FORMAL-RATIONAL FUNCTIONS ON HOMOGENEOUS SPACES

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Introduction

Extension of analytic or algebraic objects like functions or subvarieties is a very classical subject. A prototypical example is Severi's result [Se] from 1932 that any meromorphic function defined in a small complex connected neighbourhood of a hypersurface in $\mathbb{P}^n(\mathbb{C})$ can be extended to all of $\mathbb{P}^n(\mathbb{C})$ and therefore is rational. As far as we know there was no progress at all on this type of extension problem until the sixties. First Van de Ven and Remmert extended another result of Severi, proving that every holomorphic function defined in a small complex connected neighbourhood of an irreducible positive dimensional subvariety of a complex projective homogeneous space with second Betti number equal to 1 is constant (see [R-VdV]). Then Barth [Ba] generalized the above-mentioned result of Severi to arbitrary subvarieties of $\mathbb{P}^n(\mathbb{C})$. On the other hand, Hironaka [H] extended Severi's result to arbitrary smooth projective varieties X and smooth hypersurfaces Y with ample normal bundle. In fact his approach is algebraic and he considers the extension of formal-rational functions along Y. Fix an algebraically closed ground field k and let X be an irreducible projective variety over k. Then $K(X)$ will denote the field of rational functions of X, and $K(\hat{X}_{/Y})$ will denote the ring of formal rational functions of X along a closed subscheme Y of X. For X normal and Y connected of positive dimension $K(\hat{X}_{/Y})$ is a field. The field $K(X)$ can be considered naturally as a subfield of $K(\hat{X}_{/Y})$ and the extension problem of formal-rational functions asks when can any formal-rational function on X along Y be extended to a rational function on X, i.e. when do we have $K(X) = K(\hat{X}_{/Y})$?

Formal functions have been introduced and studied systematically in algebraic geometry by Zariski in [Z]. In particular, Zariski used them to prove his famous connectivity theorem. Later on this theory has been considerably extended and deepened by Grothendieck in the framework of formal schemes, by providing powerful tools (such as Grothendieck's existence theorem, see [EGA III], theorem (5.1.4)) to study related questions (see [SGA 1], [SGA 2]).

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In a basic paper Hironaka and Matsumura [H-M] showed that $K(\mathbb{P}^n) = K(\hat{\mathbb{P}}^n_{/Y})$ for all positive dimensional connected subvarieties $Y \subset \mathbb{P}^n$ and proved that for a subvariety $Y \subset A$, A an abelian variety, the extension problem can be solved precisely if Y generates A as a group and the canonical map $\mathrm{Alb}(Y) \to A$ has connected fibres. Other important contributions in connection to the extension problem of formal-rational functions are due to Hartshorne [Ha2], Chow [Ch2], Faltings [Fa1], [Fa2], [Fa4], and others.

There is an analogous problem in the complex-analytic setting. Given a pair (X, Y), consisting of a complex irreducible projective variety X and a closed subvariety Y of X, let U be a connected open neighbourhood of Y in X (in the complex topology), and denote by $\mathcal{M}(U)$ the $\mathbb{C}$-algebra of all meromorphic functions on U. We have the field extensions $K(X) \subseteq \mathcal{M}(U) \subseteq K(\hat{X}_{/Y})$ and therefore the extension problem for meromorphic functions can be solved i.e. $K(X) = \mathcal{M}(U)$ if it can be solved for formal-rational functions, i.e. $K(X) = K(\hat{X}_{/Y})$. Therefore the above mentioned result of Hironaka-Matsumura about extending formal-rational functions along a subvariety $Y \subset \mathbb{P}^n$ to $\mathbb{P}^n$ implies the results of Barth. Important contributions to the extension problem of meromorphic functions or analytic objects are due to Severi [Se], Remmert-Van de Ven [R-vdV], Barth [Ba], Chow [Ch1], Rossi [Ro] and others.

This paper grew out of an attempt to understand Chow's last paper [Ch2] concerning the extension problem of formal rational functions on projective homogeneous spaces. His result is: Let X be a projective homogeneous space and let Y be an irreducible subvariety which generates X. Then the field extension $K(X) \subseteq K(\hat{X}_{/Y})$ is algebraic. Chow uses non-archimedian function theory to somehow mimic his proof of the analogous analytic problem, which he had solved already in 1969 [Ch1]. The object of this paper is to give a proof of Chow's theorem in algebraic geometric terms. A clue to this had been sketched by Chow at the end of his paper following a suggestion of the referee. The main novel result is the following result which is very general and powerful with some surprising applications even beyond homogeneous spaces.

Theorem. *Let Y be a closed subvariety of the complete irreducible variety X such that $K(\hat{X}_Y)$ is a field. Assume that there exists a nonsingular variety T of positive dimension and a closed irreducible subvariety Z of $X \times T$ with the following properties:*

(i) The restriction $q : Z \to X$ to Z of the projection $X \times T \to X$ is dominant and $\dim(Z) > \dim(X)$.

(ii) The restriction $p : Z \to T$ to Z of the projection $X \times T \to T$ is a surjective morphism with connected fibres, $\dim(Z) > \dim(T)$, and there is a point $t_0 \in T$ such that $q(Y') = Y$ and $q|Y' : Y' \to Y$ is an isomorphism, where $Y' := p^{-1}(t_0)_{\mathrm{red}}$.

(iii) The generic fibre F of p is geometrically irreducible.

Then $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension.

The proof of it uses a beautiful idea of the referee of Chow's paper [Ch2], which - roughly speaking - consists in using a sort of "incidence variety" to reduce the problem to Grothendieck's existence theorem, to an argument involving faithfully flat descent theory and an algebraicity criterion (see theorem (2.3) below).

