# Hypersurfaces of constant special Lagrangian curvature in quasi-Fuchsian hyperbolic ends

Graham Smith

8 April 2008

Max Planck Institute for Mathematics,
Vivatsgasse 7.,
53111 Bonn,
GERMANY

**Abstract:** We show the existence of foliations of open subsets of quasi-Fuchsian hyperbolic ends by immersed hypersurfaces of constant special Lagrangian curvature, and we establish conditions for these foliations to cover the whole manifold.

**Key Words:** special Lagrangian, immersions, foliations, hyperbolic ends, quasi-Fuchsian

**AMS Subject Classification:** 30F40 (35J60, 53C21, 53C42, 58J05)

# 1 - Introduction.

The special Lagrangian equation has been of growing interest since the landmark paper [6] of Harvey and Lawson concerning calibrated geometries. In its classical form, the special Lagrangian operator is a second order, highly non-linear partial differential operator of determinant type. Explicitly:

$$SL(f) = \arctan(\mathrm{Hess}(f)) = \mathrm{Arg}(\mathrm{Det}(\mathrm{Id} + i\mathrm{Hess}(f))).$$

This operator is closely related to the Monge-Ampère operator, which is among the archetypical highly non-linear partial differential operators studied in detail in most standard works on nonlinear PDEs ([1] and [2] to name but two). Recent progress was made in the study of the special Lagrangian operator with the publication of a Bernstein Theorem ([8] and [20]) for convex solutions to the special Lagrangian equation. This result forms a counterpart to the more classical Bernstein Theorem ([7], [3], [15]) of Jörgens, Calabi and Pogorelov for the Monge-Ampère operator (the theorem being proven respectively in 2, 3 and 4 and then all higher dimensions). In [17], the author showed how the Bernstein theorem for solutions to the special Lagrangian equation may be transplanted to the differential geometric setting, yielding a compactness result for special Lagrangian (and Legendrian) submanifolds which are positive in a sense first described in [19] by Smoczyk.

Since it is defined in terms of an invariant function on the space of symmetric matrices, the special Lagrangian operator may be used to define a notion of curvature for immersed hypersurfaces. Indeed, if $M$ is an $(n+1)$ dimensional Riemannian manifold, if $\Sigma = (S, i)$ is an immersed hypersurface in $M$, and if $A$ is the shape operator of $\Sigma$, then we define $SL_A^r$ by:

$$SL_A^r = \arctan(rA).$$

In fact, it is more useful to turn this on its head and define the special Lagrangian curvature in terms of an implicit function derived from this function. We say that $\Sigma$ is strictly convex if and only if $A$ is positive definite. In this case, we consider the following function:

$$SL_A : \mathbb{R}^+ \to ]0, n\pi/2[; \qquad r \mapsto \theta = \arctan(rA).$$

This function is smooth, surjective and strictly increasing in $r$. It is thus invertible, and the inverse function depends smoothly on $A$ and $\theta$. We thus define a smooth function $\rho_\theta$ over the space of positive definite, symmetric matrices such that:

$$\arctan(\rho_\theta(A) \cdot A) = \theta.$$

We call $\rho_\theta(A)$ the $\theta$-special Lagrangian curvature of $\Sigma$. This notion of curvature generalises more classical notions, and has an especially simple form when $\theta$ is a half integer multiple of $\pi$. Indeed, when $n = 2$ and $\theta = \pi/2$, the square of the special Lagrangian curvature is the reciprocal of the Gauss (extrinsic) curvature, and when $n = 3$ and $\theta = \pi$, the square of the special Lagrangian curvature is the mean curvature divided by the Gauss curvature, and so on. In fact, continuing the analogy with the Monge-Ampère operator, the special Lagrangian curvature is merely the counterpart to the Gauss curvature.

In [17] we also showed how the compactness result that we obtained for special Legendrian submanifolds may be adapted to yield a slightly weaker compactness result for convex hypersurfaces of constant special Lagrangian curvature. This compactness result may then be applied to yield existence results such as the main theorems of this paper, which treat foliations of quasi-Fuchsian manifolds and hyperbolic ends.

Let $N$ be a compact, hyperbolic manifold of dimension $n$. The universal cover of $N$ is $\mathbb{H}^n$. Let $\pi_1(N)$ be the fundamental group of $N$. This may be identified with a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{PSO}(n,1)$ such that:

$$N \cong \mathbb{H}^n/\pi_1(N).$$

Let $i : \mathbb{H}^n \to \mathbb{H}^{n+1}$ be a totally geodesic embedding of $\mathbb{H}^n$ into $\mathbb{H}^{n+1}$. There exists a unique homomorphism $\theta_0 : \mathrm{Isom}(\mathbb{H}^n) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ with respect to which $i$ is equivariant. The image of $\pi_1(N)$ under $\theta_0$ acts properly discontinuously on $\mathbb{H}^{n+1}$. We say that a homomorphism $\theta : \pi_1(N) \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is Fuchsian if it quotients through $\theta_0$. Let $\tilde{N}$ be the quotient of $\mathbb{H}^{n+1}$ under the action of $\theta(\pi_1(N))$. Since $i$ is equivariant under the action of $\theta_0$, it quotients down to a totally geodesic immersion from $N$ into $\tilde{N}$:

$$i : N \to \tilde{N}.$$

We refer to the pair $(\tilde{N}, i)$ as the extension of $N$. We say that an $(n+1)$-dimensional hyperbolic manifold is Fuchsian when it is the extension of a compact $n$-dimensional hyperbolic manifold. Let $\mathrm{End}(N)$ provisionally denote the closure of one of the connected components of $\tilde{N} \setminus N$. This is a hyperbolic manifold with smooth, totally geodesic boundary and is the archetypical model for a hyperbolic end.

A quasi-Fuchsian hyperbolic end as an $(n+1)$-dimensional hyperbolic manifold with concave, pleated boundary $(\hat{M}, \partial\hat{M})$ which is isotopic to $(\mathrm{End}(N), N)$ for some hyperbolic $N$ (a precise definition is given in Sections 3 and 4). As a special case, a quasi-Fuchsian manifold is a complete $(n+1)$-dimensional hyperbolic manifold which is isotopic to the extension of a Fuchsian manifold (see section 5). Let $M$ be a quasi-Fuchsian manifold. We recall in section 5 that $M$ contains a canonical, compact, convex set which we call the Nielsen kernel of $M$ and denote by $K$. In the Fuchsian case, $K$ is merely the copy of $N$ embedded in $\tilde{N}$. The complement of $K$ in $M$ consists of two non-compact connected components homeomorphic to $N \times \mathbb{R}$. Each of these components is then a quasi-Fuchsian hyperbolic end.

Much is known about three dimensional quasi-Fuchsian manifolds. In particular, the Ahlfors/Bers isomorphism (cf. [14]) parametrises the moduli space of such manifolds by the Cartesian product of two copies of Teichmüller space. This notion may be generalised to higher dimensions in at least two directions. In the first case, one may continue to study quasi-Fuchsian manifolds as the quotients of small deformations of Fuchsian representations of the fundamental groups of compact hyperbolic manifolds (cf. [9]). This approach is complicated in many higher dimensional cases by rigidity. Alternatively, one may remove one half of the quasi-Fuchsian manifold, and deform the remaining hyperbolic end. In particular, the space of hyperbolic ends with pleated boundaries may be naturally identified with the space of conformal structures over a compact manifold (cf. [11]).

In this paper, we prove first the following result, which partly generalises to higher dimensions the result [13] of Labourie:

**Theorem 1.1**

*Let $M$ be a quasi-Fuchsian manifold, and let $K$ be the Nielsen kernel of $M$. Let $\Omega$ be a connected component of $\hat{M} \setminus K$. Then, for all $\theta \in ](n-1)\pi/2, n\pi/2[$, there exists a unique foliation $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty[}$ of $\Omega$ consisting of convex immersed hypersurfaces such that:*

*(i) for all $r$, $\Sigma_{r,\theta}$ is a convex graph,*

*(ii) for all $r$, $\Sigma_{r,\theta}$ is of constant $\theta$-special Lagrangian curvature equal to $r$,*

*(iii) $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty[}$ tends to $\partial K$ in the Hausdorff sense as $r$ tends to $+\infty$, and*

*(iv) $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty[}$ tends to $\partial_\infty\Omega$ in the Hausdorff sense as $r$ tends to $\tan(\theta/n)$.*

The notion of a convex graph is explained in Section 10.

Using analogous methods, we then obtain the following, slightly weaker, result in the more general setting of quasi-Fuchsian hyperbolic ends:

**Theorem 1.2**

*Let $M$ be a quasi-Fuchsian hyperbolic end. Let $K$ be the boundary of $M$. There exists a unique open subset $\Omega \subseteq M \setminus K$ and a unique foliation $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty}$ of $\Omega$ consisting of convex immersed hypersurfaces such that:*

*(i) for all $r$, $\Sigma_{r,\theta}$ is a convex graph,*

*(ii) for all $r$, $\Sigma_{r,\theta}$ is of constant $\theta$-special Lagrangian curvature equal to $r$, and*

*(iii) $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty}$ tends to $K$ in the Hausdorff sense as $r$ tends to $+\infty$.*

*Moreover, if the minimal dimension of pleats in $K$ is at least $n/2$, or if the complement of the image of the developing map of the $\mathrm{PSO}(n+1,1)$ structure associated to $M$ has non-trivial interior, then:*

*(iv) $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty[}$ tends to $\partial_\infty\Omega$ in the Hausdorff sense as $r$ tends to $\tan(\theta/n)$.*

It is shown in Section 4 how a $\mathrm{PSO}(n+1,1)$ structure is associated to a hyperbolic end.

*Remark:* We observe in passing that these results allow us to construct a large family of non-trivial special Legendrian submanifolds in the unitary bundles of quasi-Fuchsian manifolds and quasi-Fuchsian hyperbolic ends.

These results may naturally be placed in a larger context. They are both proven in three stages. The first two stages comprise Lemma 7.3, which is a compactness result, and Lemma 9.1, which shows that an equivariant immersion may be smoothly, equivariantly deformed to follow any smooth deformation of the homomorphism with respect to which it is equivariant. We work with these lemmata inside $\mathbb{H}^n$, and they are both valid for all equivariant immersions of constant special Lagrangian curvature. It is only in the third stage where the geometric properties specific to quasi-Fuchsian manifolds and quasi-Fuchsian hyperbolic ends are finally used to localise the immersions in space, and thus allow us to apply the continuity method, which is well known in the theory of PDEs. It follows that similar results may also be obtained in any other situation where the geometric properties of the ambient spaces allow us to localise the immersions in an analogous manner. One potential application of these techniques could be an alternative approach to the study of $\mathrm{PSL}(2,\mathbb{C})$ structures carried out in [5] by Gallo, Kapovich and Marden.

These results provoke a number of questions, of which the two following are perhaps the most interesting:

(i) Can Theorem 1.2 be completed, either by showing that the foliation is complete for all quasi-Fuchsian hyperbolic ends or by proving the existence of hyperbolic ends whose corresponding foliations are not complete?

(ii) In the spirit of [13] and [10], can these foliations be used to study the geometric properties of the moduli spaces of different types of ambient hyperbolic manifolds?

The paper is arranged as follows:

(a) In Section 2, we show how the result is trivial in the Fuchsian case.

(b) In Sections 3, 4 and 5, we describe the ambient spaces in which we will be working. We thus define hyperbolic ends, pleated immersions, flat conformal structures and hyperbolic manifolds.

(c) In Section 6, we introduce the notion of Cheeger/Gromov convergence for manifolds and immersed submanifolds. This is used in Section 7 to present the compactness result that forms one of the main motors of this paper.

(d) In Section 8, we study the functional analytic properties of the special Lagrangian curvature operator. We use these properties in Section 9 to obtain the local deformation result that constitutes the second main motor of this paper.

(e) In Sections 10, 11, 12 and 13, we derive the consequences of the geometric properties of the specific ambient manifolds that we are studying. These allow us to obtain in Section 14 a stronger compactness result, which then allows us to prove Theorems 1.1 and 1.2 in Section 15.

