of $\mathcal{M}$ to be

(3.2.8) $\underline{\text{supp}}\, \mathcal{M} :=$ the closed complex subspace of $\underline{X}$ defined by $F_0(\mathcal{M})$.

The underlying topological space of $\underline{\text{supp}}\, \mathcal{M}$ is $\text{supp}\, \mathcal{M} := \{x \in X \mid \mathcal{M}_x \neq 0\}$; for this, just tensorize (3.2.6) at $x \in X$ with $\mathbb{C} \cong \mathcal{O}_{X,x} | \mathfrak{m}_x$.

Remark. If $\text{Ann}(\mathcal{M})$ is the annihilator ideal of $\mathcal{M}$, then $F_0(\mathcal{M}) \subseteq \text{Ann}(\mathcal{M}) \subseteq \sqrt{F_0(\mathcal{M})}$. The first inclusion is by elementary linear algebra, whereas the second one lies considerably deeper and follows from the Rückert Nullstellensatz 5.3.1.

i) Image spaces. Let $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ be a morphism in $\underline{cpl}$. Then $\overline{\text{im}(f)} = \text{supp}(f_* \mathcal{O}_X)$ settheoretically, so if $f_* \mathcal{O}_X$ happens to be an admissible $\mathcal{O}_Y$-module, $\text{supp}(f_* \mathcal{O}_X)$ has a natural structure as a closed complex subspace of $\underline{Y}$ via $F_0(f_* \mathcal{O}_X)$ in view of a). We call this space the complex image space of $\underline{f}$, denoted $\underline{\text{im}}(\underline{f})$ or $\underline{f}(\underline{X})$.

3.3. The Equivalence Theorem.

The Equivalence Theorem asserts the equivalence of the "geometric" category of complex space singularities with the "algebraic" category of local analytic $\mathbb{C}$-algebras. Its explicit formulation seems to be due to Grothendieck ([64], Exposé 13).

We begin with describing the morphisms of a complex space $\underline{X}$ to $\underline{\mathbb{C}}^n$. If $R \in \underline{\underline{la}}$, $R/m_R \cong \mathbb{C}$ canonically via the augmentation mapping induced by the $\mathbb{C}$-algebra-structure; hence, if $\underline{X} \in \underline{\underline{cpl}}$, any section $f \in \mathcal{O}_X(X)$ defines a function $[f] : X \longrightarrow \mathbb{C}$ via

(3.3.1) $$\forall x \in X : [f](x) := f_x \bmod m_x \quad .$$

Proposition 3.3.1. If $\underline{X} \in \underline{\underline{cpl}}$, we get a bijection

$$\begin{aligned} \mathrm{Hom}_{\underline{\underline{cpl}}}(\underline{X},\underline{\mathbb{C}}^n) &\longrightarrow \mathcal{O}_X(X)^n \\ \underline{f} &\longmapsto (f^0_X(z_1),\dots,f^0_X(z_n)) \quad , \end{aligned}$$

where $f^0_X : (f^{-1}\mathcal{O}_{\mathbb{C}^n})(X) = \mathcal{O}_{\mathbb{C}^n}(\mathbb{C}^n) \longrightarrow \mathcal{O}_X(X)$.

Sketch of proof.

(i) Injectivity: Since $z_j \circ f = [f^0(z_j)]$, the $f^0(z_j)$ determine the settheoretic map $f : X \longrightarrow \mathbb{C}^n$. Now, if $\underline{f}, \underline{g} \in \mathrm{Hom}_{\underline{\underline{cpl}}}(\underline{X},\underline{\mathbb{C}}^n)$ have $f^0(z_j) = g^0(z_j)$ for $1 \le j \le n$, then $f = g$, and $f^0_x, g^0_x : \mathcal{O}_{\mathbb{C}^n,y} \longrightarrow \mathcal{O}_{X,x}$, where $y := f(x) = g(x)$, agree on the z_j for $1 \le j \le n$. But then they agree on $\mathcal{O}_{\mathbb{C}^n,y}$, since $\mathcal{O}_{\mathbb{C}^n,y} \cong \mathbb{C}\{z_1,\dots,z_n\}$ is a free object in $\underline{\underline{la}}$ by Theorem 1.3.4.

(ii) Surjectivity: Let $(f_1,\dots,f_n) \in \mathcal{O}_X(X)^n$ be given. First suppose X is a local model space in some open $U \subseteq \mathbb{C}^n$, and the f_j are induced by holomorphic functions $F_j : U \longrightarrow \mathbb{C}$ for $1 \le j \le n$. Then $F := (F_1,\dots,F_n) : U \longrightarrow \mathbb{C}^n$ induces a morphism $\underline{f} : \underline{X} \longrightarrow \underline{\mathbb{C}}^n$ with $f^0(z_j) = f_j$ for $1 \le j \le n$. In the general case cover $\underline{X}$ with local models and glue the local morphisms obtained on the overlaps by means of (i).

Definition 3.3.2 (Germs of complex spaces).

(i) A complex spacegerm, or singularity, is a tuple $(\underline{X},x)$ with $X \in \underline{\underline{cpl}}$ and $x \in X$.

(ii) A morphism of complex spacegerms , or complex mapgerm, is a morphism $\underline{f} : \underline{U} \longrightarrow \underline{V} \in \underline{\underline{cpl}}$ of an open neighbourhood $\underline{U}$ of x into an open neighbourhood $\underline{V}$ of y with $f(x) = y$, where one identifies those morphisms which coincide after restriction to possibly smaller neighbourhoods.

The complex space germs with their morphisms form a category, which I will denote $\underline{\underline{cpl}}_0$. If $(\underline{X},x) \in \underline{\underline{cpl}}_0$, and U is any open neighbourhood of x in X , $(\underline{X},x) = (\underline{U},x)$ up to isomorphism in $\underline{\underline{cpl}}_0$, and I will refer to this as "possibly shrinking $\underline{X}$ " .

There is a canonical contrafunctor

$$\mathcal{O} : \underline{\underline{cpl}}_0 \longrightarrow \underline{\underline{la}}$$

mapping $(\underline{X},x) \in \underline{\underline{cpl}}_0$ to $\mathcal{O}_{X,x}$ and $\underline{f} : (\underline{X},x) \longrightarrow (\underline{Y},y)$ to $f_x^0 : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$.

Theorem 3.3.3 (The Equivalence Theorem; Grothendieck [64], Exposé 13). $\mathcal{O} : \underline{\underline{cpl}}_0^{opp} \longrightarrow \underline{\underline{la}}$ is an equivalence of categories.

Sketch of proof. We have to show two properties:

(i) essential surjectivity on objects: For $R \in \underline{\underline{la}}$ there exists $(\underline{X},x) \in \underline{\underline{cpl}}_0$ with $\mathcal{O}_{X,x} \cong R$.

(ii) bijectivity on morphisms:

$$\begin{array}{ccc} \mathrm{Hom}_{\underline{\underline{cpl}}_0}((\underline{X},x),(\underline{Y},y)) & \longrightarrow & \mathrm{Hom}_{\underline{\underline{la}}}(\mathcal{O}_{Y,y},\mathcal{O}_{X,x}) \\ f & \longmapsto & f_x^0 \end{array}$$

is a bijection.

(i): is trivial from the constructions.

(ii): Since the question is local, we may assume, after possibly shrinking $\underline{X}$ and $\underline{Y}$, that $\underline{X} \hookrightarrow \underline{U} \subseteq \mathbb{C}^m$, $\underline{Y} \hookrightarrow \underline{V} \subseteq \mathbb{C}^n$ are local models, where U and V are open, and $x = 0 \in \mathbb{C}^m$, $y = 0 \in \mathbb{C}^n$.

<u>Injectivity</u>: We may assume $\underline{Y} = \underline{\mathbb{C}}^n$; the claim then follows from Proposition 3.3.1.

<u>Surjectivity</u>: Let $\varphi : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x} \in \underline{\underline{la}}$ be given. By Theorem 1.3.4 there is a commutative diagram

$$(3.3.2) \qquad \begin{array}{ccc} \mathcal{O}_{\mathbb{C}^n,0} & \overset{\psi}{\dashrightarrow} & \mathcal{O}_{\mathbb{C}^m,0} \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{O}_{Y,y} & \underset{\varphi}{\longrightarrow} & \mathcal{O}_{X,x} \end{array} \qquad .$$

Let $(F_j)_0 := \psi(z_j) \in \mathcal{O}_{\mathbb{C}^m,0}$, $1 \leq j \leq n$; after possibly shrinking U , we may assume the $(F_j)_0$ have representatives $F_j : U \longrightarrow \mathbb{C}$, which together define the holomorphic map

$$F := (F_j,\ldots,F_n) : U \longrightarrow \mathbb{C}^n \qquad .$$

Let $\underline{X}$ be defined by $g^1,\ldots,g^k \in \mathcal{O}_{\mathbb{C}^m}(U)$ and $\underline{Y}$ by $h^1,\ldots,h^\ell \in \mathcal{O}_{\mathbb{C}^n}(V)$. Define the $\mathcal{O}_U$-ideals

$$I := (g^1,\ldots,g^h)\cdot\mathcal{O}_U$$

$$J := (h_0^1 \circ F,\ldots,h_0^\ell \circ F)\cdot\mathcal{O}_U \qquad .$$

Then $J_0 \subseteq I_0$ because of the commuative diagram (3.3.2). By Lemma 3.1.1. we may therefore assume $J \subseteq I$. But then F induces a morphism $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ by Definition 3.1.2.(ii), and $f_x^0 = \varphi$ by construction.

<u>Corollary 3.3.4</u>. <u>$\underline{\underline{cpl}}$ is a full subcategory of the category $\underline{\underline{lrsp}}$ of spaces locally ringed in $\mathbb{C}$-algebras</u>.

For the same proof as in 3.3.3. shows the injectivity of $\mathrm{Hom}_{\underline{\underline{lrsp}}_0}((\underline{X},x),(\underline{Y},y)) \longrightarrow \mathrm{Hom}_{\underline{\underline{la}}}(\mathcal{O}_{Y,y},\mathcal{O}_{X,x})$.

Corollary 3.3.5. *Morphisms* $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^n,0)$ *correspond one-to-one to* m_x*-sequences* $(f_1,\ldots,f_n)$ (i.e. sequences $(f_1,\ldots,f_n)$ with $f_j \in m_x$ for $1 \leq j \leq n$).

Remark 3.3.6. By Corollary 3.3.5, special morphisms of germs should correspond to m_x-sequences with special properties. We will see instances of this principle later on (4.4.2, 6.2.3., 6.3.1.).

3.4 The analytic spectrum.

For later use, we shortly discuss a further application of Proposition 3.3.1.

Let A be a finitely generated $\mathbb{C}$-algebra. Picking generators $a_1,\ldots,a_n \in A$ gives an epimorphism

$$\varphi : \mathbb{C}[z_1,\ldots,z_n] \longrightarrow\!\!\!\!\!\!\rightarrow A \quad .$$

Let I be the kernel of φ , and $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{C}^n}$ the ideal sheaf generated by I . $\mathcal{I}$ defines a closed complex subspace $\underline{Z} \hookrightarrow \underline{\mathbb{C}}^n$, and there is a canonical homomorphism $\zeta : A \longrightarrow \mathcal{O}_Z(Z)$, such that for given $a \in A$ the germ $\zeta(a)_z$ at a given $z \in Z$ is the germ induced by $f_z \in \mathcal{O}_{\mathbb{C}^n,z}$ where f is any preimage of a under φ . We then have the following generalization of Proposition 3.3.1.

Proposition 3.4.1. *The pair* $(\underline{Z},\zeta)$ *represents the functor* $\underline{\underline{\text{cpl}}}^{\text{opp}} \longrightarrow \underline{\underline{\text{sets}}}$ *given by* $\underline{X} \longmapsto \text{Hom}_{\underline{\underline{\text{cpl}}}}(\underline{X},\underline{Z})$, *in other words, the canonical map*

$$\text{Hom}_{\underline{\underline{\text{cpl}}}}(\underline{X},\underline{Z}) \longrightarrow \text{Hom}_{\underline{\underline{\mathbb{C}\text{-alg}}}}(A,\mathcal{O}_X(X))$$
$$f \longmapsto f_X^0 \circ \zeta$$

induces a natural equivalence of functors.

Here, f_X^0 is the homomorphism $\mathcal{O}_Z(Z) \longrightarrow \mathcal{O}_X(X) = (f_*\mathcal{O}_X)(Z)$ given by the sheaf map $f^0 : \mathcal{O}_Z \longrightarrow f_*\mathcal{O}_X$.

The proof of the Proposition is simple, using 3.3.1, and left to the reader. For the general formalism of representable functors see [64], Exposé 11, by Grothendieck.

It follows that the pair $(\underline{Z},\zeta)$ is unique up to unique isomorphism, and so the following definition makes sense:

Definition 3.4.2. If A is a finitely generated $\mathbb{C}$-algebra, the pair $(\underline{Z},\zeta)$, or the complex space $\underline{Z}$ alone when ζ is understood, constructed above is called the analytic spectrum of A and denoted Specan(A).

§ 4. Local Weierstrass Theory II: Finite morphisms.

Classically finite maps arose naturally by solving systems of polynomial equations via Kronecker's elimination theory (see e.g. [51]); successively eliminating indeterminates by forming resultants of polynomials turns some indeterminates into free parameters, which can be varied arbitrarily and whose number should be thought of as the dimension of the solution variety; the rest of the indeterminates become algebraic functions of these parameters. Geometrically, this amounts to representing the solution variety as a finite branched cover of an affine space, and algebraically to the fact that the coordinate ring of the solution variety is a finite integral extension of a polynomial ring. This is nowadays known as "Noether normalization", and fairly easy to prove, without using elimination theory.

This picture remains true locally in the complex analytic case, but this is much harder to prove. As already mentioned before, the main reason for the applicability of local algebra to local complex analysis is the fact that, under the equivalence 3.3.3, finite mapgerms will correspond to finite, and hence integral, ring extensions of local analytic algebras, and so a kind of "relative algebraic situation" emerges. This will be the subject of the main result of this paragraph, the Integrality Theorem 4.4.1. Fundamental for it is the famous Finite Mapping Theorem 4.3.1. of Grauert and Remmert; in the proof of it, the elimination procedure of the algebraic case is mimicked geometrically by a sequence of linear projections along a line.

4.1. Finite morphisms.

From now on, all topological spaces under consideration will be Hausdorff, locally compact, and paracompact. For general facts of topology quoted in the sequel see [7], and also [14].

A continuous map $f : X \longrightarrow Y$ of topological spaces is called proper if the inverse image of a compact subset of Y is compact in X. This is equivalent to the requirement that f is closed (i.e. maps closed sets to closed sets) and has compact fibres. A proper map with finite fibres is called finite, so a map is finite iff it is closed with finite fibres. Finally, a morphism $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ of complex spaces is called finite if the underlying map $f : X \longrightarrow Y$ of topological spaces is so. Elementary considerations from topology show that any $y \in Y$ has a neighbourhood basis consisting of open neighbourhoods V such that $f^{-1}V = \coprod_{x \in f^{-1}(y)} U_x$ for open neighbourhoods U_x of x in X and $f|U_x : U_x \longrightarrow V$ is finite. Thus, there are canonical homomorphisms for a sheaf $\mathcal{M}$ on X,

$$\text{(4.1.1)} \qquad \varepsilon_y : (f_*\mathcal{M})_y \longrightarrow \bigoplus_{x \in f^{-1}(y)} \mathcal{M}_x \quad , \quad \text{for all } y \in Y ,$$

induced from $\mathcal{M}(f^{-1}V) \longrightarrow \bigoplus_{x \in f^{-1}(y)} \mathcal{M}(U_x)$ via $s \longmapsto \sum_{x \in f^{-1}(y)} s|U_x$, and one gets:

Theorem 4.1.1. *Let $f : X \longrightarrow Y$ be a finite morphism of complex spaces. Let $\mathcal{O}_X$-mod and $\mathcal{O}_Y$-mod denote the category of $\mathcal{O}_X$-modules and $\mathcal{O}_Y$-modules respectively. Then:*

(i) *The homomorphisms ε_y in (4.1.1) are isomorphisms for all $\mathcal{M} \in \mathcal{O}_X$-mod;*

(ii) *the functor $f_* : \mathcal{O}_X\text{-mod} \longrightarrow \mathcal{O}_Y\text{-mod}$ is exact.*

4.2. Weierstrass maps (see [28]).

These are the prototypes of finite morphisms in local complex analytic geometry and play a prominent rôle in what follows, since any finite morphism locally will embed in a Weierstrass map. So ultimatively basic properties of finite morphisms will be proved using Weierstrass maps.

Let $\omega^{(j)} \in \mathcal{O}_{\mathbb{C}^n,0}[w_j]$ be monic polynomials

(4.2.1) $$\omega^{(j)} = w_j^{b_j} + \sum_{\nu=0}^{b_j-1} a_\nu^{(j)}(z) w_j^\nu \quad , \quad 1 \leq j \leq k \; ,$$

$a_\nu^{(j)} \in \mathcal{O}_{\mathbb{C}^n,0}$, and $b_j \geq 1$, for $1 \leq j \leq k$. Let $B \subseteq \mathbb{C}^n$ be a domain containing $0 \in \mathbb{C}^n$ such that the $\omega^{(j)}$ have representatives, also called $\omega^{(j)}$, defined on B . We get the closed subspace $\underline{A} := \underline{N}(\omega^{(1)},\ldots,\omega^{(k)}) \hookrightarrow \underline{B} \times \underline{\mathbb{C}}^k$, and the projection $\underline{pr}_B : \underline{B} \times \mathbb{C}^k \longrightarrow \underline{B}$ defines

(4.2.2) $$\underline{\pi} := \underline{pr}_B \,|\, A : \underline{A} \longrightarrow \underline{B} \quad .$$

We call $\underline{\pi}$ a <u>Weierstrass map</u>.

Given $z \in B$, the equations (4.2.1) have only finitely many solutions. Moreover, if $\omega = w^b + \sum_{\nu=0}^{b-1} a_\nu(z) w^\nu \in \mathcal{O}_{\mathbb{C}^n,0}[w]$ and $\omega(z_0,w_0) = 0$, we have the simple estimate

$$|w_0| \leq \max\left(1, \sum_{\nu=0}^{b-1} |a_\nu(z_0)|\right) \quad ,$$

which shows that the inverse image of a bounded set is bounded. Hence:

<u>Proposition 4.2.1</u>. <u>A Weierstrass map is finite</u>.

Somewhat deeper lies:

<u>Proposition 4.2.2</u>. <u>A Weierstrass map is open</u>.

This is implied by the following easy but very useful consequence of the Weierstrass Preparation Theorem:

<u>Lemma 4.2.3</u> (Hensel's Lemma).

<u>Let</u> $\omega := \omega(z,w) = w^b + \sum_{\nu=0}^{b-1} a_\nu(z) w^\nu \in \mathcal{O}_{\mathbb{C}^n,0}[w]$ <u>be a monic polynomial of degree</u> $b \geq 1$. <u>Let</u> $\omega(0,\omega) = (w-c_1)^{b_1} \cdot \ldots \cdot (w-c_r)^{b_r}$. <u>Then there exist unique monic polynomials</u> $\omega_1,\ldots,\omega_r \in \mathcal{O}_{\mathbb{C}^n,0}[w]$; $\deg \omega_j = b_j$ for $1 \leq j \leq r$, <u>such that</u> $\omega = \omega_1 \cdot \ldots \cdot \omega_r$.

For the proof of 4.2.3, one just applies the Preparation Theorem successively in the rings $\mathcal{O}_{\mathbb{C}^n,0}[w-c_1]$, $\mathcal{O}_{\mathbb{C}^n,0}[w-c_2]$, and so on.

Now the Weierstrass map (4.2.2) clearly is open at $0 \in A$ since the equations (4.2.1) have a solution for any $z \in B$, but by Hensel's Lemma the germ $\underline{\pi} : (\underline{A},a) \longrightarrow (\underline{B},\pi(a))$ is locally around any $a \in A$ a Weierstrass map, so π is open at all $a \in A$, and so is open, which proves Proposition 4.2.2.

4.3. The Finite Mapping Theorem.

The following theorem is the fundamental result in local complex analytic geometry, and is due to Grauert and Remmert ([24], Satz 27). Recall the notion of an admissible module (Definition 3.2.2.).

Theorem 4.3.1 (The Finite Mapping Theorem). *Let $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ be a finite morphism of complex spaces. Then, if $\mathcal{M}$ is an admissible $\mathcal{O}_X$-module, $f_*\mathcal{M}$ is an admissible $\mathcal{O}_Y$-module.*

Corollary 4.3.2. *If $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ is a finite morphism of complex spaces, the complex image space $\underline{\text{im}}(\underline{f})$ in the sense of 3.2.i) exists.*

This Corollary is an obvious consequence of the Theorem.

The proof of this basic result is done in various steps. The details are in [28], Chapter 3, but since the full machine of coherence is employed there, I will give an outline, indicating the minor modifications which are necessary when not invoking the notion of coherence.

In the first step, one considers the special case where $\underline{f}$ is a Weierstrass map $\underline{\pi} : \underline{A} \longrightarrow \underline{B}$. Let the notation be as in 4.2. Let $\mathbb{N}^{n+k} = \overline{\Delta} \amalg \coprod_{j=1}^{k} \Delta_j$ be the decomposition given by the monomials $w_j^{b_j}$ according to Theorem 2.3.2; hence

$$(4.3.1) \qquad \overline{\Delta} = \mathbb{N}^n \times \overline{\Delta}_0 \quad \text{with} \quad \overline{\Delta}_0 := \left\{ B \in \mathbb{N}^k \mid 0 \le B^i < b_i \quad \text{for} \quad 1 \le i \le k \right\} .$$

Let $\mathcal{O}_B^{\overline{\Delta}_0}$ be the $\mathcal{O}_B$-module defined by

(4.3.2) $\mathcal{O}_B^{\bar{\Delta}_0}(U) := \left\{ \sum_{B \in \bar{\Delta}_0} f_B w^B \mid f_B \in \mathcal{O}_{\mathbb{C}^n}(U) \right\}$

for $U \subseteq B$ open. There is a natural $\mathcal{O}_B$-module homomorphism

(4.3.3) $\overset{0}{\pi} : \mathcal{O}_B^{\bar{\Delta}_0} \longrightarrow \pi_* \mathcal{O}_A$

given as follows: If $U \subseteq B$ is open, $\sum_{B \in \bar{\Delta}_0} f_B w^B$ is defined on $\mathcal{O}_{B \times \mathbb{C}^k}(U \times \mathbb{C}^k) = \mathcal{O}_B(\pi^{-1}U)$; this defines

(4.3.4) $\overset{0}{\pi}_U : \mathcal{O}_B^{\bar{\Delta}_0}(U) \longrightarrow \mathcal{O}_{B \times \mathbb{C}^k}(\pi^{-1}U) \xrightarrow{\text{restriction}} \mathcal{O}_A(\pi^{-1}U)$,

and so (4.3.3). The following theorem substantially generalizes Corollary 2.6.2:

<u>Theorem 4.3.3.</u> $\overset{0}{\pi}$ <u>is an isomorphism of</u> $\mathcal{O}_B$-<u>modules</u>.

This in turn is an immediate consequence of the following parametrized generalization of the Division Theorem:

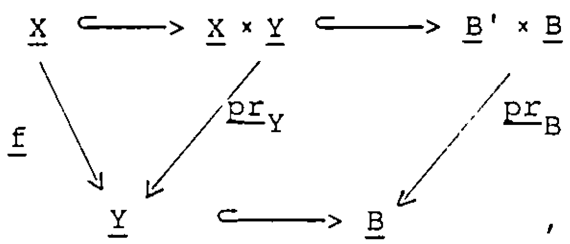
<u>Theorem 4.3.4</u> (The Generalized Division Theorem). <u>Let the notation be as in 4.2. Let</u> $y \in B$, <u>and let, for all</u> $x_j \in \pi^{-1}(y)$, <u>germs</u> $f_j \in \mathcal{O}_{\mathbb{C}^{n+k}, x_j}$ <u>be given. Then there exist unique germs</u> $g_{\alpha j} \in \mathcal{O}_{\mathbb{C}^{n+k}, x_j}$, $\alpha = 1, \ldots, k$, <u>and a unique polynomial</u> $h \in \mathcal{O}_{\mathbb{C}^n}[w_1, \ldots, w_k]$ <u>of the form</u> $h = \sum_{A \in \mathbb{N}^k} h_A w^A$ <u>with</u> $0 \leq A^i < b_i$ <u>for</u> $1 \leq i \leq k$ <u>such that for all</u> $x_j \in \pi^{-1}(y)$

$$f_j = g_{1j} \omega_{x_j}^{(1)} + \ldots + g_{kj} \omega_{x_j}^{(k)} + h_{x_j} \quad \text{in} \quad \mathcal{O}_{\mathbb{C}^{n+k}, x_j} .$$

The main point of this theorem is that <u>one</u> h works for <u>all</u> x_j . The proof is a formal consequence of the Division Theorem and Hensel's Lemma 4.2.3., and I refer to [28] for it.

Theorem 4.3.3. is then proved as follows: By Theorem 4.1.1. (i), $(\pi_* \mathcal{O}_A)_y \cong \bigoplus_{x_j \in \pi^{-1}(y)} \mathcal{O}_{A, x_j}$, so any element s_y of $(\pi_* \mathcal{O}_A)_y$ is represented by a family $(f_{x_j})_{x_j \in \pi^{-1}(y)}$, $f_{x_j} \in \mathcal{O}_{\mathbb{C}^{n+k},x_j}$. Dividing the f_{x_j} by $\omega^{(1)}_{x_j},\dots,\omega^{(k)}_{x_j}$ via Theorem 4.3.4 shows there is an unique $h_y \in \mathcal{O}^{\overline{\Delta}0}_{B,y}$ mapping to s_y , so (4.3.3) is bijective on stalks, and so bijective by Theorem 4.1.1. (ii).

The second step reduces the general case to the case of a linear projection. For this, one observes that the statement of Theorem 4.3.1 is local in the sense that any $x \in X$ has an open neighbourhood U such that $\pi|U : U \longrightarrow \pi(U)$ is again finite, and so we may assume that $\underline{X} \hookrightarrow \underline{B}'$, $\underline{Y} \hookrightarrow \underline{B}$, where $B' \subseteq \mathbb{C}^n$ and $B \subseteq \mathbb{C}^k$ are domains. One gets a commutative diagram



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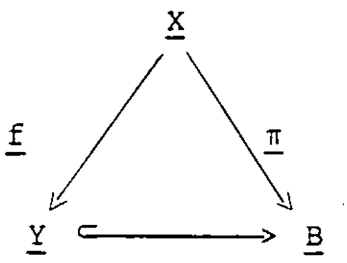
where the horizontal arrows in the upper row are closed immersions, the left hand triangle is defined by the graph construction (3.2.5), and the right hand square is defined by the closed immersions $\underline{X} \hookrightarrow \underline{B}'$, $\underline{Y} \hookrightarrow \underline{B}$. Identifying $\underline{X}$ with its image in $\underline{B}' \times \underline{B}$ we may assume we have a commutative diagram



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where π is given by the restriction of a linear projection to $\underline{X}$ which is finite, or, as I will say, where π is a finite linear projection. One now has the following lemma.

<u>Lemma 4.3.5.</u> <u>Let</u> $X \in \underline{cpl}$, <u>$\underline{Y} \overset{i}{\hookrightarrow} \underline{X}$ a closed complex subspace, and</u> <u>$\mathcal{M}$ an $\mathcal{O}_Y$-module. Then $\mathcal{M}$ is an admissible $\mathcal{O}_Y$-module if and only if</u> <u>$i_*\mathcal{M}$ is an admissible $\mathcal{O}_X$-module</u>.

The proof is a simple diagram chase and left to the reader.

This lemma shows that it suffices to prove Theorem 4.3.1 for $\underline{\pi}$.

The last step reduces now everything to the first step. We may assume that $\underline{f}$ is a finite linear projection. We may even assume that $k = 1$, for we can factor $\underline{f}$ successively into a sequence of projections along lines, and Corollary 4.3.2 and Lemma 4.3.5 reduce everything to that case.Then choose a nonzero $g \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ which vanishes on X near 0 ; after possibly shrinking $\underline{X}$ and $\underline{B}$ we may assume g is a Weierstrass polynomial by Theorem 2.6.3. We then have the commutative triangle

$$\begin{array}{ccc} \underline{X} & \overset{i}{\hookrightarrow} & \underline{N}(g) =: A \\ & \underline{f} \searrow \quad \swarrow \underline{\pi} & \\ & \underline{B} & \end{array}$$

and, again by Lemma 4.3.5, we are reduced to prove Theorem 4.3.1 for the Weierstrass map $\underline{\pi}$. Now let $\mathcal{M}$ be an admissible $\mathcal{O}_A$-module; after shrinking A and B, we may assume there is an exact sequence

$$\mathcal{O}_A^q \longrightarrow \mathcal{O}_A^p \longrightarrow \mathcal{M} \longrightarrow 0 \quad ,$$

so there is an exact sequence, since π_* is exact by 4.1.1. (ii):

$$\left(\pi_*\mathcal{O}_A\right)^q \longrightarrow \left(\pi_*\mathcal{O}_A\right)^p \longrightarrow \pi_*\mathcal{M} \longrightarrow 0 \quad ,$$

(note π_* commutes with direct sums). But $\pi_*\mathcal{O}_A \cong \mathcal{O}_B^b$ for some b by Theorem 4.3.3, hence Theorem 4.3.1 follows.

As a corollary of the proof we obtain:

Corollary 4.3.6. Let $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ *be quasifinite at* $x \in X$ (i.e. x is an isolated point of the fibre $f^{-1}f(x)$) . *Then* x *has a neighbourhood* U *and* $f(x)$ *a neighbourhood* V *with* $f(U) \subseteq V$ *such that* $\underline{f}|\underline{U} : \underline{U} \longrightarrow \underline{V}$ *is finite.*

The proof is identical with the reduction procedure in the above proof, reducing it to the case of a Weierstrass map, which is finite by Proposition 4.2.1.

4.4. The Integrality Theorem.

Recall the equivalence of categories

$$\mathcal{O} : \underline{\underline{cpl}}_0^{opp} \longrightarrow \underline{\underline{la}}$$

given by the Equivalence Theorem 3.3.3. We are now in a position to describe which homomorphisms in $\underline{\underline{la}}$ correspond to the finite mapgerms in $\underline{\underline{cpl}}_0$, and this will finally allow to describe algebraic invariants of local analytic algebras in geometric terms of $\underline{\underline{cpl}}_0$.

<u>Theorem 4.4.1</u> (The Integrality Theorem). <u>Let $f : (\underline{X},x) \longrightarrow (\underline{Y},y)$ be a holomorphic mapgerm; recall that by Theorem 3.3.3 this is equivalent to having a homomorphism</u> $\varphi : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ <u>of local analytic algebras. The following statements are equivalent</u>:

(i) <u>f is quasifinite, i.e.</u> x <u>is isolated in</u> $f^{-1}f(x)$ <u>for some (or any) representative of</u> f .

(ii) <u>f is finite, i.e. some representative of f is a finite morphism of complex spaces</u>.

(iii) φ <u>is quasifinite, i.e</u> $\mathcal{O}_{X,x} / m_Y \cdot \mathcal{O}_{X,x}$ <u>is a finite dimensional complex vectorspace</u>.

(iv) φ <u>is finite, i.e.</u> $\mathcal{O}_{X,x}$ <u>is a finite</u> $\mathcal{O}_{Y,y}$<u>-module via</u> φ .

We can visualize this situation by the following diagram:

(i) f quasifinite at x	$\overset{\Longleftrightarrow}{\scriptstyle\mathcal{O}}$	(iii) $\dim_{\mathbb{C}} \mathcal{O}_{X,x}/m_Y\mathcal{O}_{X,x} < \infty$
$\Updownarrow$		$\Updownarrow$
(ii) f finite near x	$\overset{\Longleftrightarrow}{\scriptstyle\mathcal{O}}$	(iv) $\mathcal{O}_{X,x}$ finite over $\mathcal{O}_{Y,y}$
equivalence in $\underline{\underline{cpl}}_0$		equivalence in $\underline{\underline{la}}$

I will give a bare outline of the argument, following the diagram clockwise via (i) ⇒ (iii) ⇒ (iv) ⇒ (ii) ⇒ (i) . Arguing as in the last section, I may assume throughout $\underline{f}$ is represented by a finite linear projection, $\underline{Y} = \underline{B} \subseteq \mathbb{C}^n$ is a domain containing $y = 0 \in \mathbb{C}^n$; $\underline{X}$ is defined in $\underline{Y} \times \underline{V}$, $\underline{V}$ a domain in $\mathbb{C}^k$, by a finitely generated ideal $I \subseteq \mathcal{O}_{\mathbb{C}^{n+k}}(Y \times V)$, $x = 0 \in \mathbb{C}^{n+k}$; and $\underline{f}$ is induced by the projection $\underline{pr}_Y : \underline{Y} \times \underline{V} \longrightarrow \underline{Y}$. (See Figure 2). Let $R := \mathcal{O}_{X,x}/m_y\mathcal{O}_{X,x}$.

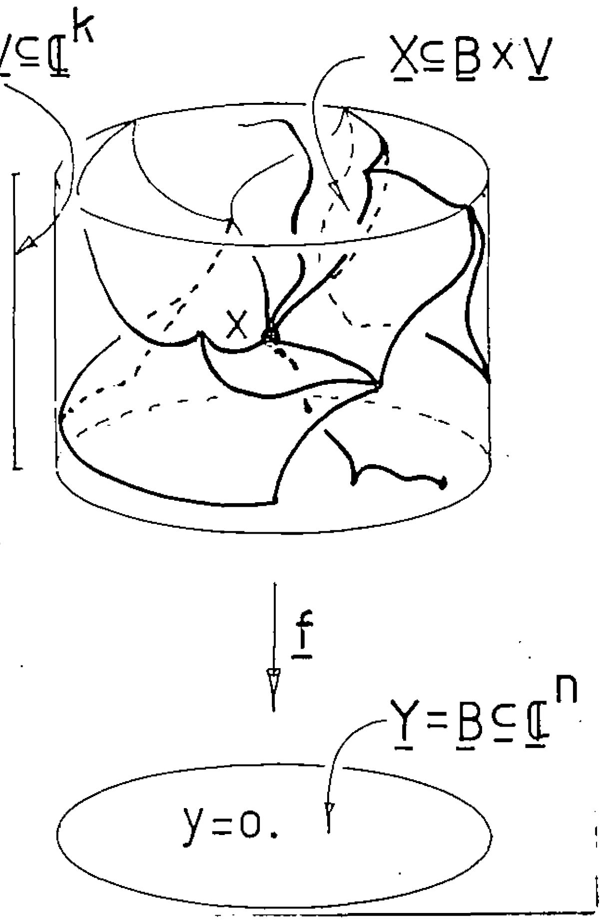


Fig. 2

(i) ⇒ (iii). The fibre $\underline{f}^{-1}(\underline{y})$ is defined by the ideal $m_y \cdot \mathcal{O}_X$, by 3.2.c). The Corollary 4.3.6 then shows $\mathcal{O}_{\underline{f}^{-1}(\underline{y}),x}$ is an admissible $\mathcal{O}_{\underline{Y},y}$-module by Theorem 4.3.1 , where $\underline{y} = (\{y\}, \mathcal{O}_{\mathbb{C}^d,y}/m_d)$.

(iii) ⇒ (iv). (iii) means that there is an integer $b \geq 1$ with $m_R^b = 0$. This implies that, after possibly shrinking X and Y , there are integers b_j , $1 \leq j \leq k$, and $g_\nu^j \in \mathcal{O}_{\mathbb{C}^{n+k}}(Y \times V)$, $\nu = 1,\ldots,n$, such that

$$(4.4.1) \qquad \omega^j(z,w) := w_j^{b_j} + \sum_{\nu=1}^{n} g_\nu^j(z,w) \cdot z_\nu \in I \qquad \text{for} \quad 1 \leq j \leq k ,$$

where $I \subseteq \mathcal{O}_{\mathbb{C}^{n+k}}(Y \times V)$ defines X . One can then show that there is a positive linear form Λ with $w_1 <_\Lambda \dots <_\Lambda w_k <_\Lambda z_1 <_\Lambda \dots <_\Lambda z_n$ such that

$$LM_\Lambda(\omega^j) = w^{b_j} \tag{4.4.2}$$

for $1 \le j \le k$. Given any $f \in \mathcal{O}_{\mathbb{C}^{n+k},0}$, divide it by $\omega^1, \dots, \omega^k$ according to the Division Theorem 2.3.2:

$$f = g_1\omega^1 + \dots + g_k\omega^k + h \tag{4.4.3}$$

with $\operatorname{supp}(h) \subseteq \overline{\Delta}$. Because of (4.4.2), h can be written as

$$h = \sum_{A \in \overline{\Delta}_0} h_A(z) w^A \tag{4.4.4}$$

with $\overline{\Delta}_0 := \left\{A \in \mathbb{N}^k \mid \forall_j : 0 \le A^j < b_j\right\}$, $h_A(z) \in \mathcal{O}_{\mathbb{C}^n,0} = \mathcal{O}_{Y,y}$. Taking (4.4.3) mod I, we see by (4.4.4) that the monomials w^A for $A \in \overline{\Delta}_0$ generate $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$.

<u>(iv) ⇒ (ii)</u>. Since $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{Y,y}$, there are integral equations

$$\omega^j(z,w) := w_j^{b_j} + \sum_{\nu=0}^{b_j-1} a_j^{(\nu)}(z) w_j^\nu \in I \tag{4.4.5}$$

for the w_j as elements in $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$. After possibliy shrinking X and Y, this gives the commutative diagram

$$\begin{array}{ccc} \underline{X} & \overset{\underline{i}}{\hookrightarrow} & \underline{A} := \underline{N}(\omega^1, \dots, \omega^k) \\ & {\scriptstyle \underline{f}} \searrow \quad \swarrow {\scriptstyle \underline{\pi}} & \\ & \underline{B} & \end{array} \tag{4.4.6}$$

where $\underline{i}$ is a closed immersion and $\underline{\pi}$ a Weierstrass map. $\underline{\pi}$ is finite by Proposition 4.2.1, hence so is $\underline{f}$.

<u>(ii) ⇒ (i)</u>. This is clear.

<u>Corollary 4.4.2.</u> (i) <u>Let</u> $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^n,0)$ <u>be defined by the elements</u> $f_j \in m_x$, $j = 1,\dots,n$. <u>Then</u>

a) $\underline{f}$ <u>is finite if and only if</u> $(f_1,\ldots,f_n)$ <u>generates an</u> m_x-<u>primary ideal of</u> $\mathcal{O}_{X,x}$.

b) $\underline{f}$ <u>is a local immersion if and only if</u> $(f_1,\ldots,f_n)$ <u>generates</u> m_x.

(ii) $(\underline{X},x) \in \underline{cpl}_0$ <u>is smooth, i.e.</u> $(\underline{X},x) \cong (\underline{\mathbb{C}}^n,0)$ <u>for some</u> n, <u>if and only if</u> $\mathcal{O}_{X,x}$ <u>is a regular local ring</u>.

<u>Sketch of proof</u>.

<u>(i) a)</u>: is clear because of Theorem 4.4.1.

<u>(i) b)</u>: If $(f_1,\ldots,f_n)$ generate m_x, $\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_n\cdot\mathcal{O}_{X,x}) = 1 < \infty$; hence $\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{\mathbb{C}^n,0}$ via f_x^0, and Nakayama's Lemma tells us that $f_x^0 : \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow \mathcal{O}_{X,x}$ is surjective. So it factors as $\mathcal{O}_{\mathbb{C}^n,0} \twoheadrightarrow \mathcal{O}_{Y,0} \cong \mathcal{O}_{X,x}$ with $(\underline{Y},0) \subseteq (\underline{\mathbb{C}}^n,0)$ defined by the ideal $I := \operatorname{Ker} f_x^0$. Conversely, if $\underline{f}$ is a local immersion, f_x^0 factors as $\mathcal{O}_{\mathbb{C}^n,0} \twoheadrightarrow \mathcal{O}_{Y,0} \cong \mathcal{O}_{X,x}$, hence is surjective, and so $f_x^0(m_n) = m_x$.

<u>(ii)</u>. If $(\underline{X},x)$ is smooth, $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{C}^n,0}$, which is regular. If $\mathcal{O}_{X,x}$ is regular, a regular system of parameters of $\mathcal{O}_{X,x}$ gives a homomorphism $\varphi : \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow \mathcal{O}_{X,x}$ such that $\hat{\varphi} : \hat{\mathcal{O}}_{\mathbb{C}^n,0} \longrightarrow \hat{\mathcal{O}}_{X,x}$ is an isomorphism. This implies φ is injective and $\dim_{\mathbb{C}}(\mathcal{O}_{X,x}/m_n\,\mathcal{O}_{X,x}) = \dim_{\mathbb{C}}(\hat{\mathcal{O}}_{X,x}/\hat{m}_n\hat{\mathcal{O}}_{X,x}) = 1$, so φ is finite and hence surjective by Nakayama's Lemma again. Hence φ is also surjective, hence an isomorphism, which implies $(\underline{X},x) \cong (\underline{\mathbb{C}}^n,0)$ by the Equivalence Theorem 3.3.3.

<u>Exercise</u>. <u>Prove 4.4.2. without passing to the completion (use 2.6.2)</u>.

<u>§ 5. Dimension and Nullstellensatz</u>.

Pursueing the analogy with elimination theory further, it is shown that a complex spacegerm has a well-defined local dimension, given as the minimal number of free parameters such that in the system of holomorphic equations defining the germ the rest of the unknowns are algebraic functions of them (this will be geometrically and algebraically exploited in 6.1 and 6.2). This local dimension coincides with the Chevalley dimension of the corresponding local ring. We introduce active elements, providing good inductive proofs for the dimension, and give a short proof of the Rückert Nullstellensatz, from which we deduce that the decomposition of a complex space germ into irreducible analytic setgerms corresponds in a one-to-one fashion to the minimal primes of the corresponding local analytic algebra.

5.1. Local dimension.

Recall that by Corollary 3.3.5 mapgerms $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^n,0) \in \underline{\underline{cpl}}_0$ correspond in a one-to-one fashion to sequences $(f_1,\dots,f_n)$ with $f_1,\dots,f_n \in m_x$.

Proposition and Definition 5.1.1 (Local dimension). Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$. The following integers are the same:

$$\min\left\{n \mid \exists\, f_1,\dots,f_n \in m_x : x \text{ is isolated in } N(f_1,\dots,f_k)\right\} ,$$

$$\min\left\{n \mid \exists \text{ finite mapgerm } \underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^n,0)\right\} ;$$

their common value is called the (local) dimension of X at x and denoted $\dim_x \underline{X}$.

This is immediate from the Integrality Theorem 4.4.1. We list the following properties:

Proposition 5.1.2. The local dimension has the following properties:

(i) $\dim_x\underline{X} \le n$ if and only if $(\underline{X},x)$ admits a finite holomorphic mapgerm $(\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^n,0)$.

(ii) $\underline{f} : (\underline{X},x) \longrightarrow (\underline{Y},x)$ finite $\Rightarrow \dim_x\underline{X} \le \dim_y\underline{Y}$.

(iii) If $(\underline{X},x) \in \underline{\underline{cpl}}_0$, define $(\underline{X}_{red},x) \hookrightarrow (\underline{X},x)$ as the subgerm corresponding to the projection $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}|N_x$, where N_x is the nilradical of $\mathcal{O}_{X,x}$, via the Equivalence Theorem 3.3.3. Then $\dim_x\underline{X} = \dim_x\underline{X}_{red}$.

(iv) If $\underline{X} \in \underline{\underline{cpl}}$, $x \longmapsto \dim_x \underline{X}$ is upper semicontinuous.

(v) If $(\underline{Y},x) \subseteq (\underline{X},x)$ is a subgerm and $\dim_x \underline{Y} < \dim_x \underline{X}$, $(Y,x) \neq (X,x)$ as germs of sets.

(vi) x is isolated in X if and only if $\dim_x \underline{X} = 0$.

(vii) $\dim_x \underline{X} = \dim \mathcal{O}_{X,x}$, the Chevalley dimension of the local ring $\mathcal{O}_{X,x}$.

Of this, (i) - (vi) are immediate from the definitions, only (vii) deserves a comment. Recall that the Chevalley dimension of a noetherian local ring R is defined to be the minimal length of a system $\underline{f} := (f_1,\ldots,f_n)$ of elements which generate an m_R-primary ideal; the latter condition is in our case equivalent to $\dim_{\mathbb{C}}(R/\underline{f}R) = \text{lenght}_R(R/\underline{f}R) < \infty$. Then the claim (vii) follows directly from the Integrality Theorem 4.4.1.

5.2. Active elements and the Active Lemma.

Active elements generalize nonzerodivisors. The main result is the Active Lemma 5.2.2, which makes inductive proofs work. Since, as we will see, activity of an element of a local analytic algebra restricts only its behaviour on the irreducible components of the corresponding complex space germ and not its behaviour on the embedded ones, it is a more flexible notion than that of nonzerodivisors.

Proposition and Definiton 5.2.1. Let R be a noetherian local ring. Then $f \in R$ is called active iff it satisfies one of the following equivalent conditions:

(i) $\forall\, \mathfrak{p} \in \mathrm{Min}(R) : f \notin \mathfrak{p}$.

(ii) $\forall\, g \in R : f \cdot g \in \mathfrak{n}_R \Rightarrow g \in \mathfrak{n}_R$, where $\mathfrak{n}_R$ is the nilradical of R .

(iii) f is a nonzero-divisor in the reduction $R_{red} := R/\mathfrak{n}_R$.