Here are some applications. In order to state them precisely we recall the following notions, due to Hironaka and Matsumura. According to [H-M], a subvariety $Y \subset X$ is called $G1$ in X if $H^0(\mathcal{O}_{\hat{X}_{/Y}}) = k$. Y is called $G2$ in X if $K(X) \subseteq K(\hat{X}_{/Y})$ is a finite algebraic field extension. Finally Y is called $G3$ in X if $K(X) = K(\hat{X}_{/Y})$.

First, we give a scheme-theoretic proof of the following fundamental result of Chow:

Theorem. *(Chow [Ch2]) Let X be a projective homogeneous space over k, and let Y be a closed irreducible subvariety of X. If Y generates X then the natural map $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension.*

Using this theorem we can get the following $G3$-results.

Theorem. *Let X be a projective homogeneous space over k, and let Y be a closed irreducible subvariety of X that generates X. Assume furthermore that one of the following two conditions is satisfied:*

(i) X is a rational variety, or

(ii) $Y \cdot Y \neq 0$ (in the Chow ring of cycles modulo numerical equivalence).

Then the natural map $K(X) \to K(\hat{X}_{/Y})$ is an isomorphism, i.e. Y is $G3$ in X.

This confirms a conjecture of Debarre (see [De1], conjecture (2.9)) in the case when X is a rational homogeneous space. These two theorems imediately imply the above mentioned $G2$ and $G3$ results for subvarieties of projective spaces or of abelian varieties. In particular every irreducible subvariety Y of dimension d, $d \geq n/2$ of a simple abelian variety A of dimension n is $G3$ in A. This contains the corresponding result in the analytic category due to Barth [Ba].

Corollary. *Let Y be a closed irreducible subvariety of a rational projective homogeneous space X. Then the following conditions are equivalent:*

(i) Y generates X.

(ii) Y is $G2$ in X.

(iii) The extension $K(X) \to K(\hat{X})$ is algebraic.

(iv) Y is $G1$ in X.

(v) Y is $G3$ in X.

(vi) $\mathrm{cd}(X \setminus Y) \leq \dim(X) - 2$, i.e. $H^q(X \setminus Y, \mathcal{F}) = 0$ for every $q \geq \dim(X) - 1$, and for every coherent sheaf $\mathcal{F}$ on $X \setminus Y$.

When does a subvariety Y generate the homogeneous space X? This happens if Y has large dimension compared to the ampleness $\mathrm{amp}(X)$ of the tangent bundle of X. Letting $\mathrm{ca}(X) = \dim(X) - \mathrm{amp}(X)$ denote the coampleness of X we have the following fact.

Fact. *Let $X = G/H$ be a projective homogeneous space and let $Y \subset X$ be an irreducible subvariety. If $\mathrm{codim}(Y) < \mathrm{ca}(X)$ then Y generates X.*

For a rational homogeneous space $X = G/P$, G semi-simple, P parabolic we have the estimate $\min \mathrm{rk}(G_i) \leq \mathrm{ca}(X)$, where G_i are the simple factors of G. As a corollary we therefore get immediately the following result due to Faltings [Fa1], [Fa3]:

Corollary. *(Faltings [Fa1], [Fa3]) Let Y be a closed irreducible subvariety of a rational projective homogeneous space $X = G/P$ over $\mathbb{C}$. If* $\operatorname{codim}_X(Y) < \operatorname{ca}(X)$ *then Y is $G3$ in X.*

The above results contain, to our knowledge, all known extension results for subvarieties of projective homogeneous spaces. Moreover we can deduce the following result, giving a bit more evidence to a problem of Hartshorne.

Proposition. *Let Y be a smooth closed subvariety of a rational homogeneous projective variety X of dimension n such that* $\dim(Y) \geq \frac{n}{2}$ *and the normal bundle $N_{Y|X}$ is ample. Then Y is $G3$ in X.*

It can also be used to reprove a $G3$ result due to Faltings for not necessarily proper intersections of subvarieties of $\mathbb{P}^n$. Faltings' proof uses local methods and it was believed that global methods would not be suitable to achieve this goal. Moreover Faltings' connectedness theorem can be improved to a $G3$ result.

Theorem. *Let X be a rational projective homogeneous space over $\mathbb{C}$, and let $f : V \to X \times X$ be a proper morphism from an irreducible variety V. If* $\operatorname{codim}_{X\times X}(f(V)) < \operatorname{ca}(X)$ *then $f^{-1}(\Delta)$ is $G3$ in V.*

Finally we considerably extend results of Ogus [Og] by showing the following.

Theorem. *Let Y be a closed irreducible subvariety of the rational projective homogeneous space X, which is locally $\mathbf{S}_2$ and satisfies* $\operatorname{codim}(Y) < \frac{1}{2} \cdot \operatorname{ca}(X)$. *Then every stratified vector bundle on Y is trivial (see [Gr], [Sp2], or [B] for the definition of stratified vector bundles).*

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1. Preliminary remarks and results on homogeneous spaces

(1.1) Let X be a projective homogeneous space over an algebraically closed field k of characteristic zero, acted on transitively by the connected algebraic group G, and let Y be a closed irreducible subvariety of X. Fix a point $p \in Y$ and consider the morphism $\varphi : G \to X$ defined by $\varphi(g) = g \cdot p$. Then the set $G_{Y,p} := \varphi^{-1}(Y) = \{g \in G | g \cdot p \in Y\}$ is a closed subset of G. Denote by G_Y the subgroup of G generated by $G_{Y,p}$. Then G_Y is a closed subgroup of G (because $G_{Y,p}$ is closed in G) and G_Y does not depend of the choice of the reference point $p \in Y$.

Definition (1.2). (Chow [Ch1]) One says that the subvariety Y generates the homogeneous space X if $G_Y = G$, i.e. if $G_{Y,p}$ generates G (in the group-theoretical sense).