I would like to thank Kirill Krasnov, François Labourie and Jean-Marc Schlenker for encouraging me to study this problem, and, in the case of the last two, for useful suggestions and observations on earlier drafts of this paper. I am also grateful to Werner Ballmann for a helpful conversation concerning hyperbolic manifolds, as well as Ursula Hamenstaedt for drawing attention to an error in an earlier version of this paper. Finally, I would like to thank the Max Planck Institutes for Mathematics in Bonn and for Mathematics in the Sciences in Leipzig for providing the conditions allowing me to carry out this work.

# 2 - The Fuchsian Case.

We construct the foliation in the Fuchsian case. Let $N$ be a compact manifold of dimension $n$ and of sectional curvature equal to $-1$, and let $(M, i)$ be the extension of $N$. Let $\Omega$ be a connected component of $M \setminus N$, and let $\mathsf{N}$ be the unit normal vector field over $(N, i)$ pointing into $\Omega$. For all $t \in \mathbb{R}^+$, we define $i_t : N \to M$ by:

$$i_t(x) = \mathrm{Exp}(t\mathsf{N}(x)),$$

where Exp is the exponential mapping of $\Omega$. We make the following observations:

(i) The family $(i_t)_{t\in\mathbb{R}^+}$ defines a smooth foliation of $\Omega$. Moreover, if we denote by $N_t$ the image of $N$ under $i_t$, then $N_t$ tends to $N$ and $\partial_\infty\Omega$ as $t$ tends to 0 and $+\infty$ respectively.

(ii) By Lemma 8.1, the $\theta$-special Lagrangian curvature of $(N, i_t)$ is constant and satisfies:

$$\rho_\theta(i_t) = \tan(\theta/n)/\tanh(t).$$

It follows that the $\theta$-special Lagrangian curvature takes values between $\tan(\theta/n)$ and $+\infty$. Moreover $\rho_\theta(i_t)$ tends to $+\infty$ and $\tan(\theta/n)$ as $t$ tends to 0 and $+\infty$ respectively.

Theorems 1.1 and 1.2 are thus trivially true in the case of Fuchsian manifolds.

# 3 - Hyperbolic Ends.

Here we define hyperbolic ends and describe the topology of their moduli spaces.

For all $m$, let $\mathbb{H}^{m+1}$ be $(m+1)$-dimensional hyperbolic space. Let $U\mathbb{H}^{m+1}$ be the unitary bundle over $\mathbb{H}^{m+1}$. Let $K$ be a convex subset of $\mathbb{H}^{m+1}$. We define $\mathcal{N}(K)$, the set of normals over $K$ by:

$$\mathcal{N}(K) = \left\{v_x \in U\mathbb{H}^{m+1} \text{ s.t. } x \in \partial K \text{ and } v_x \text{ is a supporting normal to } K \text{ at } x.\right\}$$

$\mathcal{N}(K)$ is a $C^1$ submanifold of $U\mathbb{H}^{m+1}$. Let $\Omega$ be an open subset of $\mathcal{N}(K)$. We define $\mathcal{E}(\Omega)$, the end over $\Omega$ by:

$$\mathcal{E}(\Omega) = \left\{\mathrm{Exp}(tv_x) \text{ s.t. } t \geqslant 0, v_x \in \Omega\right\}.$$

We say that a subset of $\mathbb{H}^{m+1}$ has concave boundary if and only if it is the end of some open subset of the set of normals of a convex set.

We extend the concept of convex boundary to more general manifolds. Let $(M, \partial M)$ be a smooth manifold with continuous boundary. A hyperbolic end over $M$ is an atlas $\mathcal{A}$ such that:

(i) every chart of $\mathcal{A}$ has convex boundary, and

(ii) the transition maps of $\mathcal{A}$ are isometries of $\mathbb{H}^{m+1}$.

We can construct hyperbolic ends using continuous maps into $U\mathbb{H}^{m+1}$. Let $M$ be an $m$-dimensional manifold without boundary. Let $i : M \to U\mathbb{H}^{m+1}$ be a continuous map. We say that $i$ is a convex immersion if and only if for every $p$ in $M$, there exists a neighbourhood $\Omega$ of $p$ in $M$ and a convex subset $K \subseteq \mathbb{H}^{m+1}$ such that the restriction of $i$ to $\Omega$ is a homeomorphism onto an open subset of $\mathcal{N}(K)$. In this case, we define the mapping $I : M \times [0, \infty[ \to \mathbb{H}^{m+1}$ by:

$$I(p, t) = \mathrm{Exp}(ti(p)).$$

$I$ is a local homeomorphism from $M \times ]0, \infty[$ into $\mathbb{H}^{m+1}$. If $g$ is the hyperbolic metric over $\mathbb{H}^{m+1}$, then $I^*g$ defines a hyperbolic metric over this interior. $I^*g$ degenerates over the boundary, and we identify points that may be joined by curves of zero length. We denote this identity by $\sim$ and we define $\mathcal{E}(i)$, the end of $i$ by:

$$\mathcal{E}(i) = (M \times ]0, \infty[) \cup (M/\sim).$$

Trivially, every hyperbolic end homeomorphic to $(M \times [0, +\infty[, M)$ may be constructed in this manner. Thus, if $\hat{M}$ is an end, and if $i : M \to U\mathbb{H}^{m+1}$ is a convex immersion such that $\hat{M} = \mathcal{E}(i)$, then we say that $i$ is the boundary immersion of $\hat{M}$.

Let $\Gamma$ be a discrete subgroup of the isometry group of $M$. We denote by $\mathrm{End}(M, \Gamma)$ the space of pairs $(i, \alpha)$ where:

(i) $\alpha : \Gamma \to \mathrm{Isom}(\mathbb{H}^{m+1})$ is a representation of $\Gamma$ in $\mathrm{Isom}(\mathbb{H}^{m+1})$ and

(ii) $i : M \to U\mathbb{H}^{m+1}$ is a proper, convex immersion from $M$ into $U\mathbb{H}^{m+1}$ which is equivariant with respect to $\alpha$.

The set $\mathrm{End}(M, \Gamma)$ is thus a subset of the set of continuous maps from $M \cup \Gamma$ into $\mathbb{H}^{m+1} \cup \mathrm{Isom}(\mathbb{H}^{m+1})$. We furnish this set with the compact/open topology, which is to say, the topology of local uniform convergence.

Suppose that $\Gamma$ acts properly discontinously on $M$. We may trivially extend the action of $\Gamma$ over $M$ to an action over $\mathcal{E}(i)$. By taking quotients, we see that every element of $\mathrm{End}(M, \Gamma)$ defines a hyperbolic end over $M/\Gamma \times [0, \infty[$ which we denote by $\mathcal{E}(i, \alpha)$. We identify elements of $\mathrm{End}(M, \Gamma)$ with the corresponding hyperbolic ends. We observe that, although $i$ is only $C^1$, we obtain a smooth hyperbolic structure over the interior of $M/\Gamma \times [0, \infty[$.

Let $M$ be an $m$ dimensional compact, hyperbolic manifold. We identify the universal cover of $M$ with $\mathbb{H}^m$ and we identify the fundamental group of $M$ with a discrete subgroup $\Gamma$ of $\mathrm{Isom}(\mathbb{H}^m)$. Let $i_0 : \mathbb{H}^m \to \mathbb{H}^{m+1}$ be the canonical immersion and let $\alpha_0 : \Gamma \to \mathrm{Isom}(\mathbb{H}^{m+1})$ be the canonical homomorphism. We call $\mathcal{E}(i_0, \alpha_0)$ the Fuchsian end of $M$. We define quasi-Fuchsian ends in the following section.

# 4 - Pleated Immersions and Flat Conformal Structures.

Here we define pleated immersions and show how they may be derived from conformal structures. This allows us to define quasi-Fuchsian hyperbolic ends.

We define a pleat to be the convex hull in $\mathbb{H}^{m+1}$ of a subset of $\partial_\infty \mathbb{H}^{m+1}$ containing at least two distinct points. Trivially, if $K$ is the convex hull of a subset of $\partial_\infty \mathbb{H}^{m+1}$, then $\partial K$ is a union of pleats.

Let $i : M \to U\mathbb{H}^{m+1}$ be a convex immersion. We say that $i$ is pleated if and only if it is proper, and, for all $p$ in $M$, there exists a neighbourhood $\Omega$ of $p$ in $M$, a convex subset $K \subseteq \mathbb{H}^{m+1}$ and an open subset $U$ of $\mathcal{N}(K)$ such that:

(i) $\partial K$ is the convex hull of a subset of $\partial_\infty \mathbb{H}^{m+1}$,

(ii) $U$ is a union of inverse images of pleats, and

(iii) the restriction of $i$ to $\Omega$ is a homeomorphism onto $U$.

We denote by $\mathrm{Pleat}(M,\Gamma)$ the subset of pairs $(i,\alpha)$ in $\mathrm{End}(M,\Gamma)$ where $i$ is pleated.

Let $M$ be an $m$ dimensional manifold and let $i : M \to U\mathbb{H}^{m+1}$ be a convex immersion. We define $\varphi_i$ by:

$$\varphi_i = \overrightarrow{n} \circ i.$$

We call $\varphi_i$ the developing map of $i$. This mapping is a local homeomorphism from $M$ into $\partial_\infty \mathbb{H}^{m+1}$ and thus defines a $\mathrm{PSO}(m+1,1)$ structure over $M$. Likewise (see [11]), if $\varphi$ is the developing map of a $\mathrm{PSO}(m+1,1)$ structure over $M$, then there exists a unique convex immersion $i$ such that:

(i) $i$ is pleated, and

(ii) $\varphi$ is the developing map of $i$.

We thus obtain an equivalence between developing maps of $\mathrm{PSO}(m+1,1)$ structures over $M$ and pleated, convex immersions, which we identify with hyperbolic ends with concave, pleated boundaries.

Let $\Gamma$ be a discrete subgroup of the group of isometries of $M$. We define $\mathrm{Dev}(M,\Gamma)$ to be the set of all pairs $(\varphi,\alpha)$, where:

(i) $\alpha : \Gamma \to \mathrm{Isom}(\mathbb{H}^{m+1})$ is a representation of $\Gamma$ in $\mathrm{Isom}(\mathbb{H}^{m+1})$, and

(ii) $\varphi : M \to \partial_\infty \mathbb{H}^{m+1}$ is a local homeomorphism which is equivariant with respect to $\alpha$.

The set $\mathrm{Dev}(M,\Gamma)$ is thus a subset of the set of continuous maps from $M \cup \Gamma$ into $\partial_\infty \mathbb{H}^{m+1} \cup \mathrm{Isom}(\mathbb{H}^{m+1})$. We thus furnish it with the compact/open topology, which is to say, the topology of local uniform convergence.

The construction [11] of Kulkarni and Pinkall defines a bijective mapping $\Phi : \mathrm{Dev}(M,\Gamma) \to \mathrm{Pleat}(M,\Gamma)$, which sends a developable $\mathrm{PSO}(m+1,1)$ structure to its associated pleated, convex immersion. However, since this construction depends on the global properties of the developing map, it is not necessarily a homeomorphism. Indeed, in the case where $\Gamma$ is trivial, it is easy to construct a continuous family of $\mathrm{PSO}(m+1,1)$ structures whose associated pleated, convex immersions do not form a continuous family.