Lemma 5.2.2 (The Active Lemma). Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$ and $f \in m_x$ be active. Then

$$\dim_x \underline{N}(f) = \dim_x \underline{X} - 1 \quad .$$

<u>Idea of proof</u>. It suffices to show $\dim_x \underline{N}(f) \leq \dim_x \underline{X} - 1$. Let $d := \dim_x \underline{X}$ and $\underline{\pi} : (\underline{X}, x) \longrightarrow (\underline{\mathbb{C}}^d, 0)$ be a finite holomorphic mapgerm; then f satisfies an integral equation

$$f^k + a_{k-1} f^{k-1} + \ldots + a_1 f + a_0 = 0$$

with $a_j \in \mathcal{O}_{\mathbb{C}^d,0}$, $0 \leq j \leq k-1$, by the Integrality Theorem 4.4.1. By activity, we may assume $a_0 \neq 0$. Then (5.2.1) induces the commutative diagram of complex space germs

$$\begin{array}{ccc} (\underline{N}(f), x) & \hookrightarrow & (\underline{X}, x) \\ \underline{\pi}|\underline{N}(f) \downarrow & & \downarrow \underline{\pi} \\ (\underline{N}(a_0), 0) & \hookrightarrow & (\underline{\mathbb{C}}^d, 0) \end{array} \quad ,$$

where the horizontal arrows are closed immersions, and so $\underline{\pi}|\underline{N}(f)$ is a finite holomorphic mapgerm. Hence $\dim_x \underline{N}(f) \leq \dim_0 \underline{N}(a_0)$ by Proposition 5.1.2 (ii). But since $a_0 \neq 0$, there is a line $L \subseteq \mathbb{C}^d$ such that 0 is isolated in $N(a_0) \cap L$ by the Identity Theorem for holomorphic functions in one variable, which easily implies $\dim_0 \underline{N}(a_0) \leq d - 1$. This proves the Active Lemma.

The Active Lemma has numerous consequences as we will see in the next sections. Immediate is the following one:

<u>Corollary 5.2.3</u>. $\dim_0 \mathbb{C}^n = n$.

<u>Remark 5.2.4</u>. If $\dim_x X > 0$, active elements do exist in $\mathcal{O}_{X,x}$ (see [28], p. 99).

<u>5.3. The Rückert Nullstellensatz</u>.

If $\mathbb{k}$ is an algebraically closed field and A a finitely generated $\mathbb{k}$-algebra, elements $f \in A$ define regular functions $[f] : X \longrightarrow \mathbb{k}$

on the variety $X = \operatorname{Spec} A$ (we consider only the closed points). The famous Hilbert Nullstellensatz states that $[f]$ is zero as a function if and only if f is a nilpotent element of A, or, what is equivalent in this case, nilpotent in all local rings $\mathcal{O}_{X,x}$, $x \in X$. The proof of the Nullstellensatz is rather easy in this algebraic case: One proves (i) the "weak Nullstellsatz" that any ideal $I \neq 1$ in $\mathbb{k}[X_1,\ldots,X_n]$, $n \geq 1$, has a zero, and then (ii) applies the Rabinowitsch trick (see [71], § 121). Usually (i) is proven by means of Noether normalization, which is easy in the algebraic case but hard in the complex analytic case (in fact it is our final aim in this chapter to prove it as the Local Representation Theorem in § 6); there are even more elementary proofs using the Division Algorithm in polynomial rings (which is similar to Theorem 2.3.2, but much easier to establish), see [3] for the Divison Algorithm and [46] for the Division Algorithm and the Nullstellensatz.

Although the Nullstellensatz remains true in the complex analytic case, the above approach will not work because (ii) fails; the result lies considerably deeper in this case, and was first proved by Rückert in his fundamental paper [59], in which for the first time algebraic methods were systematically introduced into Local Complex Analytic Geometry. In the treatment here, it will be a consequence of the Active Lemma.

Theorem 5.3.1 (Rückert Nullstellensatz). Let $X \in \underline{\mathrm{cpl}}$, $f \in \mathcal{O}_X(X)$, and $[f] : X \longrightarrow \mathbb{C}$ the function defined by f (see (3.3.1)). Then $[f] = 0$ if and only if $f_x \in \mathcal{O}_{X,x}$ is nilpotent for all $x \in X$.

Idea of proof.

The "if"-part is clear. For the "only if"-part, let $x \in \underline{X}$ be given; one decomposes the nilradical $N_x \subseteq \mathcal{O}_{X,x}$:

$$(5.3.1) \qquad N_x = \bigcap_{\mathfrak{p} \in \operatorname{Min}(\mathcal{O}_{X,x})} \mathfrak{p} \quad .$$

For $\mathfrak{p} \in \operatorname{Min}(\mathcal{O}_{X,x})$, let the immersion $(\underline{X}_{\mathfrak{p}},x) \hookrightarrow (\underline{X},x)$ of germs correspond to the projection $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{p}$ via the Equivalence Theorem 3.3.3. Then $\mathcal{O}_{X_{\mathfrak{p}},x} = \mathcal{O}_{X,x}/\mathfrak{p}$ is an integral domain, and so $f_{\mathfrak{p}} := f|\underline{X}_{\mathfrak{p}}$ is either 0 or active in $\mathcal{O}_{X_{\mathfrak{p}},x}$. But it cannot be active, since then by the Active Lemma $\dim_{\underline{x}}\underline{N}(f_{\mathfrak{p}}) < \dim_{\underline{x}}\underline{X}_{\mathfrak{p}}$, and so it would not vanish near x on $X_{\mathfrak{p}}$, which it must since f vanishes on X by assumption. So $(f_{\mathfrak{p}})_x = 0$ in $\mathcal{O}_{X_{\mathfrak{p}},x}$, which means $f_x \in \mathfrak{p}$. Since this holds for all $\mathfrak{p} \in \mathrm{Min}\,\mathcal{O}_{X,x}$, $f_x \in \mathcal{N}_x$ by (5.3.1).

There are other useful formulations of this result:

Corollary 5.3.2. The following statements are equivalent to Theorem 5.3.1 and do therefore hold:

(i) Let $\underline{X} \in \underline{\mathrm{cpl}}$, $\underline{Y} \hookrightarrow \underline{X}$ a closed complex subspace defined by the locally finite ideal $\mathcal{I} \subseteq \mathcal{O}_X$. Let $\mathcal{J}_Y$ be the ideal defined as $\mathcal{J}_Y(U) := \{f \in \mathcal{O}_X(U) \mid [f]|Y = 0\}$ for $U \subseteq X$ open. Then $\mathcal{J}_Y = \sqrt{\mathcal{I}}$ (this is the traditional formulation of the Nullstellensatz).

(ii) Let $\mathcal{M}$ be an admissible $\mathcal{O}_X$-module, and let $f \in \mathcal{O}_X(X)$ be such that it vanishes on $\mathrm{supp}(\mathcal{M})$ as a function, i.e. $[f] \mid \mathrm{supp}\,\mathcal{M} = 0$. Then any $x \in X$ has an open neighbourhood such that $f^t \cdot \mathcal{M} = 0$ for some integer $t \geq 1$.

For 5.3.1 $\Rightarrow$ (ii) see [28], p. 67, Corollary (use $F_0(\mathcal{M})$ instead of $\mathrm{Ann}(\mathcal{M})$ there and the fact $F_0(\mathcal{M}) \subseteq \mathrm{Ann}(\mathcal{M})$). The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ 5.3.1 are easy.

5.4. Analytic sets and local decomposition.

Let $\underline{X}$ be a complex space. A subset $A \subseteq X$ is called analytic iff it is locally around any $x \in X$ the null set of finitely many sections of $\mathcal{O}_X$ defined near x. The ideal $\mathcal{J}_A \subseteq \mathcal{O}_X$ with $\mathcal{J}_A(U) := \{f \in \mathcal{O}_X(U) \mid [f]|A = 0\}$ is called the vanishing ideal of A.

If $A \subseteq X$ is analytic, it has a well-defined local dimension at $a \in A$: Since $\mathcal{O}_{X,a}$ is noetherian by the Rückert Basissatz 1.3.2, a has an open neighbourhood U such that $A \cap U$ is, the underlying set of a closed complex subspace of U defined by a finitely generated $\mathcal{O}_U$-ideal $\mathcal{I}$ which is such that $\mathcal{I}_a = \mathcal{J}_{A,a}$, and two such ideals coincide locally near a by Lemma 3.1.1. So there is, up to isomorphy, a well-defined germ $(\underline{A},a) \in \underline{\mathrm{cpl}}_0$ defined by any such $\mathcal{I}$ in U, and we put $\dim_a A := \dim_a \underline{A}$. Especially, X is an analytic set in $\underline{X}$, and

$J_{X,x} = N_x$, the nilradical of $\mathcal{O}_{X,x}$, by the Rückert Nullstellensatz 5.3.1, and so $\dim_x \underline{X} = \dim_x X$ by Proposition 5.1.2 (iii). If $x \in X$, we have the usual notion of the germ of an analytic set at x, denoted (A,x), which is the equivalence class of an analytic set A defined in an open neighbourhood of x with respect to the equivalence relation which identifies two locally defined analytic sets when they coincide near x. We call such germs analytic setgerms. Unions of analytic germs are well-defined and so there is the notion of an irreducible germ, this being one which cannot be written as a nontrivial union. It is then an easy exercise to show that an analytic setgerm has a unique decomposition into irreducible ones which corresponds to the associated primes of their vanishing ideal; this is a consequence of the Rückert Basissatz (see [28], Chapter 4, § 1.). Together with the Nullstellensatz we get the following result:

Proposition 5.4.1 (Local decomposition). Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$. If $I \subseteq \mathcal{O}_{X,x}$ is any ideal, let the inclusion $(\underline{X}_I,x) \hookrightarrow (\underline{X},x)$ of complex spacegerms be defined by the projection $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/I$ via the Equivalence Theorem 3.3.3. Then:

(i) The complex space subgerms of $(\underline{X},x)$ correspond bijectively to the ideals of $\mathcal{O}_{X,x}$ under $I \mapsto (\underline{X}_I,x)$, and the analytic setgerms to the radical ideals of $\mathcal{O}_{X,x}$ under $I \mapsto (X_I,x)$.

(ii) $(X,x) = \bigcup_{\mathfrak{p} \in \mathrm{Min}(\mathcal{O}_{X,x})} (X_{\mathfrak{p}},x)$ is the unique decomposition of the analytic setgerm (X,x) into irreducible ones.

I refer to the decomposition in (ii) as the local decomposition of X at x into irreducible components.

I call the $X_{\mathfrak{p}}$ the local irreducible components of X at x (they are called prime components in [28]). Germs with exactly one irreducible component are called irreducible.

Using the Active Lemma, one proves the following result (see [28], p. 103), which is a converse to Proposition 5.1.2. (v) and which will be needed in § 6.

Theorem 5.4.2. Let Y be a closed complex subspace of the complex space $\underline{X}$, $x \in \underline{Y}$, and suppose $\dim_x \underline{Y} = \dim_x \underline{X}$. Then $\underline{X}$ and $\underline{Y}$ have a common local irreducible component at x .

Corollary 5.4.3 (Lemma of Ritt). Let $\underline{X}$ be a complex space, $\underline{Y} \hookrightarrow \underline{X}$ a closed complex subspace. The following statements are equivalent:

(i) $\dim_y \underline{Y} < \dim_y \underline{X}$ for all $y \in Y$.

(ii) Y is nowhere dense in X .

The proof is left as an exercise (use 5.2.2 (v) and 5.4.2).

§ 6. The Local Representation Theorem for complex space germs (Noether normalization).

In this paragraph, we are finally in a position to interprete geometrically the concepts of dimension and of a system of parameters for a local analytic algebra and to see that they give rise locally to a situation identical with Noether normalization in the algebraic case, as described at the beginning of § 4. The dimension turns out to be the unique integer d that the complex space germ corresponding to the given local analytic algebra lies spread out finitely over a germ $(\mathbb{C}^d,0)$, and these finite branched covering mapgerms are precisely those given by a system of parameters according to Corollary 3.3.5.

6.1. Openness and dimension.

We now can give a geometric characterization of the local dimension. The geometric characterization in question is the openness of a map at a point; here, a continuous map $f : X \longrightarrow Y$ of topological spaces is said to be open at a point $x \in X$ iff it maps every neighbourhood of x in X onto a neighbourhood of $f(x)$ in Y .

Lemma 6.1.1 (Open Lemma I). Let $\underline{f} : (\underline{X},x) \longrightarrow (\underline{Y},y) \in \underline{\underline{\text{cpl}}}_0$ be finite. Then f is open at x if and only if each element of $\text{Ker}(f_x^0 : \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x})$ is nilpotent.

Proof. Since f is finite, $f(X)$ is an analytic set in Y by Corollary 4.3.2. f is open at x if and only if $(f(X),y) = (Y,y)$ as germs of sets at $y \in Y$, which means $\mathcal{J}_{f(X),y} = \mathcal{J}_{Y,y}$, where

$J_{f(X)}$ and J_Y are the vanishing ideals of the analytic sets $f(X)$ and Y in $\underline{Y}$. But $J_{f(X),y} = (f_x^0)^{-1}N_{X,x}$ and $J_{Y,y} = N_y$ by the Nullstellensatz, and so

$$f \text{ open at } x \iff (f_x^0)^{-1}N_x = N_y$$
$$\iff \operatorname{Ker} f_x^0 \subseteq N_y \quad ,$$

which proves the claim.

<u>Lemma 6.1.2</u> (Open Lemma II). <u>Let</u> $\underline{f} : (\underline{X},x) \longrightarrow (\underline{Y},y) \in \underline{\underline{cpl}}_0$ <u>be finite</u>.

(i) f <u>open at</u> $x \Rightarrow \dim_x \underline{X} = \dim_y \underline{Y}$.

(ii) <u>If</u> Y <u>is locally irreducible at</u> y (i.e. Y has only one local irreducible component at y , see 5.4), <u>then</u> $\dim_x \underline{X} = \dim_y \underline{Y} \Rightarrow f$ <u>open at</u> x .

<u>Proof</u>. (<u>i</u>): We may assume $\dim_x \underline{X} > 0$. After possibly shrinking $\underline{X}$ and $\underline{Y}$, we may assume there is $g \in \mathcal{O}_Y(Y)$ which is active at y such that $f_x^0(g) =: g'$ is active at y by the so-called Lifting Lemma (see [28], p. 99; the proof there actually does not need the assumptions that X and Y are reduced). This gives the commutative diagram

$$\begin{array}{ccccc}
\underline{N}(g') & =: & \underline{X}' & \hookrightarrow & \underline{X} \\
\underline{f}|\underline{N}(g') =: \underline{f}' & & \Big\downarrow & & \Big\downarrow \underline{f} \\
\underline{N}(g) & =: & \underline{Y}' & \hookrightarrow & \underline{Y}
\end{array}$$

with $\underline{f}'$ finite and open, and this allows to induct over $\dim_x X$.

(ii) $\dim_x X \leq \dim_y f(X) \leq \dim_y Y$ by Proposition 5.1.2, hence $\dim_y f(X) = \dim_y Y$, and the claim follows from Theorem 5.4.2.

<u>6.2. Geometric interpretation of the local dimension and a system of parameters; algebraic Noether normalization.</u>

Combining the results in 6.1. gives immediately:

Theorem 6.2.1 (Dimension Theorem). Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$. If $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^n,0)$ is a finite holomorphic mapgerm, the following statements are equivalent:

(i) $f_x^0 : \mathcal{O}_{\mathbb{C}^n,0} \longrightarrow \mathcal{O}_{X,x}$ is injective,

(ii) f is open at x,

(iii) $n = \dim_x \underline{X}$.

Corollary 6.2.2. Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$. Then $\dim_x \underline{X}$ is the unique integer n such that $(\underline{X},x)$ admits a finite mapgerm to $(\underline{\mathbb{C}}^n,0)$ which is open at x.

Corollary 6.2.3. Let $R \in \underline{\underline{la}}$, $f_1,\ldots,f_n \in m_R$. Then $(f_1,\ldots,f_n)$ is a system of parameters for R if and only if the mapgerm $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^n,0) \in \underline{\underline{cpl}}_0$ corresponding to $(f_1,\ldots,f_n)$ via Corollary 3.3.5 is finite and open at x.

Corollary 6.2.4 (Algebraic Noether normalization). Let $R \in \underline{\underline{la}}$, and let $(f_1,\ldots,f_d)$ be a system of parameters for R. Then the analytic subring generated by $f_1,\ldots,f_d$ is isomorphic to $\underline{\mathbb{C}}\{X_1,\ldots,X_d\}$, and R is finite over it.

Proof. If $R \in \underline{\underline{la}}$ and $f_1,\ldots,f_k \in m_R$, the simplest way to define the analytic subring generated by them is to declare it to be the image of the homomorphism $\varphi : \mathcal{O}_{\mathbb{C}^k,0} \longrightarrow R$ defined by mapping z_i to f_i for $1 \leq i \leq k$ according to Theorem 1.3.4. By the way φ is defined, this subring should consist of the (in R) convergent infinite series $\sum_{A \in \mathbb{N}^k} c_A f^A$, $c_A \in \mathbb{C}$, and in fact one can put a topology on R, the topology of analytic convergence (see [26]) so that this statement makes sense and is true; this analytic subring then is just the closure of the subring generated by the f_i in the algebraic sense. The claim of 6.2.4 is immediate from 6.2.1 and Theorem 4.4.1 (iv).

6.3. The Local Representation Theorem; geometric Noether normalization.

We now can more fully exploit the geometry of a system of parameters of a local analytic algebra R or, what is the same according to Corollary 6.2.3, of a finite open mapgerm of a complex space germ onto a number space germ. We have already proven the "algebraic Noether normalization", namely that the system of parameters generate a subalgebra which is a convergent power series ring over which R is finite. It now will turn out that locally this implies the same geometric situation that we have in the algebraic case, where the variety corresponding to a finite $\mathbb{k}$-algebra R is a branched covering over an affine space of dimension $\dim R$, but this time the proof is substantially more difficult and needs the whole machinery described up to now.

Anyway, the following local description of a complex space germ holds, which is a kind of geometric Noether normalization:

Theorem 6.3.1 (The Local Representation Theorem). Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$, $d = \dim_x \underline{X}$, and let $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ be a finite holomorphic mapgerm; such mapgerms exist by the definition of the local dimension, and they correspond to systems of parameters for $\mathcal{O}_{X,x}$. Then $\underline{f}$ has arbitrarily small representatives $\underline{f} : \underline{X} \longrightarrow \underline{B}$, where B is a domain in $\underline{\mathbb{C}}^d$, such that the following holds:

(i) There exists a closed complex subspace $\underline{\Delta} \hookrightarrow \underline{B}$ which is nowhere dense and has the property that $X - f^{-1}(\Delta)$ is dense in $X_0 := \{x' \in X \mid \dim_{x'} \underline{X} = d\}$. $\underline{\Delta}$ can be chosen to be a hypersurface, i.e. $\underline{\Delta} = \underline{N}(\delta)$ for a nonzero $\delta \in \mathcal{O}_{\mathbb{C}^d}(B)$.

(ii) $f|X - f^{-1}(\Delta) : X - f^{-1}(\Delta) \longrightarrow B - \Delta$ is a topological covering map.

(iii) If, in addition, $\underline{X}$ is reduced at x, i.e. the nilradical $\mathcal{N}_x$ of $\mathcal{O}_{\underline{X},x}$ is zero, $\underline{f}|\underline{X} - f^{-1}(\Delta) . \underline{X} - f^{-1}(\Delta) \longrightarrow \underline{B} - \Delta$ is a holomorphic covering of complex manifolds.

We call these representatives <u>good representatives</u>. $\underline{\Delta}$ is called a <u>discriminant locus</u> for $\underline{f}$.

I will not give the detailed proof here , but describe the main ingredients, so that the rest of it is a careful exploitation on the basis of the results described until now.

It is clear that it suffices to prove (i) and (iii) for a germ reduced at X , for we can pass from $(\underline{X},x)$ to $(\underline{X}_{red},x) \hookrightarrow (\underline{X},x)$ defined by $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathcal{N}_x$.

First, one treats the case of a Weierstrass map $\underline{\pi} : \underline{A} \longrightarrow \underline{B}$ (see 4.2.) with the additional property that the defining monic polynomials $\omega^{(j)} \in \mathcal{O}_{\mathbb{C}^d}(B)[w_j]$, $1 \le j \le k$, have no multiple factors.

Put $\tilde{\delta} := \prod_{j=1}^{k} \operatorname{discr}(\omega^{(j)})$, where $\operatorname{discr}(\omega^{(j)}) \in \mathcal{O}_{\mathbb{C}^d}(B)$ is the discriminant of $\omega^{(j)}$, and let $\underline{\Delta}(\pi) := \underline{N}(\tilde{\delta})$. Then Hensel's Lemma 4.2.3 tells us that around $z_0 \in B - \Delta(\pi)$ we can write

$$(6.3.1) \qquad \omega^{(j)}(z,w_j) = \prod_{\nu=1}^{b_j} (w_j - c_\nu^{(j)}(z)) \ , \ 1 \le j \le k$$

for holomorphic functions $c_\nu^{(j)}$ defined near z_0 . If $a = (z_0,c) \in A - \pi^{-1}(\Delta)$ and, for $1 \le j \le k$, ν_j is such that $c_{\nu_j}^{(j)}(z_0) = c_j$, this forces

$$(6.3.2) \qquad \mathcal{O}_{A,a} = \mathcal{O}_{\mathbb{C}^{d+k},a} / (w_1 - c_{\nu_1}^{(1)}(z), \ldots, w_k - c_{\nu_k}^{(k)}(z)) \quad ,$$

so that clearly $\pi_a^0 : \mathcal{O}_{\mathbb{C}^d,z_0} \longrightarrow \mathcal{O}_{A,a}$ is isomorphic. Hence $\underline{\pi}$ is locally isomorphic over $B - \Delta(\pi)$ by the Equivalence Theorem 3.3.3. This shows (iii).

(i) follows from the fact that $\tilde{\delta} \neq 0$ since the $\omega^{(j)}$ have no multiple factors, hence $\Delta(\pi)$ is nowhere dense in B by the identity theorem for holomorphic functions, and so $\pi^{-1}(\Delta(\ ))$ is nowhere dense in A , since π is open by Proposition 4.2.2.

For the general case of a reduced $(\underline{X},x)$ we may assume $\underline{f}$ is induced by a linear projection $pr : \mathbb{C}^{d+k} \longrightarrow \mathbb{C}^d$. With the notation of

4.4. we get the embedding (4.4.6) of $\underline{f}$ into a Weierstrass map, which, in addition, we may assume to be of the above type, since $\mathcal{O}_{X,x}$ has no nilpotents. Let

(6.3.3.) $$(X,x) = \bigcup_{\mathfrak{p}\in \mathrm{Min}(\mathcal{O}_{X,x})} (X_{\mathfrak{p}},x)$$

(6.3.4) $$(A,0) = \bigcup_{\mathfrak{p}\in \mathrm{Min}(\mathcal{O}_{A,0})} (A_{\mathfrak{q}},0)$$

be the decompositions into locally irreducible components according to 5.4.

Let $M_0 := \{\mathfrak{p}\in \mathrm{Min}(\mathcal{O}_{X,x}) \mid \dim_x X_{\mathfrak{p}} = d\} = \mathrm{Assh}(\mathcal{O}_{X,x})$ and $M_1 := \mathrm{Min}(\mathcal{O}_{X,x}) - M_0 = \{\mathfrak{p}\in \mathrm{Min}(\mathcal{O}_{X,x}) \mid \dim_x X_{\mathfrak{p}} < d\}$ (equality by Proposition 5.1.2 (ii)). Choosing X,A,B small enough one can achieve:

1) for each $\mathfrak{p}\in M_0$ there is exactly one $\mathfrak{q} =: \mathfrak{q}(\mathfrak{p}) \in \mathrm{Min}\,\mathcal{O}_{A,0}$ with $X_{\mathfrak{p}} = A_{\mathfrak{q}(\mathfrak{p})}$; this is by Theorem 5.4.2;

2) for all $\mathfrak{p}\in M_1$ and all $x'\in X_{\mathfrak{p}}$: $\dim_{x'} X_{\mathfrak{p}} < d$; this is by upper semicontinuity of dimension (Proposition 5.1.2 (iv));

3) $\Delta(\pi) \cup \bigcup_{\substack{\mathfrak{q},\mathfrak{q}'\in \mathrm{Min}(\mathcal{O}_{A,0})\\ \mathfrak{q}\neq\mathfrak{q}'}} \pi(A_{\mathfrak{q}} \cap A_{\mathfrak{q}'}) \cup (\bigcup_{\mathfrak{p}\in M_1} f(X_{\mathfrak{p}})) =: \Delta(f)$ is an analytic subset of B ; this is by Corollary 4.3.2.

4) $N(f) \subseteq N(\delta)$ for a nonzero $\delta \in \mathcal{O}_{\mathbb{C}^d}(B)$.

One checks that for this $\Delta := N(\delta)$ the conditions (i) and (iii) of the Local Representation Theorem hold; the main ingredient is the Open Lemma II,6.1.2.

<u>Remark 6.3.2.</u> For small enough representatives, $\Delta(f)$ can in fact be defined as a complex subspace since $\underline{\Delta}(\pi)$, $\underline{\pi}(\underline{A}_{\mathfrak{q}} \cap \underline{A}'_{\mathfrak{q}'})$ and $\underline{f}(\underline{X}_{\mathfrak{p}})$ exist naturally as complex subspace germs at $0\in B$, and so their union exists as a complex subspace germ defined by the intersection of the corresponding ideals in $\mathcal{O}_{B,0}$. Moreover

(6.3.5) $$\bigcup_{\substack{q,q'\in \mathrm{Min}(\mathcal{O}_{A,0}) \\ q\neq q'}} \pi(A_q \cap A_{q'}) \subseteq \Delta(\pi) \quad ,$$

(see II 2.2.1), so

(6.3.6) $$\Delta(f) = \Delta(\pi) \cup \bigcup_{p\in M_1} f(X_p) \quad ,$$

the natural choice.

<u>Remark 6.3.3</u>. Finally, I have to make a short remark on <u>prime germs</u>, i.e. $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$ with $\mathcal{O}_{X,x}$ an integral domain, so that especially $(\underline{X},x)$ is locally irreducible (see 5.4.). Let $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ be as in the Local Representation Theorem 6.2.1, then $f^0_x : \mathcal{O}_{\mathbb{C}^d,0} \hookrightarrow \mathcal{O}_{X,x}$ is an integral ring extension by the Integrality Theorem 4.4.1. Let $h \in \mathcal{O}_{X,x}$ be a primitive element for the corresponding field extension and form, for a suitably small representative $\underline{f} : \underline{X} \longrightarrow \underline{B}$:

$$\omega(z,t) := \prod_{x'\in f^{-1}(z)} (t - h(x')) \in \mathcal{O}_B(B - \Delta)[t] \quad .$$

Then $\omega(z,t)$ extends over Δ since Δ is nowhere dense in B by the classical Riemann Extension Theorem (for a nice proof of the latter see [30],p.9),and gives a monic irreducible polynomial $\omega \in \mathcal{O}_B(B)[t]$. The homomorphism

$$v^0_x := \varphi : \mathcal{O}_{B\times\mathbb{C},0} \longrightarrow \mathcal{O}_{X,x} \quad ,$$

which maps z_i to $f^0_x(z_i)$ for $1 \leq i \leq n$ and t to h_x ,annihilates ω , and so defines a morphism, via the Equivalence Theorem 3.3.3,

$$\begin{array}{ccccc}
(\underline{X},x) & \xrightarrow{\underline{v}} & (\underline{Y},y) := (\underline{N}(\omega),0) & \hookrightarrow & (\underline{B}\times\underline{\mathbb{C}},0) \\
 & \underline{f}\searrow & \swarrow\underline{\pi} & & \\
 & & (\underline{B},0) & &
\end{array}$$

from $\underline{f}$ into the Weierstrass mapgerm $\underline{\pi}$ given by the irreducible monic polynomial ω . It can be shown that v is isomorphic outside a nowhere dense closed subspace of $\underline{B}$ for suitable representatives (exercise; for a direct proof not using 6.3.1 see [40], § 46). If we replace $\underline{\Delta}$ of 6.3.1 with this subspace, $Y - \pi^{-1}(\Delta)$ is connected since ω is irreducible, and so we get

Corollary 6.3.4. *If, in the situation of* 6.3.1, $(\underline{X},x)$ *is a prime germ, i.e. reduced and locally irreducible,* $X - f^{-1}(\Delta)$ *is connected*.

§ 7. Coherence.

7.1. Coherent sheaves.

Definition 7.1.1. (i). Let R be a ring. A finitely presentable R-module M is called coherent if all its finitely generated submodules are also finitely presentable. R is called coherent if it is coherent as a module over itself, i.e. if every finitely generated ideal is finitely presentable.

(ii). Let $(X, \mathcal{O}_X)$ be a ringed space. An admissible $\mathcal{O}_X$-module $\mathcal{M}$ is called coherent if all its locally finitely generated submodules are also admissible. $\mathcal{O}_X$ is called coherent if it is coherent as a module over itself, i.e. if every locally finitely generated $\mathcal{O}_X$-ideal is admissible.

I discuss the notion of coherence for sheaves; the discussion for modules over a ring is analogous. The coherent $\mathcal{O}_X$-modules over a ringed space $(X, \mathcal{O}_X)$ form a good category Coh/X in the sense that it is stable under various operations on sheaves (called the "yoga of coherent sheaves", see [28], Annex). From this yoga one infers:

Lemma 7.1.2. Let $(X, \mathcal{O}_X)$ be a ringed space, $\mathcal{O}_X$ a coherent sheaf of rings. Then an $\mathcal{O}_X$-module is coherent if and only if it is admissible.

So in this case the admissible modules are the right category to work with, and, given a ringed space, the question is basic whether its structure sheaf is coherent. For complex spaces, the answer is given by the following famous theorem.

Theorem 7.1.3 (Oka's Coherence Theorem). For every complex space $(X, \mathcal{O}_X)$, $\mathcal{O}_X$ is a coherent sheaf of rings.

For a nice proof, which deduces this from the Weierstrass isomorphism 4.3.3, see [28], 2.5. Other proofs are in [64], Exposé 18, and [40], where it is deduced immediately, but in a not very enlightening way, from the classical Weierstrass Preparation Theorem 2.6.3.

So from now on we identify admissible and coherent $\mathcal{O}_X$-modules on a complex space.

7.2. Nonzerodivisors.

Oka's Coherence Theorem immediately entails:

Proposition 7.2.1. Let X be a complex space, $f \in \mathcal{O}_X(X)$. Then, if f is a nonzerodivisor at x , it is a nonzerodivisor near x .

See [28], p. 68, (or just look at the kernel of $\mathcal{O}_X \xrightarrow{\cdot f} \mathcal{O}_X$) .

7.3. Purity of dimension and local decomposition.

Let $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, and let

$$(7.3.1) \qquad (X,x) = \bigcup_{\mathfrak{p} \in \mathrm{Min}(\mathcal{O}_{X,x})} (X_{\mathfrak{p}},x)$$

be its decomposition into local irreducible components according to Proposition 5.4.1.

Definition 7.3.1 (X,x) is called equidimensional (or pure dimensional) if and only if $\dim_x X_{\mathfrak{p}} = \dim_x X$ for all $\mathfrak{p} \in \mathrm{Min}(\mathcal{O}_{X,x})$.

In terms of local algebra this means $\mathrm{Assh}(\mathcal{O}_{X,x}) = \mathrm{Min}(\mathcal{O}_{X,x})$.

Theorem 7.3.2 (Purity of dimension). Let the complex space X be equidimensional at x . Then it is equidimensional near x .

The proof is left as an exercise. For it, assume $\underline{X}$ is reduced at x and represent $(\underline{X},x)$ via $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^d,0)$ as in the Representation Theorem 6.3.1. Then $f_x^0(\delta)$ is a nonzerodivisor at x ; apply 7.2.1 and Ritt's Lemma 5.4.4. to conclude $X_0 = X$ near x .

Corollary 7.3.3 (Open Mapping Lemma). Let $\underline{f} : \underline{X} \longrightarrow \underline{B}$ be a finite morphism from the complex space $\underline{X}$ to an open subspace $\underline{B} \subset \mathbb{C}^d$. If $\underline{f}$ is open at $x \in \underline{X}$, and $\underline{X}$ is equidimensional at x , $\underline{f}$ is open near x .

This follows from the Purity Theorem 7.3.2. and the Dimension Theorem 6.2.1.

Corollary 7.3.4. In the decomposition (7.3.1), for suitably small representatives, $X_{\mathfrak{p}} \cap X_{\mathfrak{p}'}$ is nowhere dense in $X_{\mathfrak{p}}$ and $X_{\mathfrak{p}'}$, for all $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Min}(\mathcal{O}_{X,x})$ with $\mathfrak{p} \neq \mathfrak{p}'$.

Proof. Exercise; use 7.3.2 to conclude $\dim_{x'}(X_{\mathfrak{p}} \cap X_{\mathfrak{p}'}) < \dim_{x'}(X_{\mathfrak{p}})$ for x' near x.

7.4. Reduction.

The significance and importance of the notion of coherence cannot be described by a few words; they manifest themselves in the numerous results they imply. From this point on, coherence is indisputable for the further developments of the theory, which comprise coherence of the sheaf of nilpotents (Cartan's Coherence theorem), theory of reduction, analyticitiy of the singular locus, normalization . For this, see the book [28].

Theorem 7.4.1 (Cartan's Coherence Theorem). For every complex space $(X, \mathcal{O}_X)$, the nilradical $\mathcal{N}_X \subseteq \mathcal{O}_X$ is coherent.

For proofs see [28],[40], [64], and the sketch below.

Corollary 7.4.2. If A is an analytic set in the complex space $\underline{X}$, the vanishing ideal $\mathcal{J}_A$ (see I, 5.4.) is coherent and endowes A with the canonical structure of a reduced complex space. Especially the analytic set X has a canonical structure as a reduced complex space and is called the reduction $\underline{X}_{red}$ of $\underline{X}$; one has $\mathcal{O}_{X_{red}} = \mathcal{O}_X/\mathcal{N}_X$ by the Rückert Nullstellensatz 5.3.1.

Here a complex space is called reduced if all its local rings have no nilpotents.

Sketch of proof of 7.4.1.

The assertion is local; so let $(\underline{X},x) \in \underline{\underline{cpl}}_0$, and we must show that there is a representative $\underline{X}$ such that $\mathcal{N}_X$ is locally finite.

Assume first $\underline{X}$ is reduced at x . Choose a representative $\underline{X}$ and a finite map $\underline{f} : \underline{X} \longrightarrow \underline{B}$ as in the Local Representation Theorem 6.3.1. Let

$$I_0 := \bigcap_{\mathfrak{p} \in \mathrm{Assh}(\mathcal{O}_{X,x})} \mathfrak{p}$$

$$I_1 := \bigcap_{\mathfrak{p} \in \mathrm{Ass}(\mathcal{O}_{X,x}) - \mathrm{Assh}(\mathcal{O}_{X,x})} \mathfrak{p} \quad .$$

After possibly shrinking X , these define locally finite ideal sheaves $I_j \subseteq \mathcal{O}_{X,x}$ and so two closed complex subspaces $\underline{X}_j \overset{g_j}{\hookrightarrow} \underline{X}$ for $j = 0,1$. Then, and here Oka's Coherence Theorem comes in, $I_0 \cap I_1$ is locally finite; hence, since $(I_0 \cap I_1)_x = I_0 \cap I_1 = (0)$, we may assume $I_0 \cap I_1 = 0$ after eventually shrinking $\underline{X}$, by Lemma 3.1.1. Further shrinking $\underline{X}$ we may assume $\dim_{x'} \underline{X}_0 = d$ for all $x' \in X_0$ and $\dim_{x'} X_1 < d$ for all $x' \in X_1$ by Theorem 7.3.2 and Proposition 5.2.2 (iv).

Let $\underline{\Delta} = \underline{N}(\delta)$ be as in 6.3.1, with $\delta \in \mathcal{O}_{\mathbb{C}^d}(B)$. We may choose $\underline{X}$ so small that $f^0(\delta)$ is a nonzerodivisor in $\mathcal{O}_{X_0,x'}$ at any $x' \in X_0$, because it is a nonzerodivisor in $\mathcal{O}_{X_0,x}$, and we then apply Proposition 7.2.1. I then propose to show $N_X = 0$.

Let $x' \in X$. Choose, after possibly shrinking X , a locally finite ideal $J \subseteq N_X$ with $J_{x'} = N_{X,x'}$. Then $\mathrm{supp}\, J \subseteq \mathrm{supp} N_X \subseteq N(f^0(\delta))$, and so there is $t \in \mathbb{N}$, $t \geq 1$, such that $f^0(\delta)^t \cdot J = 0$ near x' by the Rückert Nullstellensatz in the form of Corollary 5.3.2
(ii). Hence $N_{X,x'} = J_{x'}$ is contained in

$I_{0,x'} = \mathrm{Ker}(\mathcal{O}_{X,x'} \xrightarrow{g^0_{0,x'}} \mathcal{O}_{X_0,x'})$, so $N_X \subseteq I_0$ near x' . Since $I_1 = \mathrm{Ker}(\mathcal{O}_X \xrightarrow{g^0_1} \mathcal{O}_{X_1})$, $N_X \cap \mathrm{Ker}\, g^0_1 = N_X \cap I_1 \subseteq I_0 \cap I_1 = 0$, and so g^0_1 injects N_X into N_{X_1} . Since $\dim_{x'} \underline{X}_1 < d$ for all $x' \in X_1$, $N_{X_1} = 0$ by the induction assumption, and so $N_X = 0$.

Finally, if $(\underline{X},x)$ is arbitrary, choose, after shrinking $\underline{X}$, a locally finite ideal $J \subseteq N_X$ with $J_x = N_{X,x}$. Let $\underline{Y}$ be the closed complex subspace of $\underline{X}$ defined by J . Then $\underline{Y}$ is reduced at x and so $N_Y = 0$ by what we proved above. But $N_Y = \sqrt{J}/J$, and so $N_X = J$, which is locally finite.

II. GEOMETRIC MULTIPLICITY.

The concept of multiplicity arises as a natural generalization of the multiplicity of a solution to a polynomial equation in one indeterminate. Consider a system

(1) $$f_j(z_1,\dots,z_n) = 0 \quad , \quad j = 1,\dots,k$$

of holomorphic equations, and suppose $0 \in \mathbb{C}^n$ is a solution. Heuristically, the multiplicity of 0 as a solution should be the number of solutions "concentrated near 0", i.e. the algebraic number m of distinct generic solutions arbitrarily near to 0 (cf. [51], p. 17, Definition). Symbolically:

(2) $$m = \varinjlim_{U} (\sup \# \{z \in U \mid f_j(z) = 0 \ , \ j = 1,\dots,k \ \text{"distinct" solutions}\}) \quad ,$$

where U runs over the neighbourhoods of 0 in $\mathbb{C}^n$, and the solutions are properly counted. In modern terms, the $f_1,\dots,f_k$ define an ideal $I \in \mathcal{O}_{\mathbb{C}^n,0}$ and so a germ $(\underline{X},x) \in \underline{\mathrm{cpl}}_0$, and the multiplicity in question is called <u>the multiplicity of</u> x <u>on</u> $\underline{X}$, <u>denoted</u> $m(\underline{X},x)$.

To clarify what this means, consider the corresponding algebraic situation, where the f_j above are polynomials in $\Bbbk[z_1,\dots,z_n]$ for some field $\Bbbk$. Kronecker's elimination theory ([43], [42], and [51], which is, in a sense, still quite readable and has become a classic) represents the solutions, after a general linear coordinate transformation, as algebraic functions of some of the coordinates, $z_1,\dots,z_d$ say, which act as free parameters. The correct definition of the global multiplicity, i.e. the algebraic number of distinct generic solutions, was debated quite a time after Kronecker's 1882 paper [43] (see e.g. [42]) and found 30 years later by Macaulay [50]. In modern terms:

(3) $$\begin{aligned} M &:= \dim_K K \otimes_{\Bbbk} R \\ &= \sum_{\mathfrak{p} \in \mathrm{Assh}(R)} \mathrm{length}(R_{\mathfrak{p}}) \cdot [R/\mathfrak{p} : K] \end{aligned}$$

with $K := \mathbb{k}(z_1,\ldots,z_d)$ and $R := \mathbb{k}[z_1,\ldots,z_n]/(f_1,\ldots,f_k)$, a natural generalization, after all, of the case of one variable. (It is interesting to look at the attempts in [42] to define the correct coefficient of $[R/\mathfrak{p}:K]$ via the degrees of the factors of the resolvent and Macaulay's criticism of it in [50]. This is a good lesson how painfully and slowly concepts developed which nowadays are considered to be utterly self-explanatory and trivial. This applies equally well to primary decomposition and the notion of local multiplicity below).

Geometrically, this corresponds to representing the solution variety $X \subseteq A^n$ as branched cover

(4) $$\pi : X \longrightarrow \mathbb{A}^d \quad , \quad d = \dim X = \dim R$$

with π induced by a generic projection, and putting

(5) $$M := \text{algebraic global mapping degree of } \pi$$
$$= \sum_\lambda \ell_\lambda \cdot \#(\pi^{-1}(z) \cap X_\lambda) \quad ,$$

where the X_λ are the irreducible components of X , $\ell_\lambda = \text{length}\,\mathcal{O}_{X,X_\lambda}$ and $z \in \mathbb{A}^d$ is any point outside the image of the branching locus (a "generic" z). (That (3) and (5) agree will be proved, in a local version, in 5.1.4 below).

The local multiplicity $m(X,x)$ of X at x , then, should be the <u>local</u> mapping degree of a generic projection. This means one wishes to take a small neighbourhood U around x such that $\pi(U)$ is open in $\mathbb{A}^d$ and $\pi^{-1}\pi(x) \cap U = \{x\}$; then $m(X,x)$ should be

(6) $$m(X,x) = \sum_\lambda \ell_\lambda \cdot \#(\pi^{-1}(z) \cap U_\lambda)$$

where the U_λ are the local branches of X at x and ℓ_λ the length of a maximal primary chain starting at the primary defining U_λ , which measures the multiplicity of the generic solution on U_λ .

Unfortunately, there are no small neighbourhoods in the algebraic situation, and so it took several decades to master the concept of multiplicity. There are three ways out of this difficulty:

(i) One tries to make sense out of the limit process in (1) algebraically, i.e. out of the concept of "solutions coming together at 0 ". This leads to the theory of specialization multiplicity of v.d. Waerden and Weil ([72], [73], and [74]). This will not be touched further upon here.

(ii) One passes to formal ("infinitesimal") neighbourhoods via completion; then the analogue of the local mapping degree makes sense. This leads to the definition of Chevalley ([9], [10]; see also Chapter 1, (6.7), and 5.1.5 and 5.1.8 below).

(iii) One uses the sophisticated approach to define multiplicity via the highest coefficient of the Hilbert function of the associated graded ring; this is the definitive and commonly accepted definition of Samuel [60]. It has the advantage of being concise, and it works very well in the practice of algebraic manipulations. (Ultimately, it leads via Serre's notes [67] and the paper of Auslander and Buchsbaum on codimension and multiplicity (Ann. of Math. 68 (1958), 625-657, esp. Theorem 4.2) to the definition presented in Chapter I, (1.2).) Although the geometric significance of this definition must have been known to the experts, it seems to have been rarely explicited (it was already known to Macaulay, see [50], footnotes on p.82 and 115, and [37], which makes quite a tense reading). It corresponds, geometrically, to approximating X at x by its tangent cone and taking the local multiplicity of the tangent cone at its vertex; for cones, the problem of small neighbourhoods does not pose itself, since the local and global mapping degree of a projection of a cone agree, due to the latter's homogeneous structure.

Fortunately, small neighbourhoods do exist in Complex Analytic Geometry, and so the definition of multiplicity as the local mapping degree of a generic projection makes perfect sense; this must have been, in the reduced case, folklore ever since (cf. [13], [38] and [75]). This formalism is set up in the first three paragraphs of this part II. To handle the nonreduced case, we make use of the properties of compact Stein neighbourhoods to relate the properties of nearby analytic local rings to those of one algebraic object, the coordinate ring of the compact Stein neighbourhood; this guarantees the constancy of the numbers ℓ_λ in (6) along the local brances U_λ . This is exposed in § 1. In

§ 2, we define the local mapping degree, and in § 3 the geometric multiplicity $m(\underline{X},x)$ of $(\underline{X},x) \in \underline{\underline{cpl}}_0$. In § 4, we explain the geometry of Samuel multiplicity alluded to above, and in the last paragraph we prove that the local mapping degree definition of the multiplicity of $(\underline{X},x) \in \underline{\underline{cpl}}_0$ coincides with the Samuel multiplicity $e(\mathcal{O}_{X,x})$ of the corresponding local ring.

This geometric description of multiplicity will then be put to work in the next chapter, since it is basic for geometric proofs of equimultiplicity results due to Hironaka, Lipman, Schickhoff, and Teissier.

§ 1. Compact Stein neighbourhoods.

1.1. Coherent sheaves on closed subsets.

Let $\underline{X}$ be a complex space and $A \subseteq X$ a closed set.

Definition 1.1.1. A coherent module on A is a sheaf of the form $\tilde{\mathcal{M}}|A$, where $\tilde{\mathcal{M}}$ is a coherent $\mathcal{O}_V$-module on some open neighbourhood V of A.