(1.3) From now on it will be convenient to fix a nonsingular point p of Y as the reference point. Since for all $x \in X$ the fibre $\varphi^{-1}(x)$ is isomorphic to the isotropy subgroup $G(x)$ of x, we infer that the morphism $G_{Y,p} \to Y$ is smooth, and $G_{Y,p}$ is irreducible if G acts on X by connected isotropy subgroups (because then the fibres of φ are all irreducible and of the same dimension). It follows that the point $e \in G(p) \subseteq G_{Y,p}$ (with e the unit element of G) is a nonsingular point of $G_{Y,p}$. In particular, there is a unique irreducible component, call it H, of $G_{Y,p}$ passing through the point e. By using general dimensionality arguments, Chow observes in [Ch1] that Y generates X if and only if H generates the group G.

(1.4) In the above notations, fix a natural number $i \geq 1$ and consider the morphism $\varphi_i : H^{2i} \to G$ (where $H^i = H \times \cdots \times H$ (i times)) defined by $\varphi_i(g_1, \ldots, g_{2i}) = g_{2i}g_{2i-1}^{-1} \cdots g_t^{(-1)^t} \cdots g_1^{-1}$. Since $\varphi_i(g_1, \ldots, g_{2i}) = \varphi_{i+1}(g_1, \ldots, g_{2i}, e, e)$, it follows that $\varphi_i(H^{2i}) \subseteq \varphi_{i+1}(H^{2(i+1)})$ for every $i \geq 1$. Moreover, because $g = \varphi_1(e, g)$ $\forall g \in G$, $H \subseteq \varphi_1(H^2)$. From the definition of φ_i it also follows that if $g \in \varphi_i(H^{2i})$ then $g^{-1} \in \varphi_i(H^{2i})$, and that $\varphi_i(H^{2i}) \cdot \varphi_j(H^{2j}) \subseteq \varphi_{i+j}(H^{2(i+j)})$ $\forall i, j \in \mathbb{N}$. It follows that the subgroup $< H >$ generated by H coincides with $\bigcup_{i=1}^{\infty} \varphi_i(H^{2i})$.

On the other hand, the ascending sequence

$$\overline{\varphi_1(H^2)} \subseteq \overline{\varphi_2(H^4)} \subseteq \ldots \subseteq \overline{\varphi_i(H^{2i})} \subseteq \ldots$$

of closed irreducible subsets (where the overline means taking closure) has to become stationary, i.e. there is an $n \geq 1$ such that $\overline{\varphi_n(H^{2n})} = \overline{\varphi_{n+p}(H^{2(n+p)})}$ for every $p \geq 1$. Set $F := \overline{\varphi_n(H^{2n})}$. Then F is the closure in G of the subgroup $< H >$, and so F is a closed (connected) subgroup of G. Since by a general result of Chevalley $\varphi_n(H^{2n})$ contains a non-empty open subset U of F, we infer that $< H >= \bigcup_{g \in < H >} gU$, and in particular, $< H >$ is an open subgroup of F, whence $< H >= F$. In other words, we proved the well known fact that the subgroup $< H >$ generated by a closed irreducible subvariety H passing through e is a closed connected group. This also shows that there exists an $n \geq 1$ such that the morphism $\varphi_n : H^{2n} \to < H >$ is dominant. The reason because we recalled these facts here is that they will be explicitly used later on.

(1.5) Let Ω be a universal field, i.e. an algebraically closed field which is an extension of infinite transcendence degree of k. Let ϵ (resp. η) be the generic point of H (resp. of Y). Recall that an Ω-valued generic point u of H is a point $u \in H(\Omega) = \mathrm{Hom}(\mathrm{Spec}(\Omega), H)$ given by a homomorphism of k-extensions $\alpha : k(\epsilon) \to \Omega$, where $k(\epsilon) = \mathcal{O}_{H,\epsilon}$ is the residue field of ϵ. We will say that a given system of Ω-valued generic points $u_1, \ldots, u_m$ ($m \geq 1$) is independent over k if the corresponding k-homomorphisms $\alpha_i : k(\epsilon) \to \Omega$ have the property that the subfields $\alpha_i(k(\epsilon))$ ($i = 1, \ldots, m$) are algebraically independent k-subextensions of Ω. If $u_1, \ldots, u_m$ are independent Ω-valued generic points of H then $(u_1, \ldots, u_m) \in H^m(\Omega) = H(\Omega) \times \cdots \times H(\Omega)$ (m times) is a Ω-valued generic point of H^m (and conversely).

If $u_1, \ldots, u_m$ are m independent Ω-valued generic points of H and z is an Ω-valued generic point of Y then the symbol $u_m^{(-1)^m} \cdots u_1^{-1}$ will denote the image of $(u_m, \ldots, u_1) \in H^m(\Omega) = H(\Omega)^m$ under the map $H^m(\Omega) \to G(\Omega)$ induced by the group operation morphism $H^m \subset G^m \to G$, and the symbol $u_m^{(-1)^m} \cdots u_1^{-1}(z)$ is the image of $(u_m^{(-1)^m} \cdots u_1^{-1}, z)$ under the map $G(\Omega) \times X(\Omega) = (G \times X)(\Omega) \to X(\Omega)$ induced by the action of G on the homogeneous space X ($Y(\Omega) \subseteq X(\Omega)$).