Nonetheless, we can obtain results concerning continuity. Let $g$ be any metric over $\partial_\infty \mathbb{H}^{m+1} \cong S^{m+1}$ which is compatible with the conformal structure. Let $\varphi : M \to \partial_\infty \mathbb{H}^{m+1}$ be an equivariant local homeomorphism (i.e. a developing map) from $M$ into $\partial_\infty \mathbb{H}^{m+1}$. Following [11], we define $\overline{M}_\varphi$, the Möbius completion of $M$, to be the metric completion of $M$ with respect to $\varphi^* g$. Since any two metrics over $S^{m+1}$ are uniformly equivalent, this definition is independant of the choice of $g$ when viewed as a topological space, although not when viewed as a metric space. We thus fix $g$ in the sequel. The mapping $\varphi$ may then be uniquely extended to a continuous mapping from $\overline{M}_\varphi$ to $\partial_\infty\mathbb{H}^{m+1}$. We have the following result:

**Lemma 4.1**

*Let $(\varphi_n, \alpha_n)_{n\in\mathbb{N}}$ and $(\varphi_0, \alpha_n)$ be elements of $\mathrm{Dev}(M, \Gamma)$. Let $p$ be any point of $M$. Suppose that:*

*(i) $(\overline{M}_{\varphi_n}, p)_{n\in\mathbb{N}}$ converges to $(\overline{M}_{\varphi_0}, p)$ in the pointed Gromov/Hausdorff sense, and*

*(ii) $(\varphi_n)_{n\in\mathbb{N}}$ converges locally uniformly to $\varphi_0$ when these are viewed as mappings from $(\overline{M}_{\varphi_n})_{n\in\mathbb{N}}$ and $\overline{M}_{\varphi_0}$ into $\partial_\infty\mathbb{H}^{m+1}$ respectively.*

*The sequence $(\Phi(\varphi_n, \alpha_n))_{n\in\mathbb{N}}$ converges to $\Phi(\varphi_0, \alpha_0)$ in $\mathrm{Pleat}(M, F)$.*

**Proof:** The hypotheses of this lemma ensure that the maximal balls defined in [11] converge. It is these balls that are used to construct a pleated, convex immersion out of a developing map. $(\Phi(\varphi_n, \alpha)_n)$ thus converges, and the result now follows. $\square$

As before, let $M$ be an $m$-dimensional, compact, hyperbolic manifold, and identify the universal cover of $M$ with $\mathbb{H}^m$ and the fundamental group of $M$ with a discrete subgroup $\Gamma$ of $\mathrm{Isom}(\mathbb{H}^m) = \mathrm{PSO}(m, 1)$. Let $i_0 : \mathbb{H}^m \to \mathbb{H}^{m+1}$ be the canonical immersion and let $\alpha_0 : \mathrm{PSO}(m, 1) \to \mathrm{PSO}(m + 1, 1)$ be the unique homomorphism with respect to which $i_0$ is equivariant. Let $\mathsf{N}_0 : \mathbb{H}^m \to U\mathbb{H}^{m+1}$ be the exterior unit normal over $i_0$. We define $\varphi_0 : \mathbb{H}^m \to \partial_\infty\mathbb{H}^{m+1}$ by:

$$\varphi_0 = \overrightarrow{n} \circ \mathsf{N}_0.$$

The mapping $\varphi_0$ is equivariant with respect to $\alpha_0$ and defines the developing map of a $\mathrm{PSO}(m+1, 1)$ structure over $\mathbb{H}^m$ which quotients down to a $\mathrm{PSO}(m+1, 1)$ structure over $M$. We call this $\mathrm{PSO}(m + 1, 1)$ structure over $\mathbb{H}^m$ the Fuchsian structure of $\mathbb{H}^m$ and also of $M$.

Let $(\varphi_0, \alpha_0)$ be an element of $\mathrm{Dev}(\mathbb{H}^m, \Gamma)$. We say that this $\mathrm{PSO}(m + 1, 1)$ structure is quasi-Fuchsian if and only if there exists a continuous curve $\gamma : [0, 1] \to \mathrm{Dev}(\mathbb{H}^m, \Gamma)$ joining $(\varphi, \alpha)$ to the Fuchsian structure, $(\varphi_0, \alpha_0)$, such that $\Phi \circ \gamma$ is also a continuous curve in $\mathrm{Pleat}(\mathbb{H}^m, \Gamma)$. We say that an element $(i, \alpha)$ of $\mathrm{Pleat}(\mathbb{H}^m, \Gamma)$ is quasi-Fuchsian if and only if it is the image of a Fuchsian $\mathrm{PSO}(m + 1, 1)$ structure. In particular, we observe that a quasi-Fuchsian hyperbolic end defined in this manner always has a pleated boundary.

# 5 - Quasi-Fuchsian Manifolds.

An interesting special case of quasi-Fuchsian hyperbolic ends is that of quasi-Fuchsian manifolds. As before, for all $m$, let $\mathbb{H}^m$ be $m$-dimensional hyperbolic space. We identify $\mathrm{Isom}(\mathbb{H}^m)$ with $\mathrm{PSO}(m, 1)$. Let $M$ be a compact $n$-dimensional, hyperbolic manifold. We view $\pi_1(M)$ as a subgroup $\Gamma$ of $\mathrm{Isom}(\mathbb{H}^n)$.

We denote by $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ the space of pairs $(\varphi, \alpha)$, where:

(i) $\alpha : \Gamma \to \mathrm{Isom}(\mathbb{H}^{n+1})$ is a properly discontinous representation of $\Gamma$ in $\mathrm{Isom}(\mathbb{H}^{n+1})$, and

(ii) $\varphi : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^{n+1}$ is an injective, continuous mapping which is equivariant with respect to $\alpha$.

The set $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ is a subset of the set of continuous mappings from $\partial_\infty \mathbb{H}^n \cup \Gamma$ into $\partial_\infty \mathbb{H}^{n+1} \cup \mathrm{Isom}(\mathbb{H}^{n+1})$. We thus furnish this set with the compact/open topology, which is to say, the topology of local uniform convergence..

For all $n$, $\mathbb{H}^n$ embeds totally geodesically into $\mathbb{H}^{n+1}$. This induces a homeomorphism $\alpha_0 : \mathrm{PSO}(n,1) \to \mathrm{PSO}(n+1,1)$ and an injective continuous mapping $\varphi_0 : \partial_\infty \mathbb{H}^n \to \partial_\infty \mathbb{H}^{n+1}$ which is equivariant with respect to $\alpha_0$. The connected component of $\mathrm{Rep}(\mathbb{H}^n, \Gamma)$ which contains $(\varphi_0, \alpha_0)$ is called the quasi-Fuchsian component. The pair $(\varphi, \alpha)$ is then said to be quasi-Fuchsian if and only if it belongs to the quasi-Fuchsian component.

Let $(\varphi, \alpha)$ be quasi-Fuchsian. Since $\alpha(\Gamma)$ is properly discontinuous, it defines a quotient manifold $\hat{M}_\alpha = \mathbb{H}^{n+1}/\alpha(\Gamma)$. In the sequel, we identify a quasi-Fuchsian pair and its quotient manifold, and we say that a manifold is quasi-Fuchsian if and only if it is the quotient manifold of a quasi-Fuchsian pair. In this case it may be isotoped to the extension of a compact, hyperbolic manifold.

When $n$ is equal to 2, the space of quasi-Fuchsian manifolds is well understood and is parametrised through the Ahlfors-Bers isomorphism by the Cartesian product of two copies of Teichmüller space (cf. [14]). In higher dimensions, much remains unknown about quasi-Fuchsian manifolds, although one established construction technique involves bending hyperbolic manifolds about totally geodesic hypersurfaces (cf. [9]).

Let $(\varphi, \alpha)$ be quasi-Fuchsian. The image of $\partial_\infty \mathbb{H}^n$ under the action of $\varphi$ is a higher dimensional Jordan curve, and thus, by the higher dimensional Jordan curve theorem (cf. [4]), it divides $\partial_\infty \mathbb{H}^{n+1}$ into two open, simply connected, connected components. The group $\alpha(\Gamma)$ acts properly discontinuously on each of these connected components. The quotient of each component is thus homeomorphic to $M$, and the union of these two quotients forms the ideal boundary of $\hat{M}_\alpha$.

Let $K$ be the convex hull in $\mathbb{H}^{n+1}$ of $\varphi(\partial_\infty \mathbb{H}^n)$. This is the intersection of all closed sets with totally geodesic boundary whose ideal boundary does not intersect $\varphi(\partial_\infty \mathbb{H}^n)$. This set is equivariant under the action of $\alpha$ and thus quotients down to a compact, convex subset of $\hat{M}_\alpha$ which we refer to as the Nielsen kernel of $\hat{M}_\alpha$ and which we also denote by $K$. We observe that the boundary of $K$ is a pleated hypersurface. Trivally $M \setminus K$ consists of two quasi-Fuchsian hyperbolic ends, and it follows that quasi-Fuchsian manifolds may be studied as a special case of the quasi-Fuchsian hyperbolic ends defined in the preceeding section.

# 6 - Immersed Submanifolds and the Cheeger/Gromov Topology.

Let $M$ be a smooth Riemannian manifold. An immersed submanifold is a pair $\Sigma = (S, i)$ where $S$ is a smooth manifold and $i : S \to M$ is a smooth immersion. A pointed immersed submanifold in $M$ is a pair $(\Sigma, p)$ where $\Sigma = (S, i)$ is an immersed submanifold in $M$ and $p$ is a point in $S$. An immersed hypersurface is an immersed submanifold of codimension 1. We give $S$ the unique Riemannian metric $i^*g$ which makes $i$ into an isometry. We say that $\Sigma$ is complete if and only if the Riemannian manifold $(S, i^*g)$ is.

A pointed Riemannian manifold is a pair $(M, p)$ where $M$ is a Riemannnian manifold and $p$ is a point in $M$. Let $(M_n, p_n)_{n\in\mathbb{N}}$ be a sequence of pointed Riemannian manifolds. For all $n$, we denote by $g_n$ the Riemannian metric over $M_n$. We say that the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ converges to the pointed manifold $(M_0, p_0)$ in the Cheeger/Gromov sense if and only if for all $n$, there exists a mapping $\varphi_n : (M_0, p_0) \to (M_n, p_n)$, such that, for every compact subset $K$ of $M_0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$:

(i) the restriction of $\varphi_n$ to $K$ is a $C^\infty$ diffeomorphism onto its image, and

(ii) if we denote by $g_0$ the Riemannian metric over $M_0$, then the sequence of metrics $(\varphi_n^* g_n)_{n\geqslant N}$ converges to $g_0$ in the $C^\infty$ topology over $K$.

We refer to the sequence $(\varphi_n)_{n\in\mathbb{N}}$ as a sequence of convergence mappings of the sequence $(M_n, p_n)_{n\in\mathbb{N}}$ with respect to the limit $(M_0, p_0)$. The convergence mappings are trivially not unique.

Let $(\Sigma_n, p_n)_{n\in\mathbb{N}} = (S_n, p_n, i_n)_{n\in\mathbb{N}}$ be a sequence of pointed immersed submanifolds in $M$. We say that $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ converges to $(\Sigma_0, p_0) = (S_0, p_0, i_0)$ in the Cheeger/Gromov sense if and only if the sequence $(S_n, p_n)_{n\in\mathbb{N}}$ of underlying manifolds converges to $(S_0, p_0)$ in the Cheeger/Gromov sense, and, for every sequence $(\varphi_n)_{n\in\mathbb{N}}$ of convergence mappings of $(S_n, p_n)_{n\in\mathbb{N}}$ with respect to this limit, and for every compact subset $K$ of $S_0$, the sequence of functions $(i_n \circ \varphi_n)_{n\geqslant N}$ converges to the function $(i_0 \circ \varphi_0)$ in the $C^\infty$ topology over $K$.