Here, $\tilde{\mathcal{M}}|A$ is the restriction in the sense of sheaves of abelian groups, in other words, for $U \subseteq A$ open in A, $(\tilde{\mathcal{M}}|A)(U)$ are the continuous sections of the "espace êtalê" associated to $\tilde{\mathcal{M}}$ over A. It is not to be confused with the coherent $\mathcal{O}_A$-module $i^*\tilde{\mathcal{M}}$ if $\underline{i}$: $\underline{A} \hookrightarrow \underline{V}$ happens to be a closed complex subspace, so in this case one has to distinguish between "coherent modules on A" and "coherent $\mathcal{O}_A$-modules". Especially, we have to distinguish $\mathcal{O}|A := \mathcal{O}_X|A$ and $\mathcal{O}_A$ in this case.

Directly from the definitions and the "yoga of coherent sheaves" the following simple lemma follows:

Lemma 1.1.2. If $\mathcal{M}, \mathcal{N}$ are coherent modules on A, and $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ is a homomorphism of $\mathcal{O}|A$-modules, then $\mathcal{K}er(\alpha)$ and $\mathcal{C}oker(\alpha)$ are coherent modules on A.

1.2. Stein subsets.

In the following I assume known the simplest properties of sheaf cohomology groups for sheaves of abelian groups. They can be defined as the higher right derived functors of the section functor. On paracompact spaces they can be computed by the Čech procedure (based on alternating cochains), and on complex manifolds by the Dolbeault cohomology of (p,q)-forms (see [39], [27], [40], and [30], at least in the locally free case).

The notion of Stein subsets is closely related to the following three statements, which have their traditional names. Let $A \subseteq X$ be a closed set in a complex space.

"Theorem A". Any coherent module on A is generated by its global sections.

"Theorem B". $H^q(A,\mathcal{M}) = 0$ for all coherent modules $\mathcal{M}$ on A and all $q \geq 1$.

"Theorem F". If $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ is a surjective homomorphism of coherent modules on A , $\alpha_A : \mathcal{M}(A) \longrightarrow \mathcal{N}(A)$ is surjective.

The long exact cohomology sequence gives immediately:

Proposition 1.2.1. Theorem B implies Theorem A and Theorem F.

Definition 1.2.2. Let X be a complex space. A closed subset $A \subseteq X$ is called a Stein subset if and only if Theorem B holds for A .

In a sense, a Stein subset should be thought of as the analogue of an affine set in the case of algebraic varieties, so there should be a correspondence between coherent modules on them and modules over the coordinate ring. For this however, we have to make an additional compactness assumption, which we do in the following section.

1.3. Compact Stein subsets and the Flatness Theorem.

Let now $A = K \subseteq X$ be a compact subset. It is then easy to see that in this case the coherent modules on K are just the finitely presented $\mathcal{O}|K$-modules. Using this and standard arguments based on Proposition 1.2.1, one gets the following proposition, which states that compact Stein neighbourhoods are the appropriate analogues of the affine subsets in the algebraic case. Let $\mathcal{O}(K) := \Gamma(K,\mathcal{O}_X)$.

Proposition 1.3.1. Let X be a complex space, $K \subseteq X$ a compact Stein subset. Let $coh(K)$ be the category of coherent modules on K , and $adm(\mathcal{O}(K))$ the category of admissible, i.e. finitely presented,

$\mathcal{O}(K)$-modules. Then:

(i) $\mathcal{O}(K)$ is a coherent ring (cf. I 7.1.1. (i));

(ii) the section functor induces a natural equivalence:

(1.3.1) $$\Gamma : \underline{\underline{\mathrm{coh}}}(K) \longrightarrow \underline{\underline{\mathrm{adm}}}(\mathcal{O}(K))$$, which has

(1.3.2) $$(-) \otimes_{\mathcal{O}(K)} (\mathcal{O}|K) : \underline{\underline{\mathrm{adm}}}(\mathcal{O}(K)) \longrightarrow \underline{\underline{\mathrm{coh}}}(K)$$ as an inverse.

Theorem 1.3.2 (Flatness Theorem). Let K be a Stein compact subset in the complex space X. Then, for any $x \in K$, the natural morphism

(1.3.3) $$\lambda_x : \mathcal{O}(K) \longrightarrow \mathcal{O}_{X,x}$$
$$f \longmapsto f_x$$

is flat.

This follows from Proposition 1.3.1, because the section functor is exact by Theorem B, and hence so is $(-) \otimes_{\mathcal{O}(K)} (\mathcal{O}|K)$.

Remark 1.3.3. In the case where X is an algebraic variety (by this I mean an algebraic scheme of finite type over a field) and K is an affine set, the analogue of Theorem 1.3.2 is immediate, since λ_x is just the algebraic localization of $\mathcal{O}(K)$ with respect to the prime ideal corresponding to x. In this case, the local rings $\mathcal{O}_{X,x}$ are "semiglobal" in the sense that any element is a quotient of two sections defined on the whole of K. In the complex analytic case, λ_x does not arise by this simple construction, and, moreover, one has to work with compact Stein subsets, which makes the result much harder; we are going to show in the next section that sufficiently small compact Stein neighbourhoods always exist.

1.4. Existence of compact Stein neighbourhoods.

The theory of Stein spaces is concerned with various criteria which characterize Stein subsets (or Stein spaces). The basic reference for this is the book [27], of which I will need only the first three chapters. Fundamental for the theory is the following Theorem 1.4.1, which goes back to Cartan and Serre; it directly implies the existence of compact Stein neighbourhoods (Corollary 1.4.2) needed for the applications of Theorem 1.3.2 in the sequel, e.g. for Definition 2.2.6 and for the proofs of Theorem 5.1.4 and Theorem 5.2.1.

A <u>compact stone</u> in $\mathbb{C}^n$ with coordinates $(z_1,\dots,z_n)$ will be a compact interval in the space $\mathbb{R}^{2n}$ with coordinates $(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \dots, \mathrm{Re}\, z_n, \mathrm{Im}\, z_n)$.

<u>Theorem 1.4.1</u>. <u>A compact stone in</u> $\mathbb{C}^n$ <u>is a Stein subset</u>.

A detailed and clear proof of this is in Chapter III of [27]. Since the result is so basic, I give a short summary of the strategy of the proof. It is considerably more difficult than the proof of the corresponding statement for affine sets, which ultimately rests on localization of rings, a technique which one has not at its disposal in Complex Analytic Geometry, since the coherent sheaves on smaller open subsets of Stein subsets do not arise by localization. Complex analysis ultimately shows up by solving the $\bar\partial$-equation.

<u>1^{st} Step</u>. There are two basic Vanishing Theorems for compact stones. One is elementary and uses simple combinatorical arguments on subdivisions of stones together with alternating Čech cochains to show that $\exists\, q_0 = q_0(n)$ with $H^q(Q,\mathcal{S}) = 0$ for $q \geq q_0$ and all sheaves $\mathcal{S}$ on Q . The other lies deeper and uses Dolbeault cohomology; by explicitely solving the $\bar\partial$-equation (in the so-called $\bar\partial$-Poincaré-Lemma due to Grothendieck, see [27], II, § 3) one shows that $H^q(Q,\mathcal{O}) = 0$ for $q \geq 1$. These two Vanishing Theorems show that Theorem A implies Theorem B for compact stones, and so it suffices to show Theorem A for compact stones. ([27], III, § 3.2).

<u>2nd Step</u>. Theorem A is proven by induction on the <u>real</u> dimension d of the compact stone Q . If A_d , B_d, and F_d are the statements of Theorem A, Theorem B, and Theorem F for compact stones of dimension $\leq d$, it suffices by the first step and Proposition 1.2.1. to prove

$$A_{d-1} \text{ and } F_{d-1} \Rightarrow A_d \qquad (1.4.1)$$

3rd Step. Since sections of sheaves over a compact set extend over an open neighbourhood, one easily sees that by subdividing a one dimensional side of the d-dimensional stone Q into sufficiently small pieces the claim follows if we are able to deal with the following situation. Suppose $Q = Q^- \cup Q^+$ arises by cutting Q into two halves by a section orthogonal to a one-dimensional side (see Figure 3).

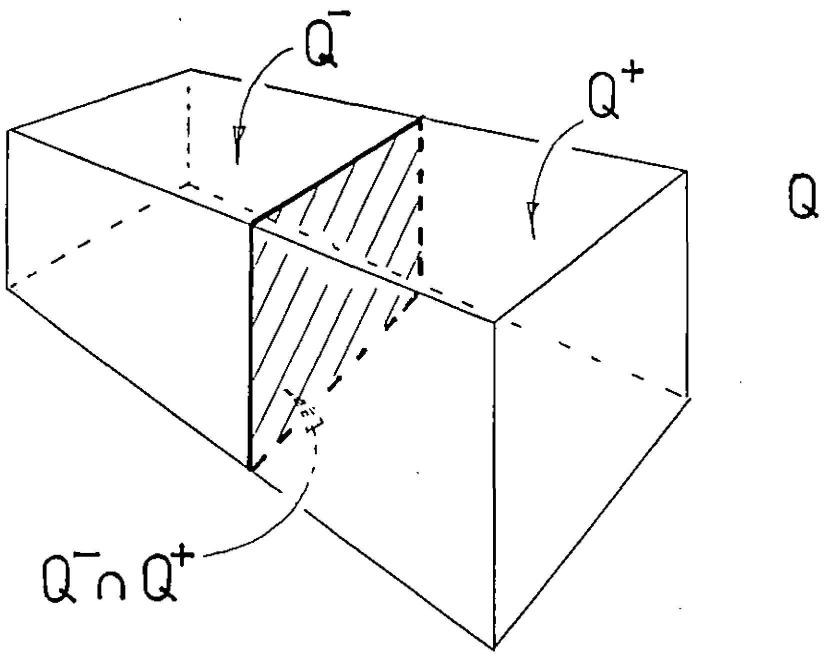


Fig. 3

Let $\mathcal{M}$ be a coherent module on Q, $\mathcal{O} := \mathcal{O}|Q$, and suppose there are given $\mathcal{O}$-module epimorphisms $h^- : \mathcal{O}^p|Q^- \twoheadrightarrow \mathcal{M}|Q^-$, $h^+ : \mathcal{O}^q|Q^+ \twoheadrightarrow \mathcal{M}|Q^+$ such that the images of h^- and h^+ generate the same subsheaf of $\mathcal{O}^p|Q^- \cap Q^+$. We then want to glue h^- and h^+ into an $\mathcal{O}$-module epimorphism $\mathcal{O}^{p+q} \twoheadrightarrow \mathcal{M}$; this will then complete step 2. Let $t_1^-, \dots, t_p^- \in \mathcal{M}(Q^-)$ and $t_1^+, \dots, t_q^+ \in \mathcal{M}(Q^+)$ be the sections defining h^- and h^+. Then one can write

$$(1.4.2) \qquad \begin{pmatrix} t_1^- \\ \cdot \\ \cdot \\ \cdot \\ t_p^- \end{pmatrix}^T \Bigg|_{Q^- \cap Q^+} = \begin{pmatrix} t_1^+ \\ \cdot \\ \cdot \\ \cdot \\ t_q^+ \end{pmatrix}^T \Bigg|_{Q^- \cap Q^+} \cdot A$$

with a matrix $A \in M(q \times p, \mathcal{O}|Q^- \cap Q^+)$. Now suppose we could find holomorphic invertible matrices $C^\pm \in GL(Q^\pm, \mathcal{O})$ such that

(1.4.3) $\quad \mathbb{1}_p \mid Q^- \cap Q^+ = (C^- \mid Q^- \cap Q^+) \cdot (C^+ \mid Q^+ \cap Q^+)$, where

$\mathbb{1}_p \in GL(p,\mathcal{O})$ is the identity matrix. This would imply

(1.4.4)
$$\begin{pmatrix} t_1^- \\ \vdots \\ t_p^- \end{pmatrix}^T \cdot C^- \Big|_{Q^- \cap Q^+} = \begin{pmatrix} t_1^+ \\ \vdots \\ t_q^+ \end{pmatrix}^T \Bigg|_{Q^{-1} \cap Q^+} \cdot A \cdot (C^+)^{-1} \Big|_{Q^- \cap Q^+}$$

So, if we then define new sections $\tilde{t}_1^-,\ldots,\tilde{t}_p^- \in \mathcal{M}(Q^-)$ via

$$\begin{pmatrix} \tilde{t}_1^- \\ \vdots \\ \tilde{t}_p^- \end{pmatrix}^T := \begin{pmatrix} t_1^- \\ \vdots \\ t_p^- \end{pmatrix}^T \cdot \bar{C} ,$$

they still define an epimorphism $\tilde{h}^- : \mathcal{O}^p|Q^- \twoheadrightarrow \mathcal{M}|Q^-$, since C^- is invertible. Now make the

(1.4.5) <u>assumption</u>: A extends over Q .

Then one could extend the sections $\tilde{t}_1^-,\ldots,\tilde{t}_p^-$ to sections $\tilde{t}_1,\ldots,\tilde{t}_p$ over Q by (1.4.4), and this would give an $\mathcal{O}$-homomorphism $\tilde{h}^- : \mathcal{O}^p \longrightarrow \mathcal{M}$ which restricts to an epimorphism over Q^- . In the same way one would produce an $\mathcal{O}$-homomorphism $\tilde{h}^+ : \mathcal{O}^q \longrightarrow \mathcal{M}$ which restricts to an epimorphism over Q^+ . Then $h := \tilde{h}^- \oplus \tilde{h}^+ : \mathcal{O}^{p+q} \twoheadrightarrow \mathcal{M}$ would be the desired epimorphism.

<u>Last Step</u>. (1.4.5) does not hold in general. One has to approximate A by a holomorphic matrix $\hat{A}$ defined on Q , which can be done via an appproximation theorem of Runge; this then forces to have a decomposition (1.4.3) not only of 1_p ,but of holomorphic $p \times p$ - matrices close to $\mathbb{1}_p$. That this can be done is the content of the famous Cartan Patching Lemma [27],III, § 1,3. This Lemma is, by a delicate interation procedure, reduced to an additive decomposition of holomorphic functions on an open polycylinder which itself is a union of two open polycylinders, the so-called Cousin Patching Lemma [27],III, § 1,1.

This Lemma, finally, is proven by explicitely solving the $\bar{\partial}$-equation. All details are in §§ 1 and 2 of Chapter III of [27].

Corollary 1.4.2. Let X be a complex space. Then any $x \in X$ has a neighbourhood basis consisting of compact Stein subsets. For this, one can take the compact sets in the inverse image of the system of compact stones 0 in $\mathbb{C}^n$ under any local immersion $(X,x) \overset{i}{\hookrightarrow} (\mathbb{C}^n,0)$.

Proof. Let $\underline{X} \hookrightarrow \underline{U}$ be a closed complex subspace of an open set $U \subseteq \mathbb{C}^n$, $x = 0 \in X \subseteq U$. Let K be a compact polydisc centered at 0 . Let M be a coherent module on $K \cap X$. After possibly shrinking U , we may assume M is the restriction of a coherent $\mathcal{O}_X$-module $\widetilde{M}$. Then $i_*\widetilde{M}$ is a coherent $\mathcal{O}_U$-module, and so $H^p(X \cap K, M) = H^p(K, i_*\widetilde{M}) = 0$ for $p \geq 1$, since K is Stein by Theorem 1.4.1.

§ 2. Local mapping degree.

In this paragraph, I assign to each finite mapgerm $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^d,0)$ a local mapping degree $\deg_x \underline{f} \in \mathbb{N}_{>0}$, which counts the algebraic number of preimages of a "general" point of $\mathbb{C}^d$ close to 0 . This will be basic for the definition of multiplicity.

2.1. Local decomposition revisited.

In order to count the number of preimages of such an $\underline{f}$ as above algebraically, I have to weight a preimage point lying on a local irreducible component where $\underline{X}$ is possibly not reduced by a certain positive number, which will appear as the value of some locally constant function along a generic subset of that component; here, I call a subset of a topological space generic if it contains an open dense subset. It is the purpose of this section to exhibit such generic subsets.

First I introduce some terminology. Let $\underline{X}$ be a complex space, $x \in X$. Define the germ $(\underline{X}_{red},x)$ as in I, 5.1.2 (iii). We then have the following loci:

$$(2.1.1)\qquad X_{reg} := \left\{x \in X \mid (\underline{X}_{red}, x) \text{ is smooth}\right\} = \left\{x \in X \mid \mathcal{O}_{X,x}/\mathcal{N}_x \text{ is regular}\right\}$$

$$(2.1.2)\qquad X_{ir} := \left\{x \in X \mid (\underline{X}, x) \text{ is irreducible}\right\} = \left\{x \in X \mid \mathcal{O}_{X,x}/\mathcal{N}_x \text{ is an integral domain}\right\} .$$

Obviously,

$$(2.1.3)\qquad X_{reg} \subseteq X_{ir} .$$

Now let $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, and let $\underline{X}$ be a good representative, i.e. there should be a finite map from $\underline{X}$ to $\underline{B}$, a domain in $\mathbb{C}^d$ satisfying the Local Representation Theorem I 6.3.1. Let

$$(2.1.4)\qquad X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

be the local decomposition of $(\underline{X},x)$ into irreducible components as in I 5.4. This decomposition has the following properties:

Proposition 2.1.1. There are arbitrarily small good representatives X such that the following statements hold:

(i) $X_\lambda \cap X_\mu$ is nowhere dense in X_λ for all $\lambda \in \Lambda$ and all $\mu \in \Lambda$ with $\mu \neq \lambda$.

(ii) X is locally reducible at all points of $\bigcup_{\substack{\lambda,\mu \in \Lambda \\ \lambda \neq \mu}} (X_\lambda \cap X_\mu)$.

Proof.
(i) is just I 7.3.4., and (ii) follows from (i) and elementary properties of the local decomposition of analytic sets (see [28], p. 108).

Corollary 2.1.2. Let the notations be as in Proposition 2.1.1. Put

(2.1.5) $$X_\lambda^0 := X - \bigcup_{\mu \neq \lambda} X_\mu \quad .$$

Then, for all $\lambda \in \Lambda$:

(2.1.6) X_λ^0 is connected, open and dense in X_λ , and open in X ;

(2.1.7) $X_\lambda \cap X_{ir} = (X_\lambda^0)_{ir}$ is connected, and this set is generic in X_λ ;

(2.1.8) $$X_{ir} = \coprod_{\lambda \in \Lambda} (X_\lambda \cap X_{ir}) \quad .$$

Proof. X_λ^0 is clearly open both in X and X , since $\bigcup_{\mu \neq \Lambda} X_\mu$ is closed as a finite union of analytic sets. It is dense by Proposition 2.1.1 (i). Let $\underline{f}_\lambda : \underline{X}_\lambda \longrightarrow \underline{B}_\lambda$ satisfy the assumption of the Local Representation Theorem I 6.3.1. So, after possibly shrinking $\underline{f}_\lambda$, $\underline{f}_\lambda$ is open by the Open Mapping Lemma I 7.3.3, and therefore $f_\lambda^{-1}(\Delta_\lambda)$ is nowhere dense in X_λ , as Δ_λ is nowhere dense in B_λ . This shows that $X_\lambda - f_\lambda^{-1}(\Delta_\lambda)$ is open and dense in X_λ , and it is connected by I 6.3.4. Since $X_\lambda - f_\lambda^{-1}(\Delta_\lambda) \subseteq X_\lambda^0$ for some $\Delta_\lambda \subseteq X_\lambda^0$, this shows X_λ^0 is connected, and dense in X_λ . Finally, $X_\lambda \cap X_{ir} = (X_\lambda^0)_{ir}$ follows from Proposition 2.1.1 (ii), and so $X_\lambda \cap X_{ir}$, containing $X_\lambda - f_\lambda^{-1}(\Delta_\lambda)$, is generic in X_λ , and connected. (2.1.8) finally is obvious from $X = \bigcup_{\lambda \in \Lambda} X_\lambda$.

Remark 2.1.3. One has, again by Proposition 2.1.1 (ii), that $X_\lambda \cap X_{reg} = (X_\lambda^0)_{reg}$, and that $(X_\lambda^0)_{reg}$, containing $X_\lambda - f_\lambda^{-1}(\Delta_\lambda)$, is generic in X_λ . Using the Jacobian criterion for regularity one may show it is the complement of a nowhere dense analytic set in X_λ . It follows that $X_{reg} = \coprod_{\lambda \in \Lambda} (X_\lambda^0)_{reg}$ is the complement of a nowhere dense analytic set in X . This implies that for any $\underline{X} \in \underline{cpl}$ the locus X_{reg} is also the complement of a nowhere dense analytic set.

Remark 2.1.4. Using the local results above, one can show the following. Let $\underline{X}$ be any complex space. Decompose X_{reg} into connected components:

$$X_{reg} = \coprod_{\lambda \in \Lambda} Z_\lambda$$

and put $X_\lambda := \overline{Z_\lambda}$. The decomposition

$$X = \bigcup_{\lambda \in \Lambda} X_\lambda$$

then will satisfy Corollary 2.1.2. Moreover, this decomposition is unique and characterized by the fact that it is a decomposition of X into irreducible analytic sets, i.e. analytic sets which cannot be written as a proper union of analytic sets. We call this decomposition the decomposition of X into (global) irreducible components. Locally this decomposition induces the decomposition given by the local decomposition into irreducible analytic setgerms. (See [40], § 49). So in the local situation above, the decomposition (2.1.4) is indeed the decomposition into global irreducible components and we will call it so, but we will make use only of the properties in Corollary 2.1.2.

2.2. Local mapping degree.

We first introduce the weights with which to count preimage points. Let R be a noetherian ring, $Ac(R)$ the set of active elements. Since

(2.2.1) $$Ac(R) = \bigcap_{\mathfrak{p} \in Min(R)} (R - \mathfrak{p})$$

by I 5.2.1, $Ac(R)$ is a multiplicative subset, and we can form the localization of R with respect to $Ac(R)$.

Definition 2.2.1. $\widetilde{Quot}(R) := (Ac)^{-1}R$ is called the modified ring of fractions of R .

Lemma 2.2.2. $\widetilde{Quot}(R)$ has the following properties:

(i) $\widetilde{\mathrm{Quot}}(R)$ <u>is artinian, and</u> $\mathrm{length}(\widetilde{\mathrm{Quot}}(R)) = \sum_{\mathfrak{p}\in \mathrm{Min}(R)} \mathrm{length}(R_{\mathfrak{p}})$;

(ii) <u>if</u> R <u>has no embedded primes</u>, $\widetilde{\mathrm{Quot}}(R) = \mathrm{Quot}(R)$, <u>the usual total ring of fractions of</u> R .

<u>Proof</u>. <u>(i)</u>: All primes of $\widetilde{\mathrm{Quot}}(R)$ are minimal by construction, so $\widetilde{\mathrm{Quot}}(R)$ is artinian. By the well-known structure of artinian rings (see [6], Chapter IV, § 2.5, Corollary 1 of Proposition 9).

$$S := \widetilde{\mathrm{Quot}}(R) \cong \prod_{\tilde{\mathfrak{p}}\in \mathrm{Min}(S)} S_{\tilde{\mathfrak{p}}} = \prod_{\mathfrak{p}\in \mathrm{Min}(R)} R_{\mathfrak{p}} ,$$

and so $\mathrm{length}(\widetilde{\mathrm{Quot}}(R)) = \sum_{\mathfrak{p}\in \mathrm{Min}(R)} \mathrm{length}(R_{\mathfrak{p}})$.

<u>(ii)</u>: In this case, $\mathrm{Ac}(R) = R - \bigcup_{\mathfrak{p}\in \mathrm{Ass}(R)} \mathfrak{p}$ is the set of nonzero-divisors of R .

<u>Proposition 2.2.3</u>. <u>Let</u> $\underline{X}$ <u>be a complex space</u>, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ <u>the decomposition into irreducible components. Then for any</u> $x \in X_{ir}$ <u>the modified ring of fractions</u> $\widetilde{\mathrm{Quot}}(\mathcal{O}_{X,x})$ <u>is of finite length</u>, <u>and the function</u> $x \longmapsto \mathrm{length}(\widetilde{\mathrm{Quot}}(\mathcal{O}_{X,x}))$ <u>is, for each</u> λ , <u>constant along the generic subset</u> $X_\lambda \cap X_{ir}$ of X_λ .

<u>Proof</u>. $\widetilde{\mathrm{Quot}}(\mathcal{O}_{X,x})$ is artinian by Lemma 2.2.2, so is of finite lenght. Since $x \in X_{ir}$, $\widetilde{\mathrm{Quot}}(\mathcal{O}_{X,x}) = (\mathcal{O}_{X,x})_{\mathcal{N}_x}$. So, because of (2.1.7), it suffices to prove that the function $x \longmapsto \mathrm{length}((\mathcal{O}_{X,x})_{\mathcal{N}_x})$ is locally constant. Let $x \in X_{ir}$ and fix a compact Stein neighbourhood K of x according to Corollary 1.4.2. From the construction there one sees that one can take K so that it has a fundamental system of open neighbourhoods $(U_\alpha)_{\alpha\in A}$ such that each U_α is irreducible and $U_\alpha \subseteq X_\lambda$, where λ is the unique $\mu \in \Lambda$ such that $x \in X_\mu^0$ by (2.1.8). Since $x \in X_\lambda^0$, and X_λ is open in X , we may, replacing $\underline{X}$ by a small open subspace contained in X_λ^0 , forget about λ and assume $X = X_\lambda$. Now, by I Corollary 7.4.2, X has the structure of a complex space $\underline{X}_{red}$ by putting $\mathcal{O}_{X_{red}} := \mathcal{O}_X/\mathcal{N}_X$. Let N be the $\mathcal{O}(K)$-ideal $\mathcal{N}_X(K) = \Gamma(K,\mathcal{N}_X)$. I claim N is prime. Since the section functor is exact by Proposition 1.3.1 (ii) (or Theorem B),

$$\Gamma(K,\mathcal{O}_{X_{red}}) = \Gamma(K,\mathcal{O}_X) / \Gamma(K,\mathcal{N}_X) \quad .$$

But

$$\Gamma(K,\mathcal{O}_{X_{red}}) = \varinjlim_{\alpha\in A} \Gamma(U_\alpha,\mathcal{O}_{X_{red}}) \quad ,$$

and the $\Gamma(U_\alpha,\mathcal{O}_{X_{red}})$ are integral domains because the U_α are irreducible, so $\Gamma(K,\mathcal{O}_{X_{red}})$ is an integral domain, and N is indeed prime. Now the natural morphism

(2.2.2) $$\lambda_{x'} : \Gamma(K,\mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x'}$$

is flat for all $x'\in K\cap X_{ir}$ by Theorem 1.3.2. The ideal N generates in $\mathcal{O}_{X,x'}$ the ideal $\mathcal{N}_{x'}$ via $\lambda_{x'}$ because of Proposition 1.3.1. Localizing (2.2.2) at N gives that

(2.2.3) $$(\lambda_{x'})_N : \Gamma(K,\mathcal{O}_X)_N \longrightarrow (\mathcal{O}_{X,x'})_{\mathcal{N}_{x'}}$$

is flat, since flatness localizes. Hence (2.2.3) is faithfully flat, being a flat local morphism of local rings. Pushing composition series of $\Gamma(K,\mathcal{O}_X)_N$ to $(\mathcal{O}_{X,x'})_{\mathcal{N}_{x'}}$ then shows by standard arguments

(2.2.4) $$\text{length}((\mathcal{O}_{X,x'})_{\mathcal{N}_{x'}}) = \text{length}(\Gamma(K,\mathcal{O}_X)_N)$$

(see the following Lemma 2.2.4). But the right hand side does not depend on x' , and this shows the Proposition.

From the literature, I cite the following lemma.

<u>Lemma 2.2.4</u>. ([31], Chapter O, Corollary (6.6.4)). <u>Let</u> $\rho : A \longrightarrow B$ <u>be a local flat homomorphism of local rings,</u> M <u>an</u> A-<u>module. Then</u>

$$\text{length}_B(M \otimes_A B) = \text{length}_A(M)\cdot\text{length}(B/m_A B)$$

in the sense that the left side is finite if and only if the right hand side is finite, and then the equality holds.

We now consider finite mapgerms $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^d,0)$ and choose a good representative $\underline{f} : \underline{X} \longrightarrow \underline{B}$, which here is defined to mean

(i) B is a domain in $\mathbb{C}^d$;

(ii) if $\dim_x \underline{X} < d$, we choose $\underline{f} : \underline{X} \longrightarrow \underline{B}$ so small that $\dim_{x'} X < d$ for all $x' \in X$ (which can be done by I 5.1.2, (iv)); put $\underline{\Delta} := \underline{\text{im}}(\underline{f})$ (then Δ is nowhere dense in B);

(iii) if $\dim_x \underline{X} = d$, $\underline{f}$ should have the properties of the Local Representation Theorem I 6.3.1 ;

(iv) Proposition 2.1.1 and Corollary 2.1.2 hold for $\underline{X}$.

Note that always $\dim_x \underline{X} \leq d$ by I 5.1.2, (iv), and that we may take good representatives to be arbitrarily small, i.e. we are allowed to shrink them when necessary.

Proposition 2.2.5. Let $\underline{f} : \underline{X} \longrightarrow \underline{B}$ be a good representative for the finite mapgerm $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^d,0)$ in $\underline{\text{cpl}}_0$ with discriminant locus $\underline{\Delta}$. Then the number $\sum_{x' \in f^{-1}(y)} \text{length}(\widetilde{\text{Quot}}(\mathcal{O}_{X,x'}))$ does not depend on the choice of $y \in B - \Delta$.

Proof. Let $y \in B - \Delta$. Then $X - f^{-1}(\Delta) \subseteq X_{ir}$, and so all the $x' \in f^{-1}(y)$ are in X_{ir} . The claim then follows from the fact that $f : X - f^{-1}(\Delta) \longrightarrow B - \Delta$ is a covering map and from Proposition 2.2.3.

I can now make the main definition:

Definition 2.2.6. Let $\underline{f} : (\underline{X},x) \longrightarrow (\mathbb{C}^d,0)$ be a finite mapgerm in $\underline{\text{cpl}}_0$, and $\underline{f} : \underline{X} \longrightarrow \underline{B}$ be a good representative with discriminant locus Δ. Then the well-defined number,

$$\deg_x \underline{f} := \sum_{x' \in f^{-1}(y)} \operatorname{length}(\widetilde{\operatorname{Quot}}(\mathcal{O}_{X,x'})) \quad ,$$

y any point in $B - \Delta$, is called the local mapping degree of the germ $\underline{f}$.

Remark 2.2.7. Since $\operatorname{length}(\widetilde{\operatorname{Quot}}(\mathcal{O}_{X,x}))$ may be difficult to compute, one hopes for a nicer formula. In fact, one may show that, in the situation of Definition 2.2.6, one can find a nowhere dense subspace $\Delta' \subseteq B$ such that $\underline{X} - f^{-1}(\Delta')$ is Cohen-Macaulay at all x lying over $B - \Delta'$ (see Theorem 2.2.11); consequently

$$\deg_x \underline{f} = \sum_{x' \in f^{-1}(y)} \operatorname{length}(\operatorname{Quot}(\mathcal{O}_{X,x})) \quad .$$

$$= \sum_{x' \in f^{-1}(y)} \dim_{\mathbb{C}} (\mathcal{O}_{X,x'}/m_y \cdot \mathcal{O}_{X,x'}) \quad ,$$

for all $y \in B - \Delta'$, where m_y is the maximal ideal of $\mathcal{O}_{\mathbb{C}^d,y}$.

We have the following simple but important fact:

Theorem 2.2.8 (Degree Formula). Let $\underline{f} : \underline{X} \longrightarrow \underline{B}$ be as in Definition 2.2.6. Then

$$\deg_x \underline{f} = \sum_{x' \in f^{-1}(y)} \deg_{x'} \underline{f}$$

for all $y \in B$.

This follows from the geometry of Definition 2.2.6. An algebraic proof will appear below, cf. 5.1.7. Theorem 2.2.8. has the important application that multiplicity will be upper semi-continuous along complex spaces, see Theorem 5.2.4.

Exercise 2.2.9. In the situation of Defintion 2.2.6

$$(2.2.5) \qquad \deg_x \underline{f} = \sum_{x' \in f^{-1}(y)} \dim_{\mathbb{C}} (\mathcal{O}_{X,x'}/m_y \cdot \mathcal{O}_{X,x'})$$

for $y \in B - \Delta$ and Δ a suitable nowhere dense analytic set in B.

For this, proceed as follows:

(i) Show by means of Fitting ideals that for an admissible module M on a reduced complex space Y the set $LF(M) := \{y \in Y | M$ is locally free at $y\}$ is the complement of a nowhere dense analytic set (cf.[28], Chapter 4, § 4).

(ii) Let now $\underline{f}$ be as in Definition 2.2.6; choose Δ in such a way that $f_*\mathcal{O}_X$ is locally free on $B - \Delta$.

Exercise 2.2.10. Use 2.2.9 (ii) to prove the following

Theorem 2.2.11. Let $\underline{X}$ be a complex space. Then the Cohen-Macaulay-locus $X_{CM} := \{x \in \underline{X} | \mathcal{O}_{X,x}$ is Cohen-Macaulay$\}$ is the complement of a nowhere dense analytic set.

What is with the smooth locus $X_{sm} := \{x \in X | \mathcal{O}_{X,x}$ is regular$\}$?

§ 3. Geometric multiplicity.

We now use the notion of the local mapping degree of a finite mapgerm to define the geometric multiplicity $m(\underline{X},x)$ of a complex space germ $(\underline{X},x) \in \underline{\underline{cpl}}_0$.

Geometric multiplicity in the reduced case is discussed in [13], [38], [61], [70] and [75].

3.1. The tangent cone.

Let $(\underline{X},x) \in \underline{\underline{\text{cpl}}}_0$, and $gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) := \bigoplus_{k \geq 0} \mathfrak{m}_x^k/\mathfrak{m}_x^{k+1}$, which is a finitely generated $\mathbb{C}$-algebra. Recall the notion of the analytic spectrum of a finitely generated $\mathbb{C}$-algebra in I 3.4.

Definition 3.1.1. $\underline{C}(\underline{X},x) := \underline{\text{Specan}}(gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x}))$, the tangent cone of $(\underline{X},x) \in \underline{\underline{\text{cpl}}}_0$.

To describe it in a more concrete way, choose generators $f_1,\ldots,f_n$ of $\mathfrak{m}_x$, i.e. an embedding $(\underline{X},x) \hookrightarrow (\mathbb{C}^n,0)$ by I 4.4.2. This gives a surjection

$$\varphi : \mathbb{C}[z_1,\ldots,z_n] = gr_{\mathfrak{m}_n}(\mathcal{O}_{\mathbb{C}^n,0}) \twoheadrightarrow gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x}) \quad ,$$

and so $\underline{C}(\underline{X},x)$ is defined in $\underline{\mathbb{C}}^n$ by the homogeneous ideal $\text{Ker}(\varphi)$, hence is a cone . If the ideal $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ defines $(\underline{X},x)$, one can show that $\text{Ker}(\varphi) = L(I)$, the ideal generated by the leitforms $L(f)$ of all the $f \in I$. So if I is generated by finitely many polynomials, the standard base algorithm discussed in I Remark 2.4.4,gives finitely many equations which define $\underline{C}(\underline{X},x)$.

Proposition 3.1.2. $\text{Dim}_x \underline{C}(\underline{X},x) = \dim_x \underline{X} = \dim gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$.

Proof. A geometric proof is somewhat involved (see Proposition 3.1.3 (iii) below), so we use the elementary properties of dimension of local rings. Now $gr_{M_x^+}(\mathcal{O}_{C(\underline{X},x),x}) = gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$, where M_x^+ is the irrelevant maximal ideal of $gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$. Since these two rings have the same Hilbert function, the result follows from the well-known main result of dimension theory of local rings (see e.g. [1], Theorem 11.14.) and the fact that this Hilbert function is just the Hilbert function of $\mathcal{O}_{X,x}$.

We now shortly touch upon another, more geometric description of the tangent cone, which puts it into a flat deformation of $(\underline{X},x)$; this appears in [45], [70], and is a special case of Fulton's and Macpherson's "deformation to the normal cone" (see [17] for the algebraic case; the analytic case is analogous):

Let $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ be defined by the ideal $I \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. For $f \in I$, let $f^* \in \mathcal{O}_{\mathbb{C}^n \times \mathbb{C},0}$ be defined by

$$f^*(z,t) := \frac{1}{t^{\nu(f)}} \cdot f(tz) \quad ,$$

where $\mathbb{C}^n$ has coordinates z and $\mathbb{C}$ has coordinate t, and $\nu(f)$ is the order of f (I(1.1.3)). Let $I^* \subseteq \mathcal{O}_{\mathbb{C}^n \times \mathbb{C},0}$ be the ideal generated by the f^* for $f \in I$. It defines a germ $(\underline{\mathcal{X}},0) \hookrightarrow (\underline{\mathbb{C}}^n \times \underline{\mathbb{C}},0)$, and the projection $\underline{\mathbb{C}}^n \times \underline{\mathbb{C}} \longrightarrow \underline{\mathbb{C}}$ defines a morphism $\underline{p} : (\underline{\mathcal{X}},0) \longrightarrow (\underline{\mathbb{C}},0)$ and so $\underline{p} : \underline{\mathcal{X}} \longrightarrow \underline{B}$, where $\underline{B} \subseteq \underline{\mathbb{C}}$ is an open disk around 0 (in fact, it is easy to see that $\underline{p}$ is defined over $\underline{\mathbb{C}}$). Then the following statements do hold:

<u>Proposition 3.1.3</u> (Deformation to the tangent cone).

(i) $(\underline{p}^{-1}(t),(0,t)) \cong (\underline{X},x)$ <u>for all</u> $t \neq 0$.

(ii) $(\underline{p}^{-1}(0),(0,0)) \cong (\underline{C}(\underline{X},x),x)$.

(iii) $p_x^0(t-p(x))$ <u>is a nonzerodivisor in</u> $\mathcal{O}_{\mathcal{X},x}$ <u>for all</u> $x \in \mathcal{X}$, <u>and so</u> $\underline{p}$ <u>is flat; especially</u> $\dim_x \underline{C}(\underline{X},x) = \dim_x \underline{X}$.

(iv) $\overline{\mathcal{X} - p^{-1}(0)} = \mathcal{X}$.

<u>Corollary 3.1.4</u>.

$$C(\underline{X},x) = \cup \left\{ \ell \mid \ell = \lim_{\substack{x \to x' \\ x \neq x'}} \overline{xx'} \right\} \quad ,$$

where $\overline{xx'}$ is the complex line through x and x', and the limit is taken in $\mathbb{P}^{n-1}$.

In other words, settheoretically is $C(\underline{X},\underline{x})$ the union of limits of secants of $\underline{X}$ through x , whence the name "tangent cone".

3.2. Multiplicity.

Let now $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, $d := \dim_x \underline{X}$. We fix generators $f_1,\ldots,f_n \in \mathfrak{m}_x$, so an embedding $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$, and so an embedding $\underline{C}(\underline{X},x) \hookrightarrow \underline{\mathbb{C}}^n$ as in 3.1. Note that $d = n$ implies $(\underline{X},x) \cong (\underline{\mathbb{C}}^n,0)$ by I 4.4.2. We now consider finite linear projections of $(\underline{X},x)$ onto $(\underline{\mathbb{C}}^d,0)$.

Definition 3.2.1. Let $\mathrm{Grass}^d(\mathbb{C}^n)$ denote the Grassmannian of d-codimensional linear subspaces $L \subseteq \mathbb{C}^n$ (see e.g. [30], Chapter 1, Section 5). Let $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$. Then $L \in \mathrm{Grass}^d(\mathbb{C}^n)$ is called good for $(\underline{X},x)$ if and only if x is isolated in $L \cap X$, and excellent for $(\underline{X},x)$ if and only if it is good for $(C(\underline{X},x),x)$, i.e. $L \cap C(\underline{X},x) = \{x\}$.

We put

$$(3.2.1) \qquad P_g^d(\underline{X},x) := \left\{L \in \mathrm{Grass}^d(\mathbb{C}^n) \mid L \quad \text{good for} \quad (\underline{X},x)\right\} ,$$

$$(3.2.2) \qquad P_e^d(\underline{X},x) := \left\{L \in \mathrm{Grass}^d(\mathbb{C}^n) \mid L \quad \text{excellent for} \quad (\underline{X},x)\right\} ,$$

and use the notations

$$(3.2.3) \qquad L \pitchfork_x X :\iff L \in P_g^d(\underline{X},x)$$

$$(3.2.4) \qquad L \pitchfork_x C(\underline{X},x) :\iff L \in P_e^d(\underline{X},x) .$$

If $L \in \mathrm{Grass}^d(\mathbb{C}^n)$, choose coordinates $(z_1,\ldots,z_n)$ so that L is $\mathbb{C}^{n-d}$ with coordinates $(z_{d+1},\ldots,z_n)$; then the projection $\pi_L : \mathbb{C}^n \longrightarrow \mathbb{C}^d$ along L defines the linear projection $\underline{p}_L := \underline{\pi}_L \mid (\underline{X},x) : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$. Then Corollary I 4.3.6 immediately implies

<u>Proposition 3.2.2</u>. <u>If</u> $L \in P^d_g(X,x)$, p_L <u>is finite</u>.

We now show that there is an ample supply of these finite projections p_L .

For this, we exploit the transversality condition algebraically; the following observation seems to be due to Lipman [49], see also [69].

Let $\underline{f} : (\underline{X},x) \longrightarrow (\underline{Y},y)$ be a mapgerm; then $\underline{f}$ induces $gr_{\mathfrak{m}}(f^0_x) : gr_{\mathfrak{m}_y}(\mathcal{O}_{Y,y}) \longrightarrow gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$, so by localizing at the irrelevant maximal ideal a homomorphism $\mathcal{O}_{\underline{C}(\underline{Y},y),y} \longrightarrow \mathcal{O}_{\underline{C}(\underline{X},x),x}$, and hence a mapgerm

$$d_x\underline{f} : (\underline{C}(\underline{X},x),x) \longrightarrow (\underline{C}(\underline{Y},y),y)$$

called the <u>differential of</u> f <u>at</u> x .

<u>Proposition 3.2.3.</u> <u>Let</u> $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ <u>be a mapgerm</u>, $d = \dim_x \underline{X}$. <u>The following conditions are equivalent</u>:

(i) $d_x\underline{f} : (\underline{C}(\underline{X},x),x) \longrightarrow (\mathbb{C}^d,0)$ <u>is finite</u> ;

(ii) <u>the ideal</u> $\mathfrak{q}_x := f^0_x(\mathfrak{m}_d)\cdot\mathcal{O}_{X,x}$ <u>is a minimal reduction of</u> $\mathfrak{m}_x$.

<u>In particular, then</u>, $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ <u>is finite</u>.

<u>Proof</u>. Let $\underline{f}$ be defined by $f_1,\ldots,f_d \in \mathfrak{m}_x$. To simplify notation, let $G := gr_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$, and let $M^+ \subseteq G$ be the irrelevant maximal

ideal, $M^+ := \bigoplus_{k>0} G_k$. Let f_j^* be the image of f_j in $G_1 = \mathfrak{m}_x/\mathfrak{m}_x^2$, $j = 1,\dots,d$, and $Q := (f_1^*,\dots,f_d^*)\cdot G$. Let $\mathfrak{q}_x^* := (f_1^*,\dots,f_d^*)\cdot \mathcal{O}_{C(\underline{X},x),x}$. Consider the injections

$$G/Q \xrightarrow[\varphi]{} (G/Q)_{M^+\cdot(G/Q)} \xrightarrow[\psi]{} \mathcal{O}_{C(\underline{X},x),x}/\mathfrak{q}_x^* \quad .$$

If $\dim_{\mathbb{C}}(\mathcal{O}_{C(\underline{X},x),x}/\mathfrak{q}_x^*) < \infty$, it follows that $\dim_{\mathbb{C}}(G/Q) < \infty$. Conversely, if $\dim_{\mathbb{C}}(G/Q) < \infty$, G/Q is artinian, and so, since it is graded, must be local, so that in fact φ is an isomorphism. Now ψ is faithfully flat by (4.1.3), and so by Lemma 2.2.4 we get $\dim_{\mathbb{C}}((G/Q)_{M^+\cdot(G/Q)}) = \dim_{\mathbb{C}}(\mathcal{O}_{C(\underline{X},x),x}/\mathfrak{q}_x^*)$. Consequently, $\dim_{\mathbb{C}}(\mathcal{O}_{C(\underline{X},x),x}/\mathfrak{q}_x^*) = \dim_{\mathbb{C}}(G/Q)$ hence is finite. It follows that $\dim_{\mathbb{C}}(\mathcal{O}_{C(\underline{X},x),x}/\mathfrak{q}_x^*) < \infty$ is equivalent to $\dim_{\mathbb{C}}(gr_{\mathfrak{m}}(\mathcal{O}_{X,x})/Q) < \infty$. But the first inequality means $d_x\underline{f}$ is finite by the Integrality Theorem I 4.4.1, and the second one that $\mathfrak{q}$ is a minimal reduction of $\mathfrak{m}_x$ by Chapter II, Theorem (10.14) and Corollary (10.15). Especially, $\mathfrak{q}_x$ is $\mathfrak{m}$-primary, and so $\dim_{\mathbb{C}}\mathcal{O}_{X,x}/\mathfrak{q}_x < \infty$, whence $\underline{f}$ is finite by the Integrality Theorem I 4.4.1.

We now get:

<u>Proposition 3.2.4</u>.

(i) <u>If</u> $L \in \mathrm{Grass}^d(\mathbb{C}^n)$, $L \pitchfork_x C(\underline{X},x)$ <u>implies</u> $L \pitchfork_x X$, <u>and so</u> $P_e^d(\underline{X},x) \subseteq P_g^d(\underline{X},x)$.

(ii) $P_e^d(\underline{X},x)$, <u>and so a fortiori</u> $P_g^d(\underline{X},x)$, <u>is generic in</u> $\mathrm{Grass}^d(\mathbb{C}^n)$.