If we take $m = 2n$ then the description of $< H >$ given in (1.4) allows one to deduce the following:

Lemma (1.6). *In the above hypotheses, Y generates the homogeneous space X if and only if there are an integer $n \geq 1$, $2n$ independent Ω-valued generic points $u_1, \dots, u_{2n}$ of H and an Ω-valued generic point z of Y such that $u_{2n}u_{2n-1}^{-1} \cdots u_t^{(-1)^t} \cdots u_1^{-1}(z)$ is an Ω-valued generic point of X.*

Proposition (1.7). *Let X be a projective homogeneous space, and let Y be a closed irreducible subvariety of X. Then Y generates X if and only if Y intersects every irreducible hypersurface of X.*

Proof. This result was proved by Speiser in the case when X is an abelian variety (see [Sp1]), and by Debarre in general in [De], (2.1). The proof which follows is an adaptation of Speiser's proof to the general case. Assume that Y generates X and that there is an irreducible hypersurface D of X such that $Y \cap D = \emptyset$. If $\varphi : G \to X$ is the morphism $\varphi(g) = g \cdot p$ from the introduction, set $D' := \varphi^{-1}(D)$.

Claim 1. For every $g \in G$ we have either $gG_{Y,p} \cap D' = \emptyset$, or $gG_{Y,p} \subseteq D'$.

Indeed, since $G_{Y,p} = \varphi^{-1}(Y)$ the conclusion of the claim is equivalent to the following: either $gY \cap D = \emptyset$, or $gY \subseteq D$. Assume that this does not hold, i.e. $gY \cap D \neq \emptyset$ and $gY \not\subseteq D$. It follows that $Y \cap g^{-1}D \neq \emptyset$ and $Y \not\subseteq g^{-1}D$. Hence the cycles (on Y) $Y \cdot D = 0$ and $Y \cdot g^{-1}D > 0$ are numerically equivalent, which is not possible since Y is a projective variety. This proves claim 1.

Claim 2. For every $g, h \in G_{Y,p}$ one has $D'h^{-1}g \subseteq D'$.

To prove claim 2, let $g, h \in G_{Y,p}$ and $\delta \in D'$ arbitrary elements. Then $\delta \in \delta h^{-1}G_{Y,p} \cap D'$ (because $h \in G_{Y,p}$), and so, by claim 1 we get $\delta h^{-1}G_{Y,p} \subseteq D'$. In particular, $\delta h^{-1}g \in D'$, and since δ was arbitrary in D', we get $D'h^{-1}g \subseteq D'$.

By applying claim 2 repeatedly we get that for any $n \geq 1$ and for every $g_1, g_2, \dots, g_{2n} \in G_{Y,p}$ one has

$$D'g_1^{-1}g_2g_3^{-1}g_4 \cdots g_{2n-1}^{-1}g_{2n} \subseteq D',$$

which implies $D' \cdot < G_{Y,p} > \subseteq D'$. So far we didn't use the hypothesis that Y generates X. This means that $G =< G_{Y,p} >$. Therefore $D' \cdot G \subseteq D'$, which is obviously impossible because D' is a hypersurface of G.

The converse is immediate. Indeed, if Y does not generate X, set $X = G/P$, with P the isotropy subgroup of the reference point $p \in Y$, and denote by P' the (closed) subgroup of G generated by $G_{Y,p}$, and by $\pi : X = G/P \to X' := G/P'$ the canonical (surjective) morphism. Then $\pi(Y)$ is a point, whence there exists an irreducible hypersurface H of X' such that $\pi(Y) \notin H$. Then any component of $\pi^{-1}(H)$ is an irreducible hypersurface which does not meet Y. $\square$

Corollary (1.8). *Let X be a projective homogeneous space, and let Y be a closed irreducible subvariety of X of dimension ≥ 2. Let $\mathcal{O}_X(1)$ be a very ample line bundle on X. If Y generates X then $Y' := Y \cap H$ also generates X, for every general $H \in |\mathcal{O}_X(1)|$.*

Proof. Let Z be an arbitrary hypersurface. By proposition (1.7) it will be sufficient to show that $Y' \cap Z \neq \emptyset$. But since Y generates X, by proposition (1.7), $Y \cap Z \neq \emptyset$, and since $\dim(Y) \geq 2$, it follows that $\dim(Y \cap Z) \geq 1$ (because the ambient space is nonsingular). This implies that $H \cap (Y \cap Z) \neq \emptyset$ (since H is hyperplane section), whence $Y' \cap Z \neq \emptyset$. $\square$

2. Algebraicity criteria for formal-rational functions

Troughout this section the ground field k will be assumed to be algebraically closed of arbitrary characteristic.

(2.1) Fix a normal complete variety X and a closed irreducible subvariety Y, and denote by $\hat{X} := \hat{X}_{/Y}$ and by $\alpha : K(X) \to K(\hat{X})$ the canonical map. Then by [H-M] (1.4) the ring $K(\hat{X})$ is actually a field, so that α makes $K(\hat{X})$ a field extension of $K(X)$. We want to characterize those formal functions $\zeta \in K(\hat{X})$ which are algebraic over $\alpha(K(X))$, or simply over $K(X)$.

Let $f : X' \to X$ be a proper surjective morphism, and let Y be a closed subvariety of X. Then by [H-M] there is a canonical morphism $\hat{f} : \hat{X'}_{/f^{-1}(Y)} \to \hat{X}_{/Y}$ of formal schemes which is compatible with f, and which induces a unique map of k-algebras $\hat{f}^* : K(\hat{X}_{/Y}) \to K(\hat{X'}_{/f^{-1}(Y)})$ making the following diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{f^*} & K(X') \\ \alpha\Big\downarrow & & \Big\downarrow\alpha' \\ K(\hat{X}_{/Y}) & \xrightarrow[\hat{f}^*]{} & K(\hat{X'}_{/f^{-1}(Y)}) \end{array}$$

commutative.