# 7 - Compactness.

Let $M$ be an oriented Riemannian manifold and let $UM$ be its unitary bundle. Let $\Sigma = (S, i)$ be an oriented, immersed hypersurface in $M$ and let $\mathsf{N} : S \to UM$ be the exterior normal vector field over $i$ in $M$. We define $\hat{\Sigma} = (S, \hat{\imath})$, the Gauss lifting of $\Sigma$, to be the immersed submanifold in $UM$ given by:

$$\hat{\Sigma} = (S, \hat{\imath}) = (S, \mathsf{N}).$$

In [17] we prove the following compactness result:

**Theorem 7.1**

*Let $M$ be a complete Riemannian manifold. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $M$ converging to $p_0$. Let $\theta \in ]0, n\pi/2[$ be an angle and let $r > \tan(\theta/n)$. For all $n$, let $(\Sigma_n, q_n) = (S_n, i_n, q_n)$ be a pointed immersed hypersurface such that:*

*(i) $i_n(q_n) = p_n$,*

*(ii) $\Sigma_n$ is convex and of constant $\theta$-special Lagrangian curvature equal to $r$, and*

*(iii) $\hat{\Sigma}_n$ is a complete submanifold of $UM$.*

*There exists a complete, pointed, immersed submanifold* $(\hat{\Sigma}_0, q_0) = (S_0, \hat{\imath}_0, q_0)$ *in* $UM$ *such that, after extraction of a subsequence,* $(\hat{\Sigma}_n, q_n)_{n\in\mathbb{N}}$ *converges to* $(\hat{\Sigma}_0, q_0)$ *in the pointed Cheeger/Gromov sense.*

*Moreover, if* $\theta$ *is not a half integer multiple of* $\pi$*, then there exists a convex, immersed hypersurface* $\Sigma_0$ *in* $M$ *of constant* $\theta$*-special Lagrangian curvature equal to* $r$ *such that* $\hat{\Sigma}_0$ *is the Gauss lifting of* $\Sigma_0$*. In other words, if* $\pi : UM \to M$ *is the canonical projection, then* $\pi \circ \hat{\imath}_0$ *is an immersion.*

This allows us to deduce the following corollary:

**Corollary 7.2**

*Let* $M$ *be a complete, hyperbolic manifold with injectivity radius bounded below by* $\epsilon > 0$*. Let* $\theta \in ]0, \pi/2[$ *be an angle that is not a half-integer multiple of* $\pi$*. For all* $r > 0$*, there exists* $B > 0$ *which only depends on* $\epsilon$*,* $\theta$ *and* $r$ *(and the dimension of* $M$*) such that, if* $\Sigma = (S, i)$ *is a complete, convex, immersed hypersurface of* $M$ *of* $\theta$*-special Lagrangian curvature equal to* $r$ *and if* $A$ *is the shape operator of* $\Sigma$*, then:*

*(i)* $\|A\| \leqslant B$*, and*

*(ii) the injectivity radius of* $\Sigma$ *is greater than* $1/B$*.*

**Proof:** We only prove $(i)$, since the proof of $(ii)$ is almost identical. We suppose the contrary and construct a sequence $(\Sigma_n, q_n) = (S_n, i_n, q_n)$ of complete, convex, immersed submanifolds of $M$ of $\theta$-special Lagrangian curvature equal to $r$ such that, if $A_n$ is the shape operator of $\Sigma_n$, then $(\|A_n(q_n)\|)_{n\in\mathbb{N}}$ tends to infinity. For all $n$, let $\hat{\Sigma}_n$ be the Gauss lifting of $\Sigma_n$ and let $p_n = \hat{\imath}_n(q_n)$. Trivially, $\hat{\Sigma}_n$ is complete. Since the injectivity radius of $M$ is bounded below by $\epsilon$, there exists $(M_0, p_0)$ such that, after taking a subsequence, $(M, p_n)_{n\in\mathbb{N}}$ converges to $(M_0, p_0)$ in the pointed Cheeger/Gromov sense. By Theorem 7.1, there exists a pointed immersed submanifold $(\hat{\Sigma}_0, q_0)$ in $UM_0$ such that $(\hat{\Sigma}_n, p_n)_{n\in\mathbb{N}}$ converges to $(\hat{\Sigma}_0, p_0)$ in the pointed Cheeger/Gromov sense. Since $\theta$ is not a half-integer multiple of $\pi$, there exists a convex, immersed submanifold, $\Sigma_0$ in $M_0$ such that $\hat{\Sigma}_0$ is the Gauss lifting of $\Sigma_0$. Trivially, $(\|A_n(p_n)\|)_{n\in\mathbb{N}}$ converges to the norm of the shape operator of $\Sigma_0$ at $p_0$. This is absurd and the second result thus follows. $\square$

*Remark:* This proof trivially also works in the much more general setting of Riemannian manifolds of bounded geometry.

We thus obtain compactness for the immersed hypersurfaces themselves rather than just their Gauss liftings:

**Lemma 7.3**

*Let* $M$ *be a complete, hyperbolic manifold. Let* $(p_n)_{n\in\mathbb{N}}$ *be a sequence in* $M$ *converging to* $p_0$*. Let* $\theta \in ]0, \pi/2[$ *be an angle that is not a half-integer multiple of* $\pi$*. Choose* $r > \tan(\theta/n)$ *and for all* $n$*, let* $(\Sigma_n, q_n) = (S_n, i_n, q_n)$ *be a pointed, complete, convex, immersed hypersurface of* $M$ *of constant* $\theta$*-special Lagrangian curvature equal to* $r$ *such that* $i_n(q_n) = p_n$*. There exists a pointed, complete, convex, immersed hypersurface* $(\Sigma_0, q_0)$ *of* $M$ *such that, after extraction of a subsequence,* $(\Sigma_n, p_n)_{n\in\mathbb{N}}$ *converges to* $(\Sigma_0, p_0)$ *in the pointed Cheeger/Gromov sense.*

**Proof:** For all $n$, let $\hat{\Sigma}_n$ be the Gauss lifting of $\Sigma_n$. By Theorem 7.1, there exists an immersed submanifold $(\hat{\Sigma}_0, q_0)$, such that, after extraction of a subsequence, $(\hat{\Sigma}_n, q_n)_{n\in\mathbb{N}}$ converges to $(\hat{\Sigma}_0, q_0)$ in the pointed Cheeger/Gromov sense. Moreover, since $\theta$ is not a half-integer multiple of $\pi$, there exists an immersed hypersurface $\Sigma_0$ in $M$ such that $\hat{\Sigma}_0$ is the Gauss lifting of $\Sigma_0$. Let $g$ and $\hat{g}$ be the metrics of $M$ and $UM$ respectively. Let $A_0$ be the shape operator of $\Sigma_0$. The matrix of $\hat{\imath}_0^*\hat{g}$ with respect to $i_0^*g$ is $\mathrm{Id} + A_0^2$. This is trivially bounded below by 1. By Corollary 7.2, it is also bounded above by $1 + B^2$. Thus $i_0^*g$ and $\hat{\imath}_0\hat{g}$ are uniformly equivalent. It follows that $\Sigma_0$ is complete and that $(\Sigma_n, q_n)_{n\in\mathbb{N}}$ converges to $(\Sigma_0, q_0)$ in the Cheeger/Gromov sense. The result now follows. □

# 8 - The Derivative of the SL-Curvature Operator.

Let $N$ and $M$ be Riemannian manifolds of dimensions $n$ and $(n+1)$ respectively. The special Lagrangian curvature operator sends the space of smooth immersions from $N$ into $M$ into the space of smooth functions over $N$. These spaces may be viewed as infinite dimensional manifolds (strictly speaking, they are the intersections of infinite sequences of Banach manifolds). Let $i$ be a smooth immersion from $N$ into $M$. Let $\mathsf{N}$ be the unit exterior normal vector field of $i$ in $M$. We identify the space of smooth functions over $N$ with the tangent space at $i$ of the space of smooth immersions from $N$ into $M$ as follows. Let $f : N \to \mathbb{R}$ be a smooth function. We define the family $(\Phi_t)_{t\in\mathbb{R}} : N \to M$ by:

$$\Phi_t(x) = \mathrm{Exp}(tf(x)\mathsf{N}(x)).$$

This defines a path in the space of smooth immersions from $N$ into $M$ such that $\Phi_0 = i$. It thus defines a tangent vector to this space at $i$. Every tangent vector to this space may be constructed in this manner.

Let $A$ be the shape operator. This sends the space of smooth immersions from $N$ into $M$ into the space of sections of the endomorphism bundle of $TN$. We have the following result:

**Lemma 8.1**

*Suppose that $M$ is of constant sectional curvature equal to $-1$, then the derivative of the shape operator at $i$ is given by:*

$$D_iA \cdot f = f\mathrm{Id} - \mathrm{Hess}(f) - fA^2,$$

*where $\mathrm{Hess}(f)$ is the Hessian of $f$ with respect to the Levi-Civita covariant derivative of the metric induced over $N$ by the immersion $i$.*

**Proof:** This is an elementary calculation. Details may be found in the proof of proposition 3.1.1 of [12]. □

For $r \in \mathbb{R}^+$, we consider the operator $SL_r$ given by:

$$SL_r(i) = \arctan(rA(i)).$$

Trivially, $SL_r(i)=\theta$ if and only if $\rho_\theta(i)=r$. Using Lemma 8.1, we obtain:

**Lemma 8.2**

*Suppose that $M$ is of constant sectional curvature equal to $-1$, then the derivative of $SL_r$ at $i$ is given by:*

$$(1/r)D_iSL_r=-\mathrm{Tr}((\mathrm{Id}+r^2A^2)^{-1}\mathrm{Hess}(f))+\mathrm{Tr}((\mathrm{Id}-A^2)(\mathrm{Id}+r^2A^2)^{-1})f.$$

This operator is trivially elliptic. We wish to establish when this operator is invertible. We first require the following technical result:

**Lemma 8.3**

*Let $0<n<m$ be positive integers. If $t\in]0,\pi/2]$, then:*

$$n\mathrm{sin}^2(t/n)\geqslant m\mathrm{sin}^2(t/m),$$

*With equality if and only if $n=1$, $m=2$ and $t=\pi/2$.*

**Proof:** The function $\sin^2(\pi t/2)$ is strictly convex over the interval $[0,\pi/4]$. Thus, for all $0<x<y\leqslant\pi/4$:

$$(1/x)\mathrm{sin}^2(x)<(1/y)\mathrm{sin}^2(y).$$

Thus, for $m>n\geqslant 2$, we obtain:

$$n\mathrm{sin}^2(t/n)>m\mathrm{sin}^2(t/m).$$

We treat the case $n=1$ separately. For $t\leqslant\pi/4$, the result follows as before. We therefore assume that $t>\pi/4$. Since the function $\sin^2(\pi t/2)$ is strictly concave over the interval $[\pi/4,\pi/2]$, it follows that $\sin^2(t)\geqslant 2t/\pi$, with equality if and only if $t=\pi/2$. However:

$$\mathrm{sin}^2(\pi/4)=1/2=(2/\pi)(\pi/4).$$

Since $m\geqslant 2$, it follows by concavity that:

$$m\mathrm{sin}^2(t/m)\leqslant\mathrm{sin}^2(t),$$

with equality if and only if $m=2$ and $t=\pi/2$. The result now follows. □

We now use Lagrange multipliers to determine critical points, and we obtain:

**Lemma 8.4**

*If $\theta\geqslant(n-1)\pi/2$ and $r>\tan(\theta/n)$, then the coefficient of the zeroth order term is non-negative:*

$$\mathrm{Tr}((\mathrm{Id}-A^2)(\mathrm{Id}+r^2A^2)^{-1})\geqslant 0.$$

*Moreover, this quantity reaches its minimum value of $0$ if and only if $r=\tan(\theta/n)$ and $A$ is proportional to the identity matrix.*

**Proof:** For all $m$, we define the functions $\Phi_m$ and $\Theta_m$ over $\mathbb{R}^m$ by:

$$\Phi_m(x_1,...,x_m) = \sum_{i+1}^m \frac{1-x_i^2}{1+r^2x_i^2}, \qquad \Theta_m(x_1,...,x_m) = \sum_{i=1}^m \arctan(rx_i).$$

Since the derivative of $\Theta_m$ never vanishes, $\Theta_m^{-1}(\theta)$ is a smooth submanifold of $\mathbb{R}^m$. Let $(\tilde{x}_1,...,\tilde{x}_m)$ be a critical point of the restriction of $\Phi_m$ to this submanifold. For all $i$, let $\tilde{\theta}_i \in [0,\pi/2[$ be such that:

$$\tan(\tilde{\theta}_i) = r\tilde{x}_i.$$

Using Lagrange multipliers, we find that there exists $\eta \in [0,\pi/2]$ such that, for all $i$:

$$\tilde{\theta}_i \in \{\eta, \pi/2-\eta\}.$$

Let $k$ be the number of values of $i$ such that $\tilde{\theta}_i > \pi/4$. Since $\theta \geqslant (n-1)\pi/2$:

$$k > m/2.$$

Choose $\eta > \pi/4$. Since $\tilde{\theta}_1 + ... + \tilde{\theta}_m = \theta$:

$$\eta = \frac{\theta - k\pi/2}{2k-m} = \frac{m(\theta/m) - 2k(\pi/4)}{m-2k}.$$

If $\tilde{\Phi}_m$ is the value acheived by $\Phi_m$ at this point, then:

$$\tilde{\Phi}_m = r^{-2}(1+r^2)(m-2k)\cos^2(\eta) + kr^{-2}(1+r^2) - mr^{-2}.$$

Since the function $\cos^2$ is concave in the interval $[\pi/4,\pi/2]$, we have:

$$\cos^2(\eta) \geqslant \frac{m\cos^2(\theta/m) - 2k\cos^2(\pi/4)}{m-2k},$$

with equality if and only if $k=0$. Thus:

$$\tilde{\Phi}_m \geqslant mr^{-2}(1+r^2)\cos^2(\theta/m) - mr^{-2},$$

with equality if and only if $\tilde{\theta}_1 = ... = \tilde{\theta}_m$. This is non-negative, and is equal to 0 if and only if $r = \tan(\theta/m)$.