<u>Proof</u>.

(i) is direct from Proposition 3.2.3.

(ii) We may assume $1 \le d \le n-1$. Put

$$R := \left\{ (L,\ell) \in \mathrm{Grass}^d(\mathbb{C}^n) \times \mathbb{P}^{n-1} \mid \ell \subseteq L \right\} ,$$

where $\mathbb{P}^k := \mathbb{P}^k(\mathbb{C})$ denotes complex projective k-space. We have the diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad q \quad} & \mathbb{P}^{n-1} \\ {\scriptstyle p}\downarrow & & \\ \mathrm{Grass}^d(\mathbb{C}^n) & & \end{array} ,$$

where p,q are projections. We now use some elementary Algebraic Geometry, as e.g. in the first chapter of [56]. This is a diagram of algebraic varieties and algebraic morphisms, and p is proper. Let $\mathbb{P}C(\underline{X},x) \subseteq \mathbb{P}^{n-1}$ be the projective tangent cone. It follows that $p(q^{-1}(\mathbb{P}C(\underline{X},x)))$ is an analytic, even algebraic, set in $\mathrm{Grass}^d(\mathbb{C}^n)$, and so is either nowhere dense in $\mathrm{Grass}^d(\mathbb{C}^n)$ or coincides with it, since $\mathrm{Grass}^d(\mathbb{C}^n)$ is a connected, smooth, and hence irreducible, variety. But equality $p(q^{-1}(\mathbb{P}C(\underline{X},x))) = \mathrm{Grass}^d(\mathbb{C}^n)$ means that any d-codimensional plane in $\mathbb{P}^{n-1}$ hits $\mathbb{P}C(\underline{X},x) \subseteq \mathbb{P}^{n-1}$, which cannot be since it has projective dimension $d-1$ by Proposition 3.1.2. Finally, note that $P_e^d(\underline{X},x) = \mathrm{Grass}^d(\mathbb{C}^n) - p(q^{-1}(\mathbb{P}C(\underline{X},x)))$, which implies (i) .

<u>Remark 3.2.5</u>. The inclusion $P_e^d(\underline{X},x) \subseteq P_g^d(\underline{X},x)$ says that if $L \in \mathrm{Grass}^d(\mathbb{C}^n)$ has $\dim_x L \cap X \geq 1$ there should be a line $\ell \subseteq L \cap C(\underline{X},x)$, which is intuitively clear, since $\dim_x L \cap X \geq 1$ tells us there are secants $\overline{xx'} \subseteq L$ with $x' \neq x$ arbitrarily close to x. So a geometric proof could be based on Corollary 3.1.4, for which, however, I did not give a complete proof. The existing geometric proofs of Proposition 3.2.4 (i) in the literature ([13], [75]) are somewhat involved. Proposition 3.2.4 (ii) is also in [75] (Chapter 7, Lemma 7N).

We are now ready for the definition of multiplicity.

<u>Definition 3.2.6</u> (Geometric multiplicity). <u>Let</u> $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, $d := \dim_x X$. <u>Fix generators</u> $f_1,\ldots,f_n$ of m_x, <u>i.e. an embedding</u> $(\underline{X},x) \hookrightarrow (\mathbb{C}^n,0)$. <u>The geometric multiplicity</u> $m(\underline{X},x)$ <u>of</u> $(\underline{X},x)$ (with respect to this embedding) <u>is defined to be</u>

$$m(\underline{X},x) := \min_{L\in P^d_g(\underline{X},x)} \{\deg_x p_L\} \quad .$$

Proposition 3.2.4 (ii) implies that this definition is not empty. $m(\underline{X},x)$ depends a priori on the embedding. It will be shown algebraically in Theorem 5.2.1 that this is not so.

Exercise 3.2.7. (i) Prove by geometric means that $m(\underline{X},x)$ depends only on the isomorphism class of $(\underline{X},x)$ in $\underline{\underline{cpl}}_0$.

Hints: First show that it suffices to compare embeddings of equal dimensions; here (2.2.5) might be of use. Then use Proposition I 3.2.1, to show that

$$(3.2.5) \qquad m(\underline{X},x) = \min_{\substack{\underline{f}:(\underline{X},x)\to(\mathbb{C}^d,0) \\ \underline{f} \text{ finite}}} \{\deg_x \underline{f}\} \quad .$$

(ii) Conclude that $m(\underline{X},x) = 1$ when $(\underline{X},x)$ is smooth. Show that conversely $(\underline{X},x)$ is smooth when $(\underline{X},x)$ is equidimensional and $m(\underline{X},x) = 1$ (Criterion of multiplicity one).

Hints: For the converse prove that a finite extension $\mathcal{O}_{\mathbb{C}^d,0} \hookrightarrow \mathcal{O}_{Y,y}$ of degree one, where $\mathcal{O}_{Y,y}$ is an integral domain, is surjective. For this, use the Local Representation Theorem I 6.3.1 and the classical Riemann Extension Theorem (see I Remark 6.3.3).

Example 3.2.8. If $L \notin P^d_e(\underline{X},x)$, it can happen that $\deg_x \underline{p}_L \neq m(\underline{X},x)$. For instance, let $\underline{X} \hookrightarrow \mathbb{C}^2$ be defined by $z_1 - z_2^2 = 0$, $L :=$ the z_2-axis, $x = 0$. Then $m(\underline{X},x) = 1$ by Exercise 3.2.7 (ii) above, but $\deg_x \underline{p}_L = 2$. However, $L \in P^d_e(\underline{X},x)$ will imply $\deg_x \underline{p}_L = m(\underline{X},x)$. See Theorem 5.2.1.

§ 4. The geometry of Samuel multiplicity.

The purpose of this paragraph is to give a geometric interpretation of the Samuel multiplicity $e(\mathfrak{q}, \mathcal{O}_{X,x})$ of an $\mathfrak{m}_x$-primary ideal $\mathfrak{q}$ in the local analytic $\mathbb{C}$-algebra $\mathcal{O}_{X,x}$; it will turn out to be canonically the geometric multiplicity $m(\underline{C},0)$, where $\underline{C}$ is the geometric affine cone corresponding to $gr_{\mathfrak{q}}(\mathcal{O}_{X,x})$, and $0 \in C$ its vertex; see Theorem 4.4.2. This has, of course, to do with very classical Algebraic Geometry, namely the fact that the Hilbert function of a projective variety determines its degree, which is the number of intersection points with a generic complementary linear subspace. This should explain, or at least motivate, the usual abstract definition of $e(\mathfrak{q}, \mathcal{O}_{X,x})$ by means of the Hilbert function of $gr_{\mathfrak{q}}(\mathcal{O}_{X,x})$. The reader who takes this definition of $e(\mathfrak{q}, \mathcal{O}_{X,x})$ for granted may skip this paragraph.

4.1. Degree of a projective variety.

Let $\underline{Z} \subseteq \mathbb{P}^{n-1}$ be a projective variety, i.e. an algebraic $\mathbb{C}$-scheme of finite type. We denote the structure sheaf of $\underline{Z}$, when $\underline{Z}$ is regarded as an algebraic variety, by $\mathcal{O}_Z^{alg}$; so $\underline{Z}$ is given by the ideal sheaf generated in $\mathcal{O}^{alg}_{\mathbb{P}^{n-1}}$ by a homogeneous ideal $I \subseteq \mathbb{C}[z_1,\ldots,z_n]$. Let $\underline{C} \subseteq \mathbb{C}^n$ be the corresponding cone; as an algebraic variety, $\underline{C} = \underline{\mathrm{Spec}}(R)$, and as a complex space, $\underline{C} = \underline{\mathrm{Specan}}(R)$, where R is the graded ring $\mathbb{C}[z_1,\ldots,z_n]/I$.

Classically, the degree $\deg(\underline{Z})$ of $\underline{Z}$ is defined to be the number of intersection points of $\underline{Z}$ with a general $(d-1)$-codimensional projective plane $P \subseteq \mathbb{P}^{n-1}$, where $d-1$ is the projective dimension of $\underline{Z}$, and hence d is the affine dimension of $\underline{C}$. One has, however, to be a little careful what "general" means, and what "number of intersection points" means when $\underline{Z}$ is not reduced.

In analogy to Proposition 3.2.3 for cones one has

<u>Proposition 4.1.1</u>. <u>The set</u> $P_e^{d-1}(\underline{Z}) := \{P \in \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1}) \mid P \cap Z$ is finite$\}$ <u>is generic in</u> $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$.

<u>Proof</u>. We use basic Projective Algebraic Geometry, see e.g. the first 72 pages of [56]. There is $P' \in \mathrm{Grass}^{d}(\mathbb{P}^{n-1})$ with $P' \cap Z = \emptyset$ since $\dim \underline{Z} = d-1$; then the linear projection $\underline{q}_{P'} : \underline{Z} \longrightarrow \underline{\mathbb{P}}^{d-1}$ along P' is finite, hence for $z \in Z$, the linear space through P' and z hits $\underline{Z}$ in only finitely many points. So $P_e^{d-1}(\underline{Z}) \neq \emptyset$. Now let $P := \{(z,L) \in \mathbb{P}^{n-1} \times \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1}) \mid z \in L\}$; it has a canonical structure as an algebraic variety $\underline{P}$, and the projection gives a fibre bundle $\underline{P} \longrightarrow \underline{\mathbb{P}}^{n-1}$, which, by pulling back via $\underline{Z} \hookrightarrow \underline{\mathbb{P}}^{n-1}$ gives us a fibre bundle $\underline{Z} \xrightarrow{p} \underline{Z}$ with $Z := \{(z,L) \in Z \times \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1}) \mid z \in L\}$. The projection $\underline{Z} \xrightarrow{q} \underline{\mathrm{Grass}}^{d-1}(\mathbb{P}^{n-1})$ is proper and finite at some point, so it is finite over a nonempty Zariski-open subset of $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$, say over $\mathrm{Grass}^{d-1}(\mathbb{C}^{n-1}) - \Delta(\underline{Z})$, where $\Delta(\underline{Z})$ is a proper Zariski-closed subset. Q.e.d.

<u>Remark 4.1.2</u>. Since $\underline{q}$ is finite outside a nowhere dense analytic set of $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$, $q_* \mathcal{O}_{\underline{Z}}$ is locally free outside a nowhere dense analytic set. One may use this to prove that the set

$$P_{CM}^{d-1}(\underline{Z}) := \{ P \in P_e^{d-1}(\underline{Z}) \mid P \cap Z \subseteq Z_{CM} \} \tag{4.1.1}$$

with $Z_{CM} := \{z \in Z \mid \mathcal{O}_{Z,z}$ is Cohen-Macaulay$\}$, is generic in $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$. Similarly, if $\underline{Z}$ is reduced, $\underline{q}$ is locally isomorphic outside a nowhere dense analytic set, and one can equally show that then

$$(4.1.2) \qquad P^{d-1}_{reg}(\underline{Z}) := \left\{ P \in P^{d-1}_{e}(\underline{Z}) \mid P \cap Z \subseteq Z_{reg} \text{ and } P \text{ is transversal to } Z_{reg} \text{ along } P \cap Z \right\}$$

is generic in $\text{Grass}^{d-1}(\mathbb{P}^{n-1})$.

Definition 4.1.3. The degree $\deg(\underline{Z})$ of $\underline{Z} \hookrightarrow \mathbb{P}^{n-1}$ is defined to be

$$\deg(\underline{Z}) := \sum_{z \in Z \cap P} \deg_{(z,P)} \underline{q}$$

where $\underline{q} : \underline{Z} \longrightarrow \text{Grass}^{d-1}(\mathbb{P}^{n-1})$ and $\Delta(\underline{Z})$ are as above, and $P \in \text{Grass}^{d-1}(\mathbb{P}^{n-1}) - \Delta(\underline{Z}) = P^{d-1}_{e}(\underline{Z})$.

That this number is independent of P can be proven as in Proposition 2.2.5, but it is simpler here, since we will see that we could have worked with the algebraic local rings, and then the local constancy of the $\deg_{(z,P)}\underline{q}$ along Z_{ir} follows without using compact Stein neighbourhoods; see Corollary 4.1.5 below.

Lemma 4.1.4. Let $\underline{Z}$ be an algebraic variety over $\mathbb{C}$. Let Z_{ir} be the locus of points where $\underline{Z}$ is locally irreducible as a complex space. Then, if $z \in Z_{ir}$, z lies on a unique irreducible component of Z as an algebraic variety, Z_{λ} say, and

$$\text{length}(\widetilde{\text{Quot}}(\mathcal{O}_{Z,z})) = \text{length}(\widetilde{\text{Quot}}(\hat{\mathcal{O}}_{Z,z})) = \text{length}(\widetilde{\text{Quot}}(\mathcal{O}^{alg}_{Z,z}))$$
$$= \text{length}(\mathcal{O}^{alg}_{Z,Z_{\lambda}}) \quad ,$$

<u>where</u> $\mathcal{O}^{alg}_{Z,Z_\lambda}$ <u>is the local ring of</u> Z <u>along</u> Z_λ . <u>In particular, it is constant along</u> $Z_\lambda \cap Z_{ir}$.

<u>Proof</u>. Consider the inclusions

$$(4.1.3) \qquad \mathcal{O}^{alg}_{Z,z} \overset{\varphi}{\hookrightarrow} \mathcal{O}_{Z,z} \overset{\psi}{\hookrightarrow} \hat{\mathcal{O}}_{Z,z} = \hat{\mathcal{O}}^{alg}_{Z,z} \quad .$$

Then, since $(\mathcal{O}_{Z,z})_{red}$ is integral, so is $(\mathcal{O}^{alg}_{Z,z})_{red}$, and z is on a unique Z_λ . Moreover, ψ and $\psi \circ \varphi$ are faithfully flat as completion morphisms, and hence so is φ .

Now it is known (and this is a nontrivial result) that for an integral local analytic $\mathbb{C}$-algebra R the completion $\hat{R}$ is integral. For this see [64], Exposé 21, Théorème 3 on p. 21-13. Or use the fact that the normalization R' of R is again a local analytic algebra ([26], Satz 2 on p. 136); since R is excellent, the minimal primes of $\hat{R}$ correspond to the maximal ideals of R' ([12], Theorem 6.5), and so $\hat{R}$ is integral. Applying this to $R := (\mathcal{O}_{Z,z})_{red}$, one has $\hat{R} = \hat{\mathcal{O}}_{Z,z} / N_z \cdot \hat{\mathcal{O}}_{Z,z}$ is integral, so $N_z \cdot \hat{\mathcal{O}}_{Z,z}$ is prime and so equals $\hat{N}_z$, the nilradical of $\hat{\mathcal{O}}_{Z,z}$. We thus get $N^{alg}_z \cdot \mathcal{O}_{Z,z} = N_z$, $N_z \cdot \hat{\mathcal{O}}_{Z,z} = \hat{N}_z$. We now can localize and get morphisms

$$(4.1.4) \qquad \widetilde{\mathrm{Quot}}(\mathcal{O}^{alg}_{Z,z}) \hookrightarrow \widetilde{\mathrm{Quot}}(\mathcal{O}_{Z,z}) \hookrightarrow \widetilde{\mathrm{Quot}}(\hat{\mathcal{O}}_{Z,z}) \quad ,$$

which are faithfully flat, and Lemma 2.2.4 gives $\mathrm{length}(\widetilde{\mathrm{Quot}}(\mathcal{O}^{alg}_{Z,z})) = \mathrm{length}(\widetilde{\mathrm{Quot}}(\hat{\mathcal{O}}_{Z,z})) = \mathrm{length}(\widetilde{\mathrm{Quot}}(\mathcal{O}_{Z,z}))$. Finally, assume $\underline{Z} = \underline{\mathrm{Spec}}(A)$ affine, where A is a finitely generated $\mathbb{C}$-alebra, with Z_λ corresponding to $\mathfrak{p} \in \mathrm{Min}(A)$, and z to a maximal ideal $\mathfrak{m}$ of $\underline{\mathrm{spec}}(A)$. Then $\mathfrak{p} \subseteq \mathfrak{m}$, and so $\widetilde{\mathrm{Quot}}(\mathcal{O}^{alg}_{Z,z}) = (A_\mathfrak{m})_\mathfrak{p} = A_\mathfrak{p} = \mathcal{O}^{alg}_{Z,Z_\lambda}$.

Corollary 4.1.5. deg($\underline{Z}$) does not depend on the choice of P. Especially, if $\underline{Z}$ is irreducible and reduced,

$$\deg(\underline{Z}) = \#(Z \cap P) \quad ,$$

where $P \in P^{d-1}_{reg}(\underline{Z})$ arbitrary (this is the classical definition).

Lemma 4.1.6. Let X be either an algebraic variety over $\mathbb{C}$ or a complex space, and let $\mathcal{O}$ denote either the algebraic or complex analytic structure. Then, for all $k \geq 0$, and $x \in X_{ir}$ (the irreducible locus with respect to the complex analytic structure), $(x,0) \in (X \times \mathbb{C}^k)_{ir}$, and

$$\text{length}(\widetilde{\text{Quot}}(\mathcal{O}_{X,x})) = \text{length}(\widetilde{\text{Quot}}(\mathcal{O}_{X\times\mathbb{C}^k,(x,0)})) \quad .$$

In particular, if $\underline{f} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ is finite, $\deg_x \underline{f} = \deg_{(x,0)}(\underline{f} \times id_{\mathbb{C}^k})$ for all k.

Proof. We may assume $k = 1$. Consider the faithfully flat extension

$$(4.1.5) \qquad \hat{\mathcal{O}}_{X,x} \longrightarrow \hat{\mathcal{O}}_{X\times\mathbb{C},(x,0)} = \hat{\mathcal{O}}_X [[t]] \quad .$$

The nilradical of $\hat{\mathcal{O}}_X[[t]]$ is $N_x \cdot \mathcal{O}_x[[t]]$, and so $(\hat{\mathcal{O}}_X[[t]])_{red} = (\hat{\mathcal{O}}_{X,x}/N_x\hat{\mathcal{O}}_{X,x})[[t]] = (\hat{\mathcal{O}}_{X,x})_{red}[[t]]$ by the proof of 4.1.4; so if N_x is prime, $\hat{N}_x = N_x \cdot \hat{\mathcal{O}}_{X,x}$ is prime, so $x \in X_{ir}$ implies $(x,0) \in (X \times \mathbb{C})_{ir}$. The claim now follows again by Lemma 2.2.4 and Lemma 4.1.4.

Proposition 4.1.7. Let $Z \hookrightarrow \mathbb{P}^{n-1}$ be a projective variety of dimension $d-1$ and with homogeneous coordinate ring R. Then for any $P \in P^{d-1}_e(\underline{Z})$ and P' a hyperplane in P with $Z \cap P' = \emptyset$:

$$\deg(\underline{Z}) = \sum_{z \in Z \cap P} \deg_z \underline{q}_{P'} \quad ,$$

<u>where</u> $\underline{q}_{P'} : \underline{Z} \longrightarrow \underline{\mathbb{P}}^{d-1}$ <u>is the projection with centre</u> P' .

(cf. (5.3) and (5.4.) in Mumford's book [56]) .

<u>Outline of proof</u>.

Let the notations be as above. Fix P and P' . Let $\mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}$ be a hyperplane containing P' and not meeting $Z \cap P$. Finally, let $\mathbb{P}^{d-1} \subseteq \mathbb{P}^{n-1}$ be such that $\mathbb{P}^{d-1} \cap P' = \emptyset$ and $\mathbb{P}^{d-1} \cap \mathbb{P}^{n-2}$ is a hyperplane in $\mathbb{P}^{d-1}$ (see Figure 4).

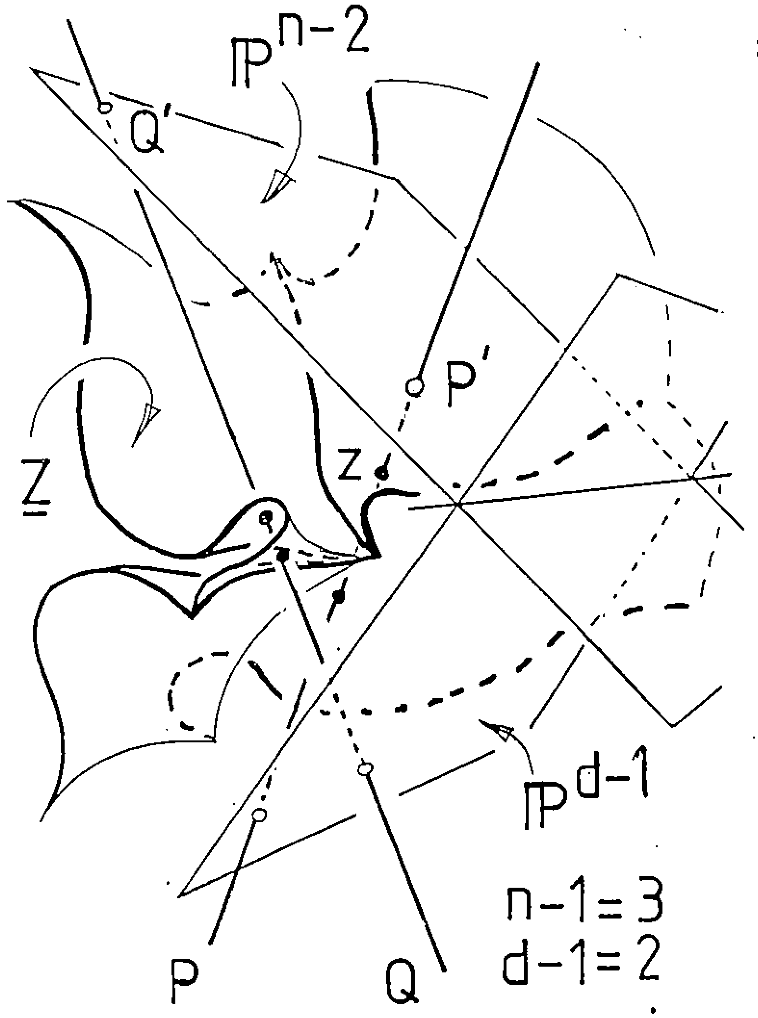


Fig. 4

We say two planes $L, L' \subseteq \mathbb{P}^{n-1}$ are <u>transversal</u>, denoted $L \pitchfork L'$, if $L \cap L'$ has minimal possible dimension. Put $\mathbb{P}_0^{d-1} := \mathbb{P}^{d-1} - (\mathbb{P}^{d-1} \cap \mathbb{P}^{n-2})$, $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})_0 := \{Q \in \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1}) \mid Q \pitchfork \mathbb{P}^{d-1}, Q \pitchfork \mathbb{P}^{n-2}, Q \pitchfork \mathbb{P}^{d-1} \cap \mathbb{P}^{n-2}\}$, and $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-2})_0 := \{Q' \in \mathrm{Grass}^{d-1}(\mathbb{P}^{n-2}) \mid Q' \pitchfork \mathbb{P}^{d-1} \cap \mathbb{P}^{n-2}\}$. These are nowhere dense Zariski-open subsets. Finally put $\underline{Z}_0 := \underline{Z} - \mathbb{P}^{n-2}$, and $\underline{Z}_0 := \underline{p}^{-1}(\underline{Z}_0) \cap \underline{q}^{-1}(\underline{\mathrm{Grass}}^{d-1}(\mathbb{P}^{n-1})_0)$ (notations as in the proof of Proposition 4.1.1). One then gets the diagram

$$
(4.1.6) \qquad
\begin{array}{ccc}
\underline{Z}_0 & \underset{\underline{g}}{\overset{\underline{f}}{\rightleftarrows}} & \underline{Z}_0 \times \underline{\mathrm{Grass}}^{d-1}(\mathbb{P}^{n-2})_0 \\
\Big\downarrow \underline{q} & & \Big\downarrow \underline{q}_{P'} \times \underline{id} \\
\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})_0 & \underset{\underline{k}}{\overset{\underline{h}}{\rightleftarrows}} & \underline{\mathbb{P}}_0^{d-1} \times \underline{\mathrm{Grass}}^{d-1}(\mathbb{P}^{n-2})_0
\end{array} ,
$$

where $\underline{f} : (z,Q) \longmapsto (z, Q \cap \mathbb{P}^{n-2})$,

$\underline{g} : (z,Q') \longmapsto (z, Q' \vee z)$, where $Q' \vee z$ denotes the plane spanned by Q' and z ,

$\underline{h} : Q \longmapsto (Q \cap \mathbb{P}^{d-1}, Q \cap \mathbb{P}^{n-2})$,

$\underline{k} : (z,Q') \longmapsto Q' \vee z$.

Then $\underline{f}$ and $\underline{g}$ are inverse to each other, and so are $\underline{h}$ and $\underline{k}$. Over $P \in \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})_0$, the diagram is commutative, and so for $z \in Z \cap P$:

$$\deg_{(z,P)} \underline{q} = \deg_{(z,P')} (\underline{q}_{P'} \times \underline{id}) = \deg_z (q_{P'}) ,$$

the last equality from Lemma 4.1.6. This proves the Proposition.

Theorem 4.1.8. Let $Z \hookrightarrow \mathbb{P}^{n-1}$ be a projective variety of dimension $d-1$ with homogeneous coordinate ring R, and let $C \hookrightarrow \mathbb{C}^n$ be the corresponding affine cone. Then

$$\deg(Z) = m(C,0) \quad ,$$

the geometric multiplicity of C at its vertex.

Proof. Let $\mathbb{C}^n$ have coordinates $(z_1,\ldots,z_n)$; we may assume $\mathbb{P}^{n-2} \subseteq \mathbb{P}^{n-1}$ in 4.1.7 is given by $z_n = 0$. Let $L' \in \mathrm{Grass}^{d-1}(\mathbb{C}^n)$ correspond to $P \in \mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$, and $\mathbb{C}^{n-1} \subseteq \mathbb{C}^n$ be the hyperplane corresponding to $\mathbb{P}^{n-2}$. Let $L := L' \cap \mathbb{C}^{n-1}$ and put $C_0 := C - \mathbb{C}^{n-1}$, where C is te affine cone corresponding to Z . Let $H_1 \subseteq \mathbb{C}^n$ be the affine hyperplane given by $z_n = 1$, and put $C_1 := C_0 \cap H_1$. Now consider the commutative diagram of morphisms of algebraic varieties

$$(4.1.7) \qquad \begin{array}{ccccccc}
C_1 & \xrightarrow{j_n} & C_1 \times \mathbb{C}^* & \xrightarrow[u]{\cong} & C_0 & \xrightarrow{\pi_n} & Z_0 \\
\downarrow {\scriptstyle p_L|H_1} & \circlearrowright & \downarrow {\scriptstyle p_L} & \circlearrowright & \downarrow {\scriptstyle p_L} & \circlearrowright & \downarrow {\scriptstyle q_{P'}} \\
\mathbb{C}^{d-1} & \xrightarrow{j_d} & \mathbb{C}^{d-1} \times \mathbb{C}^* & = & \mathbb{C}^d - \mathbb{C}^{d-1} & \xrightarrow{\pi_d} & \mathbb{P}_0^{d-1}
\end{array}$$

Here, the left horizontal arrows are inclusions via $z' \longmapsto (z',1)$, u is induced by $\mathbb{C}^{n-1} \times \mathbb{C}^* \longrightarrow \mathbb{C}^n - \mathbb{C}^{n-1}$ with $(z',\lambda) \longmapsto (\lambda z',\lambda)$, and the right horizontal arrows are induced by the canonical projection $\pi_N : \mathbb{C}^N - \{0\} \twoheadrightarrow \mathbb{P}^{N-1}$. u is isomorphic, the inverse being induced by $\mathbb{C}^n - \mathbb{C}^{n-1} \longrightarrow \mathbb{C}^{n-1} \times \mathbb{C}^*$, $z = (z',z_n) \longmapsto (z'/z_n, z_n)$ (see Figure 5).

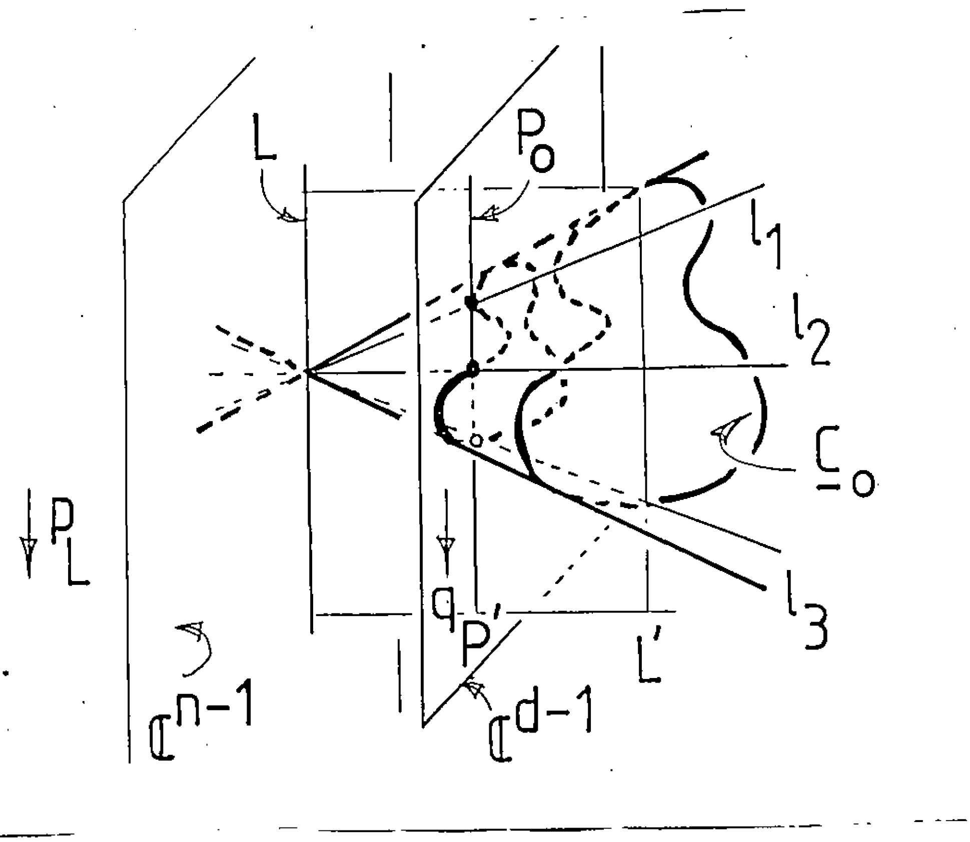


Fig. 5

From this figure, the result should be intuitively clear, since the intersection of P with Z corresponds to the intersection lines of L' with C, which in turn correspond to the intersection points of the affine plane $P_0 := P - \mathbb{C}^{n-1} = H_1 \cap L'$ with C ; but we must check the multiplicities.

The composite horizontal arrows give isomorphisms, so, since $\mathbb{C}^{n-1}$ is disjoint from $Z \cap P$, $\deg(\underline{Z}) = \sum_{w \in Z \cap P} \deg_w \underline{q}_{p'} =$

$\sum_{z' \in \underline{C}_1 \cap P_0} \deg_{z'} (\underline{p}_L | H_1)$. But this equals $\deg_0 \underline{p}_L$ by Lemma 4.1.6 and the middle square in (4.1.7.). So $\deg(\underline{Z}) = \deg_0 \underline{p}_L$ for all $L \in P^d_g(\underline{C},0) = P^d_e(\underline{C},0)$, which proves the claim.

Corollary 4.1.9. Let $Z_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Assh}(R)$, be the irreducible components of Z of dimension $d-1$, given by a homogeneous primary decomposition of 0 in R . Then

(4.1.8) $$\deg(\underline{Z}) = \sum_{\mathfrak{p}\in \mathrm{Assh}(R)} \mathrm{length}(R_{\mathfrak{p}})\cdot \deg(Z_{\mathfrak{p}}) \quad .$$

Proof. As $P \in P_e^{d-1}(\underline{Z})$ hits $Z_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Assh}(R)$, and these correspond to the maximal irreducible components of $\underline{C}_0$, it suffices to show $\mathrm{length}(\widetilde{\mathrm{Quot}}(\mathcal{O}_{C,z})) = \mathrm{length}(R_{\mathfrak{p}})$ for $z \in (C_0)_{ir}$ and $\mathfrak{p}$ corresponding to the irreducible component on which z lies. Now the affine coordinate ring of $\underline{C}_0$ is $R_{(z_n)}$, and $z_n \notin \mathfrak{p}$, since otherwise $\underline{Z} \cap \underline{P}$ would not be disjoint to $\mathbb{P}^{n-2}$. Then $\mathcal{O}^{alg}_{C,z} = (R_{(z_n)})_{\mathfrak{p}} = R_{\mathfrak{p}}$, and the claim follows from Lemma 4.1.4.

4.2. Hilbert functions.

The following result is classical; it was, at least in the reduced irreducible case, known to Hilbert ([32], p. 244), and, in general, to Macaulay [50], footnotes on pp. 82 and 115).

Theorem 4.2.1. Let R be the coordinate ring of a projective variety $\underline{Z} \hookrightarrow \mathbb{P}^{n-1}$ of dimension d. Then the Hilbert function $H(R,k) := \dim_{\mathbb{C}} R_k$ has the form

(4.2.1) $$H(R,k) = \frac{\deg(\underline{Z})}{(d-1)!}\, k^{d-1} + \text{lower terms}$$

for $k \gg 0$.

One way of geometric thinking about this goes as follows: For any projective variety $\underline{Z}$ and coherent $\mathcal{O}_Z$-module M put

(4.2.2) $$\chi(\underline{Z},M) := \sum_{i\geq 0} (-1)^i \dim_{\mathbb{C}} H^i(\underline{Z},M) \quad ,$$

where all $H^i(\underline{Z},M)$ are finite dimensional and 0 for $i > d-1$ ([65]), and one may either take analytic or algebraic sheaf cohomology ([66]).

Let M be a f.g. graded module over $\mathbb{C}[X_1,\dots,X_n]$ and $\mathcal{M}$ the corresponding coherent $\mathcal{O}_{\mathbb{P}^{n-1}}$-module. By celebrated results of [65], $H^i(\mathbb{P}^{n-1},\mathcal{M}(k)) = 0$ for $i>0$ and $k \gg 0$, and $M_k \cong \Gamma(\mathbb{P}^{n-1},\mathcal{M}(k))$ for $k \gg 0$, hence

(4.2.3) $$H(M,k) := \dim_{\mathbb{C}} M_k = \chi(\mathbb{P}^{n-1},\mathcal{M}(k)) \quad \text{for} \quad k \gg 0 \quad .$$

Now take any hyperplane $H \overset{i}{\hookrightarrow} \mathbb{P}^{n-1}$, defined by a linear form F ; then the exact sequence

(4.2.4) $$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1) \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow i_*\mathcal{O}_H \longrightarrow 0$$

induces (loc. cit. p. 277)

(4.2.5) $$0 \longrightarrow \mathcal{M}(k-1) \longrightarrow \mathcal{M}(k) \longrightarrow i_*(i^*\mathcal{M}(k)) \longrightarrow 0$$

for all k as soon as H is in general position with respect to supp $\mathcal{M}$, namely F should not belong to any prime of the homogeneous primary decomposition of M, except the possibly present irrelevant maximal ideal.

By additivity of χ, then,

(4.2.6) $$\chi(\underline{\mathbb{P}}^{n-1},\mathcal{M}(k)) = \chi(\underline{\mathbb{P}}^{n-1},\mathcal{M}(k-1)) + \chi(\underline{H},i^*\mathcal{M}(k)) \quad .$$

Applying this to M = R gives the recursion

(4.2.7) $$\chi(\underline{Z},\mathcal{O}_Z(k)) = \chi(\underline{Z},\mathcal{O}_Z(k-1) + \chi(\underline{Z}\cap\underline{H},\mathcal{O}_{Z\cap H}(k)) \quad ,$$

and by doubly inducting over k and d one gets

(4.2.8) $$\chi(\underline{Z},\mathcal{O}_Z(k)) = \sum_{j=0}^{d-1} (\underline{Z}\cap\underline{H}^{(j)},\mathcal{O}_{\underline{Z}\cap H}(j)) \cdot \binom{j+k-1}{j} \quad ,$$

where $H_1,\dots,H_{d-1}$ are hyperplanes in general position defined by linear forms $F_1,\dots,F_k$, and $\underline{H}^{(j)} := \underline{H}_1 \cap \dots \cap \underline{H}_j$. So

(4.2.9) $$H(R,k) = \chi(\underline{Z}, \mathcal{O}_Z(k)) \quad \text{for} \quad k \gg 0$$

is indeed a polynomial of degree $d-1$ in k, whose leading coefficient is $\frac{1}{(d-1)!}\chi(\underline{Z}\cap\underline{P}, \mathcal{O}_{\underline{Z}\cap\underline{P}})$, where P is a $(d-1)$-codimensional plane in general position, and $\underline{Z}\cap\underline{P}$ the scheme-theoretic intersection. But since $P_e^{d-1}(\underline{Z})$ is generic in $\mathrm{Grass}^{d-1}(\mathbb{P}^{n-1})$, we then have that, for a general choice of $H_1,\dots,H_{d-1}$, the intersection $Z\cap P$ consists of finitely many points. Then

(4.2.10) $$\mathcal{O}_{Z\cap P} = \bigoplus_{z\in Z\cap P} \mathcal{O}_{Z\cap P,z} \quad ,$$

a direct sum of artinian rings, and so

(4.2.11) $$\chi(\underline{Z}\cap\underline{P}, \mathcal{O}_{Z\cap P}) = \sum_{z\in Z\cap P} \dim_{\mathbb{C}}(\mathcal{O}_{Z\cap P,z}) \quad .$$

Choosing $P'\subseteq P$ a hyperplane in P with $P'\cap Z = \emptyset$, $q_{P'}: \underline{Z} \longrightarrow \mathbb{P}^{d-1}$ will be finite ; so $(q_{P'})_*(\mathcal{O}_Z)$ being a coherent sheaf, will be generically finite. So moving the H_j we may assume that $\mathcal{O}_{Z,z}$ is locally free over $\mathcal{O}_{\mathbb{P}^{d-1},z^1} \cong \mathcal{O}_{\mathbb{C}^{d-1},0}$ for all $z\in Z\cap P$ with $P\cap\mathbb{P}^{d-1} = \{z'\}$. But then

(4.2.12) $$\begin{aligned} \mathrm{length}(\widetilde{\mathrm{Quot}}(\mathcal{O}_{Z,z})) &= \mathrm{rank}_{\mathcal{O}_{\mathbb{C}^{d-1},0}}(\mathcal{O}_{Z,z}) \\ &= \dim_{\mathbb{C}}(\mathcal{O}_{Z,z}/m_{d-1}\mathcal{O}_{Z,z}) \quad , \\ &= \dim_{\mathbb{C}}(\mathcal{O}_{Z\cap P,z}) \quad , \end{aligned}$$

which implies $\deg(\underline{Z}) = \sum_{z\in Z\cap P} \deg_z q_{P'} = \chi(\underline{Z}\cap\underline{P}, \mathcal{O}_{Z\cap P})$. Q.e.d.

For a more classical proof which does not use sheaf cohomology see [56], p. 112 ff, which works for the case $\underline{Z}$ reduced irreducible. Since $H(-,k)$ is additive on modules,

(4.2.13) $$H(R,k) = \sum_{\mathfrak{p}\in \mathrm{Assh}(R)} \mathrm{length}(R_{\mathfrak{p}})H(R/\mathfrak{p},k) \quad ,$$

and so the general case follows also from this because of Corollary 4.1.9.

4.3. A generalization.

Let $A \in \underline{\underline{la}}$ be an artinian local $\mathbb{C}$-algebra corresponding to a one-point complex space $\underline{S} = (\{s\}, A) \in \underline{\underline{cpl}}$.

Definition 4.3.1.

(i) $\underline{\mathbb{P}}_A^{n-1} := \underline{S} \times \underline{\mathbb{P}}^{n-1}$, projective (n-1)-space over A .

(ii) A projective variety $\underline{Z}$ over A is a closed complex subspace $\underline{Z} \hookrightarrow \underline{\mathbb{P}}_A^{n-1}$ defined by a homogeneous ideal $I \subseteq A[Z_1,\dots,Z_n]$ for some n .

Remark 4.3.2. Projective varieties correspond to finitely generated graded A-algebras (positively graded, $B_0 = A$, generated by B_1). In fact if $\underline{Z}$ is as above, $R := A[Z_1,\dots,Z_n]/I$, $\underline{Z} = \text{Projan}(R)$ (see III 1.2.8), the complex space associated to the projective scheme $\text{Proj}(R)$.

Corresponding to $\underline{\mathbb{P}}_A^{n-1}$ there is affine n-space $\underline{\mathbb{A}}_A^n := \underline{S} \times \underline{\mathbb{C}}^n$ over A . Corresponding to $\underline{Z} \hookrightarrow \underline{\mathbb{P}}_A^{n-1}$ there is an affine variety $\underline{C} \hookrightarrow \mathbb{A}_A^n$, in fact $\underline{C} = \text{Specan}(R)$ as a complex space. We call again $\underline{C}$ the cone associated to $\underline{Z}$, and $\underline{Z}$ the projective cone $\underline{\mathbb{P}\,C}$ of $\underline{C}$.

Let $\underline{\mathbb{P}}^{n-1} \xrightarrow{\underline{r}} \underline{\mathbb{P}}_A^{n-1}$ be the morphism given by $A[Z_1,\dots,Z_n] \twoheadrightarrow (A/m_A)[Z_1,\dots,Z_n]$. If $\underline{Z} \hookrightarrow \underline{\mathbb{P}}_A^n$ is a projective variety over A , we put $\underline{Z}_0 =: \underline{r}^{-1}(\underline{Z})$ and

$$\deg(\underline{Z}) := (\dim_{\mathbb{C}} A)\cdot(\deg(\underline{Z}_0)) \qquad (4.3.1)$$

Now let M be a finitely generated B-module. Define again the Hilbert function $H(M,k)$ to be

$$H(M,k) := \dim_{\mathbb{C}} M_k \qquad (4.3.2)$$

Then Theorem 4.1.8 and Theorem 4.2.1 still hold with the convention (4.3.1) for $\deg(\underline{Z})$.

4.4. Samuel multiplicity.

Let now $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, $\mathfrak{q}$ an $\mathfrak{m}_x$-primary ideal of $\mathcal{O}_{X,x}$, defining a zero dimensional complex subspace of $\underline{X}$ supported on x , which we call $\underline{x}(\mathfrak{q})$.

Definition 4.4.1 (Normal cone). The normal cone of $\underline{x}(\mathfrak{q})$ in $\underline{x}$ is defined to be

$$\underline{C}(\underline{X},\underline{x}(\mathfrak{q})) := \underline{\mathrm{Specan}}(\mathrm{gr}_{\mathfrak{q}}(\mathcal{O}_{X,x})) \quad .$$

In case $\mathfrak{q} = \mathfrak{m}$, $\underline{C}(\underline{X},\underline{x}(\mathfrak{q})) = \underline{C}(\underline{X},x)$, the tangent cone.
The epimorphism $\mathrm{Sym}(\mathfrak{q}/\mathfrak{q}^2) \twoheadrightarrow \mathrm{gr}_{\mathfrak{q}}(\mathcal{O}_{X,x})$ gives an embedding $\mathbb{P}\,C(\underline{X},\underline{x}(\mathfrak{q})) \hookrightarrow \underline{\mathbb{P}}_A^{d-1}$, where $d := \dim_{\mathbb{C}}(\mathfrak{q}/\mathfrak{q}^2)$ and $A := R/\mathfrak{q}$. Taking the Hilbert function of $\mathbb{P}\,C(\underline{X},\underline{x}(\mathfrak{q}))$ with respect to this embedding we get from Theorem 4.1.8, Theorem 4.2.1 and the discussion in 4.3:

Theorem 4.4.2. $\quad e(\mathfrak{q},\mathcal{O}_{X,x}) = m(\underline{C}(\underline{X},\underline{x}(\mathfrak{q})),x)$.

Remark 4.4.3. For an extension of this to the general scheme-theoretic context see the paper [57] of C.P. Ramanujam.

§ 5. Algebraic multiplicity.

In this paragraph, I show the equality $m(\underline{X},x) = e(\mathfrak{m}_x,\mathcal{O}_{X,x})$ for a complex space germ $(\underline{X},x)$.

5.1. Algebraic degree.

I now give some algebraic formulae for the local mapping degree, which relate it to Samuel multiplicity.

Proposition 5.1.1. Let X be a complex space, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, and Z be an irreducible component of the support $\mathrm{supp}(\mathcal{M})$ of $\mathcal{M}$. Let the ideal $\mathcal{P} \subseteq \mathcal{O}_X$ define Z in X. Then

(i) For $z \in Z_{ir}$, the localization $(\mathcal{M}_z)_{\mathcal{P}_z}$ is an artinian $(\mathcal{O}_{X,z})_{\mathcal{P}_z}$-module;

(ii) the function $z \longmapsto \mathrm{length}(\mathcal{M}_z)_{\mathcal{P}_z}$ is locally constant on Z_{ir}.

Proof. The proof is analogous to the proof of Proposition 2.2.3, so the details are omitted. One proves

$$\mathrm{length}_{(\mathcal{O}_{X,z'})_{\mathcal{P}_{z'}}}((\mathcal{M}_{z'})_{\mathcal{P}_{z'}} = \mathrm{length}_{\mathcal{O}(K)_P} M_P \quad \text{for} \quad z' \in Z_{ir} \cap K ,$$

where, for given $z \in Z_{ir}$, K is a suitable compact Stein neighbourhood of z, P is the $\mathcal{O}(K)$-ideal $\Gamma(K,\mathcal{P})$, and M the $\mathcal{O}(K)$-module $\Gamma(K,\mathcal{M})$, by localizing the flat map

$$\lambda_{z'} : \mathcal{O}(K) \longrightarrow \mathcal{O}_{X,z'}$$

at P and again using Lemma 2.2.4.