Assume now that Y' is a closed subvariety of X' such that $f(Y') \subseteq Y$. Then $Y' \subseteq f^{-1}(Y)$, whence we have the canonical flat morphism $\psi : \hat{X'}_{/Y'} \to \hat{X'}_{/f^{-1}(Y)}$, which induces the canonical map $\psi^* : K(\hat{X'}_{/f^{-1}(Y)}) \to K(\hat{X'}_{/Y'})$. Therefore we get the map $\tilde{f} := \hat{f} \circ \psi : \hat{X'}_{/Y'} \to \hat{X}_{/Y}$ which induces the unique map of k-algebras $\tilde{f}^* := \psi^* \circ \hat{f}^* : K(\hat{X}_{/Y}) \to K(\hat{X'}_{/Y'})$ making the following diagram commutative

$$\begin{array}{ccc} K(X) & \xrightarrow{f^*} & K(X') \\ \alpha\Big\downarrow & & \Big\downarrow\alpha' \\ K(\hat{X}_{/Y}) & \xrightarrow[\tilde{f}^*]{} & K(\hat{X'}_{/Y'}) \end{array}$$

With these notations one has the following important result of Hironaka and Matsumura (see [H-M], theorem (2.7)), whose proof is based essentially on Grothendieck's existence theorem (see [EGA III], theorem (5.1.4)):

Theorem (2.2). *(Hironaka-Matsumura) Let $f : X' \to X$ be a proper surjective morphism of irreducible varieties. Then for every closed subvariety Y of X there is a canonical isomorphism $K(\hat{X'}_{/f^{-1}(Y)}) \cong [K(\hat{X}_{/Y}) \otimes_{K(X)} K(X')]_0$, where $[A]_0$ denotes the total ring of fractions of a commutative unitary ring A.*

Next we prove the following algebraicity criterion, which will play an important role in the sequel:

Theorem (2.3). *Let $\zeta \in K(\hat{X}_{/Y})$ be a formal function of a complete variety X along a closed subvariety Y of X such that $K(\hat{X}_{/Y})$ is a field (e.g. if X is normal and Y is connected, see [H-M], (1.4)). Then the following conditions are equivalent:*

(i) ζ is algebraic over $K(X)$.

(ii) There is a proper surjective morphism $f : X' \to X$ and a closed subvariety Y' of X' such that $f(Y') \subseteq Y$ and such that $\tilde{f}^*(\zeta)$ is algebraic over $K(X')$.

(iii) There is a proper surjective morphism $f : X' \to X$ and a closed subvariety Y' of X' such that $f(Y') \subseteq Y$ and such that $\tilde{f}^*(\zeta) \in K(X')$.

(iv) There is a finite surjective morphism $f : X' \to X$, with X' normal, and a closed subvariety Y' of X' such that $f(Y') \subseteq Y$, $f|Y' : Y' \to Y$ is an isomorphism and f is étale at every point of Y' (i.e. an étale neighbourhood of (X, Y), see [Gi]), with the property that $\tilde{f}^*(\zeta) \in K(X')$.

Proof. Let us first prove that (iii) implies (i). Assume that ζ is transcendental over $K(X)$. Since X is normal and Y is connected, by [H-M] (1.4) the ring $K(\hat{X}) := K(\hat{X}_{/Y})$ is a field. In particular, the map $\tilde{f}^*$ is injective. Then it follows that $\tilde{f}^*(\zeta)$ is transcendental over $K(X)$. If we set $\zeta' = \tilde{f}^*(\zeta)$, we get the purely transcendental field subextension $K(X) \subset E := K(X)(\zeta')$ of $K(X')$. Set $V := X \times \mathbb{P}^1$. Then V is identified with a variety whose field of rational functions is E such that the map $g^* : K(X) \to K(V) = E$ induced by the projection $g : V = X \times \mathbb{P}^1 \to X$ is just the subextension $K(X) \subset E$. Moreover, since $g^{-1}(Y) \cong Y \times \mathbb{P}^1$ is connected (because Y is so), and V is normal (because X is so), by [H-M] (1.4), the ring $K(\hat{V}_{/g^{-1}(Y)})$ is a field.

On the other hand, the extension $E = K(V) \subseteq K(X')$ (via f^*) yields a unique rational dominant map $h : X' \dashrightarrow V$ such that this extension coincides with $h^* : K(V) \to K(X')$. Let X'' be the normalization of the closure (in $X' \times V$) of the graph of h, let $h' : X'' \to V$ be the corresponding projection onto V (which is a proper surjective morphism), and let $u : X'' \to X'$ be the projection onto X'. Clearly, u is a birational morphism. Therefore, if we denote by $Y'' := u^{-1}(Y')$, the induced maps $u^* : K(X') \to K(X'')$ and $\hat{u}^* : K(\hat{X}'_{/Y'}) \to K(\hat{X}''_{/Y''})$ are both isomorphisms. The fact that u^* is an isomorphism is obvious, while the fact that $\hat{u}^*$ is an isomorphism is an immediate consequence of theorem (2.2) of Hironaka and Matsumura. So, without loss of generality we may assume that the rational dominant map h is a proper surjective morphism. In other words, we showed that we may assume that f factors as $f = g \circ h$, with $g : V \to X$ and $h : X' \to V$ proper surjective morphisms such that $K(V) = K(X)(\zeta')$, with $\zeta' \in K(X')$ transcendental over $K(X)$ (and of the form $\hat{f}^*(\zeta)$). Thus we get the following commutative diagram:

$$\begin{array}{ccccc} K(X) & \xrightarrow{g^*} & K(V) & \xrightarrow{h^*} & K(X') \\ \alpha\downarrow & & \downarrow\alpha'' & & \downarrow\alpha' \\ K(\hat{X}) & \xrightarrow[\hat{g}^*]{} & K(\hat{V}_{/g^{-1}(Y)}) & \xrightarrow[\tilde{h}^*]{} & K(\hat{X}'_{/Y'}) \end{array}$$

At this point we can apply theorem (2.2) to deduce that $K(\hat{V}_{/g^{-1}(Y)})$ is canonically isomorphic to $[K(\hat{X}) \otimes_{K(X)} K(V)]_0$. In particular, $\hat{g}^*(\zeta)$ is trancendental over $K(V)$ (because ζ is transcendental over $K(X)$). Since $K(\hat{V}_{/g^{-1}(Y)})$ is a field, the map $\tilde{h}^*$ is injective. It follows that $\tilde{h}^*(\hat{g}^*(\zeta)) = \tilde{f}^*(\zeta)$ is transcendental over $h^*(K(V))$. However, this is not possible, because by the construction of V, $\tilde{f}^*(\zeta) = \zeta' \in h^*(K(V))$. This shows that the assumption that ζ was transcendental over $K(X)$ is absurd, proving that (ii) implies (i).