We now show that $\Phi_m$ attains its minimum over $\Theta_m^{-1}(\theta)$. We treat first the case $\theta > (m-1)\pi/2$. The functions $\Phi_m$ and $\Theta_m$ extend to continuous functions over the cube $[0,+\infty]^m$. Let $(\tilde{x}_1,...,\tilde{x}_m)$ be the point in $\Theta_m^{-1}(\theta)$ where $\Phi_m$ is minimised, and suppose now that it lies on the boundary of the cube. Since $\theta > (m-1)\pi/2$, $\tilde{x}_i > 0$ for all $i$. Without loss of generality, there exists $n < m$ such that:

$$x_1,...,x_n < +\infty, \qquad x_{n+1},...,x_m = +\infty.$$

Let $(\tilde{\theta}_1, ..., \tilde{\theta}_m)$ be as before. We define $\theta'$ by:

$$\theta' = \tilde{\theta}_1 + ... + \tilde{\theta}_n.$$

Since $\tilde{\theta}_{n+1} = ... = \tilde{\theta}_m = \pi/2$, it follows that $\theta' = \theta - (m-n)\pi/2$. Moreover:

$$\Phi_m(x_1, ..., x_m) = \Phi_n(x_1, ..., x_n) - (m-n)r^{-2}.$$

Since $(\tilde{x}_1, ..., \tilde{x}_m)$ minimises $\Phi_m$ it follows that $(\tilde{x}_1, ..., \tilde{x}_n)$ is the minimal valued critical point of $\Phi_n$ in $\Theta_n^{-1}(\theta')$. Thus:

$$\Phi_m(x_1, ..., x_m) = nr^{-2}(1 + r^2)\cos^2(\theta'/n) - mr^{-2}.$$

Let $\eta \in ]0, \pi/2[$ be such that:

$$\theta = n\pi/2 - \eta.$$

We have:

$$n\cos^2(\theta'/n) = n\sin^2(\eta/n), \qquad m\cos^2(\theta/m) = m\sin^2(\eta/m).$$

It follows by Lemma 8.3 that:

$$\Phi_m(x_1, ..., x_m) > mr^{-2}(1 + r^2)\cos^2(\theta/m) - mr^{-2}.$$

It follows that $(\tilde{x}_1, ..., \tilde{x}_m)$ cannot be the minimum of $\Phi_m$ over $\Theta_m^{-1}(\theta)$, which is absurd. The result now follows in the case $\theta > (m-1)\pi/2$.

It remains to study the case $\theta = (m-1)\pi/2$. This follows as before, with the single exception that it is now possible that $\tilde{x}_1 = 0$, in which case $\tilde{x}_2 = ... = \tilde{x}_n = +\infty$. However:

$$\Phi_m(0, +\infty, ..., +\infty) = 1 - (m-1)r^{-2}.$$

Now, $r \geqslant \tan((m-1)\pi/2m)$. Thus, since $m \geqslant 2$:

$$r^{-1} \leqslant \tan(\pi/2m) \leqslant 2/m.$$

Thus:

$$\Phi_m(0, +\infty, ..., +\infty) \geqslant 1 - 4(m-1)/m^{-2} \geqslant 0,$$

The result now follows. $\square$

This yields the following Corollary:

**Corollary 8.5**

*If $\theta \geqslant (n-1)\pi/2$ and $r \geqslant \tan(\theta/n)$, then $D_i SL_r$ is invertible.*

**Proof:** This follows immediately from the preceeding lemma and the maximum principal. $\square$

Finally, we express this in terms of the derivative, $D_i\rho_\theta$, of the $\theta$-special Lagrangian curvature, $\rho_\theta$. We have the following result:

**Lemma 8.6**

*$D_i\rho_\theta$ is Fredholm. Moreover, if $\theta \geqslant (n-1)\pi/2$ and $r \geqslant \tan(\theta/n)$, then $D_i\rho_\theta$ is invertible.*

**Proof:** We trivially calculate $D_i\rho_\theta$ in terms of $D_iSL_r$. It follows by Lemma 8.2 that $D_i\rho_\theta$ is Fredholm. Invertibility follows from the preceeding corollary. $\square$

*Remark:* We therefore see that $D_i\rho_\theta$ is always invertible in exactly the cases that we wish to study.

# 9 - Deforming Equivariant Immersions.

The results of Section 8 permit us to locally deform equivariant immersions of $\mathbb{H}^n$ in $\mathbb{H}^{n+1}$. Let $\Gamma \subseteq \text{Isom}(\mathbb{H}^n)$ be a cocompact subgroup acting properly discontinuously on $\mathbb{H}^n$. Thus $\mathbb{H}^n/\Gamma$ is a compact manifold. Let $\alpha : \Gamma \rightarrow \text{Isom}(\mathbb{H}^{n+1})$ be a homomorphism. Let $\theta \in ]0, n\pi/2[$ be an angle and let $\rho \in \mathbb{R}^+$ be a positive real number. Suppose that there exists a convex immersion $i : \mathbb{H}^n \rightarrow \mathbb{H}^{n+1}$ of constant $\theta$-special Lagrangian curvature equal to $\rho$ which is equivariant with respect to $\theta$. Thus, for all $\gamma \in \Gamma$:

$$i \circ \gamma = \alpha(\gamma) \circ i.$$

We obtain the following local deformation result:

**Lemma 9.1**

*Let $(\alpha_t)_{t\in]-\epsilon,\epsilon[}$ be a smooth family of homomorphisms such that $\alpha_0 = \alpha$. If $\theta > (n-1)\pi/2$ and if $\rho \geqslant \tan(\theta/n)$, then there exists $0 < \delta < \epsilon$ and a unique smooth family of immersions $(i_t)_{t\in]-\delta,\delta[}$ such that $i_0 = i$ and, for all $t$:*

*(i) $\rho_\theta(i_t) = \rho$ and,*

*(ii) $i_t$ is equivariant with respect to $\alpha_t$.*

**Proof:** This proof may be divided into two stages:

(i) We approximate the desired family by constructing a smooth, equivariant family of deformations of $i$ which are not necessarily immersions, and not necessarily of constant $\theta$-special Lagrangian curvature. First we construct a fundamental domain for $\Gamma$. Let $p$ be a point in $\mathbb{H}^n$. Let $P \subseteq \mathbb{H}^n$ be the orbit of $p$ under the action of $\Gamma$. Thus:

$$P = \Gamma p.$$

We define $\Omega \subseteq \mathbb{H}^n$ to be the set of all points on $\mathbb{H}^n$ which are closer to $p$ than to any other point in the orbit of $p$:

$$\Omega = \{q \in \mathbb{H}^n \text{ s.t. } d(q,p) < d(q,p') \text{ for all } p' \in P \setminus \{p\}\}.$$

Trivially, $\Omega$ is a polyhedron and a fundemental domain for $\Gamma$.

Using $\Omega$, we now construct the family of deformations. For each $t$, we construct a (non-continuous) deformation be defining $i_t$ to be equal to $i$ over the interior of $\Omega$ and then extending this function to the orbit of $\Omega$ (which is almost all of $\mathbb{H}^n$) by equivariance with respect to $\alpha_t$. These deformations may trivially be smoothed along $\partial\Omega$. The only complication is that the smoothing must be performed in an equivariant manner. The following recipe allows us to achieve exactly this.

For any submanifold $X \in \mathbb{H}^n$ and for all $\epsilon > 0$, let $X^\epsilon$ be the set of all points in $X$ which are at a distance (in $X$) greater than $\epsilon$ to the boundary of $X$. That is:

$$X^\epsilon = \{p \in X \text{ s.t. } d_X(p, \partial X) > \epsilon\}.$$

Choose $\epsilon_n$ small. For all $\gamma \in \Gamma$, we define $(\tilde{\imath}_t^n)_{t\in]\epsilon,\epsilon[}$ over $\gamma\Omega^{\epsilon_n}$ by:

$$\tilde{\imath}_t^n(p) = \alpha_t(\gamma) i(\gamma^{-1}(p)).$$

This family is trivially equivariant with respect to $(\alpha_t)_{t\in]-\epsilon,\epsilon[}$.

Choose $\epsilon_{n-1}$ small. Let $F_{n-1}$ be any $(n-1)$-dimensional face of $\Omega$. We may trivially extend $(\tilde{\imath}_t^n)_{t\in]-\epsilon,\epsilon[}$ smoothly across a neighbourhood of $F_{n-1}^{\epsilon_{n-1}}$. Since every element of $\Gamma$ is of infinite order, there is no element which fixes any face of $\Omega$ (since otherwise it would permute the domains touching that face, and thus be of finite order). It follows that, by choosing $\epsilon_n$ and $\epsilon_{n-1}$ small enough, we may extend this family further to a smooth equivariant extension over every face in the orbit of $F_{n-1}$. We then continue extending this family over every face of $\Omega$ until all $(n-1)$-dimensional faces are exhausted. By working downwards inductively on the dimension of the faces, we thus obtain a smooth equivariant family $(\tilde{\imath}_t)_{t\in]-\epsilon,\epsilon[} = (\tilde{\imath}_t^0)_{t\in]-\epsilon,\epsilon[}$ which extends $i$.

(ii) We now modify this approximation to obtain the desired family of immersions. Since $\Omega$ is relatively compact, there exists $\delta < \epsilon$ such that, for $|t| < \delta$, $\tilde{\imath}_t$ is an immersion. Moreover, we may suppose that for $\eta > 0$ sufficiently small, we may extend $\tilde{\imath}_t$ smoothly along normal geodesics to a smooth equivariant immersion from $\mathbb{H}^n\times]-\eta,\eta[$ into $\mathbb{H}^{n+1}$. We thus view $(\tilde{\imath}_t)_{t\in]-\delta,\delta[}$ as a smooth family of immersions from $\mathbb{H}^n\times]-\eta,\eta[$ into $\mathbb{H}^{n+1}$.

We denote by $g$ the hyperbolic metric over $\mathbb{H}^{n+1}$. We define the family $(g_t)_{t\in]-\delta,\delta[}$ such that, for all $t$:

$$g_t = \tilde{\imath}_t^* g.$$

The action of $\Gamma$ over $\mathbb{H}^n$ trivially extends to an action of $\Gamma$ over $\mathbb{H}^n\times]-\eta,\eta[$. For all $t$, $g_t$ is equivariant under this action of $\Gamma$. We denote $M = \mathbb{H}^n/\Gamma$ and we obtain a smooth family, which we also call $(g_t)_{t\in]-\delta,\delta[}$, of hyperbolic metrics over $M\times]-\eta,\eta[$.

Let $j_0$ be the canonical immersion of $M$ into $M\times]-\eta,\eta[$. Trivially, with respect to $g_0$, $\rho_\theta(j_0) = \rho$. As in Section 8, we view $\rho_\theta$ as a second order, non-linear differential operator sending immersions of $M$ into $M\times]-\eta,\eta[$ into functions over $M$. Since infinitesimal variations of immersions may be interpreted as functions over $M$ times the normal vector field of $M$ in $M\times]-\eta,\eta[$, the derivative $D\rho_\theta$ of $\rho_\theta$ may be interpreted as a second order, linear differential operator from $C^\infty(M)$ into $C^\infty(M)$. By Lemma 8.6, the operator $D\rho_\theta$ is invertible. After reducing $\delta$ if necessary, the implicit function theorem for non-linear PDEs allows us to extend $j_0$ to a smooth family $(j_t)_{t\in]-\eta,\eta[}$ of immersions of $M$ into $M\times]-\eta,\eta[$ such that, for all $t$, the $\theta$-special Lagrangian curvature of $j_t$ with respect to $g_t$ equals $\rho$.