We now apply this, with $f : X \longrightarrow B$ as in Definition 2.2.6, to the coherent $\mathcal{O}_B$-module $f_*\mathcal{O}_X$:

Corollary 5.1.2. The number

$$(5.1.1) \qquad \sum_{x' \in f^{-1}(y)} \dim_{\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,y})} \left(\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,y}) \otimes_{\mathcal{O}_{\mathbb{C}^d,y}} \mathcal{O}_{X,x'} \right)$$

is independent of $y \in B$.

Proof. By I Theorem 4.1.1,

$$(f_*\mathcal{O}_X)_y = \bigoplus_{x'\in f^{-1}(y)} \mathcal{O}_{X,x'} \quad \text{for all} \quad y\in B$$

as an $\mathcal{O}_{\mathbb{C}^d,y}$-module. The claim now follows by Proposition 5.1.1.

Recall now Serre's notation: Let R be a local ring, $\mathfrak{q}$ an $\mathfrak{m}_R$-primary ideal, M an R-module, $d\in\mathbb{N}$ such that $\dim_R M \le d$; then put

$$(5.1.2) \qquad e_{\mathfrak{q}}(M,d) := \begin{cases} e(\mathfrak{q},M) & \text{if} \quad \dim_R M = d \\ 0 & \text{else} \end{cases}$$

(see [67], p. V-3). We then have the formula (loc.cit, or Chapter I, Theorem (1.8)):

$$(5.1.3) \qquad e_{\mathfrak{q}}(M,d) = \sum_{\dim(R/\mathfrak{p})=d} \text{length}(M_{\mathfrak{p}})\cdot e_{\mathfrak{q}}(R/\mathfrak{p},d)$$

(because of additivity of length).

Corollary 5.1.3. In the situation of Corollary 5.1.2, the number

$$(5.1.4) \qquad \sum_{x'\in f^{-1}(y)} e_{\mathfrak{q}_{x'}}(\mathcal{O}_{X,x'},d)$$

is also independent of $y\in B$, where $\mathfrak{q}_{x'}$ is the ideal in $\mathcal{O}_{X,x'}$ generated by the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{\mathbb{C}^d,y}$; in fact it equals the number (5.1.1.).

Proof. The number in question is $e_{\mathfrak{m}_y}((f_*\mathcal{O}_X)_y,d)$, which by (5.1.3) is just $\text{length}(\text{Quot}(\mathcal{O}_{\mathbb{C}^d,y})\otimes_{\mathcal{O}_{\mathbb{C}^d,y}}(f_*\mathcal{O}_{X,x})_y)$, since $R = \mathcal{O}_{\mathbb{C}^d,y}$ is regular and so $e(\mathfrak{m}_d,R)=1$. And this number is (5.1.1).

We now can characterize the local mapping degree algebraically.

Theorem 5.1.4 (Multiplicity formula). Let $\underline{f}$ be as in Definition 2.2.6. Then the following numbers are equal:

(i) the local mapping degree $\deg_x \underline{f}$;

(ii) $\dim_{\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0})} (\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0}) \otimes_{\mathcal{O}_{\mathbb{C}^d,0}} \mathcal{O}_{X,x})$;

(iii) the Samuel multiplicity $e_{\mathfrak{q}}(\mathcal{O}_{X,x}, d)$ with $\mathfrak{q} = \mathfrak{m}_d \cdot \mathcal{O}_{X,x} = (f_1,\ldots,f_d)\cdot \mathcal{O}_{X,x}$, where $(f_1,\ldots,f_d)$ define $\underline{f}$ according to I, Corollary 3.3.5.

Remark 5.1.5.

a) For a complete local ring containing a field which is an integral domain, (ii) was Chevalley's original definition of the multiplicity $e(\mathfrak{q}_x, \mathcal{O}_{X,x})$ (up to multiplying with the degree of the residue field extension, which is 1 here) in [9], § IV. Somewhat later he extended it to quasi-unmixed local rings in [10], Definition 3 on p. 13, and his definition can be shown to be again the number (ii). In other words, the philosophy behind his definition was to mimic, by passing to the completion, the notion of local mapping degree by an algebraic construction. See also Remark 5.1.8.

b) The equality of (ii) and (iii) is a special case of the Projection Formula (Theorem (6.3) in Chapter I).

Proof of Theorem 5.1.4. We may assume $\dim_x X = d$, since otherwise all numbers are 0 . The equality of (ii) and (iii) has just been seen in the proof of Corollary 5.1.3.

To prove the equality of (i) and (ii), we are reduced, by Corollary 5.1.2, to prove the equality

$$(5.1.5) \qquad \mathrm{length}(\mathrm{Quot}(\mathcal{O}_{X,x'})) = \dim_{\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,y})} (\mathrm{Quot}(\mathcal{O}_{\mathbb{C},y}) \otimes_{\mathcal{O}_{\mathbb{C}^d,y}} \mathcal{O}_{X,x'})$$

in the special case where in the diagram

$$\begin{array}{ccc} (\underline{X}_{red}, x') & \overset{\underline{i}}{\hookrightarrow} & (\underline{X}, x') \\ & {\scriptstyle \underline{g} := \underline{f} \circ \underline{i}} \searrow \quad \swarrow {\scriptstyle \underline{f}} & \\ & (\underline{\mathbb{C}}^d, y) & \end{array}$$

$\underline{g}$ is an isomorphism and where $\underline{i}$ is defined by $\mathcal{O}_{X,x'} \twoheadrightarrow \mathcal{O}_{X,x'} / \mathcal{N}_{x'}$. We thus have that in the situation

$$\mathcal{O}_{\mathbb{C}^d,y} \xrightarrow{f^0_{x'}} \mathcal{O}_{X,x'} \xrightarrow{i^0_{x'}} \mathcal{O}_{X,x'} / \mathcal{N}_{x'} \qquad ,$$

$f^0_{x'}$ is injective by I Theorem 6.2.1 and $i^0_{x'} \circ f^0_{x'}$ is an isomorphism. The claim then follows from the following Lemma.

Lemma 5.1.6. Let $R \hookrightarrow S$ be a finite extension of local analytic $\mathbb{C}$-algebras such that R is an integral domain and the nilradical $\mathfrak{n}_S$ of S is prime. Then $\mathrm{Quot}(R) \otimes_R S \cong \widetilde{\mathrm{Quot}}(S)$.

Proof. Since $\mathfrak{n}_S$ is prime, any element of S is either nilpotent or active by (2.2.1). By the argument in the proof of the Active Lemma I 5.2.2 and $t \in \mathrm{Ac}(S) = S - \mathfrak{n}_S$ satisfies an integral equation

$$t^k + r_{k-1}t^{k-1} + \ldots + r_1 t + r_0 = 0 \tag{5.1.6}$$

with $k \geq 1$, $r_j \in R$ for $0 \leq j \leq k-1$, and $r_0 \neq 0$.

Now any element of $\mathrm{Quot}(R) \otimes_R S$ can be written as a fraction s/r with $s \in S$, $r \in R - \{0\}$. Since $R - \{0\} \hookrightarrow S - \mathfrak{n}_S$, we can consider this as an element of $\widetilde{\mathrm{Quot}}(S)$, and this gives a homomorphism

$$\varphi : \mathrm{Quot}(R) \otimes_R S \longrightarrow \widetilde{\mathrm{Quot}}(S) \ . \tag{5.1.7}$$

I claim φ is an isomorphism.

Injectivity of φ : Suppose $s/r \in \mathrm{Quot}(R) \otimes_R S$ maps to 0 in $\widetilde{\mathrm{Quot}}(S)$. This means there is $t \in \mathrm{Ac}(S)$ with $t \cdot s = 0$. Multiplying

(5.1.6) with s shows $r_0 \cdot s = 0$, with $r_0 \in R - \{0\}$, hence $s/r = 0$ in $\mathrm{Quot}(R) \otimes_R S = (R - \{0\})^{-1}S$.

Surjectivity of φ: Let $s/t \in \widetilde{\mathrm{Quot}}(S)$; it suffices to produce $u \in S$ such that $tu = r \in R - \{0\}$, for then $s/t = su/r$.

Now $t \in \mathrm{Ac}(S)$, therefore (5.1.6) gives

$$t\left(t^{k-1} + r_{k-1}t^{k-2} + \ldots + r_1\right) = -r_0 ,$$

so it suffices to take $u := t^{k-1} + r_{k-1}t^{k-2} + \ldots + r_1$ and $r := -r_0$.

Remark 5.1.7. The degree formula 2.2.8. holds.

This is now immediate by 5.1.2 and 5.1.4.

Remark 5.1.8. Formula (3.2.5) can be written as

$$(5.1.8) \qquad m(\underline{X},x) = \min_{\substack{(f_1,\ldots,f_d)\text{s.o.p.} \\ \text{of } \mathcal{O}_{X,x}}} \left\{ \dim_{\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0})} \mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0}) \otimes_{\mathcal{O}_{\mathbb{C}^d,0}} \mathcal{O}_{X,x} \right\} .$$

By the proof of Lemma 4.1.4, $\mathrm{Quot}(\mathcal{O}_{X,x}) \longrightarrow \mathrm{Quot}(\hat{\mathcal{O}}_{X,x})$ is a flat morphism of local rings with residue field extension of degree 1; from this one can show

$$\dim_{\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0})} (\mathrm{Quot}(\mathcal{O}_{\mathbb{C}^d,0}) \otimes_{\mathcal{O}_{\mathbb{C}^d,0}} \mathcal{O}_{X,x}) =$$

$$\dim_{\mathrm{Quot}(\hat{\mathcal{O}}_{\mathbb{C}^d,0})} (\mathrm{Quot}(\hat{\mathcal{O}}_{\mathbb{C}^d,0}) \otimes_{\hat{\mathcal{O}}_{\mathbb{C}^d,0}} \hat{\mathcal{O}}_{X,x})$$

which is just Chevalley's definition of his $e(\mathcal{O}_{X,x};f_1,\ldots,f_d)$.
So $m(\underline{X},x)$ corresponds to taking the minimal value of these multiplicities, as asserted in the Historical Remark Chapter I, (6.7),c).

5.2. Algebraic multiplicity.

We now characterize the geometric multiplicity algebraically.

Theorem 5.2.1 (The Multiplicity Theorem). Let $(\underline{X},x) \hookrightarrow (\mathbb{C}^n,0)$ be an embedding of $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, $d := \dim_x \underline{X}$, and $L \in P^d_g(\underline{X},x)$. Then

(i) $\deg_x \underline{p}_L \geq e(m_x, \mathcal{O}_{X,x})$;

(ii) if $L \in P^d_e(\underline{X},x)$, $\deg_x \underline{p}_L = e(m_x,\mathcal{O}_{X,x})$, and if $(\underline{X},x)$ is pure dimensional , the converse holds ;

(iii) $m(\underline{X},x) = e(m_x,\mathcal{O}_{X,x})$, i.e. the geometric multiplicity of $(\underline{X},x)$ equals the Samuel multiplicity of $\mathcal{O}_{X,x}$. Especially, $m(\underline{X},x)$ does not depend on the embedding $(\underline{X},x) \hookrightarrow (\mathbb{C}^n,0)$, but only on the isomorphism class of (X,x) in $\underline{\underline{\mathrm{cpl}}}_0$.

Proof.

(i). We have $\deg_x \underline{p}_L = e(q_x,\mathcal{O}_{X,x})$ by Theorem 5.1.4, where $q_x = \underline{p}^0_{L,x}(m_d)\cdot\mathcal{O}_{X,x}$. Since $q_x \subseteq m_x$ is m_x-primary, $e(q_x,\mathcal{O}_{X,x}) \geq e(m_x,\mathcal{O}_{X,x})$ by the definition $e(q_x,\mathcal{O}_{X,x})$

(ii). If $L \in P^d_e(\underline{X},x)$, $L \pitchfork_x C(\underline{X},x)$, which means $d_x\underline{p}_L$ is quasi-finite at $x \in C(\underline{X},x)$, and hence finite as a mapgerm $d_x\underline{p}_L : (\underline{C}(\underline{X},x),x) \longrightarrow (\mathbb{C}^d,0)$ by I Corollary 4.3.6. So q_x is a minimal reduction of m_x by Proposition 3.2.3, and so $e(q_x,\mathcal{O}_{X,x}) = e(m_x,\mathcal{O}_{X,x})$ by Chapter I, Proposition (4.14.). The converse is just the Theorem of Rees (cf. [49], Theorem 1 of §1).

(iii). This is immediate from (i) and (ii). Q.e.d.

For geometric proofsof Rees's Theorem in the reduced case for the maximal ideal see [13], Th. 6.3 and [75], Chap. 7, Th. 7P. For the geometric interpretation of the general case of Rees' Theorem see III, 3.2.2.

Corollary 5.2.2. $m(\underline{X},x) = m(\underline{C}(\underline{X},x),x)$.

This gives a geometric proof of the following well-known fact:

Proposition 5.2.3. Let $(\underline{X},x) \in \underline{\underline{cpl}}_0$ be equidimensional. Then $m(\underline{X},x) = 1$ implies $(\underline{X},x)$ is smooth.

Proof. $m(\underline{X},x) = m(\underline{C}(\underline{X},x),x)$ by Corollary 5.2.2.

$= \deg(\mathbb{P}\,\underline{C}(\underline{X},x))$ by Theorem 4.1.8, where $\underline{C}(\underline{X},x) \hookrightarrow \mathbb{C}^n$ with $n = \dim_{\mathbb{C}}(m_x/m^2)$. But $\deg(\mathbb{P}\,\underline{C}(\underline{X},x)) = 1$ implies that $\mathbb{P}\,\underline{C}(\underline{X},x)$ is a (d-1)-dimensional linear space (see Exercise) and so $d = n$, since otherwise m_x could be generated by less than n elements which cannot be. This proves the claim.

Exercise: $\mathbb{P}\,\underline{C}(X,x)$ is equidimensional (Hint: Consider 3.1.3. Or blow up $\underline{X}$ at x).

As an application of 5.2.1, we now prove:

Theorem 5.2.4. (Upper Semicontinuity of Multiplicity). Let $\underline{X} \in \underline{\underline{cpl}}$. Then the function $x \longmapsto e(m_x, \mathcal{O}_{X,x})$ is upper semicontinuous, i.e. any $x \in X$ has a neighbourhood U such that $e(m_{x'}, \mathcal{O}_{X,x'}) \le e(m_x, \mathcal{O}_{X,x})$ for all $x' \in U$.

Proof. Since the claim is local, we may assume $(\underline{X},x) \hookrightarrow (\mathbb{C}^n,0)$ for some n . Let $L \in P^d_e(\underline{X},x)$ where $d = \dim_x X$, then $L \pitchfork_x X$ by Proposition 3.2.4 (i), and so $p_L : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ is quasifinite and hence finite by Proposition 3.2.2. So $L + x' \in P^d_g(\underline{X},x')$ for x' near x . Choosing U sufficiently small, we have

$$e(m_x, \mathcal{O}_{X,x}) = \deg_x \underline{p}_L \qquad \text{by Theorem 5.2.1, (ii)}$$

$$= \sum_{\tilde{x} \in \underline{p}^{-1}(x')} \deg_{\tilde{x}} \underline{p}_L \qquad \text{by Theorem 2.2.8}$$

$$\ge \deg_{x'} \underline{p}_L$$

$$\ge e(m_{x'}, \mathcal{O}_{X,x'}) \qquad \text{by Theorem 5.2.1, (i)} .$$

III. GEOMETRIC EQUIMULTIPLICITY.

As exposed in the preface of this book, one of the numerical conditions to be imposed on a subspace $\underline{Y}$ of a complex space $\underline{X}$ as to qualify for a suitable centre of blowing up is that $\underline{X}$ should have the same multiplicity along $\underline{Y}$. This condition has been studied algegraically in Chapter IV, and it is the purpose of this part to give a description of it from a geometric point of view.

The appropriate geometric property of the blowup $\underline{\tilde{X}} \xrightarrow{\pi} \underline{X}$ of $\underline{X}$ along $\underline{Y}$ which is controlled by the multiplicity in case $\underline{Y}$ is smooth is the equidimensionality of $\underline{\pi}$ restricted to the exceptional divisor. In terms of the normal cone, it is called normal pseudoflatness of $\underline{X}$ along $\underline{Y}$; in terms of local algebra, it is just the condition $ht(I) = s(I)$, where I defines $\underline{Y}$ in $\underline{X}$ locally. Normal pseudoflatness has been introduced by Hironaka in [34], and the name originates from the fact that it is just that weaker version of normal flatness which keeps the essential topological properties of the latter. The surprising fact that equimultiplicity is equivalent to normal pseudoflatness is due in the special case of a surface along a smooth curve to Zariski, and, in the general case,to Hironaka and Schickhoff.

In the first paragraph I introduce the notions of normal cone, blowup,and normal flatness and pseudoflatness for the complex analytic case. In the following section, I give a detailed account of the result of Hironaka and Schickhoff and related results of Lipman and Teissier. These results could have been,in principle, mostly derived from the corresponding algebraic results by the method of compact Stein neighbourhoods, but I have preferred to give a geometric proof more or less along the original lines. This was done partly to give an introduction to the geometric method, where multiplicity appears as a local mapping degree and which is used explicitely by the authors mentioned above, and partly to illustrate the geometric content of various other algebraic notions and methods; in particular, the relation of equimultiplicity with reduction and integral dependence, which is emphasized in the preface of this book, is commented on. The last paragraph, finally, describes more briefly the geometric content of equimultiplicity and normal flatness along a nonsmooth centre, where equimultiplicity in the former sense has to be modified to a general type of multiplicity, which however, can again be described geometrically by local mapping degrees.

My general contention is that the relation between equimultiplicity and normal pseudoflatness asserts, on the geometric level, that the local mapping degree of a linear projection of a complex spacegerm (embedded in a number space) is a measure for the contact of the kernel of the projection with the spacegerm at the intersection. In that sense, the requirement of equimultiplicity of a space X along a subspace Y puts a transversality condition on the intersection of the space with the family of projections defining the multiplicity of the space along the subspace. This transversality appears as growth conditions on the local coordinates of X in directions normal to Y, and so the relations with integral dependence and normal pseudoflatness emerge. From this the fundamental rôle of the Theorem of Rees-Böger should be apparent, and I have tried to indicate the connections with this theorem at the appropriate places.

§ 1. Normal flatness and pseudoflatness.

Here I discuss the notions of normal flatness and normal pseudoflatness of a complex space along a closed complex subspace. Basic is the result that these notions are open, and generic when the subspace is reduced. It is derived from the algebraic case by the method of compact Stein neighbourhoods, and for this some technical preparations are needed, which are supplied in 1.1. In 1.2 the notions of the analytic and projective spectrum over an arbitrary base $\underline{S} \in \underline{\underline{cpl}}$ are discussed; these constructions are fundamental for the construction of the normal cone and of the blowup. Section 1.3 contains a proof that flatness is open, and generic along a reduced base. Finally, in 1.4, we define the normal cone, the blowup, and discuss normal flatness and normal pseudoflatness.

1.1. Generalities from Complex Analytic Geometry.

In the sequel I need some general facts from Complex Analytic Geometry which I collect here.

First some notation. Let $\underline{X}$ be a complex space. If $x \in X$, $\mathfrak{p} \subseteq \mathcal{O}_{X,x}$ a prime, I put

$$(1.1.1) \qquad \mathbb{k}(\mathfrak{p}) := \mathrm{Quot}(\mathcal{O}_{X,x}/\mathfrak{p}) \quad ,$$

the residue field of the local ring $(\mathcal{O}_{X,x})_{\mathfrak{p}}$. Let M be a coherent $\mathcal{O}_X$-module, $x \in X$, then

(1.1.2) $$M(x) := M_x/\mathfrak{m}_x M_x = M_x \otimes_{\mathcal{O}_{X,x}} \mathbb{C} \quad .$$

<u>Proposition 1.1.1</u>. <u>Let the notation be as above</u>.

(i) $\dim_{\mathbb{C}} M(x) \geq \dim_{\Bbbk(\mathfrak{p})} (M_x \otimes_{\mathcal{O}_{X,x}} \Bbbk(\mathfrak{p}))$ <u>for all</u> $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_{X,x})$.

(ii) $\dim_{\mathbb{C}} M(x) \geq \dim_{\mathbb{C}} M(x')$ <u>for all</u> x' <u>near</u> x , <u>i.e. the function</u> $y \longmapsto \dim_{\mathbb{C}} M(y)$ <u>is upper semicontinuous</u>.

(iii) <u>The freenees locus</u> $LF(M) := \{x \in X \mid M$ <u>is locally free at</u> $x\}$ <u>is the complement of an analytic set</u> $\mathrm{Deg}(M)$.

(iv) <u>If</u> <u>X</u> <u>is reduced</u>, M <u>is locally free at</u> x <u>if and only if the function</u> $y \longmapsto \dim_{\mathbb{C}} M(y)$ <u>is constant near</u> x . <u>Further</u>, $\mathrm{Deg}(M)$ <u>is nowhere dense</u>.

<u>Proof</u>.

<u>(i)</u>. Let $m := \dim_{\mathbb{C}} M(x)$. Then m generators of M_x over $\mathcal{O}_{X,x}$ give m generators of $(M_x)_{\mathfrak{p}}$ over $(\mathcal{O}_{X,x})_{\mathfrak{p}}$. Then apply Nakayama's Lemma.

<u>(ii)</u>. Let $F_n(M)$ be the n-th Fitting ideal of M (cf. I 3.2.h)) and $\underline{X}_n(M)$ the closed complex subspace defined by it. Tensorizing the exact sequence of I (3.2.6) at x with $\mathbb{C}$ shows

(1.1.3) $$X_n(M) = \{y \in X \mid \dim_{\mathbb{C}} M(y) > n\} \quad .$$

Now, with $m = \dim_{\mathbb{C}} M(x)$, $x \in X - X_m(M)$, which is open.

<u>(iii)</u>. It is easy to see that

(1.1.4) $$M_x \ \underline{\text{is locally free of rank}} \ n \Leftrightarrow$$
$$F_n(M)_x = {}_{X,x} \quad \text{and} \quad F_{n-1}(M)_x = 0 \ .$$

Hence,

(1.1.5) $$LF(M) = X - \bigcap_{n\geq 0} (X_n(M) \cup \mathrm{supp}\, F_{n-1}(M)) \quad ,$$

and $\bigcap_{n\geq 0}(X_n(M) \cup \mathrm{supp}\, F_{n-1}(M))$ is analytic since the family $(X_n(M) \cup \mathrm{supp}\, F_{n-1}(M))_{n\in\mathbb{N}}$ becomes locally stationary.

<u>(iv)</u>. Let $r := r(M) := \min\{\dim_{\mathbb{C}} M(x) \mid x \in X\}$. Then $X(r) := X - X_r(M)$ is nonempty and open. Now all $x \in X(r)$ are in $X_{r-1}(M)$, so $F_{r-1}(M) \mid X(r) \subseteq N_X \mid X(r)$, which implies $F_{r-1}(M)_x = 0$ for $x \in X(r)$ since <u>X</u> is reduced. The claim now follows by replacing <u>X</u> with any open neighbourhood of a given $x \in X$ and applying (1.1.4).

Theorem 1.1.2 (Cartan). Let $\mathcal{M}$ be a coherent module on the complex space X and $\mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots \subseteq \mathcal{M}$ an increasing chain of coherent submodules. Then this chain is locally stationary.

For a slick elementary proof see [28] , Chapter 5, § 6; see also [14], 0.40.

Next, we set up a formalism ([5],[29],[38],[41],[63]) by which results in Algebraic Geometry can often be transferred to Complex Analytic Geometry; we will use it in 1.4 to deduce the fact that normal flatness is generic from the Krull-Seidenberg-Theorem in Chapter IV, (24.4). This idea seems to have originated from footnote 18 on p. 136 of [33]. We partly follow the presentation of [38].

In the following, $\underline{X}$ is a local model in some open set $U \subseteq \mathbb{C}^n$.

Definition 1.1.3. A distinguished compact Stein set in X is a compact neighbourhood of some $x \in X$ of the form $Q \cap X$, where Q is a compact stone in U .

By II Corollary 1.4.2, any $x \in X$ has a neighbourhood basis consisting of distinguished compact Stein subsets.

We first need a noetherian property for distinguished compact Stein subsets. The following result is a special case of a theorem due to Frisch ([16], Théorème (I, 9)) and Siu ([68], Theorem 1).

Proposition 1.1.4. Let K be a distinguished compact Stein subset in a complex space $\underline{X}$. Then $\mathcal{O}(K) = \Gamma(K,\mathcal{O}_X)$ is a noetherian ring.

Proof. We may assume $\underline{X} \overset{i}{\hookrightarrow} \underline{U}$ is a local model, where $U \subseteq \mathbb{C}^n$. Let $Q \subseteq U$ be a compact stone which defines K , i.e. $K = X \cap Q$. The surjection $\mathcal{O}_U \twoheadrightarrow i_*\mathcal{O}_X$ induces the surjection $\Gamma(Q,\mathcal{O}_U) \twoheadrightarrow \Gamma(K,\mathcal{O}_X)$ by Theorem B. So it suffices to prove $\Gamma(Q,\mathcal{O}_U)$ is noetherian. For this we induct over the real dimension d of Q .

If $d = 0$, Q is a point, and the claim is just the Rückert Basissatz, I 1.3.2. Let $d \geq 1$, and suppose the proposition is true for (d-1)-dimensional compact stones. Suppose $I \subseteq \Gamma(Q,\mathcal{O}_X)$ were not finitely generated, so we can find a sequence $f_1, f_2, f_3, \ldots$ of elements in I such that we get a strictly increasing sequence

$I_1 \subset I_2 \subset I_3 \subset \ldots$ with $I_j := (f_1,\ldots,f_j)\cdot\Gamma(Q,\mathcal{O}_X)$.

Now we may write

$$(1.1.6) \qquad Q = \bigcup_{\ell=1}^{2(d+1)} Q_\ell \amalg \mathring{Q} \quad ,$$

where the Q_ℓ are compact (d-1)-dimensional stones, and $\mathring{Q}$ is a stone which is open in the real vectorsubspace of $\mathbb{C}^n$ spanned by Q . By the induction assumption there are finitely many elements $g_1,\ldots,g_t \in \Gamma(Q,\mathcal{O}_X)$ such that $I\cdot\Gamma(Q_\ell,\mathcal{O}_X) = (g_1,\ldots,g_t)\cdot\Gamma(Q_\ell,\mathcal{O}_X)$ for $\ell = 1,\ldots,2(d+1)$. Let U be an open neighbourhood of Q in $\mathbb{C}^n$ such that $g_1,\ldots,g_t \in \Gamma(U,\mathcal{O}_X)$. Define ideal sheaves $\mathcal{I}_j \subseteq \mathcal{O}_U$ via

$$(1.1.7) \qquad \mathcal{I}_j(V) := \begin{cases} (g_1,\ldots,g_t)\cdot\mathcal{O}_V \ , & V \subseteq U - Q \ \text{open} \\ (g_1,\ldots,g_t,f_1,\ldots,f_j)\cdot\mathcal{O}_V \ , & V \subseteq U \ \text{open}, \ V \cap Q \neq \emptyset \ . \end{cases}$$

Then $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{I}_3 \subset \ldots$ is a strictly increasing sequence of coherent $\mathcal{O}_U$-ideals, so it cannot become eventually stationary on the compact set Q . This contradicts Theorem 1.1.2. Q.e.d.

A point $x \in K$ defines a character $\chi_x : \mathcal{O}(K) \longrightarrow \mathbb{C}$ via $\chi_x(f) := f(x)$, called a <u>point character</u>. Its kernel is a maximal ideal of $\mathcal{O}(K)$, denoted M_x . Let $\underline{K}$ be the ringed space $(K,\mathcal{O}|K)$, and $\underline{\text{Spec}}(\mathcal{O}(K))$ be the usual prime spectrum as a ringed space. We get a map of ringed spaces

$$(1.1.8) \qquad \underline{\phi}_K : \underline{K} \longrightarrow \underline{\text{Spec}}(\mathcal{O}(K))$$

by putting

$$(1.1.9) \qquad \phi_K(x) := M_x = \text{Ker}(\chi_x) \quad \text{for} \quad x \in K \ ,$$

and

$$(1.1.10) \qquad \begin{aligned} \phi^0_{K,D(f)} := \mathcal{O}(K)_{(f)} &\longrightarrow \Gamma(D(f),\mathcal{O}_X) \\ g/f^m &\longmapsto (x \longmapsto g(x)/f(x)^m) \end{aligned}$$

for $f \in \mathcal{O}(K)$.

We call a subset $A \subseteq K$ <u>analytic in</u> K if there is an analytic subset $\tilde{A}$ of some open neighbourhood $V \supseteq K$ such that $A = \tilde{A} \cap K$; this is the same as requiring that there is a finitely generated ideal sheaf $\mathcal{I} \subseteq \mathcal{O}|K$ such that $A = N(\mathcal{I})$. The following result is basic.

Proposition 1.1.5. If $B \subseteq \mathrm{Spec}(\mathcal{O}(K))$ is Zariski-closed, $\phi_K^{-1}(B) =: A$ is analytic in K, in fact $A = N(\mathcal{I})$, when $B = V(I)$ for $I \subseteq \mathcal{O}(K)$ an ideal and $\mathcal{I} = I \cdot \mathcal{O}|K$. In particular, ϕ_K is a morphism of ringed spaces.

Proof. Let $B = V(I)$; since $\mathcal{O}(K)$ is noetherian, $I = (f_1,\dots,f_k)\cdot\mathcal{O}(K)$ for some $f_1,\dots,f_k \in \mathcal{O}(K)$, and $\mathcal{I} = (f_1,\dots,f_k)\cdot\mathcal{O}|K$. Then $x \in \phi_K^{-1}(V(I)) \iff M_x \supseteq I \iff f_1,\dots,f_k \in \mathrm{Ker}(\chi_x) \iff x \in N(\mathcal{I})$.

Remark 1.1.6.

(i) The sheaf morphism ϕ_K^0 is regular on the stalks. From this one may deduce the openness of certain analytic loci, e.g. the regular locus, the Cohen-Macaulay locus, or the normal locus of a complex space, from the corresponding scheme-theoretic results, which, as a rule, are easier to prove; see [38].

(ii) One may use Proposition 1.1.5 to deduce the openness of the flatness locus of a coherent $\mathcal{O}_X$-module $\mathcal{M}$ with respect to a morphism $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ of complex spaces from the corresponding algebraic result (Theorem of Frisch); see [41].

1.2. The analytic and projective analytic spectrum.

This section generalizes I, 3.4 and II, 4.3 to the case of families of affine respectively projective varieties parametrized by a complex space.

Definition 1.2.1. Let $\underline{S} \in \underline{\mathrm{cpl}}$, $\mathcal{A}$ a sheaf of $\mathcal{O}_S$-algebras. $\mathcal{A}$ is called an admissible $\mathcal{O}_S$-algebra, or an $\mathcal{O}_S$-algebra locally of finite presentation, if every $x \in S$ has an open neighbourhood U such that there are sections $g_1,\dots,g_\ell \in \mathcal{O}_S(U)[T_1,\dots,T_k]$ and an epimorphism

(1.2.1) $$\psi_U : \mathcal{O}_U[T_1,\ldots,T_k] \twoheadrightarrow A|U$$

of $\mathcal{O}_U$-algebras such that $Ker(\psi_U)$ is the ideal generated by $g_1,\ldots,g_\ell$.

Now consider the category $\underline{\underline{cpl}}/\underline{S}$ of complex spaces over $\underline{S}$, whose objects are the morphisms $\underline{\varphi} : \underline{W} \longrightarrow \underline{S}$ in $\underline{\underline{cpl}}$ and whose morphisms are the commutative diagrams

(1.2.2) $$\begin{array}{ccc} \underline{W} & \xrightarrow{\underline{f}} & \underline{W}' \\ & {\scriptstyle \underline{\varphi}} \searrow \quad \swarrow {\scriptstyle \underline{\varphi}'} & \\ & \underline{S} & \end{array}$$

Then an $\mathcal{O}_S$-algebra A induces a contrafunctor

(1.2.3) $$Hom_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(A,-) : \underline{\underline{cpl/S}} \longrightarrow \underline{\underline{sets}}$$

as follows: It assigns to an object $\underline{\varphi} : \underline{W} \longrightarrow \underline{S}$ in $\underline{\underline{cpl/S}}$ the set $Hom_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(A,\underline{\varphi}_*\mathcal{O}_W)$, and to the commuative triangle (1.2.2) the map

(1.2.4) $$\begin{array}{ccc} Hom_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(A,\underline{\varphi}'_*\mathcal{O}_{W'}) & \longrightarrow & Hom_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(A,\underline{\varphi}_*\mathcal{O}_W) \\ \alpha & \longmapsto & \underline{\varphi}'_*(f^0)\circ\alpha \end{array} .$$

<u>Theorem 1.2.2</u> (see [64], Exposé 19). <u>If</u> A <u>is an admissible</u> $\mathcal{O}_S$-<u>algebra</u>, <u>the functor</u> (1.2.3) <u>is representable in</u> $\underline{\underline{cpl/S}}$.

This means the following: There is an object $\underline{\pi}_X : \underline{X} \longrightarrow \underline{S}$ in $\underline{\underline{cpl/S}}$ and an element $\zeta_X \in Hom_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(A,(\underline{\pi}_X)_*\mathcal{O}_X)$ such that the natural transformation

(1.2.5) $$Hom_{\underline{\underline{cpl/S}}}(-,\pi_X) \longrightarrow Hom_{\mathcal{O}_S\underline{\underline{alg}}}(A,-) ,$$

which assigns to $\underline{\varphi} : \underline{W} \longrightarrow \underline{S} \in \underline{\underline{cpl/S}}$ the map

(1.2.6) $$\mathrm{Hom}_{\underline{\underline{cpl/S}}}(\underline{\varphi},\underline{\pi}_X) \longrightarrow \mathrm{Hom}_{\mathcal{O}_S\text{-}\underline{\underline{alg}}}(\mathcal{A},\underline{\varphi}_*\mathcal{O}_W)$$

$$\begin{array}{ccc} \underline{W} & \xrightarrow{\underline{f}} & \underline{X} \\ & {\scriptstyle\underline{\varphi}}\searrow \quad \swarrow{\scriptstyle\underline{\pi}_X} & \\ & \underline{S} & \end{array} \longmapsto (\underline{\pi}_X)_*(\underline{f}^0) \circ \zeta_X ,$$

is a natural equivalence of functors.

As usual, the pair $(\underline{\pi}_X,\zeta_X)$ is unique up to unique isomorphism.

The universal property together with the glueing construction I 3.2 a) reduces the proof to the case $\mathcal{A} = \mathcal{O}_S[T_1,\ldots,T_k]/\mathcal{I}$, where $\mathcal{I}$ is generated by sections $g_1,\ldots,g_\ell \in \mathcal{O}_S(S)[T_1,\ldots,T_k]$. Now there is a natural morphism $\mathcal{O}_S(S)[T_1,\ldots,T_k] \longrightarrow \mathcal{O}_{S\times\mathbb{C}^k}(S\times\mathbb{C}^k)$, hence $g_1,\ldots,g_\ell$ generate an ideal $\mathcal{J} \subseteq \mathcal{O}_{S\times\mathbb{C}^k}$, and one defines $\underline{\pi}_X$ via

(1.2.7) $$\begin{array}{ccc} \underline{X} := \underline{N}(\mathcal{J}) & \xhookrightarrow{\underline{i}} & \underline{S}\times\underline{\mathbb{C}}^k \\ {\scriptstyle\underline{\pi}_X}\downarrow & \circlearrowleft & \swarrow{\scriptstyle\underline{pr}_S} \\ \underline{S} & & \end{array}$$

The homomorphism $\zeta : \mathcal{O}_S(S)[T_1,\ldots,T_k] \hookrightarrow \mathcal{O}_{S\times\mathbb{C}^k}(S\times\mathbb{C}^k) \xrightarrow{i_X^0} \mathcal{O}_X(\underline{\pi}^{-1}S)$,

factors through $\mathcal{I}$ and restricts over any open $U \subseteq S$, defining ζ_X . Details are left to the reader.

<u>Definition 1.2.3.</u> <u>The pair</u> $(\underline{\pi}_X,\zeta_X)$, <u>or, if no confusion is possible, the complex space</u> $\underline{X}$ <u>over</u> $\underline{S}$, <u>is called the analytic spectrum of the admissible</u> $\mathcal{O}_S$<u>-algebra</u> $\mathcal{A}$ <u>and denoted</u> <u>Specan</u>$(\mathcal{A})$.

We also write, 'par abus de languague', $\underline{\pi}_{\mathcal{A}}$: <u>Specan</u>$(\mathcal{A}) \longrightarrow \underline{S}$ for $\underline{\pi}_X : \underline{X} \longrightarrow \underline{S}$.

The analytic spectrum has the expected functional properties, see [64], Exposé 19. We mention here:

Proposition 1.2.4 (Base change). Let $\mathcal{A}$ be an admissible $\mathcal{O}_S$-algebra, $\underline{\varphi} : \underline{T} \longrightarrow \underline{S} \in \underline{cpl}$. Let $\underline{\psi} : \underline{Specan}(\underline{\varphi}^*\mathcal{A}) \longrightarrow \underline{Specan}(\mathcal{A}) \in \underline{cpl}$ correspond to the canonical morphism $\mathcal{A} \longrightarrow \underline{\varphi}_*\underline{\varphi}^*\mathcal{A}$ via (1.2.6). Then the diagram

(1.2.8)
$$\begin{array}{ccc} \underline{Specan}(\underline{\varphi}^*\mathcal{A}) & \xrightarrow{\underline{\psi}} & \underline{Specan}(\mathcal{A}) \\ \downarrow \underline{\pi}_{\underline{\varphi}^*\mathcal{A}} & \circlearrowright & \downarrow \underline{\pi}_{\mathcal{A}} \\ \underline{T} & \xrightarrow[\underline{\varphi}]{} & \underline{S} \end{array}$$

is cartesian, i.e. $\underline{Specan}(\underline{\varphi}^*\mathcal{A}) = \underline{Specan}(\mathcal{A}) \times_{\underline{S}} \underline{T}$.

From this we see the following: Let $\mathcal{A}_s$ be the stalk of $\mathcal{A}$ at $s \in S$, $\mathfrak{m}_s \subseteq \mathcal{O}_{S,s}$ the maximal ideal, and put

(1.2.9) $$\mathcal{A}(s) := \mathcal{A}_s/\mathfrak{m}_s \cdot \mathcal{A}_s = \mathcal{A}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C} ,$$

which is a finitely generated $\mathbb{C}$-algebra. Then in 1.2.7.

(1.2.10) $$\underline{X}_s := \underline{\pi}^{-1}(s) = \underline{Specan}(\mathcal{A}(s))$$

by base change, i.e. we may think informally of $\underline{X} = \underline{Specan}(\mathcal{A})$ as a family of affine varieties (considered as complex spaces) parametrized by the points of the complex space $\underline{S}$ via $\underline{\pi} : \underline{X} \longrightarrow \underline{S}$. This motivates the following result, which I just quote:

Proposition 1.2.5 ([64], Exposé 19, Prop. 3 and 4).

(i) The points of $\underline{X}_s$ correspond bijectively to the elements of $Vm(\mathfrak{m}_s\mathcal{A}_s) := \{\mathfrak{n} \in Specm(\mathcal{A}_s) \mid \mathfrak{n} \supseteq \mathfrak{m}_s\mathcal{A}_s\}$ under $x \in \underline{X}_s \longmapsto Ker(\mathcal{A}_s \xrightarrow{\zeta^0_{X,s}} \mathcal{O}_{X,x})$

(ii) Let $\mathfrak{n} \in Vm(\mathfrak{m}_s\mathcal{A}_s)$ correspond to $x \in \underline{X}_s$. Then $\zeta^0_{X,s}$ factors as $\mathcal{A}_s \longrightarrow (\mathcal{A}_s)_\mathfrak{n} \xrightarrow{\varphi_x} \mathcal{O}_{X,x}$, and

(1.2.11) $$\hat{\varphi}_x : (\mathcal{A}_s)_\mathfrak{n} \longrightarrow \hat{\mathcal{O}}_{X,x}$$

is an isomorphism.

We now come to the projective analytic spectrum.

Definition 1.2.6. Let $S \in \underline{cpl}$. An admissible graded $\mathcal{O}_S$-algebra is an admissible $\mathcal{O}_S$-algebra such that

(i) A is positively graded, i.e. $A = \bigoplus_{n\geq 0} A_n$, and locally generated by A_1 as $\mathcal{O}_S$-algebra.

(ii) The local representations (1.2.1) can be so chosen that ψ_U is a graded homomorphism of degree zero, where $T_1,\ldots,T_k$ have degree one.

Proposition 1.2.7 ([47], 1.4). Let A be a graded $\mathcal{O}_S$-algebra which is locally finitely generated as $\mathcal{O}_S$-algebra. Then the following statements are equivalent:

(i) A is an admissible graded $\mathcal{O}_S$-algebra.

(ii) A_k is a coherent $\mathcal{O}_S$-module for all $k \geq 0$.

Since the reference may be not easily accessible, I give a short idea of the proof.

(i) ⇒ (ii): Consider (1.2.1); $\mathcal{K}er(\psi_U)$ is a locally finite $\mathcal{O}_U$-module, so $A_k|U \cong (\mathcal{O}_U[T_1,\ldots,T_n])_k/\mathcal{K}er(\psi_U)_k$ is coherent.

(ii) ⇒ (i): The question is local, so we may assume we have an epimorphism

$$(1.2.12) \qquad \mathcal{O}_S[T_1,\ldots,T_k] \xrightarrow{\psi} A$$

of graded $\mathcal{O}_S$-algebras. Let $K := \mathcal{K}er(\psi)$, and put for $n \in \mathbb{N}$

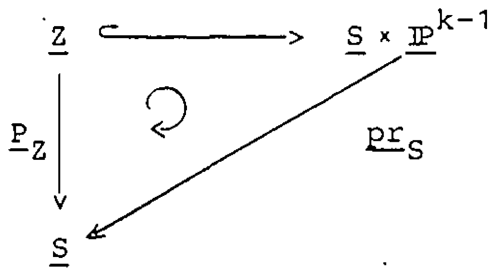
$$(1.2.13) \qquad A^{(n)} := \mathcal{O}_S[T_1,\ldots,T_k]/\bigoplus_{k\leq n} K_k \quad .$$

Then $A^{(0)} \twoheadrightarrow A^{(1)} \twoheadrightarrow \ldots$ is a decreasing tower of admissible $\mathcal{O}_S$-algebras. This gives us an increasing chain of coherent $\mathcal{O}_{S\times\mathbb{C}^k}$-ideals

$I^{(0)} \subseteq I^{(1)} \subseteq \ldots$, where $I^{(h)}$ defines $\underline{X}^{(n)} := \underline{\text{Specan}}(A^{(n)}) \subseteq \underline{S} \times \underline{\mathbb{C}}^k$. The claim then follows from Theorem 1.1.2.

If A is an admissible graded $\mathcal{O}_S$-algebra, we have local representations (1.2.1) with $Ker(\psi_U)$ homogeneous. Therefore, in the local construction of $\underline{\text{Specan}}(A)$ in diagram (1.2.7), the $\mathcal{O}_S$-homogeneous ideal J defines a closed complex subspace $\underline{Z} \hookrightarrow \underline{S} \times \underline{\mathbb{P}}^{k-1}$, and we get the commutative diagram

(1.2.14)



The $\underline{p}_Z$ glue well because of the functorial properties of the Specan-construction; so, for any admissible graded $\mathcal{O}_S$-algebra, we have constructed a complex space $p_{\underline{Z}} : \underline{Z} \longrightarrow \underline{S}$ over $\underline{S}$.

Definition 1.2.8. The space so obtained is called the projective analytic spectrum of A and denoted $\underline{p}_A : \underline{\text{Projan}}(A) \longrightarrow \underline{S}$, or $\underline{\text{Projan}}(A)$ for short.

Remark 1.2.9. As in 1.2.4, base change holds for the Projan-construction.

1.3. Flatness of admissible graded algebras.

Definition 1.3.1. Let $\underline{S} \in \underline{cpl}$, A an admissible $\mathcal{O}_S$-algebra. Then A is called flat along S at $s \in \underline{S}$ if and only if A_s is a flat $\mathcal{O}_{S,s}$-module. A is called flat along S if and only if it is flat along S at all $s \in S$.

Remark 1.3.2. If A is flat along S , $(A_s)_n$ is $\mathcal{O}_{S,s}$-flat for all s and all $\mathfrak{n} \in \mathrm{Specan}(A_s)$, hence $\widehat{(A_s)_{\mathfrak{n}}} = \hat{\mathcal{O}}_{X,x}$ is $\hat{\mathcal{O}}_{S,s}$ flat for all $s \in S$ and $x \in \underline{\pi}_X^{-1}(s)$, where $\underline{\pi}_X : \underline{X} \longrightarrow \underline{S}$ is $\mathrm{Specan}(A)$, by Proposition 1.2.5. It follows that $\underline{\pi}_X : \underline{X} \longrightarrow \underline{S}$ is a flat morphism.

Proposition 1.3.3. Let $S \in \underline{\mathrm{cpl}}$ be reduced, A an admissible graded $\mathcal{O}_S$-algebra. The following statements are equivalent:

(i) A is a flat $\mathcal{O}_S$-algebra.

(ii) The functions $s \longmapsto \dim_{\mathbb{C}} A_k(s)$ (see (1.2.9)) are locally constant for all k .