Of course, (iv) implies (iii) and (iii) implies (ii).

We prove now that (i) implies (iv). Since ζ is algebraic over $K(X)$, $K(X) \subset K(X)(\zeta)$ is a finite field extension; in particular, $K(X)(\zeta)$ is a function field. Then take as $f : X' \to X$ the normalization of X in $K(X)(\zeta)$. By construction, $\zeta \in K(X')$. It remains to check that the immersion $Y \subset X$ lifts to X', and that, if Y' is the image of Y in X' then X' is étale at each point of Y'. Although not stated explicitly, this is done in [Gi], theorem (4.3), whose proof works in our situation perfectly well. Indeed, the only change which has to be made in the proof of theorem (4.3) of [Gi] is to take $K' = K(X)(\zeta)$ (instead of $K' = K(\hat{X})$). So with this minor change the proof of Gieseker of theorem (4.3) of [Gi] yields exactly what we need.

Therefore the conditions (i), (iii) and (iv) are all equivalent, and (iii) implies (ii). It remains to prove that (ii) implies (iii). But by the implication (i) implies (iv) (applied to X' and $\tilde{f}^*(\zeta)$), the fact that $\tilde{f}^*(\zeta)$ is algebraic over $K(X')$ yields the existence of a finite surjective morphism $u : Z \to X'$ and a closed subset W of Z such that $u(W) = Y'$ such that $\tilde{u}^*(\tilde{f}^*(\zeta)) \in K(Z)$. Then taking the proper surjective morphism $g := f \circ u : Z \to X$ we get $g(W) \subseteq Y$ and $\tilde{g}^*(\zeta) \in K(Z)$, hence (ii) implies (iii) . □

As a consequence of theorem (2.3) we mention the following result (which was proved independently also by E. Szabó):

Corollary (2.4). *Let X be a projective irreducible variety, and let Y be a closed irreducible subvariety of X. The following two conditions are equivalent:*

(i) For every proper surjective morphism $f : X' \to X$, $f^{-1}(Y)$ is connected.

(ii) $K(\hat{X}_{/Y})$ is a field and $K(X)$ is algebraically closed in $K(\hat{X}_{/Y})$.

Proof. Assume that (i) holds. To prove that $K(\hat{X}_{/Y})$ is a field let $f : Z \to X$ be the normalization morphism. Since f is birational, by theorem (2.2), $K(\hat{X}_{/Y}) \cong K(\hat{Z}_{/f^{-1}(Y)})$. Since by (i), $f^{-1}(Y)$ is connected, it follows that $K(\hat{Z}_{/f^{-1}(Y)})$ is a field (taking into account that Z is normal and applying [H-M], (1.4)). This proves that $K(\hat{X}_{/Y})$ is a field. On the other hand, if there would exist an algebraic formal function $\zeta \in K(\hat{X}_{/Y})$ over $K(X)$ which does not belong to $K(X)$ then by theorem (2.3) there is a finite morphism $f : X' \to X$ of degree > 1 (equal to the degree of the minimal polynomial of ζ over $K(X)$) and a closed subvariety Y' satisfying the condition (iii) of theorem (2.3). It follows that Y' is a connected component of $f^{-1}(Y)$, and since $\deg(f) > 1$, it does not coincide with $f^{-1}(Y)$. In other words, $f^{-1}(Y)$ is not connected, contradicting (i).

Conversely, let $f : X' \to X$ be an arbitrary proper surjective morphism. Then by [EGA IV] (4.3.1) the ring $K(\hat{X}_{/Y}) \otimes_{K(X)} K(X')$ has a unique minimal prime ideal p

(because we assumed (ii)). Then the localization $(K(\hat{X}_{/Y}) \otimes_{K(X)} K(X'))_p$ (which is a local ring) coincides with $[K(\hat{X}_{/Y}) \otimes_{K(X)} K(X')]_0$, and this latter ring is by theorem (2.2) isomorphic to $K(\hat{X}'_{/f^{-1}(Y)})$. Thus $K(\hat{X}'_{/f^{-1}(Y)})$

is a local ring, which implies that $f^{-1}(Y)$ is connected. $\square$

(2.5) Let $f : X' \to X$ be a dominant (not necessarily proper) morphism of irreducible varieties into the complete variety X. Let Y (resp. Y') be a closed subvariety of X resp. of X') such that $f(Y') \subseteq Y$.

We claim that if Y' is proper over k then there still exists a canonical map of k-algebras $\tilde{f}^* : K(\hat{X}_{/Y}) \to K(\hat{X}'_{/Y'})$ having the properties stated in (2.1). Indeed, one can use Nagata's compactification theorem to find an irreducible complete variety V containing X' as an Zariski open subset. Then $f : V \dashrightarrow X$ becomes a rational map on V. Let W be the closure in $V \times X$ of the graph of the morphism $f : X' \to X$, and let $g : W \to X$ be the restriction to W of the projection $V \times X \to X$. Then g is a proper morphism, W contains X' as a Zariski open subset (and in particular, $Y' \subset W$), and $g|X' = f$. Since f is dominant, g is also dominant, and therefore surjective because g is proper. Then set $\tilde{f}^* := \tilde{g}^*$ ($\tilde{g}^* : K(\hat{X}_{/Y}) \to K(\hat{W}_{/Y'}) = K(\hat{X}'_{/Y'})$). Since Y' is complete, the map $\tilde{f}^*$ is independent of the choice of the compactification morphism $g : W \to X$.