For all $t$, let $\tilde{\jmath}_t$ be the lift of $j_t$ from $\mathbb{H}^n$ into $\mathbb{H}^{n+1}$. We now define $i_t = \tilde{\imath}_t \circ \tilde{\jmath}_t$. Trivially, $(i_t)_{t\in]-\delta,\delta[}$ is the desired family of immersions, and existence follows.

Let $(i'_t)_{t\in]-\delta,\delta[}$ be another family of immersions having the desired properties. For $\delta$ sufficiently small, the image of $i'_t$ is contained in the image of $\tilde{\imath}_t$. For all $t$, we thus project $\tilde{\jmath}'_t = \tilde{\imath}_t \circ i'_t$ to an immersion $j'_t$ of $M$ into $M\times]-\eta,\eta[$. By the uniqueness part of the implicit function theorem for non-linear PDEs, for all sufficiently small $t$, $j'_t$ coincides with $j_t$. Uniqueness now follows by a standard open/closed argument. $\square$

# 10 - Graphs.

We aim to obtain uniform diameter bounds. We first make the following definition:

**Definition 10.1**

*Let $\hat{M} = M \times [0,\infty[$ be a quasi-Fuchsian hyperbolic end. Let $\Sigma = (S,i)$ be an immersed hypersurface in $\hat{M}$. We say that $\Sigma$ is a convex graph if and only if there exists $f : M \to ]0,\infty[$ such that:*

*(i) $\Sigma$ coincides with the graph of $f$, and*

*(ii) the set $\{(x,t) \text{ s.t. } x \in M \;\&\; t \leqslant f(x)\}$ is a convex subset of $\hat{M}$.*

*In this case, we define* $\mathrm{Int}(\Sigma)$ *by:*

$$\mathrm{Int}(\Sigma) = \{(x,t) \text{ s.t. } x \in M \;\&\; t \leqslant f(x)\},$$

*and we call* $\mathrm{Int}(\Sigma)$ *the interior of* $\Sigma$.

*Remark:* Observe that we insist that $f$ be strictly positive. This is because hyperbolic ends are singular at $M \times \{0\}$.

This condition is preserved by continuous deformation:

**Lemma 10.2**

*Let $(\hat{M}_n)_{n\in\mathbb{N}}, \hat{M}_0$ be quasi-Fuchsian hyperbolic ends such that $(\hat{M}_n)_{n\in\mathbb{N}}$ converges to $\hat{M}_0$. For all $n$, let $\Sigma_n = (S_n, i_n)$ be convex, compact immersed hypersurfaces in $\hat{M}_n$ such that $\Sigma_n$ converges to $\Sigma_0$. Then $\Sigma_0$ is a graph if and only if $\Sigma_n$ is for all sufficiently large $n$.*

**Proof:** For all $n \in \mathbb{N} \setminus \{0\}$, let $i_n : \mathbb{H}^m \to \mathbb{H}^{m+1}$ be the boundary immersion of $\hat{M}_n$. By definition $(i_n)_{n\in\mathbb{N}}$ converges locally uniformly to $i_0$. Throughout the rest of the proof, we identify hyperbolic ends and their submanifolds with the corresponding lifts in $\mathbb{H}^{m+1}$.

Suppose first that $\Sigma_0$ is a graph. Then it is transversal to the foliation of $\hat{M}_0 = M_0\times]0,+\infty[$ by vertical geodesics. It thus follows that $\Sigma_n$ is also a graph for all sufficiently large $n$.

Suppose that $\Sigma_n$ is a graph for all large $n$. We first show that $\Sigma_0$ does not intersect the boundary of $\hat{M}_0$. Indeed, suppose that it did, then, by continuity, $\partial\hat{M}_0$ is an interior tangent to $\Sigma_0$. However, this is not possible, since $\partial\hat{M}_0$ is pleated and $\Sigma_0$ is convex. It now remains to show that $\Sigma_0$ is a graph. We suppose the contrary, then there exists a point

$p \in \Sigma$ and a vertical geodesic $\gamma : [0, +\infty[ \rightarrow \hat{M}_0$ such that $\gamma$ is tangent to $\Sigma_0$ at $p = \gamma(t)$. We may assume that $t$ is the lowest such point. Thus, by continuity, $\gamma([0, t[) \subseteq \text{Int}(\Sigma_0)$. However, $\Sigma_0$ is convex, and $\gamma$ is thus an exterior tangent to $\Sigma_0$ at $p$. This is absurd, and the result now follows. $\square$

# 11 - The Geometric Maximum Principal.

The geometric maximum principal allows us to control the location of each leaf of the foliation. We have the following result concerning positive definite symmetric matrices:

**Lemma 11.1**

*Let $A$ be a positive definite symmetric matrix of rank $n$. If $0 < \lambda_1 \leqslant ... \leqslant \lambda_n$ are the eigenvalues of $A$ arranged in ascending order, then, for all $k$:*

$$\lambda_k = \mathop{\text{Inf}}_{\text{Dim}(E)=k} \mathop{\text{Sup}}_{v \in E \setminus \{0\}} \|Av\|/\|v\|.$$

**Proof:** Let $e_1, ..., e_n$ be the eigenvectors of $A$. We define $\hat{E}$ by:

$$\hat{E} = \langle e_1, ..., e_k \rangle.$$

Let $\pi$ be the orthogonal projection onto $\hat{E}$. Let $E$ be a subspace of $\mathbb{R}^n$ of dimension $k$. For all $v$ in $E$:

$$\|A\pi(v)\|^2 \cdot \|v\|^2 \leqslant \|Av\|^2 \cdot \|\pi(v)\|^2.$$

If the restriction of $\pi$ to $E$ is an isomorphism, then it follows that:

$$\lambda_k = \mathop{\text{Sup}}_{v \in \hat{E} \setminus \{0\}} \|Av\|/\|v\| \leqslant \mathop{\text{Sup}}_{v \in E \setminus \{0\}} \|Av\|/\|v\|.$$

Otherwise, there exists a non-trivial $v \in E$ such that $\pi(v) = 0$, in which case:

$$\|Av\| \geqslant \lambda_{k+1}\|v\| \geqslant \lambda_k\|v\|.$$

The result now follows. $\square$

This yields:

**Corollary 11.2**

*Let $A$,$A'$ be two symmetric, positive definite matrices of rank $n$ such that $A' \geqslant A$. If $\lambda_1, ..., \lambda_n$ and $\lambda'_1, ..., \lambda'_n$ are the eigenvalues of $A$ and $A'$ respectively arranged in ascending order, then, for all $k$:*

$$\lambda'_k \leqslant \lambda_k.$$

This result allows us to deduce a geometric maximum principal for hypersurfaces of constant special Lagrangian curvature:

**Lemma 11.3**

*Let $M$ be a Riemannian manifold and let $\Sigma = (S, i)$ and $\Sigma' = (S', i')$ be convex, immersed hypersurfaces in $M$. For $\theta \in ]0, n\pi/2[$, let $\rho_\theta$ and $\rho'_\theta$ be the $\theta$-special Lagrangian curvatures of $\Sigma$ and $\Sigma'$ respectively. If $p \in S$ and $p' \in S'$ are such that $q = i(p) = i'(p')$, and $\Sigma'$ is an interior tangent to $\Sigma$ at $q$, then:*

$$\rho_\theta(p) \geqslant \rho'_\theta(p').$$

**Proof:** If $A$ and $A'$ are the shape operators of $\Sigma$ and $\Sigma'$ respectively, then:

$$A'(p') \geqslant A(p).$$

It follows that:

$$\arctan(\rho(p)A'(p')) \geqslant \arctan(\rho(p)A(p)) = \theta = \arctan(\rho'(p')A'(p')).$$

The result now follows since the mapping $\rho \mapsto \arctan(\rho A'(p'))$ is strictly increasing. $\square$

# 12 - Upper and Lower Bounds.

Let $M$ be a quasi-Fuchsian hyperbolic end and let $K$ be its boundary. For all $d$, let $K_d$ be the hypersurface in $M$ at a distance $d$ from $K$. Let $\Sigma = (S, i)$ be a $C^0$ hypersurface in $M$. We say that $\Sigma$ is convex immersed if and only if for all $p \in S$, there exists an open, convex set $\Omega$ such that $i$ sends a neighbourhood of $p$ homeomorphically onto an open subset of $\partial\Omega$. We now make the following definition:

**Definition 12.1**

*Let $M$ be a manifold and let $\Sigma = (S, i)$ be a $C^0$ convex, immersed hypersurface in $M$. Let $p$ be a point in $S$, let $\theta \in ]0, n\pi/2[$ be an angle and let $r > \tan(\theta/n)$ be a positive real number. The $\theta$-special Lagrangian curvature of $\Sigma$ at $p$ is said to be at least (resp. at most) $r$ in the weak sense if and only if there exists a smooth, convex, immersed submanfold $\Sigma' = (S, i')$ of $\theta$-special Lagrangian curvature equal to $r$ such that $\Sigma'$ is an exterior (resp. interior) tangent to $\Sigma$ at $p$.*

The geometric maximum principal can trivially be generalised to incorporate the case of $C^0$ convex, immersed hypersurfaces. The following result allows us to obtain upper bounds:

**Lemma 12.2**

*Let $\theta \in ]0, n\pi/2[$ be an angle. For all $d > 0$, the $\theta$-special Lagrangian curvature of $K_d$ is at least $\tan(\theta/n)/\tanh(d)$ in the weak sense.*

**Proof:** Let $N$ be a hyperbolic manifold and let $i : \mathbb{H}^m \to \mathbb{H}^{m+1}$ an equivariant, convex immersion of the universal cover of $N$ such that $M$ is the end associated to $i$. Throughout the rest of the proof, we identify hypersurfaces in $M$ with the corresponding lifts in $\mathbb{H}^{m+1}$.

For all $d \geqslant 0$, we define $i_d : \mathbb{H}^m \to \mathbb{H}^{m+1}$ by:

$$i_{d(p)} = \mathrm{Exp}(di(p)).$$

Trivially $K_d = (\mathbb{H}^m, i_d)$. Let $q_0$ be a point in $\mathbb{H}^m$. Let $\pi : U\mathbb{H}^{m+1} \to \mathbb{H}^{m+1}$ be the canonical projection and let $P_0$ be the totally geodesic hypersurface normal to $(\pi \circ i)(p_0)$. Let $P_d$ be the immersed hypersurface at a distance $d$ from $P_0$. $K_d$ is trivially an interior tangent to $P_d$. by Lemma 8.1 the shape operator of $P_d$ is equal to $\tanh(d)\mathrm{Id}$. Its $\theta$-special Lagrangian curvature at this point is therefore equal to $\tan(\theta/n)/\tanh(d)$. The result now follows. □

Lower bounds are more involved. Let $M$ be a quasi-Fuchsian manifold, and let $K$ be its Nielsen kernel. Let $\Omega$ be one of the connected components of $M \setminus K$ and let $K'_0$ be the component of $\partial K$ which does not intersect $\Omega$ (i.e. $K'_0$ is the boundary component of $K$ lying on the other side of $K$ from $\Omega$). For all $d > 0$, let $K'_d$ be the level hypersurface in $\Omega \cup K$ at a distance of $d$ from $K'_0$. We have the following result:

**Lemma 12.3**

*Let $\theta \in ]0, n\pi/2[$ be an angle. For all $d > 0$, the $\theta$-special Lagrangian curvature of $K_d$ is at most $\tan(\theta/n)/\tanh(d)$ in the weak sense.*

**Proof:** This is identical to the proof of Lemma 12.2. □

In the general case of hyperbolic ends, we obtain the following weaker result:

**Lemma 12.4**

*Let $\theta \in ]0, n\pi/2[$ be a angle. For all $0 < m \leqslant n$, there exists a function $\kappa_m : [0, +\infty[ \to [0, +\infty[$ such that, if $m$ is the minimal dimension of pleats in $\partial M$, then the $\theta$-special Lagrangian curvature of $K_d$ is at most $\tan(\theta/n)/\tanh(d)$ in the weak sense. Moreover:*