Proof. A is a flat $\mathcal{O}_S$-algebra if and only if A_k is a flat $\mathcal{O}_S$-module for all k . But each A_k is a coherent $\mathcal{O}_S$-module by Proposition 1.2.7. The claim then follows from Proposition 1.1.1 (iv), since over a local ring, to be flat means to be free.

We now have the following theorem, which has been stated by Hironaka in [33], p. 136, and proved by means of Proposition 1.1.3 in [38], and by other means in [47].

Theorem 1.3.4 (Flatness is generic). Let A be an admissible graded $\mathcal{O}_S$-algebra on the complex space $\underline{S}$. Then the set $F(A) := \{s \in S \mid A_s$ is a flat $\mathcal{O}_{S,s}$-module$\}$ is the complement of an analytic set. If S is reduced, $S - F(A)$ is nowhere dense.

Proof. The question is local. Let $K \subseteq S$ be a distinguished compact Stein subset, and let $A_k := \Gamma(K, A_k)$, $A = \bigoplus_{k \geq 0} A_k$, $R := \Gamma(K, \mathcal{O}_X) = \mathcal{O}(K)$; R is noetherian by Proposition 1.1.4. Let $s \in K$. Then

(1.3.1) A_s is $\mathcal{O}_{S,s}$-flat $\Longleftrightarrow$ $\forall k \geq 0 : (A_k)_s$ is $\mathcal{O}_{S,s}$-flat

$\Longleftrightarrow$ $\forall k \geq 0 : (A_k)_{M_s}$ is R_{M_s}-flat, since $R_{M_s} \longrightarrow \mathcal{O}_{S,s}$ is faithfully flat by II 1.3.2.

$\Longleftrightarrow$ A_M is R_{M_s}-flat .

Hence $K \cap F(A) = \phi_K^{-1}(F(A))$. The first claim now follows by the Krull-Seidenberg-Grothendieck - Theorem (Chapter IV, (24.4)) and by Proposition 1.1.5. The second claim follows from Proposition 1.1.1 (iv) and 1.3.3. (ii):

(1.3.2) $$S - F(A) = \bigcup_{k>0} \mathrm{Deg}(A_k)$$

has empty interior as a countable union of nowhere dense analytic sets by the theorem of Baire.

Remark 1.3.5. Theorem 1.3.4 can be interpreted more concretely, without using the Krull-Seidenberg-Grothendieck-Theorem, as follows, using 1.3.3.instead. Let $\underline{S}$ be reduced. Then 1.3.4 would follow from 1.3.3, if one were able to show that the Hilbert functions $H(A(s),-)$ were constant for s near s_0 , i.e. if $k \longmapsto H(A(s),k)$ were independent of s near s_0 . Note that this is a priori stronger that the statement (ii) of 1.3.3, since the neighbourhoods of s_0 on which the functions $\dim_{\mathbb{C}} A_k(s)$ are constant might depend on k .

Now it is known that each Hilbert function $k \longmapsto H(A(s),k)$ becomes a polynomial, of degree $d_0(s)-1$, say, for k above some number $k_0 = k_0(s)$, and so is determined by any $d_0(s)$ values at numbers $k > k_0(s)$. So the constancy of finitely many functions $\dim_{\mathbb{C}} A_k(s)$ near s_0 would guarantee the constancy of all of them if we were able to bound $d_0(s)$ and $k_0(s)$ near s_0 ; this would then imply 1.3.4 because of 1.3.3 (ii). So what one wants to show is:

(1.3.3) For any $s_0 \in S$, there are a neighbourhood U of s_0 and natural numbers d_0 and k_0 such that $H(A(s),k)$ is a polynomial in k for all $k > k_0$ of degree $<d_0$ for $s \in U$.

There might be two ways to establish (1.3.3). For the first one, results of Grauert and Remmert for projective morphisms over a basis in cpl (concering the vanishing of the sheaves $(R^i\underline{p})_* M(n)$ for $\underline{p} = \underline{p}_A$: Projan$(A) \longrightarrow \underline{S}$ and $\underline{M}$ a coherent module on Projan(A) and generalizing well-known facts from the scheme-theoretic case; (see [25], [2] Chapter IV)) suggest that one should have: There is a neighbourhood U of s_0 and a number k_0 such that

$$H(A(s),k) = \chi(\underline{Z}_s, \mathcal{O}_{Z_s}(k)) \quad \text{for} \quad k \geq k_0 , s \in U \quad ,$$

where $\underline{p} : \underline{Z} \longrightarrow \underline{S}$ is $\underline{\text{Projan}}(A)$, $\underline{Z}_s$ the fibre $\underline{p}^{-1}(s)$, and $\mathcal{O}_{Z_s}(k) = \mathcal{O}_Z(1)^{\otimes k}$, $\mathcal{O}_Z(1)$ the canonical linebundle on $\underline{Z}$. Then (1.3.3) holds with $d_0 = \max\{\dim \underline{Z}_s \mid s \in U\} + 1$.

The other approach might be based on a parametrized version of the division algorithm for rings of the form $\mathcal{O}_{S,s}[Z_1,\ldots,Z_\ell]$ (see [20], (1.2.7) and [62], 1.3). Applying this to the ideal $I_{s_0} \subseteq \mathcal{O}_S[Z_1,\ldots,Z_\ell]$, where $A \cong \mathcal{O}_S[T_1,\ldots,T_\ell]/I$ locally, should give a leitideal generated by monomials $\lambda_A Z^A$, where λ_A are germs in $\mathcal{O}_{S,s_0}$. Now the Hilbert function of a homogeneous ideal $I \subseteq \mathbb{C}[Z_1,\ldots,Z_\ell] =: R$ is the Hilbert function of the leitideal $LM(I)$, and so (see [53])

$$H(R/I,k) = \sum_{j=0}^{t} (-1)^k \sum_{1 \leq i_1 < \ldots < i_j \leq t} \binom{\ell-1+\deg \ell c.m(M_{i_1},\ldots,M_{i_j})+k}{\ell-1}_+ ,$$

where the monomials $M_1,\ldots,M_t$ generate $LM(I)$. From this it may be possible to see that $H(A(s),k) = H(R/I_sR,k)$ is constant outside the subspace of (S,s_0) defined by the λ_A and can only increase over there, so that $s \longmapsto H(A(s),k)$ is upper semicontinuous, and that the $H(A(s),k)$ are polynomials for all s near s_0 for k above a fixed value k_0 . (Added in proof: By oral communication of J.L. Vicente this effective approach has been worked out in complete detail in a forthcoming book of Aroca, Hironaka, and Vicente on the resolution of singularities of complex spaces).

1.4. The normal cone, normal flatness, and normal pseudoflatness.

Let $\underline{X}$ be a complex space, $\underline{Y} \xhookrightarrow{i} \underline{X}$ a closed complex subspace, defined by the locally finite ideal $I \subseteq \mathcal{O}_X$.

<u>Lemma 1.4.1.</u> <u>The graded</u> $\mathcal{O}_X$-<u>algebra</u> $B(I,\mathcal{O}_X) := \bigoplus_{k \geq 0} I^k$ <u>is an admissible graded</u> $\mathcal{O}_X$-<u>algebra</u>.

Proof. Since $I = B_1(I,\mathcal{O}_X)$ is locally finite and generates $B(I,\mathcal{O}_X)$, the $\mathcal{O}_X$-algebra $B(I,\mathcal{O}_X)$ is locally finitely generated. Moreover, I is coherent, so all I^k, $k \geq 0$, are coherent, and the claim follows from Proposition 1.2.7.

Corollary 1.4.2. The graded $\mathcal{O}_Y$-algebra

$$(1.4.1) \qquad G(I,\mathcal{O}_X) := \bigoplus_{k\geq 0} I^k/I^{k+1}$$

is an admissible graded $\mathcal{O}_Y$-algebra.

Proof. $G(I,\mathcal{O}_X) = i^* B(I,\mathcal{O}_X)$, and $B(I,\mathcal{O}_X)$ is an admissible graded $\mathcal{O}_X$-algebra.

Hence, the following definition makes sense:

Definition 1.4.3. $\pi_{G(I,\mathcal{O}_X)}$: Specan$(G(I,\mathcal{O}_X)) \longrightarrow \underline{Y}$ is called the normal cone of $\underline{Y}$ in $\underline{X}$ and denoted $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$.

For geometric applications to equimultiplicity we need a geometric description of $C(\underline{X},\underline{Y})$, which will explain the name 'normal cone'. Recall that a blowup $\underline{\pi} : \underline{\tilde{X}} \longrightarrow \underline{X}$ of $\underline{X}$ along $\underline{Y}$ is a morphism which is universal among the morphisms $\varphi : \underline{X}' \longrightarrow \underline{X}$ having the property that $\varphi^{-1}\underline{Y}$ is a hypersurface in $\underline{X}'$, i.e. locally generated by a nonzero-divisor. It is unique up to unique isomorphism.

Theorem 1.4.4. $\underline{p}$: Projan$(B(I,\mathcal{O}_X)) \longrightarrow \underline{X}$ is the blowup of $\underline{X}$ along $\underline{Y}$.

I will not prove Theorem 1.4.4, but make some remarks which I will use anyway. Let I be generated over the open subspace $\underline{U} \hookrightarrow \underline{X}$ by $g_1,\ldots,g_k \in \mathcal{O}_X(U)$, and consider the morphism

$$(1.4.2) \qquad \underline{\gamma} : \underline{U} - Y \longrightarrow \underline{\mathbb{P}}^{k-1}$$
$$x \longmapsto [g_1(x) : \ldots : g_k(x)] \quad .$$

It can then be shown that $\underline{p}|U$ above is given as

(1.4.3)
$$\begin{array}{ccc} \overline{\underline{\Gamma}_\gamma} & \hookrightarrow & \underline{U} \times \mathbb{P}^{k-1} \\ {\scriptstyle \underline{p}|U} \downarrow & \circlearrowleft & \swarrow {\scriptstyle \underline{pr}_U} \\ \underline{U} & & \end{array} ,$$

where $\underline{\Gamma}_\gamma \hookrightarrow (\underline{U} - Y) \times \mathbb{P}^{k-1}$ is the graph space of γ according to I 3.2 g), and $\overline{\underline{\Gamma}_\gamma}$ is the idealtheoretic closure of $\underline{\Gamma}_\gamma'$, i.e. the smallest closed complex subspace of $\underline{U} \times \mathbb{P}^{k-1}$ containing $\underline{\Gamma}_\gamma$ as an open subspace (for this see [14], 0.44). It is then not difficult to show, using the factorization criterion for holomorphic maps through a closed complex subspace (see [28], Chapter I, § 2.3), that (1.4.3) constitutes the blowup locally, which proves 1.4.4 by universality. (The diagram (1.4.3) coincides with the local description given by Hironaka and Rossi in [37]; consult this paper for details).

<u>Corollary 1.4.5.</u> <u>If $\pi : \tilde{\underline{X}} \longrightarrow \underline{X}$ blows up $\underline{Y}$, $\pi^{-1}(\underline{Y}) \cong \mathbb{P}\,C(\underline{X},\underline{Y})$, the projectivized normal cone.</u>

<u>Proof</u>. $\mathbb{P}\,C(\underline{X},\underline{Y})$ is defined as $\underline{\text{Projan}}(G(I,\mathcal{O}_X))$. But $G(I,\mathcal{O}_X) = i^* B(I,\mathcal{O}_X)$, where $i : \underline{Y} \hookrightarrow \underline{X}$ is the inclusion, and the claim follows by base change for Projan (Remark 1.2.9).

This gives the following description of the fibre $\nu^{-1}(y)$ of the normal cone $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ at a point $y \in Y$. Choose generators $g_1,\dots,g_k \in \mathcal{O}_{X,y}$ of the stalk I_y , where the ideal $I \subseteq \mathcal{O}_X$ defines $\underline{Y} \hookrightarrow \underline{X}$, and add elements $h_1,\dots,h_f$ such that $h_1,\dots,h_f, g_1,\dots,g_k$ generate the maximal ideal. After possibly shrinking $\underline{X}$, we may assume these generators are in $\mathcal{O}_X(X)$, and they define, according to I 4.2.2, an embedding $\underline{X} \overset{i}{\hookrightarrow} \mathbb{C}^n$, $n := f + k$, as a locally closed subspace. Then $g_1,\dots,g_k$ are induced by the coordinates $z_{f+1},\dots,z_n$ of $\mathbb{C}^n$ via $\underline{i}$. Let $K := \mathbb{C}^k \times 0$, and let $\underline{p} : \underline{\mathbb{C}}^n \longrightarrow \underline{K}$ be the projection; then $\gamma(x) = p(\overline{yx}) \subseteq K$, and (1.4.3) gives, together with

Corollary 1.4.5.

$$(1.4.4) \qquad \nu^{-1}(y) = \cup\left\{\ell \mid \ell = \lim_{\substack{x\to y\\ x\in X-Y}} p(\overline{yx})\right\} \subseteq K \ .$$

<u>Corollary 1.4.6.</u> $\dim_{\xi}\underline{C}(\underline{X},\underline{Y}) = \dim_{\nu(\xi)}\underline{X}$ <u>for all</u> $\xi \in C(\underline{X},\underline{Y})$. <u>If</u> $(\underline{X},\nu(\xi))$ <u>is equidimensional, so is</u> $(\underline{C}(\underline{X},\underline{Y}),\xi)$.

<u>Proof</u>. There is a canonical embedding $\underline{Y} \hookrightarrow \underline{C}(\underline{X},\underline{Y})$, corresponding to the augmentation homomorphism $G(I,\mathcal{O}_X) \twoheadrightarrow \mathcal{O}_X/I = i_*\mathcal{O}_Y$, where $I \subseteq \mathcal{O}_X$ defines $\underline{Y} \overset{i}{\hookrightarrow} \underline{X}$, via the universal property of the Specan-construction. In the sequel, therefore, we may view $\underline{Y}$ as being naturally embedded in $\underline{C}(\underline{X},\underline{Y})$.

Let $\xi \in C(\underline{X},\underline{Y})$. We may assume $(Y,\nu(\xi)) \neq (X,\nu(\xi))$. First, let $\xi \notin Y$, so it is not a vertex of a fibre of $\underline{\nu}$. Then ξ corresponds to a line on $C(\underline{X},\underline{Y})$, i.e. to a point $x' \in \tilde{X}$, where $\underline{\pi} : \underline{\tilde{X}} \longrightarrow \underline{X}$ is the blowup of $\underline{X}$ along $\underline{Y}$. Now $\underline{\pi}|\underline{\tilde{X}} - \pi^{-1}(Y) : \underline{\tilde{X}} - \pi^{-1}(Y) \longrightarrow \underline{X} - Y$ is isomorphic; so there are points on X_{reg} arbitrarily close to $\pi(x') =: x = \nu(\xi)$, hence $\dim_{x'}\tilde{X} = \dim_{\nu(\xi)}X$. Since $\underline{\pi}^{-1}(\underline{Y}) = \underline{\mathbb{P}C}(\underline{X},\underline{Y})$ is a hypersurface in $\underline{\tilde{X}}$, i.e. locally generated by a nonzerodivisor, $\dim_{x'}\tilde{X} = \dim_{x'}\underline{\mathbb{P}C}(\underline{X},\underline{Y}) + 1$ by the Active Lemma I 5.2.2. Thus we get $\dim_{\xi}C(\underline{X},\underline{Y}) = \dim_{x'}\tilde{X} = \dim_{\nu(\xi)}X$.

If ξ is a vertex, there are points ξ' arbitrarily close to ξ on $\underline{C}(\underline{X},\underline{Y}) - Y$, where $\dim_{\xi'}C(\underline{X},\underline{Y}) = \dim_{\nu(\xi')}X$ by the first case; this again implies $\dim_{\xi}C(\underline{X},\underline{Y}) = \dim_{\nu(\xi)}X$.
The last claim is obvious. Q.e.d.

<u>Remark</u>. For the algebraic proof, see Chapter II, Theorem (9.7).

<u>Definition 1.4.7</u> (Hironaka). <u>Let</u> $X \in \underline{cpl}$, $\underline{Y} \overset{i}{\hookrightarrow} \underline{X}$ <u>a closed complex subspace</u>, $y \in Y$. <u>Then</u> $\underline{X}$ <u>is called normally flat along</u> Y <u>at</u> y <u>if and only if</u> $G(I,\mathcal{O}_X)_y$ <u>is a flat</u> $\mathcal{O}_{Y,y}$<u>-module</u>. $\underline{X}$ <u>is called</u>

normally flat along Y if and only if it is normally flat along Y at all $y \in Y$.

The following theorem with an idea of proof was formulated by Hironaka ([33], p. 136) and proved in [38], Theorem 1.5, and in [46], Théorème 8.1.3.

Theorem 1.4.8. Let $X \in \underline{\underline{cpl}}$, $\underline{Y} \xrightarrow{i} \underline{X}$ a closed complex subspace, and let $F(X,Y) := \{y \in Y \mid X$ is normally flat along Y at $y\}$. Then $F(X,Y)$ is the complement of an analytic set in $\underline{Y}$. Moreover, when $\underline{Y}$ is reduced, $Y - F(X,Y)$ is nowhere dense.

Proof. This is immediate from Theorem 1.3.4.

We finally need the following weaker notion, whose importance was also discovered by Hironaka ([34], Definition (2.4) and Remark (2.5)). We use throughout $\dim_y \nu^{-1}(y) = \dim \nu^{-1}(y)$, cf. II, Proposition 3.1.2.

Proposition and Definition 1.4.9. Let $X \in \underline{\underline{cpl}}$, $\underline{Y} \hookrightarrow \underline{X}$ a closed complex subspace, and $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ be the normal cone. Let $\underline{X}$ be equidimensional at $y \in Y$. The following statements are equivalent.

(i) $\underline{\nu}$ is universally open near y, i.e. there is an open neighbourhood U of y in Y such that, for any base change $\underline{U}' \longrightarrow \underline{U}$ in $\underline{\underline{cpl}}$, $(\underline{\nu}|U) \times_{\underline{U}} \underline{U}'$ is an open map;

(ii) $\dim \underline{\nu}^{-1}(\underline{z})$ does not depend on $\underline{z}$ near y;

(iii) $\dim \nu^{-1}(z) = \dim_y X - \dim_y Y$.

We call X normally pseudoflat along Y at y if and only if one of these statements holds true (this clearly is an open condition on y).

The statement (iii) just means $ht(I_y) = s(I_y)$, where $I_y \subseteq \mathcal{O}_{X,y}$ defines $(\underline{Y},y) \hookrightarrow (\underline{X},y)$; see Proposition 2.2.5 below.

Outline of proof. We may assume $\underline{U} = \underline{Y}$ and $\underline{Y}$ reduced. We have the following general facts for a morphism $\underline{f} : \underline{W} \longrightarrow \underline{Z}$ in $\underline{\underline{cpl}}$:

1) $z \mapsto \dim \underline{f}^{-1}(\underline{z})$ is upper semicontinuous ([14], 3.4).

2) $\forall w \in \underline{W} : \dim_w \underline{f}^{-1}f(w) + \dim_{f(w)}\underline{Z} \geq \dim_w \underline{W}$ ([14], 3.9.) .

3) If $\underline{Z}$ is equidimensional at all points and $\underline{f}$ is open, equality holds in 2) for all $w \in \underline{W}$. Conversely, if $\underline{Z}$ is irreducible at $z = f(w)$ and equality holds in 2) for w , f is open at w ([14], 3.10 and 3.9).

We may assume $\underline{X}$ to be equidimensional of dimension d at all points of Y by I Theorem 7.3.2. Then $\underline{C}(\underline{X},\underline{Y})$ is equidimensional of dimensional d at all points by Corollary 1.4.6.

Let $(Y\ )_{\lambda \in \Lambda}$ be the irreducible components of $\underline{Y}$; we may assume Λ finite and the $\underline{Y}_\lambda$ given by the local decomposition of $(\underline{Y},y)$ by II Remark 2.1.4.

<u>(i) ⇒ (iii)</u>: Make the base change $\underline{Y}_\lambda \hookrightarrow \underline{Y}$ and get from 3)

$$\dim \nu^{-1}(y) + \dim_y Y_\lambda = d \qquad \text{for all } \lambda \quad .$$

<u>(iii) ⇒ (ii)</u>: This follows from 1) and 2).

<u>(ii) ⇒ (i)</u>: (cf. [34]) Since $\underline{C}(\underline{X},\underline{Y})$ is equidimensional, we may, through any given point $\xi \in S(\underline{X},\underline{Y})$ and for any λ find an irreducible subgerm $(\underline{W}_\lambda,\xi) \subseteq (\underline{C}(\underline{X},\underline{Y}),\xi)$ such that $\dim_\xi \underline{W}_\lambda = \dim_{\nu(\xi)}\underline{Y}_\lambda$ and $\nu|\underline{W}_\lambda : (\underline{W},\xi) \longrightarrow (\underline{Y}_\lambda,\nu(\xi))$ is finite. Then, for suitable representatives, $\nu|\underline{W}_\lambda : \underline{W}_\lambda \longrightarrow \underline{Y}_\lambda$ is universally open; for this, use the fundamental facts on open finite mappings of I, § 6. Since this holds for all λ and ξ , ν must be, after a possible shrinking, universally open.

<u>Remark 1.4.10</u>. A motivation for the definition is the following: If $\underline{X}$ is normally flat along $\underline{Y}$, the normal cone map $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ is a flat map of complex spaces by Remark 1.3.2. Now it is known that flatness is stable under base extension and that a flat map is open, hence a flat map is universally open (see [14], 3.15 and 3.19, and [36], p. 225).

This is in fact the main topological property of a flat map, which, in particular, implies that the fibres of a flat map have the expected minimal generic dimension. In this sense, normal pseudoflatness retains the topological essence of normal flatness.

Remark 1.4.11. Normal flatness of X along Y at y implies normal pseudoflatness at this point. Hence, in the situation of 1.4.9, if Y is reduced, the set $PF(\underline{X},\underline{Y}) := \{y \in Y \mid \underline{X}$ is normally pseudoflat along $\underline{Y}$ at $y\}$ is generic in $\underline{Y}$.

Proposition 1.4.12. Let the situation be as in 1.4.9. Let y be a smooth point on $\underline{Y}$. Then the following statements are equivalent:

(i) $\underline{X}$ is normally flat along $\underline{Y}$ at y.

(ii) The natural morphism

$$\underline{\nu}^{-1}(\underline{y}) \times \underline{C}(\underline{Y},y) \longrightarrow \underline{C}(\underline{X},y) \qquad (1.4.5)$$

is an isomorphism.

Proof. Since (1.4.5) corresponds to an algebraic morphism of the corresponding projectivized cones, the celebrated results of [66] imply that (1.4.5) is an isomorphism of complex spaces if and only if it is an isomorphism of algebraic schemes. In view of this, the Proposition 1.4.12 is a mere restatement of a well-known fact about the Hironaka-Grothendieck-Isomorphism (cf. [33]).

§ 2. Geometric equimultiplicity along a smooth subspace.

In this paragraph we analyse the geometric significance of a complex space X having the same multiplicity along a subspace Y near a smooth point y of Y, and give various characterizations due to Hironaka, Schickhoff, Lipman, and Teissier (see Theorem 2.2.2 below). The motivation, of course, is to understand which restrictions this requirement puts on the blowup of X along Y; see the preface of this book. The result of Hironaka-Schickhoff is that equimultiplicity in the above sense is equivalent to normal pseudoflatness, so we have the noteworthy fact that the dimension of the normal cone fibres are controlled by the multiplicity. The underlying reason why this is so is that the requirement of equimultiplicity and of the normal cone fibre having the generic minimal dimension both put a transversality condition on X along Y relating the two properties. To be more precise, let us embed X locally around y in some $\mathbb{C}^n$ so that Y becomes a linear subspace. Let $L \in P_e^d(\underline{X},y)$ be a projection centre whose corresponding projection onto $\mathbb{C}^d$ has the multiplicity $m(\underline{X},y)$ as local mapping degree. It turns out that both requirements amount to the requirement that $Y \times L$ and X intersect transversally along Y in the sense that $Y \times L \cap C(\underline{X},\underline{Y}) = Y$. If X is normally pseudoflat along Y at y, this fact comes about by blowing up X and $Y \times L$ along Y, and the various projection centres in $Y \times L$ parametrized by points of Y yield projections whose local mapping degrees are constant and give the multiplicity of X along Y. The converse direction, starting from equimultiplicity and reaching transversality, is more delicate and is essentially the geometric version of the Theorem of Rees-Böger. Inherent is the principle that multiplicity was defined as a minimal mapping degree, and this minimality forces the projection centre defining the multiplicity to be generic and hence transversal. Archetypical for this situation is $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ given by a Weierstraß equation so that the z_n-axis L has 0 as isolated intersection point with X; it is then a challenging exercise to convince oneself that the projection along L has minimal mapping degree if and only if L is transversal to the tangent cone. We end by analysing some further geometric conditions and their relationship to various algebraic characterizations of equimultiplicity, especially to the notion of reduction and integral dependence, as exposed in the first four chapters of this book. It is instructive to return again to the above Weierstraß example and to convince oneself that the transversality of L to the tangent cone is, in this case, equivalent to z_n

being, as a function on X , integrally dependent on the ideal generated in $\mathcal{O}_{X,x}$ by $z_1,\ldots,z_{n-1}$. In particular, it appears that the algebraic connection between reduction and integral dependence is reflected geometrically by the fact that the transversality condition stated above is equivalent to growth conditions on the coordinate functions of $\mathbb{C}^n$ along normal directions of Y in X .

2.1. Zariski-equimultiplicity.

Throughout this section we employ the following notation. $\underline{X}$ is a complex space, $\underline{Y}$ a closed complex subspace, $y \in \underline{Y}$ a smooth point on $\underline{Y}$, $I \subseteq \mathcal{O}_Y$ the ideal defining $\underline{Y} \overset{i}{\hookrightarrow} \underline{X}$, and $\mathfrak{p}_z \in \operatorname{Spec}(\mathcal{O}_{X,z})$ the ideal defining the subgerm $(\underline{Y},z) \subseteq (\underline{X},z)$ for $z \in Y_{ir}$. If $(R,\mathfrak{m}_R)$ is a local noetherian ring, $e(R) := e(m_R,R)$.

<u>Definition 2.1.1</u> (Zariski-equimultiplicity). <u>Let</u> $(\underline{X},\underline{Y},y)$ <u>be as above. Then</u> $\underline{X}$ <u>is called Zariski-equimultiple along</u> Y <u>at</u> y <u>if and only if the function</u> $z \mapsto m(\underline{X},z)$ <u>on</u> Y <u>is constant near</u> y .

The following result exploits this definition algebraically [38],[49]).

<u>Theorem 2.1.2</u> (algebraic characterization of equimultiplicity). <u>Let</u> $(\underline{X},\underline{Y},y)$ <u>be as stated above. The following conditions are equivalent</u>:

(i) $\underline{X}$ <u>is Zariski-equimultiple along</u> Y <u>near</u> y .

(ii) $e(\mathcal{O}_{X,y}) = e((\mathcal{O}_{X,y})_{\mathfrak{p}_y})$, <u>where</u> $\mathfrak{p}_y \in \operatorname{Spec}(\mathcal{O}_{X,y})$ <u>defines</u> $(\underline{Y},y) \hookrightarrow (\underline{X},y)$.

This will be an immediate consequence of the following proposition, which explains the geometric significance of the number $e((\mathcal{O}_{X,y})_{\mathfrak{p}_y})$.

<u>Proposition 2.1.3</u> <u>Let</u> $(\underline{W},w) \in \underline{\mathrm{cpl}}_0$, $(\underline{Z},w) \hookrightarrow (\underline{W},w)$ <u>a prime subgerm. Then, after suitably shrinking</u> $\underline{W}$:

(i) $m(\underline{W},w) \geq e((\mathcal{O}_{\underline{W},w})_{\mathfrak{p}_w})$, where $\mathfrak{p}_w \in \mathrm{Spec}(\mathcal{O}_{W,w})$ defines $(\underline{Z},w)$.

(ii) There is a nowhere dense analytic set $A \subseteq Z$ such that $m(\underline{W},z) = e((\mathcal{O}_{W,w})_{\mathfrak{p}_w})$ for all $z \in Z - A$.

In other words, $e((\mathcal{O}_{W,w})_{\mathfrak{p}_w})$ is the generic multiplicity of W along the subspace $\underline{Z} \hookrightarrow \underline{W}$ defined locally by $\mathfrak{p}_w$.

Proof of 2.1.3. Since $\underline{Z}$ is reduced at w , we may assume, after possibly shrinking $\underline{W}$, that there is a nowhere dense analytic set A such that $\underline{Z} - A$ is reduced and smooth, and $\underline{W}$ is normally flat along $\underline{Z} - A$; this follows from I 6.3.1, and 1.4.8. Now consider the chain

$$(2.1.1) \qquad m(\underline{W},y) \overset{(1)}{\geq} m(\underline{W},z) \overset{(2)}{=} e((\mathcal{O}_{W,z})_{\mathfrak{p}_z}) \overset{(3)}{=} e((\mathcal{O}_{W,w})_{\mathfrak{p}_w}) \ , \ z \in Z - A \ .$$

(1): This is just the upper semicontinuity of multiplicity in II Theorem 5.2.4.

(2): This is II Theorem 5.2.1 (iii) and Corollary (21.12) of Chapter IV.

(3): This results from the following Lemma 2.1.4.

This proves the Proposition 2.1.3.

Proof of Theorem 2.1.2. After shrinking $\underline{X}$, let $A \subseteq Y$ be such that 2.1.3 (ii) holds, so $e((\mathcal{O}_{X,y})_{\mathfrak{p}_y})$ is the generic value of $m(\underline{X},z)$, and

$$(2.1.2) \qquad m(\underline{X},y) \geq m(\underline{X},z) \geq e((\mathcal{O}_{X,y})_{\mathfrak{p}_y}) \quad ,$$

both inequalities by upper semicontinuity of multiplicity (II Theorem 5.2.4). Q.e.d.

Lemma 2.1.4 Let W be a complex space , $\mathcal{M}$ a coherent $\mathcal{O}_W$-module, and Z an irreducible component of $\mathrm{supp}\,\mathcal{M}$. Then the function $z \longmapsto e((\mathcal{M}_z)_{\mathfrak{p}_z})$ is locally constant on Z_{ir} .

Proof. This is done by the methods of compact Stein neighbourhoods and is similar to the proof of II 2.2.3, so I will be brief. Let $I \subseteq \mathcal{O}_X$ define $\underline{Z} \hookrightarrow \underline{W}$. Let $z_0 \in Z_{ir}$, and choose a compact Stein neighbourhood K of z_0 in W. Let $R := \Gamma(K, \mathcal{O}_W)$, $P := \Gamma(K, I)$, which is a prime ideal of R by II, proof of 2.2.3. Finally, put $M := \Gamma(K, M)$. If $z \in K \cap Z_{ir}$, the homomorphism

$$(2.1.2) \qquad (\lambda_z)_P : R_P \longrightarrow (\mathcal{O}_{W,z})_{\mathfrak{p}_z} ,$$

where $\mathfrak{p}_z \in \operatorname{Spec}(\mathcal{O}_{W,z})$, defines $(\underline{Z},z) \hookrightarrow (\underline{W},z)$, and is faithfully flat by II Theorem 1.3.2. Moreover,

$$(2.1.3) \qquad (M_z)_{\mathfrak{p}_z} = M_P \otimes_{R_P} (\mathcal{O}_{W,z})_{\mathfrak{p}_z} .$$

Then, for all $k \geq 0$, we get by II Lemma 2.2.4:

$$(2.1.4) \qquad \operatorname{length}((M_z)_{\mathfrak{p}_z} / \mathfrak{p}_z^k (M_z)_{\mathfrak{p}_z}) = \operatorname{lenght}(M_P / P^k M_P) ,$$

this proves the lemma.

Remark 2.1.5 If one just wants Theorem 2.1.2 without the characterization in Proposition 2.1.3, one could use the chain

$$m(\underline{X},y) \overset{(1)}{\geq} m(\underline{X},z) \overset{(2')}{\geq} e((\mathcal{O}_{\underline{X},z})_{\mathfrak{p}_z}) \overset{(3)}{=} e((\mathcal{O}_{\underline{X},y})_{\mathfrak{p}_y}) \quad \text{for } z \text{ near } y ,$$

with (2') given by Proposition (30.1) of Chapter VI.

Corollary 2.1.6. Let X be a complex space, Y a smooth closed complex subspace. If X is normally flat along Y, then X is Zariski-equimultiple along Y.

Proof. Condition (ii) of Theorem 2.1.2 holds by Corollary 2 on p. 186 of [33].

Remark 2.1.7. Corollary (21.12) in Chapter IV relates normal flatness to an equality of Hilbert functions. In fact, normal flatness can be characterized by this; this is the content of the following famous theorem.

Theorem of Bennett (complex analytic case). Let X be a complex space, $Y \hookrightarrow X$ a smooth connected closed complex subspace. The following statements are equivalent:

(i) X is normally flat along Y .

(ii) All local rings $\mathcal{O}_{X,y}$, $y \in Y$, have the same Hilbert function, i.e. $z \longmapsto H^{(0)}(\mathcal{O}_{X,z},-)$ is constant for z near y .

The algebraic analogue, the original Theorem of Bennett, is Theorem (22.24) in Chapter IV. The complex analytic version above is proven in [48], Theorem (4.11).

Remark 2.1.8. Definition 2.1.1 makes sense for (X,y) and $(Y,y) \hookrightarrow (X,y)$ arbitrary. I leave an appropriate statement of Theorem 2.1.2 in the general case to the reader.

2.2. The Hironaka-Schickhoff-Theorem.

We have seen in Corollary 2.1.6 that normal flatness along a smooth subspace implies Zariski-equimultiplicity along this subspace. It is a remarkable discovery of Hironaka and Schickhoff that normal pseudoflatness along a smooth subspace is equivalent to Zariski-equimultiplicity (see Theorem 2.2.2 below). Recall that we employ the property (ii) of Proposition 1.4.9 as the definition of normal pseudoflatness, but it is property (i) which characterizes normal pseudoflatness as the notion carrying the topological essence of normal flatness, so it is this topological essence which 'interpretes' Zariski-equimultiplicity along a smooth subspace geometrically (for Zariski-equimultiplicity along a nonsmooth subspace see § 3). Hironaka proved that normal pseudoflatness along smooth centres implies equimulitiplicity in [34], Remark (3.2). Schickhoff proved the converse in [61], p. 49; in fact he proved the stronger statement below, which is analogous to Proposition 1.4.11, and shows how much from normal flatness is lost by normal pseudoflatness. Both proofs were geometric, and I will given the outlines in the sequel; the algebraic essence of the Hironaka-Schickhoff-Theorem is Satz 2 of [77] ; using the method of compact Stein neighbourhoods, it would be possible to derive the Hironka-Schickhoff-Theorem from this algebraic result.

Before formulating the main result, I fix some terminology. Let $(\underline{X},y) \in \underline{\underline{cpl}}_0$ be a complex spacegerm of dimension d, $(\underline{Y},y) \hookrightarrow (\underline{X},x)$ a complex subspacegerm. After possibly shrinking $\underline{X}$, we may assume:

(2.2.1) (i) $\underline{X} \hookrightarrow \underline{U}$ as a closed complex subspace, where $U \subseteq \mathbb{C}^n$ is open, such that $\underline{X}$ is equidimensional at all points if $(\underline{X},y)$ was equidimensional, and $y = 0 \in U$.

(ii) $\underline{Y} \hookrightarrow \underline{X}$ is a closed complex subspace, and $\underline{Y} = \underline{X} \cap \underline{G}$, where G is the linear subspace of $\mathbb{C}^n$ given by $z_{f+1} = \ldots = z_n = 0$.

This can always be achieved by choosing generators $g_1,\ldots,g_\ell \in \mathcal{O}_{X,y} =: R$ of the ideal $I \subseteq R$ defining $(\underline{Y},y) \hookrightarrow (\underline{X},y)$ and adding elements $h_1,\ldots,h_m \in R$ such that $h_1,\ldots,h_m$, $g_1,\ldots,g_\ell$ generate the maximal ideal of R. Then $n := m + \ell$, and we write points in $\mathbb{C}^n$ as pairs (u,t), with $u = (z_1,\ldots,z_m)$ and $t = (z_{m+1},\ldots,z_n)$; the $h_1,\ldots,h_m$ are induced by $z_1,\ldots,z_m$, and the $g_1,\ldots,g_\ell$ by $z_{m+1},\ldots,z_n$.

(iii) If $(\underline{Y},y)$ is smooth, $\underline{Y}$ is connected and smooth everwhere, and $m = \dim_y Y =: f$.

Since $\underline{Y} \hookrightarrow \underline{G}$, $\underline{Y} \times \underline{\mathbb{C}}^\ell \hookrightarrow \underline{\mathbb{C}}^n$. Any $h \in \mathcal{O}_{\mathbb{C}^n,0}$, considered as an element in $\mathcal{O}_{Y\times\mathbb{C}^\ell,0} = \mathcal{O}_{Y,y}\{t_1,\ldots,t_\ell\}$, can be written as

$$(2.2.2) \qquad h = \sum_{k=\nu_Y(h)}^{\infty} h_k \quad , \quad h_k \in \mathcal{O}_{Y,y}[t_1,\ldots,t_\ell]$$

with $\nu_Y(h)$ uniquely determined by requiring $\nu_Y(h) \neq 0$. We call $\nu_Y(h)$ <u>the order of</u> h <u>along</u> Y <u>at</u> y, and $h_{\nu_Y(h)} =: L_Y(h)$ the Y-<u>leitform of</u> h. The germ $(\underline{C}(\underline{X},\underline{Y}),y) \hookrightarrow (\underline{Y} \times \underline{\mathbb{C}}^k,0)$ is then defined by the ideal generated by all $L_Y(h)$ for $h \in J$, where the ideal $J \subseteq \mathcal{O}_{\mathbb{C}^n,0}$ defines $(\underline{X},y) \hookrightarrow (\underline{\mathbb{C}}^n,0)$. This ideal is called the Y-<u>leitideal of</u> J and denoted $L_Y(J)$. It is possible to find finitely many generators $f_1,\ldots,f_s$ of J such that $L_Y(f_1),\ldots,L_Y(f_s)$ generate $L_Y(J)$; we call $\{f_1,\ldots,f_s\}$ a Y-<u>standard-base of</u> J.

After possibly shrinking $\underline{X}$, we may assume that $L_Y(f_1),\dots,L_Y(f_s)$ are defined on $\underline{Y} \times \mathbb{C}^{\ell}$; then $\underline{C}(\underline{X},\underline{Y}) \hookrightarrow \underline{Y} \times \mathbb{C}^{\ell}$, and $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ is induced by the projection $\underline{Y} \times \underline{\mathbb{C}}^{\ell} \longrightarrow \underline{\mathbb{C}}^{\ell}$.

We make all these assumptions in the sequel of this section.

Example 2.2.1.

1) $\underline{X} \hookrightarrow \underline{\mathbb{C}}^3$ given by $g(x,y,z) = z^2 - x^2y = 0$, $\underline{Y}$ the x-axis, i.e. defined by $(y,z)\cdot\mathcal{O}_{\mathbb{C}^3,0}$. Then $\nu_Y(g) = 1$, and $g_\nu = -x^2y$ defines $\underline{C}(\underline{X},\underline{Y})$. See Figure 6 .

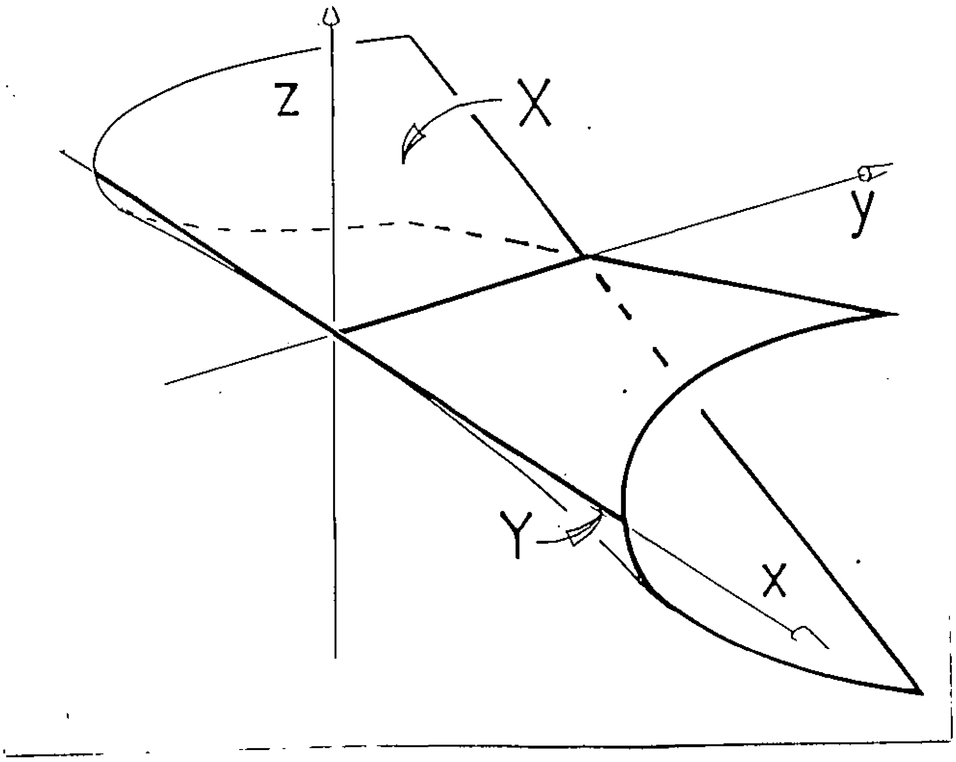


Fig. 6

2) $\underline{X} \hookrightarrow \underline{\mathbb{C}}^3$ given by $g(x,y,z) = z^2 - y^2(y+x^2) = 0$, $\underline{Y}$ again the x-axis defined by $(y,z)\cdot\mathcal{O}_{\mathbb{C}^3}$. Then $\nu_F(g) = 2$, and $g_{\nu_I(g)} = z^2 - y^2x^2$ defines $\underline{C}(\underline{X},\underline{Y})$. See Figure 7 .

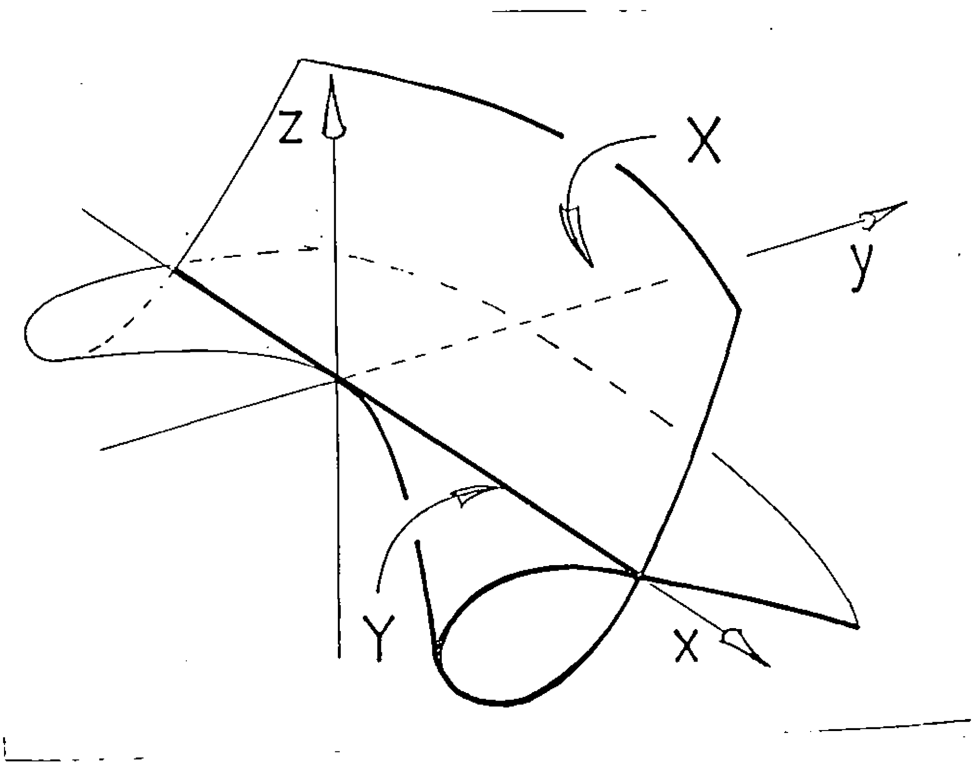


Fig. 7

The main result on the geometric significance of equimultiplicity is now the following theorem.

Theorem 2.2.2 (Geometric analysis of equimultiplicity; Hironaka-Lipman-Schickhoff-Teissier). *Let* $(\underline{Y},y) \hookrightarrow (\underline{X},y) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ *be embeddings of complex spacegerms,* $(\underline{X},x)$ *equidimensional of dimension* d , $(\underline{Y},y)$ *smooth of dimension* f , *and let* X,Y *be chosen as stated above. The following statements are equivalent.*

(i) $\underline{X}$ *is Zariski-equimultiple along* Y *at* y .

(ii) *There is* $L \in \mathrm{Grass}^d(\mathbb{C}^n)$ *and a neighbourhood* V *of* y *in* X *such that* $L_z \cap V = \{z\}$ *and* $L_z \in P_e^d(\underline{X},z)$ *for all* $z \in V \cap Y$, *where* $L_z := L + z$ ([61]).

(iii) *There is a nonempty Zariski-open subset* V *of* $\mathrm{Grass}^d(\mathbb{C}^n)$ *such that for any* $L \in V$ *there is a neighbourhood* V *of* y *in* X *such that* $L_z \cap V = \{z\}$ *for all* $z \in V \cap Y$ ([69]).

(iv) X *is normally pseudoflat along* Y *at* y , *i.e.* $\dim \nu^{-1}(y) = d - f$, ([34], [61]).