Moreover, in theorem (2.3) we can only assume that f is a dominant morphism, provided that in conditions (ii) and (iii) Y' is assumed to be complete.

The next result is the main general criterion and its proof is based on an idea of the referee of Chow's paper [Ch2] (and sketched at the end of that paper):

Theorem (2.6). *Let Y be a closed subvariety of the complete irreducible variety X such that $K(\hat{X}_Y)$ is a field. Assume that there exists a nonsingular variety T of positive dimension and a closed irreducible subvariety Z of $X \times T$ with the following properties:*

(i) The restriction $q : Z \to X$ to Z of the projection $X \times T \to X$ is dominant and $\dim(Z) > \dim(X)$.

(ii) The restriction $p : Z \to T$ to Z of the projection $X \times T \to T$ is a surjective morphism with connected fibres, $\dim(Z) > \dim(T)$, and there is a point $t_0 \in T$ such that $q(Y') = Y$ and $q|Y' : Y' \to Y$ is an isomorphism, where $Y' := p^{-1}(t_0)_{\mathrm{red}}$.

(iii) The generic fibre F of p is geometrically irreducible.

Then $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension.

Proof. Let R denote the completion of the local ring $\mathcal{O}_{T,t_0}$ with respect to its maximal ideal, then R is a regular local ring (isomorphic to a formal power series ring). Consider the base change diagram

$$\begin{array}{ccc} Z' := Z \times_T \operatorname{Spec}(R) & \xrightarrow{u} & Z \\ r\downarrow & & \downarrow p \\ \operatorname{Spec}(R) & \xrightarrow[v]{} & T \end{array}$$

where the bottom horizontal morphism corresponds to the canonical map $\mathcal{O}_{T,t_0} \to R$. Then r is a proper surjective morphism (since p is so) whose closed fiber is isomorphic to Y'.

Claim (2.6.1). Let $\varphi : \mathcal{O}_{T,t_0} \to A$ be a flat homomorphism of commutative rings such that $\operatorname{Spec}(A)$ is irreducible, and set $Z_A := Z \otimes_T A$. Then Z_A (and in particular, $Z' = Z_R$) is irreducible.

To see this, consider the cartesian diagram

$$\begin{tikzcd} Z_A \arrow[r, "u_A"] \arrow[d, "p_A"'] & Z \arrow[d, "p"] \\ \operatorname{Spec}(A) \arrow[r, "v_A"'] & T \end{tikzcd}$$

where v_A is the morphism associated to φ. Then the morphism u_A is flat since the morphism v_A is so. Therefore every irreducible component of Z_A dominates Z, and since p is surjective, every such component also dominates $\operatorname{Spec}(A)$. This implies that the generic fibre F_A of p_A contains the generic points of all irreducible components of Z_A, which in turn implies that F_A is dense in Z_A. On the other hand, $F_A \cong F \otimes_{k(T)} K_A$, where $K_A := [A]_0$ is the total ring of fractions of A. By hypothesis (iii), F_A is irreducible, whence Z_A is irreducible. This proves claim (2.6.1).

Using hypothesis (ii) and the fact that X is a complete variety (and hence p is proper), we can apply Grothendieck's existence theorem (see [EGA III], (5.1.4)) which asserts that the functor $\mathcal{F} \to \hat{\mathcal{F}}_{/Y'}$ defined on the category of all coherent sheaves on Z' into the category of all coherent sheaves on $\hat{Z}'_{/Y'} = \hat{Z}_{/Y'}$ is an equivalence of categories. As a rather straightforward consequence of this one can deduce that the canonical map $K(Z') \to K(\hat{Z}_{/Y'}) = K(\hat{Z}'_{/Y_R})$ is an isomorphism, where $Y_R := u^{-1}(Y')$. This is done by using the notion of pole sheaf on $\hat{Z}_{/Y'}$ associated to any non-zero formal function in $K(\hat{Z}_{/Y'})$, see proposition (2.2) of [H-M]. What we get is the following: start with a non-zero formal function $\zeta \in K(\hat{X}_{/Y})$, then $\tilde{q}^*(\zeta) \in K(\hat{Z}_{/Y'})$ comes actually from $K(Z')$, i.e. $\tilde{q}^*(\zeta)$ is a rational function on the scheme Z'.(Note that by hypotheses (i) and (ii) and by (2.5) the map $\tilde{q}^* : K(\hat{X}_{/Y}) \to K(\hat{Z}_{/Y'})$ is well defined.)

Claim (2.6.2). Let $\zeta \in K(\hat{X}_{/Y})$ be an arbitrary formal function. Then $\tilde{q}^*(\zeta)$ is algebraic over $K(Z)$.

Assuming claim (2.6.2), we can apply theorem (2.3) - via the observation (2.5) - to get that that ζ is algebraic over $K(X)$. Since $\zeta \in K(\hat{X}_{/Y})$ was arbitrary, we proved that $K(X) \to K(\hat{X}_{/Y})$ defines an algebraic field extension. Thus theorem (2.6) is proved modulo claim (2.6.2).

Claim (2.6.2) is proved by using faithfully flat descent theory (see [SGA 1], chapter VIII). Let S denote the strict henselization of the regular local ring $\mathcal{O}_{T,t_0}$. It is well known that S is a noetherian regular local ring, and that there are faithfully flat inclusions of local rings $\mathcal{O}_{T,t_0} \subset S \subset R$. Moreover, the fraction field $K_S = [S]_0$ consists precisely of all Laurent series of $K_R := [R]_0$ which are algebraic over $\mathcal{O}_{T,t_0}$. In particular, K_S is algebraically closed in K_R.