*(i) for all $m$, $\kappa_m(d)$ tends to $+\infty$ as $d$ tends to $0$,*

*(ii) for all $m$, $\kappa_m(d)$ tends to $\tan(\theta/n)$ as $d$ tends to $+\infty$, and*

*(iii) for all $m \geqslant n/2$, $\kappa_m(d)$ is strictly decreasing.*

**Proof:** The proof is conceptually analogous to the proof of Lemma 12.2. The only difference is that, instead of using a totally geodesic supporting hypersurface, we use the normal sphere bundle of a totally geodesic submanifold of dimension $m$. We define the function $\lambda_{m,\theta}$ by:

$$\lambda_{m,\theta}(r, d) = m\mathrm{arctan}(r\mathrm{coth}(d)) + (n - m)\mathrm{arctan}(r\mathrm{tanh}(d)).$$

We define $\kappa_m(d)$ such that:

$$\lambda_{m,\theta}(\kappa_m(d), d) = \theta.$$

If $P_d$ is the hypersurface of constant distance to such a bundle, then its $\theta$-special Lagrangian curvature is equal to $\kappa_m(d)$. Trivially, for any fixed $r$, $\lambda_{m,\theta}(r, d)$ converges to

$(n-m)\pi/2$ and $n\mathrm{arctan}(r)$ as $d$ tends to $0$ and $+\infty$ respectively. It thus follows that $\kappa_m(d)$ converges to $+\infty$ and $\tan(\theta/n)$ as $d$ tends to $0$ and $+\infty$ respectively. Finally, for $m \geqslant n/2$, elementary calculus allows us to show that, for all fixed $r$, $\lambda_{m,\theta}(r,d)$ is strictly increasing in $d$. Consequently, for $m \geqslant n/2$, $\kappa_m$ is strictly decreasing, and the result now follows. $\square$

We now obtain upper and lower bounds for the distance between a hypersurface of constant $\theta$-special Lagrangian curvature and the Nielsen kernel in a quasi-Fuchsian manifold:

**Lemma 12.5**

*Let $M$ be a quasi-Fuchsian manifold. Let $K$ be the Nielsen kernel of $M$. Let $\theta \in ](n-1)\pi/2, n\pi/2[$ be an angle. Let $D$ be the diameter of $K$. If $r \in ]\tan(\theta/n), +\infty[$ and if $\Sigma = (S,i)$ is a compact, convex immersed submanifold of constant $\theta$-special Lagrangian curvature equal to $r$, then, for all $p \in S$:*

$$\mathrm{arctanh}(r^{-1}\tan(\theta/n)) - D \leqslant d(i(p),K) \leqslant \mathrm{arctanh}(r^{-1}\tan(\theta/n)).$$

**Proof:** Since $\Sigma$ is compact, there exists a point $p \in S$ such that $d(i(p),K)$ is maximised. Let $d$ be the distance of $i(p)$ from $K$. $\Sigma$ is trivially an interior tangent to $K_d$ at $p$, and the upper bound now follows by Lemma 12.2 and the geometric maximum principal. The lower bound follows Lemma 12.3 in an analogous manner. $\square$

In the more general case of a quasi-Fuchsian hyperbolic end, we have the following result:

**Lemma 12.6**

*Let $\hat{M}$ be a quasi-Fuchsian hyperbolic end. Let $K$ be the boundary of $\hat{M}$. Let $\theta \in ](n-1)\pi/2, n\pi/2[$ be an angle. There exists a decreasing function $\delta_\theta : [\tan(\theta/n), +\infty[ \rightarrow ]0, +\infty[$ which depends on $\theta$ and $\hat{M}$ such that if $r \in ]\tan(\theta/n), \infty[$ and if $\Sigma = (S,i)$ is a compact, convex immersed submanifold of $\theta$-special Lagrangian curvature equal to $r$, then, for all $p \in S$:*

$$\delta_\theta(r) \leqslant d(i(p),K) \leqslant \mathrm{arctanh}(r^{-1}\tan(\theta/n)).$$

*Moreover, if the minimal dimension of the pleats in $K$ is at least $n/2$, then $\delta_\theta(r)$ tends to $+\infty$ as $r$ tends to $\tan(\theta/n)$.*

**Proof:** The proof of this is identical to the proof of Lemma 12.5, with the only difference being that, in the last sentence, we use Lemma 12.4 instead of 12.3. $\square$

# 13 - Diameter Bound for the Immersed Hypersurfaces.

Let $(M_n)_{n\in\mathbb{N}}, M_0$ be quasi-Fuchsian hyperbolic ends such that $(M_n)_{n\in\mathbb{N}}$ converges to $M_0$. For all $n \in \mathbb{N}\cup\{0\}$, let $K_n$ be the boundary of $M_n$. Let $\theta \in ](n-1)\pi/2, n\pi/2[$ be an angle and choose $r \in ]\tan(\theta/n), \infty[$. For $n \in \mathbb{N}$, let $\Sigma_n = (S_n, i_n)$ be a compact, convex graph in $M_n$ of $\theta$-special Lagrangian curvature equal to $r$. In this section, we obtain uniform diameter bounds for the $(\Sigma_n)_{n\in\mathbb{N}}$. We first require the following technical result which tells us that if a curve of bounded geodesic curvature never approaches the same point twice (in some sense) then its length is bounded by a function of the volume of the ambient manifold.

**Lemma 13.1**

*Let $M$ be a Riemannian manifold. Let $R$ be the Riemann curvature tensor of $M$, let $\mathrm{Inj}(M)$ be the injectivity radius of $M$, and let $\mathrm{Vol}(M)$ be the volume of $M$. Let $B \in \mathbb{R}^+$ be such that:*

$$\mathrm{Vol}(M), \|R\| \leqslant B, \qquad \mathrm{Inj}(M) \geqslant 1/B.$$

*Let $\gamma : [0, L] \to M$ be a smooth curve parametrised by unit length. Let $d$ denote the distance in $M$. There exists $\delta_0, K > 0$ which only depend on $B$ (and the dimension of $M$) such that, for all $\delta < \delta_0$ and for all $\lambda > 0$, if:*

$$|t - t'| \geqslant \lambda \Rightarrow d(\gamma(t), \gamma(t')) \geqslant \delta,$$

*then $L \leqslant K\delta^{-n}\lambda$.*

**Proof:** Let $n$ be the dimension of $M$. let $V_n$ be the volume of the Euclidean ball in $\mathbb{R}^n$ of unit radius. For all $r > 0$ and for all $p \in M$ let $B_r(p)$ be the geodesic ball of radius $r$ about $p$ in $M$. There exists $r_0 > 0$ which only depends on $B$ such that, for all $p \in M$ and for all $r < r_0$:

$$\mathrm{Vol}(B_r(p)) \geqslant \frac{1}{2}V_n r^n.$$

Choose $N \in \mathbb{N}$. If $L \geqslant N\lambda$, then there exists $t_0, ..., t_N \in [0, L]$ such that, for all $i \neq j$:

$$|t_i - t_j| \geqslant \lambda.$$

Thus, for all $i \neq j$:

$$B_{(\delta/2)}(\gamma(t_i)) \cap B_{(\delta/2)}(\gamma(t_j)) = \emptyset.$$

By reducing $\delta_0$ if necessary, we may assume that $\delta_0 < r_0$. Thus:

$$\begin{array}{rll} & \mathrm{Vol}(M) & \geqslant 2^{-(n+1)}(N+1)V_n\delta^n \\ \Rightarrow & N+1 & \leqslant (2^{n+1}/V_n)\mathrm{Vol}(M)\delta^{-n}. \end{array}$$

We choose $K$ such that $K = (2^{n+1}/V_n)\mathrm{Vol}(M)$. Thus, if $\delta < \delta_0$:

$$N + 1 \leqslant K\delta^{-n}.$$

Consequently, if $\tilde{N}$ is the maximal such $N$, then:

$$L \leqslant (\tilde{N} + 1)\lambda \leqslant K\delta^{-n}\lambda.$$

The result now follows. □

*Remark:* Strictly speaking, we have proven this theorem in the case where $M$ is complete without boundary. The case where $M$ is complete with concave boundary is proven analogously, using geodesic half balls instead of geodesic balls.

This now allows us to obtain an upper bound for the diameter of each leaf of the foliation:

**Lemma 13.2**

*There exists $R \geqslant 0$ such that, for all $n$:*

$$\mathrm{Diam}(\Sigma_n) \leqslant R.$$

**Proof:** We suppose the contrary and obtain a contradiction. Since the sequence $(M_n)_{n\in\mathbb{N}}$ of quasi-Fuchsian hyperbolic ends converges, by Lemma 12.6, for all $n$, there exists $\Omega_n \subseteq M_n$ and $B \geqslant 0$ such that:

$$\Sigma \subseteq \Omega_n, \qquad \mathrm{Vol}(\Omega_n) \leqslant B.$$

Since $(M_n)_{n\in\mathbb{N}}$ converges, we may assume that, for all $n$, the injectivity radius of $M_n$ is bounded below by $\epsilon = 1/B$. For all $n$, let $A_n$ be the shape operator of $\Sigma_n$. By Corollary 7.2, there exists $B$ such that:

$$\|A_n\| \leqslant B.$$

Moreover, we may suppose that, for all $n$, the injectivity radius of $\Sigma_n$ is bounded below by $\epsilon$.

For all $n$, let $\gamma_n$ be the minimizing geodesic in $\hat{\Sigma}_n$ of maximum length. For all $n$, let $L_n$ be the length of $\gamma_n$. By hypothesis, $(L_n)_{n\in\mathbb{N}} \to +\infty$. For all $n$, the geodesic curvature of $\gamma_n$ in $M_n$ is bounded above by $B$. Let $\delta$ be smaller than the injectivity radius of $\Sigma_n$ for all $n$. By Lemma 13.1, for sufficiently large $n$ there exist two distinct points $t_n, t'_n$ such that $|t_n - t'_n| > 2\epsilon$ and:

$$d(\gamma_n(t_n), \gamma_n(t'_n)) \leqslant \delta.$$

For any point $p \in S$, for all $n \in \mathbb{N}$ and for all $r \in \mathbb{R}^+$ let $B_n(p; r)$ be the ball of radius $r$ about $p$ in $\hat{\Sigma}_n$. For all $n$, since $\gamma_n$ is a minimising geodesic, the distance in $\hat{\Sigma}_n$ between $\gamma_n(t_n)$ and $\gamma_n(t'_n)$ is greater than $2\epsilon$. Thus $B_n(\gamma_n(t_n), \epsilon)$ and $B_n(\gamma_n(t'_n), \epsilon)$ are disjoint in $\hat{\Sigma}_n$. Moreover, since $\hat{\Sigma}_n$ is a convex graph, they cannot intersect in $M_n$. Since we can choose $\delta$ as small as we wish, we may place the centres of these two balls as close to each other as we wish, and these two balls will thus be almost parallel in $M_n$. This is absurd, again since the submanifolds are convex graphs. The result now follows. □

# 14 - Compactness.

We now obtain the compactness part of the existence result:

**Lemma 14.1**

*Let $(M_n)_{n\in\mathbb{N}}, M_0$ be quasi-Fuchsian hyperbolic ends such that $(M_n)_{n\in\mathbb{N}}$ converges to $M_0$. Let $\theta \in ]0, n\pi/2[$ be an angle, and let $r \in ]\tan(\theta/n), +\infty[$ be a real number. For all $n \in \mathbb{N}$, let $\Sigma_n = (S, i_n)$ be a convex, compact, immersed hypersurface in $M_n$ such that:*

*(i) $\Sigma_n$ is a convex graph, and*

*(ii) the $\theta$-special Lagrangian curvature of $\Sigma_n$ is equal to $r$.*

*There exists a compact, convex, immersed hypersurface $\Sigma_0 = (S, i_0)$ in $M_0$ such that:*

*(i) $\Sigma_0$ is a convex graph,*

*(ii) the $\theta$-special Lagrangian curvature of $\Sigma$ is equal to $r$,*

*and, after extraction of a subsequence, $(\Sigma_n)_{n\in\mathbb{N}}$ converges to $\Sigma_0$ in the Cheeger/Gromov sense.*

**Proof:** For all $n \in \mathbb{H} \cup \{0\}$, let $j_n : \mathbb{H}^m \to U\mathbb{H}^{m+1}$ be the boundary immersion of $\hat{M}_n$. By definition, $(j_n)_{n\in\mathbb{N}}$ converges locally uniformly to $j_0$. We obtain results for lifts of hypersurfaces in $\mathbb{H}^{m+1}$. The main technical difficulty in this proof involves showing how the results for the lifts imply the corresponding results for the hypersurfaces in the hyperbolic ends themselves.