Moreover, if one of these condition holds, one may take $V = P_e^d(\underline{X},x)$ in (iii), and then $L \in P_e^d(\underline{X},z)$ for all $L \in V$ and $z \in Y$ near y.

Addendum to Theorem 2.2.2 (cf. Teissier [69], Chapter I, 5.5).

The condition (iii) is equivalent with

(iii') There exists a nonempty Zariski-open subset $U \subseteq \mathrm{Grass}^{d-f}(\mathbb{C}^n,Y) := \{H \in \mathrm{Grass}^{d-f}(\mathbb{C}^n) \mid H \supseteq Y\}$ such that $(Y,y) = (X \cap H,y)$ as analytic setgerms for all $H \in U$.

Exercise 2.2.3. Analyse the given conditions in the two cases of Example 2.2.1.

The rest of this section is devoted to an outline of the proof, which will be geometric.

Basic is a careful setup for a finite projection $\underline{h} : (\underline{X},y) \longrightarrow (\underline{\mathbb{C}}^d,0)$, which is to give $m(\underline{X},z)$ for all z on Y near y. For this, we collect the following facts, which hold after possibly shrinking $\underline{X}$.

2.2.3.

(i). Let $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ be a complex subspacegerm, $d := \dim_x \underline{X}$, $f \in \mathbb{N}$ with $0 \leq f \leq d$. Let $K \in \mathrm{Grass}^f(\mathbb{C}^n)$. We say

(2.2.3) K weakly transverse to $\underline{X}$ at x :$\iff$
$\dim_x X \cap K = d-f$, denoted $K \pitchfork_x \underline{X}$;

K transverse to $\underline{X}$ at x :$\iff$
$\dim_x C(\underline{X},x) \cap K = d-f$, denoted $K \pitchfork \underline{C}(\underline{X},x)$;

and put

(2.2.4) $$P_g^f(\underline{X},x) := \{K \in \mathrm{Grass}^f(\mathbb{C}^n) \mid K \pitchfork_x \underline{X}\},$$

$$P_e^f(\underline{X},x) := \{K \in \mathrm{Grass}^f(\mathbb{C}^n) \mid K \pitchfork C(\underline{X},x)\}.$$

Then $P^f_e(\underline{X},x) \subseteq P^f_g(\underline{X},x)$. To see this, note that $C(\underline{X} \cap \underline{K},x) \subseteq C(\underline{X},x) \cap K$; so, if $\dim_x C(\underline{X},x) \cap K = d - f$, we have $\dim C(\underline{X} \cap \underline{K},x) = \dim_x \underline{X} \cap \underline{K} \leq d - f$; since always $\dim_x \underline{X} \cap \underline{K} \geq d - f$ (for instance by the Active Lemma, I 5.2.2), we get equality.

The set $P^f_e(\underline{X},x)$ is a nonempty Zariski-open subset of $\mathrm{Grass}^f(\mathbb{C}^n)$, so $P^f_g(\underline{X},x)$ is generic in $\mathrm{Grass}^f(\mathbb{C}^n)$. The proof is a straightforward generalization of the case $f = d$ in II, 4.1: If $\underline{Z} \hookrightarrow \underline{\mathbb{P}}^{n-1}$ is a (d-1)dimensional variety, consider the fibre bundle given by the "incidence correspondence"

$$\begin{array}{l} Z := \{(z,K) \in Z \times \mathrm{Grass}^f(\mathbb{P}^{n-1}) \mid z \in K\} \\ \Big\downarrow q \\ \mathrm{Grass}^f(\mathbb{C}^n) \quad . \end{array}$$

Then, by Elementary Algebraic Geometry, q has fibre dimension $(d-f) - 1$ outside a proper Zariski-closed subset (see e.g. [56], Chapter 3, (3.15)). Now apply this to $\underline{Z} := \underline{\mathbb{P}C}(\underline{X},x)$.

We finally define the notion of being <u>strongly transverse</u>, which is based on the following theorem.

<u>Theorem</u>. <u>Let</u> $\underline{X} \in \underline{\mathrm{cpl}}$. <u>Then the Cohen-Macaulay-locus</u> $X_{CM} := \{x \in X \mid \mathcal{O}_{X,x}$ is Cohen-Macaulay$\}$ <u>is the complement of a nowhere dense analytic set. Moreover, if</u> $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_{X,y})$ <u>defines</u> $(\underline{Y},y) \hookrightarrow (\underline{X},y)$, $(Y,y) \cap (X_{CM},y) \neq \emptyset$ <u>if and only if</u> $(\mathcal{O}_{X,y})_{\mathfrak{p}}$ <u>is Cohen-Macaulay</u>.

This can be proved by the methods of distinguished compact Stein neighbourhoods, see Remark 1.1.6 (i). For the first statement, see also II Theorem 2.2.11; the second statement can also be proved by the methods of [64], Exposé 21. We will make use only of the first statement at the moment.

Further, if $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$, and $(A,x) \subseteq (X,x)$ is an analytic set-germ with $(A,x) \neq (X,x)$, the set of $K \in \mathrm{Grass}^f(\mathbb{C}^n)$ with $(A \cap K,x) \neq (X \cap K,x)$ is generic in $\mathrm{Grass}^f(\mathbb{C}^n)$ for $0 \leq f < d$ (for this, one may assume A being defined by one equation, and then the proof is left to the reader). We define

$$(2.2.5)\qquad P^f_{gs}(\underline{X},x) := \begin{cases} \{K \in P^f_g(X,x) \mid ((X\cap K)_\lambda,x) \cap (X_{CM},x) \neq \emptyset \text{ for all irreducible components } ((X\cap K)_\lambda,x) \text{ of } (X\cap K,x) \text{ of dimension } d\}, \text{ if } f<d\ ; \\ \{K \in P^f_g(\underline{X},x) \mid K \pitchfork C(\underline{X},\underline{Y}\cap\underline{K}) \text{ if } f=d\}; \end{cases}$$

$$P^f_{es}(\underline{X},x) := P^f_e(\underline{X},x) \cap P^f_{gs}(\underline{X},x) \quad .$$

These are generic sets in $\mathrm{Grass}^f(\mathbb{C}^n)$. If $K \in P^f_{es}(\underline{X},x)$, we say K is <u>strongly transverse to</u> $(\underline{X},x)$.

We have the following lemma:

<u>Lemma</u>. <u>Let</u> $(\underline{X},x) \xrightarrow{i} (\mathbb{C}^n,0)$ <u>be in</u> $\underline{\underline{\mathrm{cpl}}}_0$, $L \in P^d_g(\underline{X},x)$, <u>and</u> $K \in P^f_{gs}(\underline{X},x)$ <u>with</u> $K \supseteq L$. <u>Let</u> $\underline{h} : (\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^d,0)$ <u>be the projection along</u> L , <u>and</u> $\underline{h}_x : (\underline{X}\cap\underline{K},x) \longrightarrow (\underline{\mathbb{C}}^d \cap \underline{K},0)$ <u>be the restriction of</u> $\underline{h}$ <u>to</u> $\underline{X}\cap\underline{K}$. <u>Then</u>

$$\deg_x \underline{h} = \deg_x \underline{h}_x$$

<u>Proof</u>. There is $b \in \mathbb{C}^d \cap K$ such that $h^{-1}(b) = h_x^{-1}(b) \subseteq X_{CM} \cap X_{reg}$. Then II Remark 2.2.7 shows that we get the same contributions to $\deg_x \underline{h}$ and $\deg_x \underline{h}_x$.

<u>Corollary</u>. $m(\underline{X},x) = m(\underline{X}\cap\underline{K},x)$.

<u>Proof</u>. Choose $L \in P^d_e(\underline{X},x)$; since $C(\underline{X}\cap\underline{K},x) \subseteq C(\underline{X},x)$, $L \in P^{d-f}_e(\underline{X}\cap\underline{K},x;K) := \{L_0 \in \mathrm{Grass}^{d-f}(K) \mid L_0 \pitchfork C(\underline{X}\cap\underline{K},x)\}$. Then $\deg_x \underline{h} = m(\underline{X},x)$, and $\deg_x \underline{h}_x = m(\underline{X}\cap\underline{K},x)$. Q.e.d.

If $K \in P^d_g(\underline{X},x)$, $K_z := K + z \in P^d_g(\underline{X},z)$ for z near x . If $K \in P^d_{es}(\underline{X},x)$, $K_z \in P^d_{gs}(\underline{X},x)$ for z near x . This follows, because the $\underline{X}_z := \underline{X}\cap\underline{K}_z$

are the fibres of the projection $\underline{p}_K \mid \underline{X}$, where $\underline{p}_K : \underline{\mathbb{C}}^n \longrightarrow \underline{\mathbb{C}}^n$ is the projection along K , and from the openness of X_{CM} .

<u>(iii)</u> Let $(\underline{X},x) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ be equidimensional of dimension d , and $K \in P_e^f(\underline{X},x)$. Then

$$C(\underline{X} \cap \underline{K},x) = C(\underline{X},x) \cap K \quad .$$

<u>Proof</u> ([61], 2.9). First remark that for any $(\underline{W},w) \in \underline{\underline{\mathrm{cpl}}}_0$, $\dim_w \underline{W} = \dim_z W_{reg}$ for z near w , the dimension of the manifold of regular points on W ; this follows from the local representation Theorem I 6.3.1 (iii), since $\dim_w \underline{W} = \dim_w W_{red}$.

Consider the deformation $\underline{p} : (\underline{X},(0,0)) \subsetneq (\underline{\mathbb{C}}^n \times \underline{\mathbb{C}},(0,0)) \longrightarrow (\underline{\mathbb{C}},0)$ to the tangent cone $C(\underline{X},y) \cong X_0 := \underline{p}^{-1}(\underline{0})$ in II Proposition 3.1.3 and the resulting description of $C(\underline{X},y)$ by II Corollary 3.1.4. From this the inclusion $C(\underline{X} \cap \underline{K},y) \subseteq C(\underline{X},y) \cap K$ is obvious. For the converse, note that X_0 is nowhere dense in X by II 3.1.3 (iv) and so $(X,(0,0))$ is equidimensional of dimension $d+1$ by the introductory remark. So $\dim_{(z,t)}(X \cap (K \times \mathbb{C})) \geq d+1-f$ for all (z,t) close to $(0,0)$, but $\dim_{(0,0)}(X_0 \cap (K \times \{0\})) = d-f$ by assumption. Hence there is the strict inclusion $(X \cap (K \times \mathbb{C}),(0,0)) \supset (X_0 \cap (K \times \{0\}),(0,0))$ of analytic setgerms, and this proves $C(\underline{X} \cap \underline{K},y) \supseteq C(\underline{X},y) \cap K$. Q.e.d.

<u>(iv)</u> Let $(\underline{Y},y) \hookrightarrow (\underline{X},y) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ be as in Theorem 2.2.2. Consider the diagram of projections

$$\begin{array}{ccc} & R := \{(K,L) \in \mathrm{Grass}^f(\mathbb{C}^n) \times \mathrm{Grass}^d(\mathbb{C}^n) \mid L \subseteq K\} & \\ \swarrow q & & r \searrow \\ \mathrm{Grass}^f(\mathbb{C}^n) & & \mathrm{Grass}^d(\mathbb{C}^n) \end{array} \quad .$$

We then define various sets:

$$P_{\lambda\mu\nu}(\underline{X},\underline{Y},y) := q^{-1}(P^f_{\lambda\mu}(\underline{X},y) \cap P^f_g(Y,y)) \cap r^{-1}(P^d_\nu(\underline{X},y))$$

where λ, ν are the letters "g","e", and μ is the blank or "s". These are generic subsets of R.

Moreover, given $L \in P^d_\mu(\underline{X},y)$, the sets $P_{\mu\mu\nu}(\underline{X},\underline{Y},y) \cap r^{-1}(L)$ are generic in $r^{-1}(L)$; so, for given $L \in P^d_\mu(\underline{X},y)$, there is $K \in P^f_{\mu\nu}(\underline{X},y)$ (for both values of ν) such that $K \supseteq L$.

Elements $(K,L) \in P_{x\mu\nu}(\underline{X},\underline{Y},y)$ now allow to perform the basic construction for the proof of Theorem 2.2.2:

Let $(K,L) \in P_{\lambda\mu\nu}(\underline{X},\underline{Y},y)$ be given. Let the coordinates $(z_1,\dots,z_n)$ on $\mathbb{C}^n$ be such that K is defined by $z_1 = \dots = z_f = 0$. We use the following notations:

(2.2.6) $\underline{P}_K : (\underline{\mathbb{C}}^n,0) \longrightarrow (\underline{Y},y)$ the projection along K;

$\underline{\rho} : \underline{X} \longrightarrow \underline{Y}$ the restriction $\underline{P}_K \mid \underline{X}$;

$\underline{X}_z := \underline{\rho}^{-1}(z) = \underline{X} \cap \underline{K}_z$ for $z \in Y$ near y, with K_z the affine plane $K + z$ parallel to K through z;

$\underline{E}$: a d-dimensional plane containing Y complementary to L;

$\underline{P}_L : \underline{\mathbb{C}}^n \longrightarrow \underline{E}$ the projection along L;

$\underline{h} : \underline{X} \longrightarrow \underline{E}$ the restriction $\underline{P}_L \mid \underline{X}$;

$\underline{h}_z : \underline{X}_z \longrightarrow \underline{E}_z := \underline{E}\ \underline{K}_z$ the restriction of $\underline{h}$ to $\underline{K}_z$ and hence the projection along L_z;

$\underline{P} := \underline{P}_y : \underline{\mathbb{C}}^n \longrightarrow \underline{K}$ the projection along $\underline{Y}$.

The following figure may illustrate the situation.

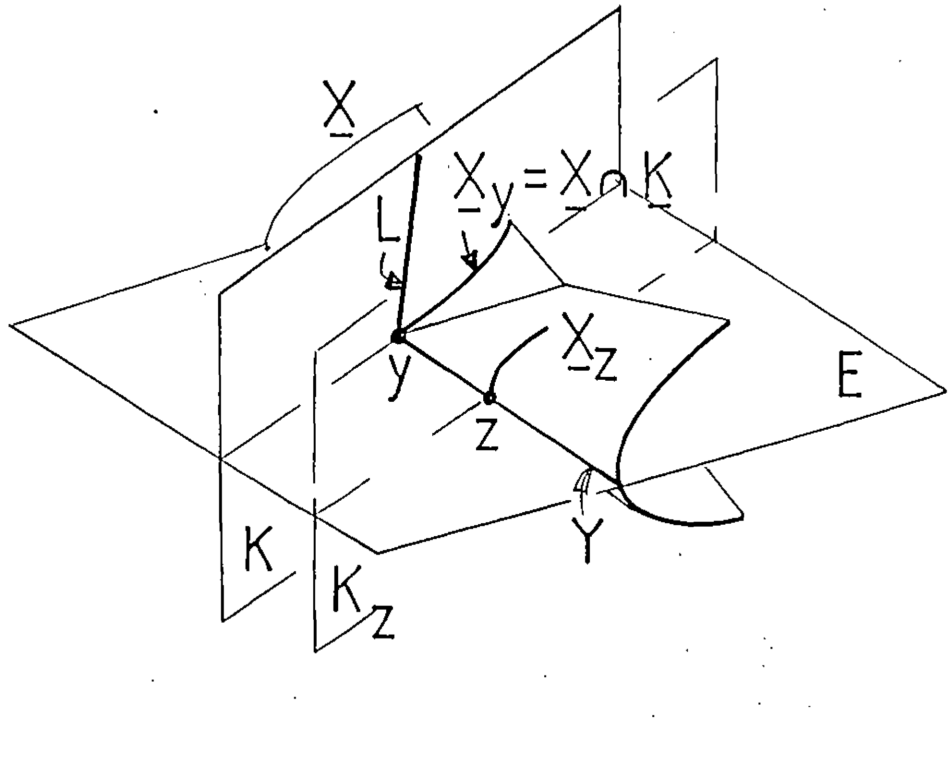


Fig. 8

We now come to the actual proof of Theorem 2.2.2. We use the notations of (2.2.6) throughout. Further, if $K \in \mathrm{Grass}^f(\mathbb{C}^n)$ is given, it defines an embedding $\underline{C}(\underline{X},\underline{Y}) \hookrightarrow \underline{Y} \times \underline{\mathbb{C}}^{n-f} \hookrightarrow \underline{\mathbb{C}}^n$ of the normal cone, with $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ induced by the projection $\underline{Y} \times \underline{\mathbb{C}}^{n-f} \longrightarrow \underline{Y}$ according to the description given in 1.4. If $K \in P^r_g(\underline{X},x)$, we have the settheoretic inclusions

(2.2.7) (i) $C(\underline{X} \cap \underline{K},y) \subseteq \nu^{-1}(y) \subseteq K$

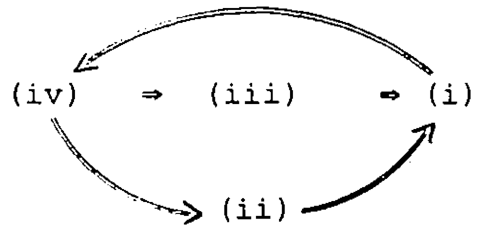
(ii) $C(\underline{X} \cap \underline{K},y) \subseteq C(\underline{X},x) \cap K$,

and, if $K \in P^f_e(\underline{X},x)$,

(iii) $C(\underline{X} \cap \underline{K},y) = C(\underline{X},x) \cap K$;

this will be used without further comment.

We proceed according to the pattern



(iv) ⇒ (iii) (cf. [34], [69]) Choose $K \in P^f_g(\underline{X},y)$. With the conventions above, K is given by $z_1 = \ldots = z_f = 0$, $\underline{p} : \underline{\mathbb{C}}^n \longrightarrow \underline{K}$ denotes the projection along Y , and $\underline{\nu}^{-1}(\underline{y})$ may be regarded as a subvariety of $\underline{K}$, which is of dimension $d-f$ by assumption. So $P^{d-f}_e(\underline{\nu}^{-1}(\underline{y}),y;K) :=$ $:= \{L \in \mathrm{Grass}^{d-f}(K) \mid L \pitchfork \nu^{-1}(y)\}$ is a nonempty Zariski-open set of $\mathrm{Grass}^{d-f}(K)$. Put $V_0(y) := \{L \in \mathrm{Grass}^d(\mathbb{C}^n) \mid p(L) \in P^{d-f}_e(\underline{\nu}^{-1}(\underline{y}),y;K\}$. This is a nonempty Zariski-open subset of $\mathrm{Grass}^d(\mathbb{C}^n)$, and the claim is that (iii) holds for $V := V_0(y)$. Suppose this were not so. We could then find an $L \in V_0(y)$ and a sequence $(x^{(j)})_{j\in\mathbb{N}}$ such that $x^{(j)} \in$ $(X-Y) \cap (L+x^{(j)}), h(x^{(j)}) \in Y$, and $x^{(j)} \longrightarrow y$. After selecting a suitable subsequence we may assume $p(\overline{x^{(j)}y})$ converges to a line ℓ in $\mathbb{P}(K)$, since $\mathbb{P}(K)$ is compact. But then $\ell \subseteq \nu^{-1}(y)$ by (1.4.4), and $\ell \subseteq p(L)$ by construction, which contradicts the fact that $p(L) \in P^{d-f}_e(\underline{\nu}^{-1}(\underline{y}),y;K)$. So we have (iii) .

Before showing (iii) ⇒ (i) , one shows the following consequence of (iv):

(2.2.8) Assume (iv) holds. Let $K \in P^f_g(\underline{X},y)$ and $L \subseteq K$ be in $P^{d-f}_e(\underline{\nu}^{-1}(\underline{y}),y;K)$. Then $L_z \in P^{d-f}_e(\underline{\nu}^{-1}(\underline{z}),z;K_z)$ for all z outside a nowhere dense analytic subset of Y .

For this, let $\underline{\pi} : \underline{\tilde{\mathbb{C}}}^n \longrightarrow \underline{\mathbb{C}}^n$ be the blowup of $\underline{\mathbb{C}}^n$ along $\underline{\mathbb{C}}^f \times \underline{0}$. The strict transforms of $\underline{X}$ and $\underline{Y} \times \underline{L}$ under $\underline{\pi}$ give the blowups $\underline{\tilde{X}}$ and $(\underline{Y} \times \underline{L})^{\sim}$ along $\underline{Y}$. Their exceptional divisors $\underline{\mathbb{P}C}(\underline{X},\underline{Y})$ and $\underline{Y} \times \underline{\mathbb{P}}(L)$ are subvarieties of $\underline{Y} \times \underline{\mathbb{P}}(K)$, and so meet in a subvariety of $\underline{Y} \times \underline{\mathbb{P}}(K)$, whose image under $\underline{Y} \times \underline{\mathbb{P}}(K) \longrightarrow \underline{Y}$ is a subvariety of $\underline{Y}$ since this map is proper. This shows (2.2.8).

(iii) ⇒ (i) (cf. loc. cit.) By Proposition 2.1.3, the function $z \longmapsto m(\underline{X},z)$ has a generic value, m say, outside a nowhere dense analytic set A in Y . By Theorem 1.4.8, we may assume $Y - A \subseteq F(X,Y)$, the flatness locus of X along Y. So (iv) holds at all points of $Y-A$. We choose $K \in P^f_{gs}(\underline{X},y)$; after shrinking Y, we may assume $K_z \in P^f_{gs}(\underline{X},z)$ for all $z \in Y$ by (2.2.3) (ii). Choose a $w \in Y-A$ and an L in the generic set $V \cap V_0(w) \cap \mathrm{Grass}^{d-f}(K)$ of $\mathrm{Grass}^{d-f}(K)$.

Since $L \in V_0(w)$, we know by (2.2.8) that $L_z \in P^{d-f}_e(\underline{\nu}^{-1}(\underline{z}),z;K_z)$ outside a nowhere dense analytic set B ; we may assume $B \supseteq A$. Since $C(\underline{X}_z,z) \subseteq \nu^{-1}(z)$ always, we have $L_z \in P^{d-f}_e(\underline{X}_z,z;K_z)$. The Lemma and Corollary of 2.2.3 (ii) imply:

(2.2.9) $$\deg_z\underline{h} = \deg_z\underline{h}_z = m(\underline{X}_z,z) = m(\underline{X},z) ,$$

so $\deg_z\underline{h}$ must have the generic value m on $Y-B$.

On the other hand, we have $L \in V$. Now the degree formula II Theorem (2.2.8), applied to $\underline{h}$, gives

(2.2.10) $$\deg_y\underline{h} = \sum_{z' \in h^{-1}h(z)} \deg_{z'}\underline{h} ,$$

for z near y. But the assumption (iii) forces $h^{-1}h(z) = \{z\}$ near y, so

(2.2.11) $$\deg_y\underline{h} = \deg_z\underline{h}$$

for z near y. This implies $\deg_y\underline{h} = m$ by (2.2.9) so we have equimultiplicity by upper semicontinuity of multiplicity (II Theorem 5.2.4).

<u>(i) ⇒ (iv)</u> (cf. [61]). Let $\underline{X}$ be equimultiple along $\underline{Y}$ at y. Let $L \in P^d_e(\underline{X},x)$ and $\underline{h} : \underline{X} \longrightarrow \underline{E}$ be the corresponding projection as in (2.2.10). Then $\deg_y\underline{h} = m(\underline{X},y)$, and so by (2.2.11), $\deg_y h \geq \deg_z\underline{h} \geq m(\underline{X},z)$ for z near y, hence we have $\deg_y h = m(\underline{X},z)$ for $z \in Y$ near y by equimultiplicity.

We will now show: If $L \in P^d_g(\underline{X},x)$ is such that for the corresponding projection we have $\deg_y h = m(\underline{X},z)$ for $z \in Y$ near y, then $L \pitchfork \nu^{-1}(y)$; this will obviously establish (i) ⇒ (iv). One proves this first in case $(\underline{X},y)$ is a hypersurface, and then for general $(\underline{X},y)$ by the classical device of reducing it to the hypersurface case via a finite projection. We let $K \in P_e^f(\underline{X},x)$ be the plane given by $z_1 = \dots = z_f = 0$ and define the normal cone $\underline{C}(\underline{X},\underline{Y}) \hookrightarrow \underline{\mathbb{C}}^n$ by this K.

So let $\underline{X}$ be a hypersurface in $\underline{\mathbb{C}}^n = \underline{\mathbb{C}}^{d+1}$. We choose coordinates $z_1,\dots,z_n$ in such a way that Y is given by $z_{f+1} = \dots = z_n = 0$. We decompose $\mathbb{C}^n = \mathbb{C}^f \times \mathbb{C}^k$ and write points in $\mathbb{C}^n$ as (z,t) with $z = (z_1,\dots,z_f)$ and $t = (z_{f+1},\dots,z_n)$. Let $g \in \mathcal{O}_{\mathbb{C}^n}(U)$ be an equation for $\underline{X}$; one can write

$$(2.2.12) \qquad g(z,t) = \sum_{A \in \mathbb{N}^k} g_A(z) \cdot t^A$$

(notation as in I, §§ 1-2)), where the $g_A(z)$ are holomorphic functions on $Y = (\mathbb{C}^f \times 0) \cap U$. The Y-leitform of g (as defined in (2.2.2)) is

$$(2.2.13) \qquad L_Y(g) = \sum_{\substack{A \in \mathbb{N}^k \\ |A| = \nu}} g_A(z) \cdot t^A \quad ,$$

where $\nu = \nu_Y(g)$ is the degree of the first nonzero monominal t^A appearing in (2.2.13) with respect to the lexicographic degree order.

Now the equimultiplicity assumption on X along Y at y implies that the $g_A(z)$ with $|A| = \nu$ cannot simultaneously vanish at $y = 0$. For suppose this were the case. The analytic set defined by the simultaneous vanishing of the $g_A(z)$ with $|A| = \nu$ is nowhere dense in Y because $L_Y(g)$ does not vanish identically on K since $K \in P_e^f(\underline{X},y)$. So there are, arbitrarily close to y, points $z_0 \in Y$ such that $g_A(z_0) \neq 0$ for at least one A with $|A| = \nu$. But then all monomials in the development of $g(z,t) \in \mathbb{C}\{z,t\}$ of (2.2.13) would have degree $> \nu$, whereas in the corresponding development of $g(z,t) \in \mathbb{C}\{z-z_0,t\}$ there would appear monomials of degree ν, and the multiplicity $m(\underline{X},y) > \nu$ would drop to $m(\underline{X},z_0) = \nu$ which cannot be by assumption (here we agree on $m(\underline{X},z_0) = 0$ if $z_0 \notin X$). Note that this argument establishes, in particular:

(2.2.14) $\nu_Y(g)$ = generic multiplicity $m(\underline{X},z)$ for $z \in Y$ near y .

It follows that $\nu_Y(g) = m(\underline{X},y)$, hence the leitform $L(g)$ is

(2.2.15) $$L(g) = \sum_{\substack{A \\ |A|=\nu}} g_A(0)\cdot t^A \quad .$$

(2.2.13) and (2.2.15) show:

(2.2.16) $$\underline{C}(\underline{X},y) = \underline{\nu}^{-1}(\underline{y}) \times \underline{C}(\underline{Y},y) \quad ,$$

and so X <u>is normally flat along</u> Y <u>at</u> y . In particular, we get

(2.2.17) $$\nu^{-1}(y) = C(\underline{X},y) \cap K = C(\underline{X} \cap \underline{K},y) \quad .$$

We now turn to $L \in P^d_g(\underline{X},y)$. In suitable coordinates $v = (v',v_n)$ of $\mathbb{C}^n$, we may assume g is a Weierstraß polynomial $g(v',v_n) = v_n^b + a_{b-1}(v')v_n^{b-1} + \ldots + a_1(v')v_n + a_0(v')$, and L is given by $v' = 0$. Then $\deg_y\underline{h} = b$, and, by assumption, $b = m(\underline{X},y) = \nu(g)$. So v_n^b appears in $L(g)$, which means $L \in P^d_e(\underline{X},y)$. So we can choose $K \in P^f_e(\underline{X},y)$ with $K \supseteq L$, and then (2.2.17) shows $L \pitchfork \nu^{-1}(y)$.

We now treat the general case . So let $\underline{Y} \hookrightarrow \underline{X} \hookrightarrow \underline{U}$ be as in Theorem 2.2.2, and let $L \in P^d_g(X,x)$ be such that

(2.2.18) $$\deg_y\underline{h} = m(\underline{X},z)$$

for all $z \in Y$ near y , $\underline{h}$ the projection along L . We want to show $L \pitchfork \nu^{-1}(y)$, where $\underline{\nu} : \underline{C}(\underline{X},Y) \longrightarrow \underline{Y}$ is the normal cone. For this, it suffices to show $L^\lambda \cap \nu^{-1}(y) = \{y\}$ for each line $L^\lambda \subseteq L$.

We may assume $\underline{X}$ is reduced. Namely, by the degree formula (II Theorem 2.2.8), we have

(2.2.19) $$\deg_y\underline{h} = \sum_{z' \in h^{-1}h(z)} \deg_{z'}\underline{h} \geq \deg_z\underline{h} \quad ,$$

so our assumption forces $h^{-1}h(z) = \{z\}$ and $\deg_z \underline{h} = \deg_y \underline{h}$ for $z \in Y$ near y. But then $\deg_z \underline{h}_{red} = \deg_y \underline{h}_{red}$ for $z \in Y$ near y. Moreover, $\deg_z \underline{h}_{red} = m(\underline{X}_{red}, z)$, and so we have our assumption on L with respect to $\underline{X}_{red}$. By the limit description (1.4.4), $\nu^{-1}(y)$ depends only on $\underline{X}_{red}$, and so it suffices to consider the case $\underline{X} = \underline{X}_{red}$.

We describe lines in L by linear forms $\lambda \in \check{L} - \{0\}$, where $\check{L} := \mathrm{Hom}(L, \mathbb{C})$ is the dual of L, in the following way: We fix $\lambda \in L - \{0\}$ and choose L^λ to be a complementary line to $\mathrm{Ker}(\lambda)$.

This gives us the following situation.

(2.2.20)
$$\begin{array}{ccc} \underline{X} & \xrightarrow{\underline{p}_\lambda} & \underline{X}_\lambda \\ & \underline{h} \searrow \quad \circlearrowleft \quad \swarrow \underline{h}_\lambda & \\ & U_E & \end{array}$$

Here, we have assumed $U = U_E \times U_L$ with U_E open in E, U_L open in L. The maps are finite projections; $\underline{p}_\lambda := \underline{\pi}_\lambda \mid X$ with $\underline{\pi}_\lambda : \underline{\mathbb{C}}^n \longrightarrow \underline{E} \oplus \underline{L}^\lambda \cong \underline{\mathbb{C}}^{d+1}$ the projection along $\mathrm{Ker}(\lambda)$, $\underline{X}_\lambda := \underline{\mathrm{im}}(\pi_\lambda)$, and $\underline{h}_\lambda$ the projection along L^λ. $\underline{X}_\lambda \hookrightarrow \underline{U}_E \times \underline{L}^\lambda$ is a hypersurface, given be the equation

(2.2.21)
$$\omega^\lambda(z,t) := \prod_{x \in h^{-1}h(z) = L+z} (t - \lambda(x-z))^{\deg_x \underline{h}} \in \mathcal{O}(U_E)[t] \quad ,$$

where we regard $\mathcal{O}(U_E)[t] \hookrightarrow \mathcal{O}(U_E \times L^\lambda)$ under $t \longmapsto \lambda$. This follows because π_λ is given by

(2.2.22)
$$\pi_\lambda(v) = (p_L(v), \rho_\lambda(v)) \quad ,$$

where $\rho_\lambda : \mathbb{C}^n \longrightarrow L^\lambda$ is the projection along $E \oplus \mathrm{Ker}(\lambda)$, and from the classical arguments involving the elementary symmetric functions in the $\lambda(x-z)$ for $x \in h^{-1}(z)$. We have

(2.2.23) $$X = \bigcap_{\lambda \in L-\{0\}} \pi_\lambda^{-1}(X^\lambda) \quad ;$$

namely, $X \subseteq \pi_\lambda^{-1}(X^\lambda)$ for all λ since X is equidimensional, and on the other hand, for any $v \in \mathbb{C}^n - X$, there is $\lambda \in \check{L} - \{0\}$ with $\lambda(x-v) \neq 0$ for all $x \in h^{-1}p_L(v)$, and so $\pi_\lambda(v) \notin X^\lambda$ by (2.2.21) and (2.2.22).

From (2.2.21), we see $\deg_z \underline{h}_\lambda = \sum_{x \in h^{-1}h(z)} \deg_x \underline{h} = \deg_z \underline{h}$ and so, putting $z = y$, in particular $m(\underline{X}^\lambda, y) = m(\underline{X}, x)$. Let $C(\underline{X},y)^\lambda := \pi_\lambda(C(\underline{X},x))$; then, since π_λ is proper, one may show, by the limit description of tangent cones,

(2.2.24) $$C(\underline{X},y)^\lambda = C(\underline{X}^\lambda, y) \quad .$$

So $\pi_\lambda : C(\underline{X},y) \longrightarrow C(\underline{X}^\lambda, y)$ is finite, and, in particular, if $K \in P_e^f(\underline{X},y)$, we have $K^\lambda := \pi_\lambda(K) \in P_e^f(\underline{X}^\lambda, y)$. If we define the normal cones of $\underline{Y}$ in $\underline{X}^\lambda$ by the K^λ, we get, by the hypersurface case proved above, that $L^\lambda \pitchfork \nu_\lambda^{-1}(y)$. Again by the properness of π_λ and the limit description of normal cones, there results

(2.2.25) $$(\nu^{-1}(y))^\lambda = \nu_\lambda^{-1}(y) \quad ,$$

where $(\nu^{-1}(y))^\lambda := \pi_\lambda(\nu^{-1}(y))$. Hence $L^\lambda \cap \nu^{-1}(y) = \{y\}$, as we wanted to show. So (i) $\Rightarrow$ (iv) is established.

Note that this proof shows, in addition ,

(2.2.26) $$C(\underline{X},y) = \nu^{-1}(y) \times C(\underline{Y},y) \quad .$$

This follows, because, by (2.2.16), we have

(2.2.27) $$C(\underline{X}^\lambda,y) = \nu_\lambda^{-1}(y) \times C(\underline{Y},y) \quad \text{for all} \quad \lambda \in \check{L} - \{0\} \quad ;$$

then, by (2.2.24) and (2.2.25), we get (2.2.26) by intersecting (2.2.27) over all λ and using (2.2.23) (for X^λ, $(\nu^{-1}(y))^\lambda$, and $(C(\underline{X},y))^\lambda$). In particular, we get

(2.2.28) $$\nu^{-1}(y) = C(\underline{X},y) \cap K = C(\underline{X} \cap \underline{K}, y)$$

for $K \in P_e^f(\underline{X},y)$ under the condition (i). This is in fact equivalent to (i) and hence to (2.2.26), because it clearly implies $\dim \nu^{-1}(y) = d-f$, so (iv) holds, and we have already (iv) ⇒ (i).

(iv) ⇒ (ii) ⇒ (i): By the proof of (iv) ⇒ (iii) and (2.2.8) we even know that (ii) holds for all $L \in V_0(y)$. The implication (ii) ⇒ (i) follows because we have (2.2.11) for the projection $\underline{h}$ along L by the same reasoning as in the step (iii) ⇒ (i); by assumption, we have $\deg_z \underline{h} = m(\underline{X},z)$ for all z near y in addition, and this shows (i).

This establishes the equivalence of (i) - (iv). For the additional statements, note that the step (iv) ⇒ (iii) showed we may take $V = V_0(y) := \{L \in \mathrm{Grass}^d(\mathbb{C}^n) \mid p(L) \in P_e^{d-f}(\underline{\nu}^{-1}(\underline{y});K)\}$. If one of the statements of Theorem 2.2.2 holds, we know all of them hold for all $z \in Y$ near y, and then (2.2.28) and (2.2.8) applied to z, show $L_z \in P_e^d(\underline{X},z)$ for all $L \in V$ and $z \in Y$ near y. This concludes the proof of Theorem 2.2.2.

The proof of the Addendum is left to the reader.

Before commenting further on the significance of the various characterizations of normal pseudoflatness, let us remark that the proof of (i) ⇒ (iv) gave further important characterizations. Recall, for $g \in \mathcal{O}_{X,y}$, the notions of the order $\nu(g)$ (I, (1.1.3)) and the order $\nu_Y(g)$ of g along Y ((2.2.2)).

<u>Theorem 2.2.2 (cont.)</u>. <u>Let $\underline{Y} \hookrightarrow \underline{X} \hookrightarrow \underline{U}$ be as in Theorem 2.2.2. Then the following statements are equivalent to (i) - (iv) of Theorem 2.2.2</u>:

(v) <u>Let</u> $I \subseteq \mathcal{O}_U$ <u>be the ideal defining</u> $\underline{X} \hookrightarrow \underline{U}$. <u>There are finitely many equations</u> $g_\lambda \in I(U)$ <u>with the following properties. Let</u> $\underline{X}_\lambda := \underline{N}(g_\lambda)$, <u>and</u> $\underline{\nu}_\lambda : \underline{C}(\underline{X}_\lambda,\underline{Y}) \longrightarrow \underline{Y}$ <u>be the normal cones for all</u> λ. <u>Then</u>:

1) $\nu(g_\lambda) = \nu_Y(g_\lambda)$ for all λ ;

2) $C(\underline{X},y) = \bigcap_\lambda C(\underline{X}_\lambda,y)$;

3) $\nu^{-1}(y) = \bigcap_\lambda \nu_\lambda^{-1}(y)$,

<u>where</u> $\nu^{-1}(y)$, $\nu_\lambda^{-1}(y)$ <u>are defined in</u> $\mathbb{C}^n$ <u>with respect to some</u> $K \in P_e^f(\underline{X},y)$.

(vi) $C(\underline{X},y) = \nu^{-1}(y) \times C(\underline{Y},y)$ <u>with respect to some</u> $K \in P_g^f(\underline{X},y)$.

(vii) $\nu^{-1}(y) = C(\underline{X} \cap \underline{K},y)$ <u>with respect to some</u> $K \in P_g^f(\underline{X},y)$.

<u>If one of the conditions</u> (i) - (vii) <u>holds</u>, (vii) <u>holds for all</u> $K \in P_g^f(\underline{X},y)$.

<u>Moreover, if</u> X <u>is a hypersurface, the following condition is also equivalent to</u> (i) - (vii):

(iv') $\underline{X}$ <u>is normally flat along</u> $\underline{Y}$ <u>at</u> y .

I leave it to the reader to show (i) ⇒ (v) ⇒ (vi); all the other implications have been mentioned above.

Conditions (v) and (vi) are particularly interesting for the relation between normal flatness and normal pseudoflatness; (v) shows algebraically, and (vi) geometrically, how much is lost when passing from normal flatness to normal pseudoflatness. For normal flatness, condition (v) would require, in addition to $\nu_Y(g_\lambda) = \nu(g_\lambda)$, that the $L_Y(g_\lambda)$ generate the normal cone $C(\underline{X},\underline{Y})$ (note that this implies that the g_λ generate the ideal defining $\underline{X} \to \underline{U}$, so $C(\underline{X},x) = \bigcap_\lambda C(X_\lambda,x)$
Condition (vi) would require $\underline{C}(\underline{X},y) = \underline{\nu}^{-1}(\underline{y}) \times \underline{C}(\underline{Y},y)$ so normal pseudoflatness keeps the geometric content of normal flatness, but looses the possibly nonreduced structure.

In order to connect Theorem 2.2.2 with the algebraic equimultiplicity results of Chapter IV of this book, we formulate the following result.

<u>Proposition 2.2.3.</u> <u>Let</u> $(\underline{X},y) \in \underline{cpl}_0$, $(\underline{Y},y) \hookrightarrow (\underline{X},y)$ <u>a complex subspacegerm defined by the ideal</u> $I \subseteq R := \mathcal{O}_{X,y}$. <u>Then</u>:

(2.2.29) $\operatorname{codim}_y \underline{Y} = \operatorname{ht}(I)$;

(2.2.30) $\dim \nu^{-1}(y) = s(I)$.

Proof. A local analytic algebra is catenary (e.g. by the Active Lemma I 5.2.2). This gives (2.2.29) by the Dimension Formula, Chapter III (18.6.1). Further, by base change for Specan, Proposition 1.2.4, $\underline{\nu}^{-1}(\underline{y}) = \underline{\text{Specan}}(\bigoplus_{k\geq 0} I^k/m_x I^k)$. This gives (2.2.30).

By 2.1.2 and 2.2.3, then, we see that the equivalence (i) ⟺ (iv) of Theorem 2.2.2 is, for local analytic ℂ-algebras, equivalent to Satz 2 of [77], thus elucidating its geometric content in this case. Conversely, (20.9) gives an algebraic proof of the Hironaka-Schickhoff-Theorem, based on 2.1.2, which used compact Stein neighbourhoods to interprete invariants of localizations of local analytic ℂ-algebras geometrically (note that the localization of $R \in \underline{\underline{la}}$ is no longer in $\underline{\underline{la}}$, so does not correspond directly to a geometric object via the Equivalence Theorem I 3.3.3). This is a particular case of the general principle that distinguished compact Stein neighbourhoods provide a systematic way of translating results from local complex analytic geometry into local algebra and vice versa. In this vein, the equivalence (iv) ⟺ (vi) ⟺ (vii) of Theorem 2.2.2 is essentially the geometric content of (ii) ⟺ (iii) ⟺ (iii') of [78] (see also the discussion in [49], § 5), and we will deduce geometric properties in $\underline{\underline{cpl}}_0$ from local algebra in 3.2. below.

Exercise 2.2.4. Try to express the statement (ii) of Theorem 2.2.2 in terms of local algebra and to show its being equivalent to the equimultiplicity condition $e(R) = e(R_{\mathfrak{p}})$ algebraically.

(ii) Try to translate the proof of Theorem 2.2.2 into an algebraic proof of Satz 2 of [77]. What do the choices of the f- and d-codimensional planes K and L mean algebraically?

I close this section by some comments on the geometric and algebraic significance of the various conditions in Theorem 2.2.2 and 2.2.2 (cont.); these will be partly, within this limited account, informal.

The equivalence (i) ⟺ (v), i.e. that the size of the normal cone is controlled by equimultiplicity, is geometrically a transversality statement, as we will see now. This should be, in a sense, not too surprising, since multiplicity was defined as a generic mapping degree, and we have already seen in II Theorem 5.2.1, that a projection has generic mapping degree if its kernel is transverse to the tangent cone.

The appropriate generalization of this is the following theorem, which we actually proved in the course of establishing (i) $\Longleftrightarrow$ (iv) of Theorem 2.2.2.

Theorem 2.2.5. Let $Y \hookrightarrow \underline{X} \hookrightarrow \underline{U}$ be as in Theorem 2.2.2, and let $L \in P^d_g(\underline{X},y)$. The following conditions are equivalent:

(i) $\deg_y \underline{h} = m(\underline{X},z)$ for $z \in Y$ near y, where $\underline{h}$ is the projection along L.

(ii) $Y \times L \pitchfork_Y X$, i.e. $Y \times L$ intersects X transversally along Y in the sense that $Y \times L \cap C(\underline{X},\underline{Y}) = Y$ ($C(\underline{X},\underline{Y})$ defined with respect to any (n-f)-dimensional plane $K \supseteq L$).

Remark 2.2.6.

1) If we put $Y = \{y\}$, we get the statement (ii) of II Theorem 5.2.1 which is the geometric form of the Theorem of Rees (cf. Theorem 1 in §1 of [49]) for reductions of the maximal ideal. For primary ideals, see Proposition 3.2.2 (ii) below. In fact, Theorem 2.2.5 is a variant of the geometric form of the Theorem of Böger (Chapter III, Theorem (19.6)) for the case of a regular prime ideal. The transversality condition in (ii) just means that the ideal generated in $\mathcal{O}_X$ via the projection $\underline{X} \longrightarrow \mathbb{C}^{d-f}$ along $Y \times L$ is a minimal reduction of the ideal generating Y. This gives a geometric picture of the meaning of a minimal reduction in this case. For the general case, see Theorem 3.2.7 below.

2) We did not use the Theorem of Rees (i.e. the important direction (i) $\rightarrow$ (ii) in Chapter III, Theorem (19.3)) to establish (i) $\Rightarrow$ (ii) above, so we really gave it a geometric proof. The direction (ii) $\Rightarrow$ (i) was also established in a geometric way, although one may object that I made use of the fact that, if $L \in P^d_e(\underline{X},y)$, one has $\deg_y \underline{h} = m(\underline{X},y)$, which was established in an algebraic way in II Theorem 5.2.1 using the theory of reduction. We will see in the proof of Theorem 2.2.8 below how to interprete this more geometrically.

I now turn to a discussion of condition (vi). Note that the equivalence (iv) $\Longleftrightarrow$ (vii) means, in particular:

Either $\nu^{-1}(y) = C(\underline{X} \cap \underline{K}, y)$ or $\dim \nu^{-1}(y) > d - f$ (where $K \in P^f_e(\underline{X}, x))$, which is, at first sight, rather surprising. Trying to understand this sheds some more light on the geometry of equimultiplicity, so I give an informal account. For this, we have to take a closer look how normal directions arise geometrically.

Proposition 2.2.7 (Existence of testarcs). Let $(\underline{X}, x) \in \underline{\underline{cpl}}_0$, and $(A, x) \subsetneqq (X, x)$ be an analytic setgerm. Then there exists a morphism $\underline{\alpha} : (\mathbb{D}, 0) \longrightarrow (\underline{X}, x)$, where $\mathbb{D} \subseteq \mathbb{C}$ is the open unit disc, such that $\alpha(\mathbb{D} - 0) \subseteq X - A$ and $\alpha(0) = x$. We call $\underline{\alpha}$ a testarc for (A, x) .