Let $\tilde{S}$ be the universal cover of $S$ and, for all $n$, let $\tilde{i}_n : \tilde{S} \to \mathbb{H}^{m+1}$ be the lift of $i_n$. We denote $\tilde{\Sigma}_n = (\tilde{S}, \tilde{i}_n)$. Let $p$ be an arbitrary point of $S$ and let $\tilde{p}$ be its lift. By Lemma 12.6 there exists $B > 0$ such that $d(i_n(p), K_n) \leqslant B$ for all $n$. Thus by Lemma 7.3, there exists a (possibly non-compact) pointed, immersed submanifold $(\tilde{\Sigma}_0, p_0) = (\tilde{S}_0, \tilde{p}_0, \tilde{i}_0)$ in $\mathbb{H}^{m+1}$ such that $(\tilde{\Sigma}_n, \tilde{p})_{n\in\mathbb{N}}$ converges to $(\tilde{\Sigma}_0, \tilde{p}_0)$ in the pointed Cheeger/Gromov sense.

By Lemma 13.2, there exists $R > 0$ such that $\mathrm{Diam}(\Sigma_n) \leqslant R$ for all $n$. Let $g$ be the Reimannian metric of $\mathbb{H}^{m+1}$. For all $n$ let $d_n$ be the distance over $\tilde{S}$ induced by $\tilde{i}_n^* g$. Let $\gamma$ be an element of $\pi_1(S)$. $\gamma$ acts isometrically on $(\tilde{S}, \tilde{i}_n^* g)$. Since there exists $R'$ such that $d_n(\tilde{p}, \gamma(\tilde{p})) < R'$ for all $n$, we may take limits to obtain an isometric action of $\gamma$ on $\tilde{S}_0$. Thus $\pi_1(S)$ acts isometrically on $\tilde{S}_0$.

By Corollary 7.2, there exists $\epsilon > 0$ such that $\mathrm{Inj}(\Sigma_n) \geqslant \epsilon$ for all $n$. Thus, for all $n$, $d_n(\tilde{p}, \gamma(\tilde{p})) \geqslant \epsilon$. By taking limits, we thus see that the action of $\pi_1(S)$ on $\tilde{S}_0$ is properly discontinous.

For all $n$, let $\alpha_n : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^{m+1})$ be the homomorphism with respect to which $\tilde{i}_n$ is equivariant. Trivially, for all $\gamma$, there exists $R'$ such that, for all $n$, $d(\tilde{i}_n(\tilde{p}), \alpha_n(\gamma)(\tilde{i}_n(\tilde{p}))) < R'$. It follows that there exists a homomorphism $\alpha_0 : \pi_1(S) \to \mathrm{Isom}(\mathbb{H}^{m+1})$ such that $(\alpha_n)_{n\in\mathbb{N}}$ converges locally uniformly to $\alpha_0$. Trivially, $\tilde{i}_0$ is equivariant with respect to $\alpha_0$.

By Lemma 10.2, $\tilde{i}_0$ is a graph over $j_0$. We thus quotient $\tilde{\Sigma}_0 = (\tilde{S}_0, \tilde{i}_0)$ down to yield an immersed submanifold $\Sigma_0 = (S_0, i_0)$ inside $\hat{M}_0$. $\Sigma_0$ is complete and of finite radius, and is thus compact. Moreover, it is a graph over $K_0$. As in the proof of Lemma 9.1, we may find a neighbourhood $\Omega$ of $\Sigma_0$ and, for sufficiently large $n$, an open subset $\Omega_n$ of $\hat{M}_n$ such that $(\Omega_n)_{n\in\mathbb{N}}$ converges to $\Omega$ in the Cheeger/Gromov sense. For sufficiently large $n$, $\Sigma_n$ lies in $\Omega_n$, and it trivially follows that, within these open sets, $\Sigma_n$ converges to $\Sigma_0$ in the Cheeger/Gromov sense. $\square$

# 15 - Proof of the Main Results.

We begin by proving existence:

**Lemma 15.1**

*Let $M$ be a quasi-Fuchsian hyperbolic end with pleated boundary. Let $\theta \in ](n-1)\pi/2, n\pi/2[$ be an angle. For all $r \in ]\tan(\theta/n), +\infty[$, there exists a convex graph $\Sigma = (S, i)$ in $\hat{M}$ of constant $\theta$-special Lagrangian curvature equal to $r$.*

**Proof:** Let $(M_t)_{t\in[0,1]}$ be a continuous family of quasi-Fuchsian hyperbolic ends such that $M_0$ is Fuchsian and $M_1 = M$. For all $t$, let $K_t$ be the boundary of $M_t$. Let $J \subseteq [0,1]$ be the subset defined such that $t \in J$ if and only if there exists a convex graph $\Sigma_t = (S_t, i_t)$

in $M_t$ of constant $\theta$-special Lagrangian curvature equal to $r$. Since $M$ is Fuchsian, $0 \in J$ and $J$ is therefore non-empty. By Lemma 9.1, $J$ is open, and by Lemmata 10.2 and 14.1, $J$ is closed. It thus follows that $J = [0,1]$ and the result now follows. □

We now prove uniqueness:

**Lemma 15.2**

*Let $M$ be a quasi-Fuchsian hyperbolic end. Let $\theta \in ](n-1)\pi/2, n\pi/2[$ be an angle, let $r \in ]\tan(\theta/n), +\infty[$ be a positive real number and let $\Sigma = (S,i), \Sigma' = (S,i')$ be convex graphs in $M$. If $\Sigma$ and $\Sigma'$ are both of constant $\theta$-special Lagrangian curvature equal to $r$, then they coincide. In otherwords, they are reparametrisations of each other.*

**Proof:** We suppose the contrary. We first observe that by using the continuity technique as in Lemma 15.1, $\Sigma$ may be extended to a smooth family $(\Sigma_s)_{s\in[r,+\infty[} = (S, i_s)_{s\in[r,+\infty[}$ such that, for all $s$, $\Sigma_s$ is a convex graph of $\theta$-special Lagrangian curvature equal to $r$. Let $K$ be the boundary of $M$. For all $s$, let $f_s$ be the function of which $\Sigma_s$ is the graph. By Lemma 12.6, $f_s$ tends to zero uniformly as $s$ tends to $+\infty$.

We define $d, d' : S \to ]0, +\infty[$ by:

$$d(p) = d(i(p), K), \qquad d'(p) = d'(i(p), K).$$

Without loss of generality, we may suppose that $\mathrm{Inf}(d') \leqslant \mathrm{Inf}(d)$. It follows that $\Sigma'$ intersects the family $(\Sigma_s)_{s\in]r,+\infty[}$ non-trivially. Let $s_0$ be the supremum of all $s$ such that $\Sigma'$ intersects $\Sigma_s$. Since $\Sigma'$ is a graph, $\mathrm{Inf}(d') > 0$, and so $s_0 < +\infty$. By compactness and continuity, $\Sigma_{s_0}$ is an interior tangent to $\Sigma'$ at some point, but this is impossible by the geometric maximum principal. The result now follows. □

These results may be now combined to prove Theorem 1.1:

**Proof of Theorem 1.1:** Existence and uniqueness follow from Lemmata 15.1 and 15.2 respecively. For all $(r,\theta)$, let $\Sigma_{r,\theta} = (S, i_{r,\theta})$ be the unique convex graph in $M$ of constant $\theta$-special Lagrangian curvature equal to $r$. In analogy to section 8, for all $r$, let $D_{i_r}\rho_\theta$ be the derivative of the $\theta$-special Lagrangian operator abour $i_{r,\theta}$. The operator $\rho_\theta$ is defined implicitely in terms of $SL_r$, and it follows from Lemmata 8.2 and 8.4 that:

$$(D_{i_{r,\theta}}\rho_\theta)\varphi = 1 \qquad \Rightarrow \qquad \varphi < 0.$$

For all $r$, let $f_r$ be the function of which $\Sigma_{r,\theta}$ is the graph. It follows that the family $(f_r)_{r\in]\tan(\theta/n),+\infty[}$ is smooth and strictly decreasing and thus that the family of hypersurfaces $(\Sigma_{r,\theta})_{r\in]\tan(\theta/n),+\infty[}$ smoothly foliates an open subset of $M \setminus K$. By Lemma 12.5 this foliation converges to $K$ and $\partial\Omega$ as $r$ tends to $+\infty$ and $\tan(\theta/n)$ respectively. □

Likewise, we obtain a proof of Theorem 1.2:

**Proof of Theorem 1.2:** The proof of this is identical to the proof of Theorem 1.1 with the exception that, in the last sentence, we use Lemma 12.6 instead of Lemma 12.5. The case where the complement of the image of the developing map has non-trivial interior may be treated by using an adapted version of Lemma 12.5. □

# 16 - Bibliography.


[1] Aubin T., *Nonlinear analysis on manifolds. Monge-Ampère equations*, Die Grundlehren der mathematischen Wissenschaften, **252**, Springer-Verlag, New York, (1982)

[2] Caffarelli L., Nirenberg L., Spruck J., The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. *Comm. Pure Appl. Math* **37** (1984), no. 3, 369–402

[3] Calabi E., Improper affine hyperspheres of convex type and a generalisation of a theorem by K. Jörgens, *Michigan Math. J.* **5** (1958), 105–126

[4] Dold A., *Lectures on algebraic topology*, Classics in Mathematics, Springer-Verlag, New York, (1995)

[5] Gallo D., Kapovich M., Marden A., The monodromy groups of Schwarzian equations on closed Riemann surfaces, *Ann. Math.* **151** (2000), no. 2, 625–704

[6] Harvey R., Lawson H. B. Jr., Calibrated geometries, *Acta. Math.* **148** (1982), 47–157

[7] Jörgens K., Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, *Math. Ann.* **127** (1954), 130–134

[8] Jost J., Xin Y.L., A Bernstein theorem for special Lagrangian graphs, *Calc. Var.* **15** (2002), 299–312

[9] Kapovich M., Kleinian Groups in Higher Dimensions, In *Geometry and Dynamics of Groups and Spaces. In memory of Alexander Reznikov*, Progress in Mathematics, **265**, Birkhäuser, Basel, (2007)

[10] Krasnov K., Schlenker J.M., On the renormalized volume of hyperbolic 3-manifolds, math.DG/0607081

[11] Kulkarni R.S., Pinkall U., A canonical metric for Möbius structures and its applications, *Math. Z.* **216** (1994), no.1, 89–129

[12] Labourie F., Un lemme de Morse pour les surfaces convexes, *Invent. Math.* **141** (2000), 239–297

[13] Labourie F., Problème de Minkowski et surfaces à courbure constante dans les variétés hyperboliques, *Bull. Soc. Math. Fr.* **119** (1991), 307-325

[14] Otal J.P., *The Hyperbolization Theorem for Fibered 3-Manifolds*, SMF/AMS Texts and Monographs, **7**, AMS, Providence, RI; SMF, Paris, (2001)

[15] Pogorelov A. V., On the improper convex affine hyperspheres, *Geometriae Dedicata* **1** (1972), 33–46

[16] Rosenberg H., Spruck J. On the existence of convex hyperspheres of constant Gauss curvature in hyperbolic space, *J. Diff. Geom.* **40** (1994), no. 2, 379–409

[17] Smith G., Special Legendrian structures and Weingarten problems, Preprint, Orsay (2005)

[18] Smith G., An Arzela-Ascoli theorem for immersed submanifolds, to appear in *Ann. Fac. Sci. Toulouse Math.*, math.DG/0510231

[19] Smoczyk K., Longtime existence of the Lagrangian mean curvature flow, *Calc. Var. Partial Differential Equations* **20** (2004), no. 1, 25–46

[20] Yuan Y., A Berstein problem for special Lagrangian equations, *Invent. Math.* **150** (2002), 117–125