Sketch of proof. If $(\underline{C}, c) \in \underline{\underline{cpl}}_0$ is onedimensional, we get $\underline{\alpha} : (\mathbb{D}, 0) \longrightarrow (\underline{C}, c)$ by parametrizing an irreducible component. This reduces the proof to the case $(\underline{X}, x) = (\mathbb{C}^d, 0)$ via the Local Representation Theorem I 6.3.1. Then just parametrize a complex line transverse to A at x . Q.e.d.

Applying this to the blowup $\pi : \underline{X} \longrightarrow \underline{X}$ of $\underline{X}$ along $\underline{Y}$, with x being a point in $\pi^{-1}(Y)$ and $A := \pi^{-1}(Y)$, we see that in the limit description (1.4.4) of $\nu^{-1}(y)$ we can restrict the limit process to testarcs for (Y, y) :

(2.2.31) $\quad \ell$ is a line in $\nu^{-1}(y) \Longleftrightarrow \ell = \lim_{t \to 0} p(\overline{y\alpha(t)})$ for some testarc $\underline{\alpha} : (\mathbb{D}, 0) \longrightarrow (\underline{X}, y)$ for (Y, y) .

Here, it is understood we have choosen $K \in P^f_g(\underline{X}, y)$, and $\underline{p} : \underline{\mathbb{C}}^n \longrightarrow \underline{K}$ is the projection along $\underline{Y}$. The normals at y now fall into two classes: Those that belong to $C(\underline{X} \cap \underline{K}, y)$, which I call ordinary normals, and those that do not, which I call excess normals. The equivalence of (vi) and (vii) says that the failure of normal pseudoflatness is due to the existence of excess normals. These are characterized as follows:

(2.2.32) $\quad \ell \subseteq K$ is an excess normal $\Longleftrightarrow \ell = (p \circ \alpha)^{\cdot}(0)$, where $\underline{\alpha} : (\mathbb{D}, 0) \longrightarrow (\underline{X}, y)$ is a testarc for (Y, y) such that $\alpha^{\cdot}(0)$ is a tangent line of $\underline{X}$ at x , but $(p \circ \alpha)^{\cdot}(0)$ is not a tangent line of $\underline{X} \cap \underline{K}$ at y .

Here I have put $\dot{\beta}(0) := \lim_{t\to 0} \overline{y\beta(t)}$ for a testarc $\underline{\beta}$.

The following picture may illustrate the situation.

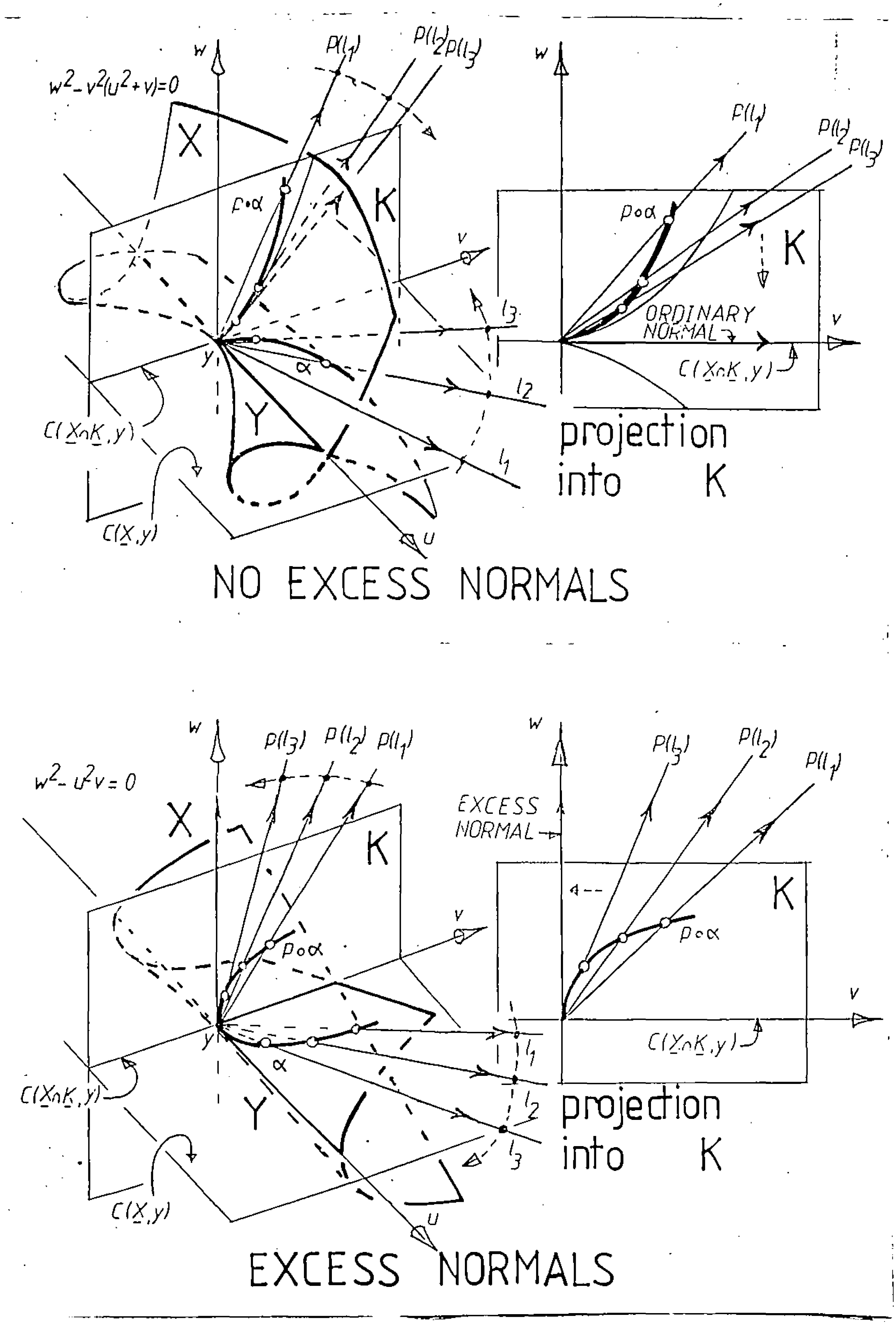


Fig. 9

So we have to analyse what it means, in terms of testarcs, that a line $\ell \subseteq \mathbb{C}^p$ is not tangent to a given $(\underline{W},w) \hookrightarrow (\mathbb{C}^p,0)$. Clearly

(2.2.33) $\qquad \ell \not\subseteq C(\underline{W},w) \iff$ for all testarcs $\underline{\alpha} :(\mathbb{D},0) \longrightarrow (\underline{W},w) : \ell \neq \dot{\alpha}(0)$

Choose coordinates $(z_1,\ldots,z_p)$ such that ℓ is given by $z_1 = \ldots = z_{p-1} = 0$. It is conceivable that the requirement $\ell \neq \dot{\alpha}(0)$ puts growth conditions on the coordinate functions $z_1,\ldots,z_d$ restricted to α , as the following picture suggests:

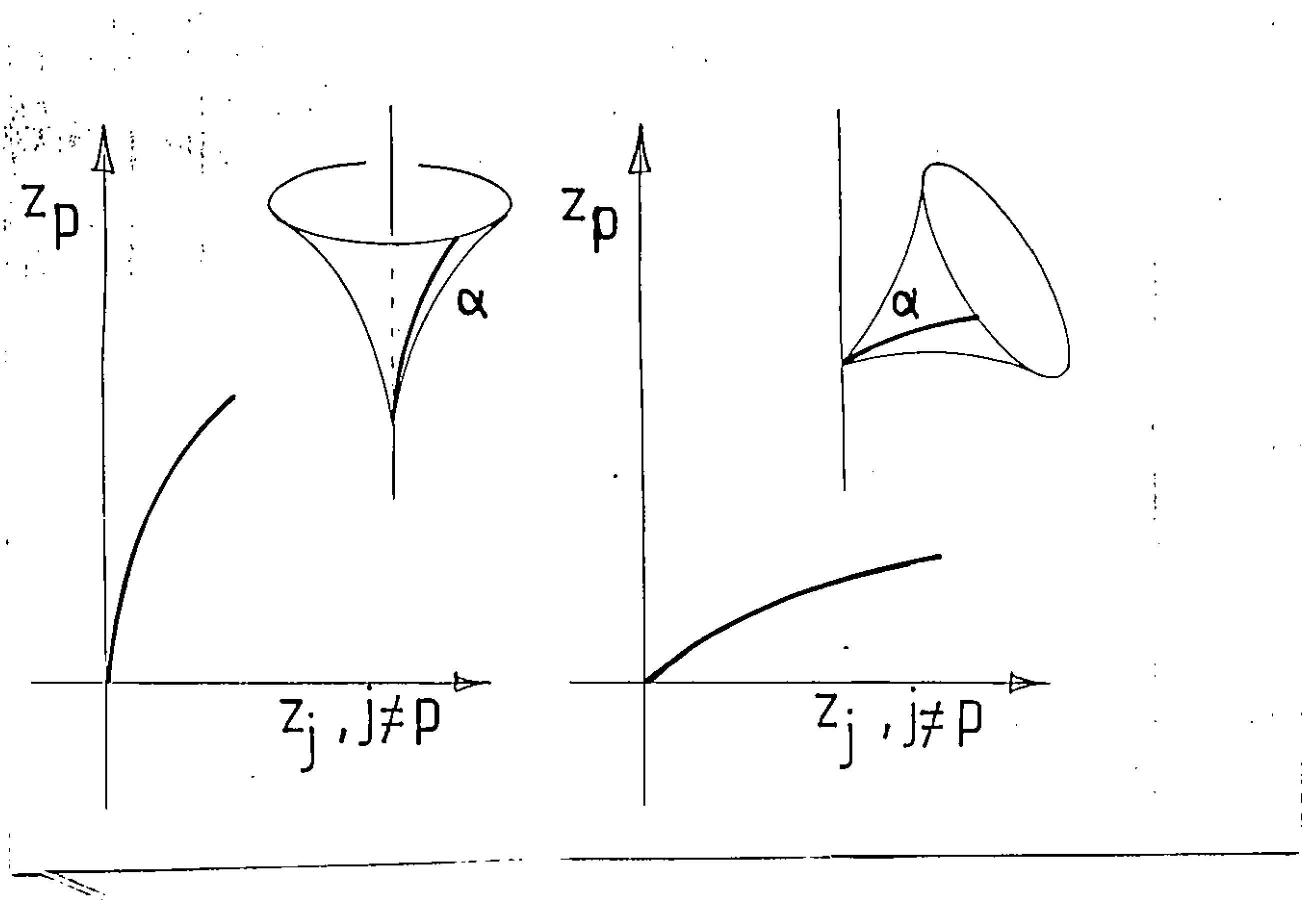


Fig. 10

It turns out that the appropriate growth conditions are:

(2.2.34) $\qquad \ell \neq \dot{\alpha}(0) \longleftrightarrow$ <u>there is a neighbourhood</u> V <u>of</u> $0 \in \mathbb{C}$ <u>and</u> $C \in \mathbb{R}_{>0}$ <u>such that</u>

$$|z_p \circ \alpha(t)| \leq C \sup_{1 \leq j \leq p-1} |z_j \circ \alpha(t)|$$

<u>for all</u> $t \in V$.

Now testarcs $\underline{\alpha}$ define valuations v_α on $R := \mathcal{O}_{W,w}$ in the sense of

Chapter I , Definition (4.18), via

$$(2.2.35) \qquad v_\alpha(f) := \nu(\alpha^0(f)) = \mathrm{ord}_0(\alpha^0(f))$$

where $f \in \mathcal{O}_{W,w}$, $\alpha^0 : \mathcal{O}_{W,w} \longrightarrow \mathcal{O}_{\mathbb{C},0}$ is given by $\underline{\alpha} : (\mathbb{D},0) \longrightarrow (\underline{W},w)$, and ord_0 denotes the order of vanishing at $0 \in \mathbb{C}$. Then the condition (2.2.3) reads

$$(2.2.3\ ) \qquad \ell \neq \dot{\alpha}(0) \iff v_\alpha(z_p) \geq v_\alpha((z_1,\ldots,z_{p-1})\mathcal{O}_{W,w}) \quad \text{for all } \alpha ,$$

and so the valuation criterion of integral dependence of Chapter I, (4.20) strongly suggests that $\ell \neq \dot{\alpha}(0)$ <u>is equivalent to</u> z_p , <u>regarded as a function on</u> W , <u>being integrally dependent on the ideal</u> $(z_1,\ldots,z_{p-1})\cdot\mathcal{O}_{W,w}$.

In fact, there is the following proposition:

<u>Proposition 2.2.8</u> ([69]). <u>Let</u> $(\underline{X},x) \in \underline{\underline{\mathrm{cpl}}}_0$, $I \subseteq \mathcal{O}_{X,x}$ <u>an ideal</u> , $f \in \mathcal{O}_{X,x}$. <u>The following statements are equivalent</u>:

(i) <u>For all testarcs</u> $\underline{\alpha} : (\underline{\mathbb{D}},0) \longrightarrow (\underline{X},x)$, $v_\alpha(f) \geq v_\alpha(I)$.

(ii) <u>For all systems of generators</u> $(g_1,\ldots,g_\ell)$ <u>of</u> I <u>there is a neighbourhood</u> V <u>of</u> x <u>in</u> X <u>and</u> $C \in \mathbb{R}_{>0}$ <u>such that</u>

$$|f(y)| \leq C \cdot \sup_{1\leq j\leq \ell} |g_j(y)|$$

<u>for</u> $y \in V$.

(iii) $f \in \overline{I}$.

(i) $\Rightarrow$ (iii) depends on the fact that in the proof of (ii) $\Rightarrow$ (i) of Proposition (4.20) of Chapter I the valuations v_α suffice, see the argument in the proof of Chapter I, 1.3.4 of [69]. (iii) $\Rightarrow$ (ii) follows because the equation of integral dependence gives the necessary estimates, and (ii) $\Rightarrow$ (i) is immediate. For the complex analytic proof see [69], Chapter I, 1.3.1 and 1.3.4.

From this results we see:

Theorem 2.2.9. Let $(\underline{W},w) \hookrightarrow (\mathbb{C}^p,0)$ be of dimension d, L the d-codimensional plane given by $z_1 = \ldots = z_d = 0$. Then $L \pitchfork C(\underline{W},w)$ if and only if $z_{d+1},\ldots,z_n \in \overline{(z_1,\ldots,z_d)\cdot \mathcal{O}_{W,w}}$.

It is in this way how the algebraic notion of integral dependence comes in when describing the geometric notion of transversality.

We can now translate the condition (vii) into algebra. We formulate (2.2.32) in the following way:

(2.2.37) There are no excess normals, i.e. (vi) holds $\Longleftrightarrow$ for all testarcs α such that $(p\circ\alpha)^{\cdot}(0)$ is not a tangent line of $\underline{X}\cap\underline{K}$ at y, $\dot{\alpha}(0)$ is not a tangent line of $\underline{X}$ at y.

This can be exploited as follows.

We first get the generalization of Theorem 2.2.9:

Theorem 2.2.10. Let $(\underline{Y},y) \hookrightarrow (\underline{X},y) \hookrightarrow (\mathbb{C}^n,0)$ be as in Theorem 2.2.2, $L \in P^d_g(\underline{X},y)$. Choose any $(n-f)$-dimensional plane $K \supseteq L$, thus $K \in P^f_g(\underline{X},y)$ (defining an embedding $\underline{C}(\underline{X},\underline{Y}) \hookrightarrow \mathbb{C}^n$). Let the coordinates $z_1,\ldots,z_d = 0$ be such that L is defined by $z_1 = \ldots = z_d$, and K be $z_1 = \ldots = z_f = 0$. Then $Y\times L \pitchfork C(\underline{X},\underline{Y})$ if and only if $\overline{(z_{f+1},\ldots,z_n)\cdot\mathcal{O}_{X,x}} = \overline{(z_{f+1},\ldots,z_d)\cdot\mathcal{O}_{X,x}}$.

This follows by applying Theorem 2.2.9 to (2.2.32), since there are no excess normals if and only if (vii) holds, i.e. we have equimultiplicity, and so (vii) is equivalent to $Y\times L \pitchfork C(\underline{X},\underline{Y})$ by Theorem 2.2.5. The geometric content of this is that transversality is equivalent to growth conditions on the coordinates of X along directions normal to Y, and this is the geometric interpretation of the fact that a (minimal) reduction is characterized by integral dependence.

Further, it is now easy to see that we have, using Theorem 2.2.5:

(2.2.38) $\underline{X}$ is equimultiple along $\underline{Y}$ at y if and only if for all $L \in P^d_g(\underline{X},y)$ we have $L \pitchfork \nu^{-1}(y) \Longleftrightarrow L_z \pitchfork \nu^{-1}(z)$ for all $z\in Y$ near y outside some nowhere dense analytic subset.

Since normal pseudoflatness holds outside a nowhere dense analytic set, so that we can apply Theorem 2.2.10 there, we get, putting together our achievments, the following theorem.

Theorem 2.2.2 (cont.) Let $Y \overset{i}{\hookrightarrow} X \hookrightarrow U$ be as in Theorem 2.2.2, and let the ideal $I \subseteq \mathcal{O}_X$ define i. The following statements are equivalent to the statements of Theorem 2.2.2.

(viii) ("Principle of specialization of (minimal) reduction"). Let $J \subseteq I$. Then J_y is a (minimal) reduction of I_y if and only if J_z is a (minimal) reduction of I_z for all $z \in Y$ near y outside a nowhere dense analytic set in Y.

(ix) ("Principle of specialization of integral dependence"; cf.[69], Chapter I, 5.1). Let $f \in \mathcal{O}_X(X)$. Then $f_y \in \overline{I}_y$ if and only if $f_z \in \overline{I}_z$ for all $z \in Y$ near y outside a nowhere dense analytic set in Y.

The discussion of (ix) is similar to that of (viii) by embedding $X \hookrightarrow \mathbb{C}^n$ in such a way that f is a coordinate on K. One can also show (viii) $\Longleftrightarrow$ (ix) directly.

§ 3. Geometric equimultiplicity along a general subspace.

If a complex space X has the same multiplicity along a smooth subspace Y, the results of the last paragraph show that this numerical condition gives control over the blowup $\pi : \tilde{X} \longrightarrow X$ of X along Y to the extent that $\pi|D : D \longrightarrow Y$ is equidimensional, where $D \hookrightarrow \tilde{X}$ is the exceptional divisor (which is the same as saying that X is normally pseudoflat along Y). This is no longer so when Y becomes singular, and it turns out that the "naive" equimultiplicity condition above has to be replaced by a more refined equimultiplicity condition in order to guarantee normal pseudoflatness. The algebraic formulation of this result is Theorem (20.5) of Chapter IV, and it is the purpose of this paragraph to survey the geometric significance of these and related results in that case.

In general, these two notions of equimultiplicty are not related. To visualize this, I give in the first section a short description of the geometric significance of the first one, a result due to Lipman. In the subsequent section I give a somewhat more detailed description of the geometric meaning of the refined equimultiplicity condition and various other equivalent geometric and algebraic conditions, including normal pseudoflatness. These are the appropriate analogues of the smooth case, formulated in Theorem 2.2.2 above, and correspond to the algebraic results of Theorem 3 in [78] . I also describe the relation with the reduction of ideals and integral dependence. The main difference to the smooth case is that one has to replace the tangent cones by the normal cones to possibly nonreduced one-point-subspaces induced in $\underline{X}$ along $\underline{Y}$ by a suitable projection, and to change the multiplicities accordingly. These are also local mapping degrees.

The underlying geometric principle is again that the local mapping degree of a projection measures the order of contact of the kernel of this projection with the spacegerm on which it is defined. Hence, the equimultiplicity condition of a space along a subspace controls the intersection behaviour of the family of this projection centres along the subspace with the space under consideration and so represents a transverality condition on the normal cone. The algebraic notion corresponding to transversality is that of the reduction of an ideal (or integral dependence), and so it is not surprising that the Theorem of Rees-Böger is fundamental to equimultiplicity considerations and contains, in a sense, the essence of it; I have made some comments on this at the end.

3.1. Zariski-equimultiplicity

The following result shows that the geometric description of Zariski-equimultiplicity in Theorem 2.2.2 (ii) can be maintained. It will, however, no longer control the dimension of the normal cone fibres, which makes this notion therefore not very interesting for the study of the blowup along a nonsmooth centre. The main reason for this is that along a general subspace the tangent cones to the ambient space are not related to the fibres of the normal cone and to the normal cones of a transverse plane section, which was the case in the smooth situation.

For the definition of Zariski-equimultiplicity see Remark 2.1.8.

Theorem 3.1.1 (Geometric analysis of Zariski-equimultiplicity; [49], Proposition (4.3)). Let $(\underline{X},y) \hookrightarrow (\mathbb{C}^n,0)$ be an equidimensional spacegerm of dimension d, $(\underline{Y},y) \hookrightarrow (\underline{X},y)$ a complex subspacegerm. The following statements are equivalent.

(i) $\underline{X}$ is Zariski-equimultiple along $\underline{Y}$ at y .

(ii) There is $L \in \mathrm{Grass}^d(\mathbb{C}^n)$ and a neighbourhood V of y in $\underline{X}$ such that $L_z \cap V = \{z\}$ and $L_z \in P_e^d(\underline{X},z))$ for all $z \in V \cap Y$.

(iii) For all $L \in P_e^d(\underline{X},y)$ there is a neighbourhood V of y in $\underline{X}$ such that $L_z \cap V = \{z\}$ and $L_z \in P_e^d(\underline{X},z)$ for all $z \in V \cap Y$.

Proof. For $L \in P_e^d(\underline{X},y)$, let $\underline{h} := \underline{p}_L : (\underline{X},y) \longrightarrow (\underline{E},0)$ be the projection along L to a d-dimensional plane $E \subseteq \mathbb{C}^n$ complementary to L . We have

(3.1.1)
$$\begin{array}{ccc} \deg_y \underline{h} = \sum\limits_{z' \in h^{-1}h(z)} \deg_z \underline{h} & \geq & \deg_z \underline{h} \\ \vee\!| & & \vee\!| \\ m(\underline{X},y) & & m(\underline{X},z) \end{array}$$

for z near y on Y .

(i) ⇒ (iii) If $L \in P_e^d(\underline{X},y)$, (3.1.1) implies $h^{-1}h(z) = \{z\}$ and $\deg_z \underline{h} = m(\underline{X},z)$ for z near y on Y . Then $L_z \in P_e^d(\underline{X},z)$ by the geometric form of the Theorem of Rees, Remark 2.2.6,1).

(iii) ⇒ (ii) This is obvious.

(ii) ⇒ (i) By (3.1.1), $m(\underline{X},z) = \deg_z \underline{h} = \deg_y \underline{h}$ for z near y on Y .

3.2. Normal pseudoflatness.

As mentioned before, if we have $(\underline{Y},y) \hookrightarrow (\underline{X},y)$, the tangent cone $\underline{C}(\underline{X},y)$ will in general not be related to the fibre $\underline{\nu}^{-1}(\underline{y})$ of the normal cone $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$, and so it cannot be expected that its dimension is controlled by the multiplicity of X along Y near y .

Recall that the geometric analysis of equimultiplicity along a smooth subspace in 2.2. depended heavily on the use of a finite projection, $\underline{h}$. It turns out that the correct cones which to replace the tangent cones with are the normal cones $\underline{C}(\underline{X},\underline{Y})$, where $\underline{Y} \hookrightarrow \underline{X}$ is the one-point-space defined in $\underline{X}$ be the primary ideal of $\mathcal{O}_{X,y}$ generated via the finite projection, and that the correct multiplicities are the sums of the multiplicities corresponding to these cones in the fibres of the projection restricted to Y. This will be described now. Since the results are a natural generalization of the smooth case, which has been exposed in detail in § 2, arguments are only sketched, or omitted. They describe the geometric content of Theorem 3, in [78].

<u>Definition 3.2.1. Let</u> $(\underline{X},x) \in \underline{\underline{cpl}}_0$, $\dim_x X =: d$, $L' \in \mathrm{Grass}^k(\mathbb{C}^n)$ <u>such that</u> x <u>is isolated in</u> $X \cap L'$ <u>with</u> $d \leq k \leq n$, <u>and</u> $q' := p^0_{L'}(m_k) \subseteq m_x$ <u>the</u> m_x<u>-primary ideal generated via the projection</u> $\underline{p}_{L'}(\underline{X},x) \longrightarrow (\underline{\mathbb{C}}^k,0)$ <u>along</u> L'. <u>Let</u> $\underline{x} \hookrightarrow \underline{X}$ <u>be the one-point complex spacegerm defined by</u> q'.

(i) $$P^d_g(\underline{X},\underline{x}) := \{L \in P^d_g(\underline{X},x) \mid L \supseteq L'\}$$

$$P^d_e(\underline{X},\underline{x}) := \{L \in P^d_g(\underline{X},\underline{x}) \mid L \pitchfork C(\underline{X},\underline{x})\}$$

$$= P^d_g(C(\underline{X},\underline{x}),\underline{x}) \quad ,$$

<u>where</u> $\underline{C}(\underline{X},\underline{x})$ <u>is the normal cone of</u> $\underline{x} \hookrightarrow \underline{X}$. (These are both generic subspaces of the grassmannian of d-codimensional planes in $\mathbb{C}^n$ containing L'.)

(ii) $$m(\underline{X},\underline{x}) := \min\{\deg_x \underline{p}_L \mid L \in P^d_g(\underline{X},\underline{x})\} \quad .$$

In generalization of II, Theorem 5.2.1, one has

<u>Proposition 3.2.2. Let the notation be as in Definition 3.2.1; in particular,</u> L', <u>or</u> q', <u>is fixed</u>.

(i) $\deg_x \underline{p}_L \geq e(q',\mathcal{O}_{X,x})$ <u>for all</u> $L \in {}^d_g(\underline{X},\underline{x})$.

(ii) (Theorem of Rees). <u>If</u> $L \in P^d_e(\underline{X},\underline{x})$, $\deg_x \underline{p}_L = e(q',\mathcal{O}_{X,x})$. <u>Conversely, if</u> $(\underline{X},x)$ <u>is equidimensional and</u> $\deg_x \underline{p}_L = e(q',\mathcal{O}_{X,x})$, <u>then</u> $L \in P^d_e(\underline{X},x)$.

<u>Notation 3.2.3</u>. We consider $(\underline{Y},y) \hookrightarrow (\underline{X},y) \hookrightarrow (\underline{\mathbb{C}}^n,0)$, $\dim_y Y =: f$, and $(\underline{X},x)$ equidimensional of dimension d . We assume the conventions (2.2.1) (i),(ii), and (iii) made at the beginning of 2.2; so we assume $\underline{Y} \hookrightarrow \underline{X} \hookrightarrow \underline{U}$ with U a domain in $\mathbb{C}^n$, and $\underline{Y} = \underline{X} \cap \underline{G}$, where G is an m-codimensional plane in $\mathbb{C}^n$ such that $\underline{Y} = \underline{X} \cap \underline{G}$, called a <u>generating plane for</u> $\underline{Y}$. Let $I \subseteq \mathcal{O}_X$ define $\underline{Y} \hookrightarrow \underline{X}$. Further, let $K \in P_g^f(\underline{Y},y)$ (cf. (2.2.4)). We let the coordinates on $\mathbb{C}^n$ be chosen in such a way that K is given by $z_1 = \dots = z_f = 0$ and G by $z_{f+1} = \dots = z_{f+m} = 0$. Let $L' := G \cap K$. Then L' $\mathrm{Grass}^{f+m}(\mathbb{C}^n)$. The projection along L' defines a finite map $\underline{h}' : \underline{X} \longrightarrow \underline{\mathbb{C}}^{f+m}$, and we will use the multiplicities induced by h' in $\underline{X}$ along $\underline{Y}$ to control the fibres of the normal cone (see Figure 11).

For this, put $\underline{y} := \underline{y}(K) := \underline{Y} \cap \underline{K} = (\underline{h}')^{-1}(0)$; the multiplicity in question is $m(\underline{X},\underline{y})$, the behaviour of which along $\underline{Y}$ is relevant for normal pseudoflatness. One has $m(\underline{X},\underline{y}) = e(I_y(\underline{x}),\mathcal{O}_{X,y})$, $\underline{x} := (z_1,\dots,z_f)$ the set of parameters of $\mathcal{O}_{X,y}$ defining K (cf. Chap.I, (3.6)). Put

$$(3.2.1) \qquad \underline{h}_K : \underline{X} \longrightarrow \underline{F}$$

to be the projection along K , where $\underline{F} = \underline{\mathbb{C}}^f \times \underline{0} \hookrightarrow \underline{\mathbb{C}}^n$. We get the commutative diagram

$$(3.2.2) \qquad \begin{array}{ccc} \underline{Y} & \hookrightarrow & \underline{X} \\ & \underline{h}' \searrow \ \circlearrowleft \ \swarrow \underline{h}_K & \\ & \underline{F} & \end{array}$$

and, for $z \in F$ near y , $(\underline{h}')^{-1}(z) = \underline{Y} \cap \underline{K}_z$. The behaviour of $m(\underline{X},\underline{y})$ along $\underline{Y}$ is as follows.

<u>Proposition 3.2.4</u>. <u>Put, for</u> $z \in F$ <u>near</u> y ,

$$(3.2.3.) \qquad m(\underline{X},\underline{Y} \cap \underline{K}_z) := \sum_{z' \in Y \cap K_z} m(\underline{X},\underline{z}') \quad .$$

<u>Then</u>

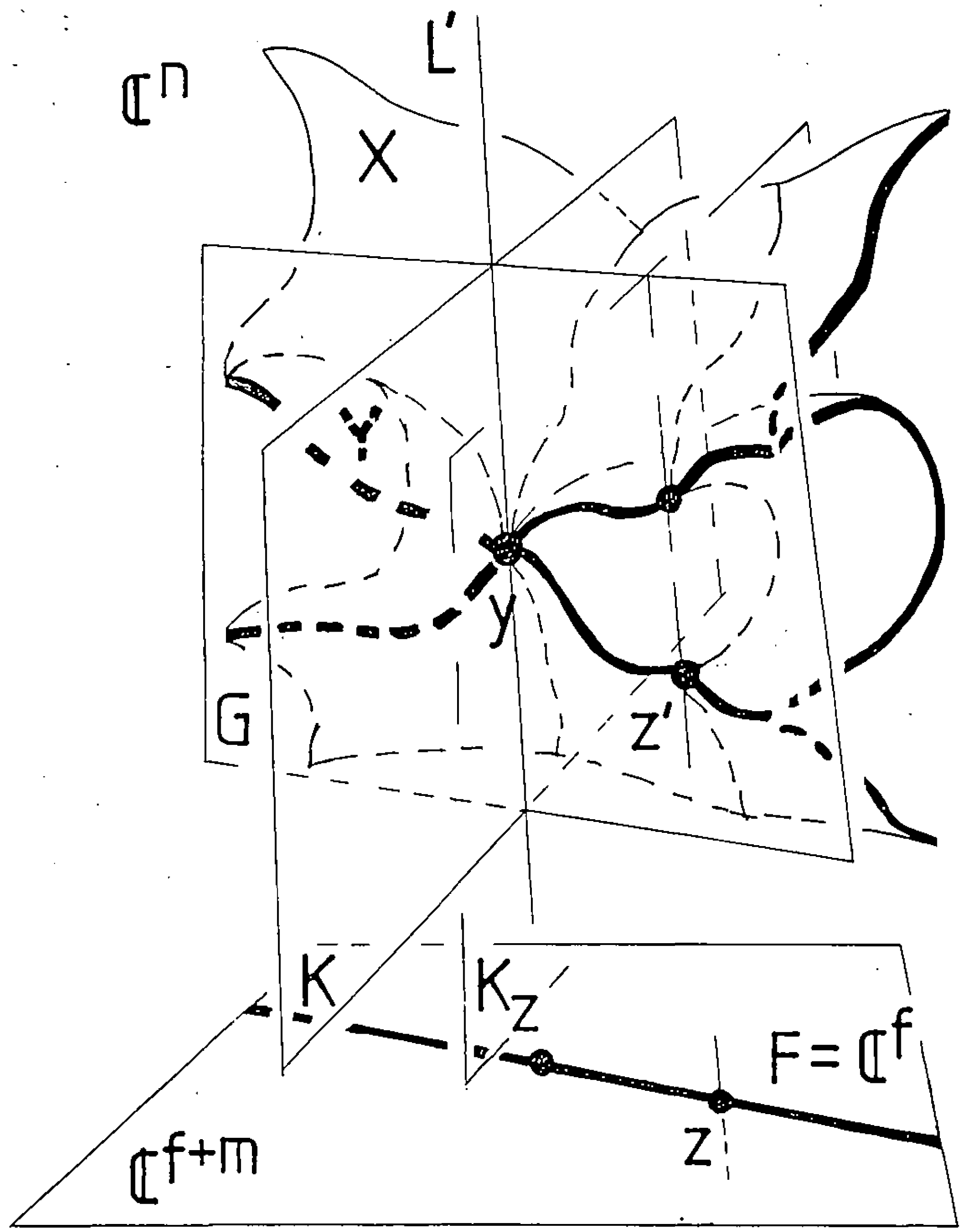


Figure 11

(i) $m(\underline{X},\underline{Y}\cap K_z) \leq m(\underline{X},\underline{y})$ for all z near y, and has a constant value for z near y outside a nowhere dense analytic subset $A \subseteq Y$, denoted $m(\underline{X},\underline{Y},\underline{K})$.

(ii) If $I \subseteq \mathcal{O}_X$ defines $\underline{Y} \hookrightarrow \underline{X}$,

$$m(\underline{X},\underline{Y},\underline{K}) = e(\underline{x},I_y,\mathcal{O}_{X,y}),$$

where $\underline{x}$ is the set $(z_1,\ldots,z_f)$ of parameters of $\mathcal{O}_{X,y}$ defining K, and $e(\underline{x},I_y,\mathcal{O}_{X,y})$ is the generalized multiplicity of Chapter I, (3.9).

The proof is similar to the proof of Theorem 2.1.2 ; one considers the admissible graded $\mathcal{O}_F$-algebra $G((\underline{h}')_* I,\mathcal{O}_F) = \bigoplus_{k\geq 0} (\underline{h}')_*(I^k/I^{k+1})$ and uses the fact that normal flatness is generic, i.e. Theorem 1.4.8. See [54]

This leads to the following definition:

Definition 3.2.5. Let $y \in \underline{Y} \hookrightarrow \underline{X} \hookrightarrow \underline{U}$ be as in 3.2.3. If $K \in P_g^f(\underline{Y},y)$, X is said to be K-equimultiple along $\underline{Y}$ at y if and only if the function $z \longmapsto m(\underline{X},\underline{Y}\cap \underline{K}_z)$ is constant for all $z \in F$ near y.

For equimultiplicity considerations, one wants to proceed as in the smooth case and choose an $L \in P_e^d(\underline{X},\underline{y})$ with $L \subseteq K$, in order to use the local mapping degree of the projection $\underline{h} =: \underline{h}_{K,L} : \underline{X} \longrightarrow \underline{E} := \mathbb{C}^d \times \underline{0}$ along $\underline{Y}$ to compare the various $m(\underline{X},\underline{Y}\cap \underline{K}_z)$. For this, one may show there is an open neighbourhood V of K in $\mathrm{Grass}^f(\mathbb{C}^n)$ such that $m(\underline{X},\underline{Y}\cap \underline{K}'_z)$ does not depend on K' for $K' \in V$ and z near y (this are grassmannian arguments similar to those employed in II, 4.1.). So, since $P_{es}^f(\underline{X},y)$ is generic in $\mathrm{Grass}^f(\mathbb{C}^n)$, we may replace K with some $K' \in P_{es}^f(\underline{Y},y)$ without affecting $m(\underline{X},\underline{Y}\cap \underline{K}_z)$ (this is the geometric content of (20.3) and (20.4) in Chapter IV). So we may always assume, for questions concerning $m(\underline{X},\underline{Y}\cap \underline{K}_z)$, that $K \in P_{es}^f(\underline{Y},y)$.

Then $C(\underline{X},\underline{y}) \cap K = C(\underline{X}\cap \underline{K},\underline{y})$, and the set $P_e^{d-f}(\underline{X}\cap \underline{K},\underline{y};K)$ $:= \{L \in \mathrm{Grass}^{d-f}(K) \mid L' \subseteq L$ and $L \pitchfork C(\underline{X},\underline{x})\}$ is generic in $\mathrm{Grass}^{d-f}(K)$, so we can always choose an $L \in P_e^{d-f}(\underline{X}\cap \underline{K},\underline{y};K)$. Then $L \in P_e^d(\underline{X},\underline{y})$, and if $\underline{h} : \underline{X} \longrightarrow \underline{E} := \mathbb{C}^d \times \underline{0}$ is the projection along L, there is the fundamental chain of inequalities for $z \in F$ near y :

$$(3.2.4) \qquad m(\underline{X},\underline{y}) = \deg_y \underline{h} = \sum_{z' \in h^{-1}(z)} \deg_{z'} \underline{h} \overset{(1)}{\geq} \sum_{z' \in (h')^{-1}(z)} \deg_{z'} \underline{h}$$

$$\overset{(2)}{\geq} \sum_{z' \in (h')^{-1}(z)} m(\underline{X},\underline{z}') = m(\underline{X},\underline{Y} \cap \underline{K}_z) \quad .$$

The inequality (1) holds because $(h')^{-1}(z) \subseteq h^{-1}(z)$, and (2) holds because $\deg_{z'} \underline{h} = e(\mathfrak{q},\mathcal{O}_{X,z'}) \geq e(\mathfrak{q}',\mathcal{O}_{X,z'}) = \deg_{z'} \underline{h}' =: m(\underline{X},z')$, where $\mathfrak{q}' \supseteq \mathfrak{q}$ are the primary ideals induced by $\underline{h}'$ and $\underline{h}$ from the maximal ideal of $\mathcal{O}_{E,z}$.

The various aspects of K-equimultiplicity of $\underline{X}$ along $\underline{Y}$ at y are now summarized in the following theorem.

Theorem 3.2.6 (Geometric analysis of equimultiplicity). Let $y \in \underline{Y} \hookrightarrow \underline{X} \hookrightarrow \underline{U} \hookrightarrow \mathbb{C}^n$ be as described in 3.2.3. Let $K \in P^f_{es}(\underline{Y},y)$. The following conditions are equivalent:

(i) $\underline{X}$ is K-equimultiple along Y at y .

(ii) There is $L \in \mathrm{Grass}^d(\mathbb{C}^n)$ and a neighbourhood V of y in X such that, for all $z \in \mathbb{C}^f \times \{0\} \cap V$, $V \cap L_z = Y \cap K_z$ and $L_{z'} \in P^d_e(\underline{X},\underline{z}')$ for all $z' \in Y \cap K_z$.

(iii) For all $L \in P^d_e(\underline{X},\underline{y})$ there is a neighbourhood V of y in X such that $V \cap L_z = Y \cap K_z$ for all $z \in \mathbb{C}^f \times \{0\} \cap V$.

(iv) $\underline{X}$ is normally pseudoflat along $\underline{Y}$ at y , i.e. $\dim \nu^{-1}(y) = d - f$, where $\underline{\nu} : \underline{C}(\underline{X},\underline{Y}) \longrightarrow \underline{Y}$ is the normal cone.

(v) There is $L \in \mathrm{Grass}^d(\mathbb{C}\ )$ such that $(G+L) \cap C(\underline{X},\underline{Y}) = Y$.

(vi) $C(\underline{X},\underline{y}) = \nu^{-1}(y) \times (\mathbb{C}^f \cap \{0\})$.

(vii) $C(\underline{X} \cap \underline{K},\underline{y}) = \nu^{-1}(y)$.

(viii) ("Principle of specialization of minimal reduction"). Let $I \subseteq \mathcal{O}_X$ define $\underline{Y} \hookrightarrow \underline{X}$. Let $J \subseteq I$. Then J_y is a (minimal) reduction of I_y if and only if J_z is a (minimal) reduction of I_z for all $z \in Y$ near y outside a nowhere dense analytic set.

(ix) ("Principle of specialization of integral dependence", cf.[69],). Let $f \in \mathcal{O}_X(X)$. Then $f_y \in \overline{I_y}$ if and only if $f_z \in \overline{I_z}$ for all $z \in Y$ near y outside a nowhere dense analytic set.

If one of these conditions holds, (i) and (vii) hold for all $K \in P_g^f(\underline{Y},y)$.

The implications (iv) ⇒ (iii) ⇒ (i) follow , analogously to 2.2, by blowing up $\underline{\mathbb{C}}^n$ along $\underline{G}$ and using (3.2.4). (i) ⟺ (ii) follows from (3.2.4) and the Theorem of Rees (Proposition 3.2.2. (ii)). I do not know of a geometric proof of (i) ⇒ (iv), but (i) ⟺ (iv) follows from the corresponding algebraic result (ii)⟺(ii) of Theorem 3 in [78]. in view of Proposition 3.2.4 (cf. [54]), and it is a useful exercise to visualize the proof of that Theorem geometrically using the geometric form of Böger's Theorem below. The equivalence (iv) ⟺ (vi) ⟺ (vii) also follows from Theorem 3 of [78]. The implications (vi) ⇒ (vii) ⇒ (iv) are also direct geometrically. The equivalence (v) ⟺ (viii) can be treated as in the smooth case, and (viii) ⟺ (ix) is left to the reader. One may also derive the equivalence (iv) ⟺ (viii) as a direct consequence of Böger's Theorem:

Theorem 3.2.7 (Theorem of Böger). Let $(\underline{Y},y) \hookrightarrow (\underline{X},y) \hookrightarrow (\underline{\mathbb{C}}^n,0)$ be as in Theorem 3.2.6. Let $L \in P_g^d(\underline{X},\underline{y})$, and $K \in P_g^f(\underline{Y},y)$ containing L . Let $G \in \mathrm{Grass}^m(\mathbb{C}^n)$ be such that $\underline{Y} = \underline{X} \cap \underline{G}$, and let $\underline{h} : \underline{X} \to \underline{\mathbb{C}}^d$ be the projection along L . The following statements are equivalent.

(i) $\deg_z \underline{h} = m(\underline{X},\underline{Y} \cap \underline{K}_z)$ for all z near Y outside a nowhere dense analytic set, and $(G+L) \cap X = Y$ near Y .

(ii) $G+L$ intersects X transversally along Y , i.e. $(G+L) \cap C(\underline{X},\underline{Y}) = Y$.

Exercise 3.2.8 (i) Derive this theorem from Böger's Theorem (cf. Theorem 1 in [78]) and show the equivalence (i) ⟺ (v) of Theorem 3.2.6.

We end our survey of Theorem 3.2.6 by establishing (iv) $\Longleftrightarrow$ (viii). The implication (viii) $\Rightarrow$ (iv) is left to the reader. For (iv) $\Rightarrow$ (viii), the "only if" statement is obvious, because I is locally finitely generated at y. For the "if"-statement, let $J \subseteq I$ be a minimal reduction, $J = (g_1,\dots,g_\ell)\cdot\mathcal{O}_X$. We may assume $\underline{X}$ is so embedded in $\mathbb{C}^n$ that $(g_1,\dots,g_\ell) = (z_{f+1},\dots,z_d)$. The assumptions then imply that condition (i) of Theorem 3.2.7 holds, and the conclusion follows from (ii) of the theorem.

An interpretation of this is that the content of Böger's Theorem, beyond the content of Rees' Theorem, is essentially the statement of the principle of specialization of integral dependence. This is also apparent from the proof of (19.6) in Chapter III.

Finally, as an application of Theorem 3.2.6 we mention the following geometric variant of proof of the result Theorem (b) of [79].

<u>Theorem. Let</u> $(\underline{X},y) \hookrightarrow (\mathbb{C}^n,0)$ <u>be in</u> $\underline{\text{cpl}}_0$, $(\underline{Y},y) \hookrightarrow (\underline{X},y)$ <u>a complex spacegerm, and let the notation be as in 3.2.3. Let</u> $K \in P_g^f(\underline{Y},y)$ <u>and suppose</u> $\underline{X}$ <u>is</u> K-<u>equimultiple along</u> $\underline{Y}$ <u>at</u> y. <u>Let</u> $\underline{\pi} : \tilde{\underline{X}} \longrightarrow \underline{X}$ <u>be the blowup of</u> $\underline{X}$ <u>along</u> $\underline{Y}$ <u>and let</u> $\tilde{y} \in \pi^{-1}(y)$. <u>Then</u>

$$m(\tilde{\underline{X}},\tilde{y}) \le m(\underline{X},\underline{Y},\underline{K}) \quad .$$

<u>Idea of proof</u>. If $(\underline{C},0) \hookrightarrow (\underline{\mathbb{C}}^p,0)$ is a cone, $m(\underline{C},c) \le m(\underline{C},0)$ for all $c \in C$ by the Degree Formula II 2.2.8. Now let the line $\ell \subseteq C(\underline{X},\underline{Y})$ correspond to $\tilde{y} \in \pi^{-1}(y)$ and let $\xi \in \ell - \{0\}$. By Theorem 3.2.6 (vii) we may assume $\xi \in \nu^{-1}(y)$. We have the chain of inequalities:

$$\begin{aligned} m(\tilde{\underline{X}},\tilde{y}) &\le m(\underline{\pi}^{-1}(\underline{y}),\tilde{y}) = m(\underline{C}(\underline{X},\underline{Y}),\xi) \\ &\le m(\underline{\nu}^{-1}(\underline{y}),y) = m(\underline{X},y) \le m(\underline{X},\underline{y}) \\ &= m(\underline{X},\underline{Y},\underline{K}) \quad , \end{aligned}$$

which proves the claim.

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