Start with an arbitrary non-zero $\zeta \in K(\hat{X}_{/Y})$. Since locally on $\hat{X} := \hat{X}_{/Y}$ the formal function ζ is a quotient of two non-zero sections of the structure sheaf $\mathcal{O}_{\hat{X}}$, there is a non-empty open subset U in X which intersects Y such that $\zeta \in \Gamma(\hat{U}, \mathcal{O}_{\hat{X}})$. Set $V := q^{-1}(U)$ and for every flat map $\varphi : \mathcal{O}_{T,t_0} \to A$ as in (2.6.1), let $V_A := V \otimes_T A$. If A is a local ring, set also $\hat{V_A} := \hat{V}_{A/Y'_A}$, with $Y'_A := p_A^{-1}(m_A) \cong Y'$, where m_A denotes the maximal ideal of A. Then we may write the sequence

$$\Gamma(\hat{U}, \mathcal{O}_{\hat{X}}) \to \Gamma(\hat{V_R}, \mathcal{O}_{\hat{Z'}}) \rightrightarrows \Gamma(\hat{V_R} \times_{\hat{U}} \hat{V_R}, \mathcal{O}_{\hat{Z'} \times_{\hat{X}} \hat{Z'}}),$$

in which the image of the first arrow is contained in the kernel of the double arrow. Since $\zeta' := \tilde{q}^*(\zeta) \in K(Z')$, we may assume that the rational function ζ' is defined in the open subset V_R (by shrinking U if necessary). Indeed, since ζ' is defined on $\hat{V_R}$ (as a formal function), the rational function ζ' is defined at each point of $V_R \cap Y'_R$. This amounts to the following purely algebraic statement: for any flat inclusion of noetherian local rings $B \subset C$ one has $B = C \cap [B]_0$, where the intersection is taken in $[C]_0$ (see e.g. [Ha1] if B is regular, [Sp1] if B is normal, and [B], page 33 in general).

Now look at the canonical double arrow

$$\Gamma(V_R, \mathcal{O}_{Z'}) \rightrightarrows \Gamma(V_R \times_U V_R, \mathcal{O}_{Z' \times_X Z'}),$$

which is compatible with the above formal double arrow. Using [EGA I], (10.9.7) and (10.8.11) we infer that the images ζ'_1 and ζ'_2 of ζ' under the latter double arrow coincide in an open neighbourhood of $(V_R \cap Y'_R) \times_{Y \cap U} (V_R \cap Y'_R) \cong V \cap Y'_R$.

On the other hand, since the morphism $Z' = Z_R \to X$ factors as $Z' = Z_R \to Z_S \to X$, we get a canonical morphism $Z' \times_{Z_S} Z' \to Z' \times_X Z'$, whence a canonical map

$$\alpha : \Gamma(V_R \times_U V_R, \mathcal{O}_{Z' \times_X Z'}) \to \Gamma(V_R \times_{V_S} V_R, \mathcal{O}_{Z' \times_{Z_S} Z'}).$$

Claim (2.6.3). $\alpha(\zeta'_1) = \alpha(\zeta'_2)$.

To prove claim (2.6.3), observe by what we said above that $\alpha(\zeta'_1)$ and $\alpha(\zeta'_2)$ coincide in an open neighbourhood of $(V_R \cap Y'_R) \times_{V_S \cap Y'_S} (V_R \cap Y'_R)$. To prove that these two sections coincide on the whole of $Z' \times_{Z_S} Z'$, it will be sufficient to show that this latter scheme is irreducible. To do this, in claim (2.6.1) we take $A = R \otimes_S R$. By claim (2.6.1), $Z_A = Z \otimes_S (R \otimes_S R) \cong Z' \times_{Z_S} Z'$ is irreducible,provided $\operatorname{Spec}(R \otimes_S R)$ is irreducible. But observe that the total fraction rings of $R \otimes_S R$ and of $K_R \otimes_{K_S} K_R$ are the same (because the ring extension $S \subset R$ is flat), whence it will be enough to check that $\operatorname{Spec}(K_R \otimes_{K_S} K_R)$ is irreducible. But this last fact follows from [EGA IV], (4.3.2) because K_S is algebraically closed in K_R (here we make essential use of the fact that S is the strict henselization of $\mathcal{O}_{T,t_0}$!). So claim (2.6.3) is proved.

By claim (2.6.3) we get that $\zeta' \in \Gamma(V_R, \mathcal{O}_{Z'})$ belongs to

$$\operatorname{Ker}(\Gamma(V_R, \mathcal{O}_{Z'}) \rightrightarrows \Gamma(V_R \times_{V_S} V_R, \mathcal{O}_{Z' \times_{Z_S} Z'})).$$

At this point, since $V_R \to V_S$ is faithfully flat, we may apply a result from the faithfully flat descent theory (see [SGA1], exp. VIII, corollary (1.7)) to deduce that $\zeta' \in \Gamma(V_S, \mathcal{O}_{Z_S})$. This implies that $\zeta' \in K(Z_S)$.

Now, recalling the definition of $Z_S = Z \otimes_T S$, we infer that

$$K(Z_S) = [K(Z) \otimes_{K(T)} K_S]_0.$$

Since K_S is an algebraic field extension of $K(T)$, this clearly implies that ζ' is algebraic over $K(Z)$, which proves claim (2.6.2), and thereby theorem (2.6). $\square$

3. Applications to homogeneous spaces

Throughout this section k will denote an algebraically closed field of characteristic zero (although many results hold in arbitrary characteristic as well). The first application of theorem (2.6) is the following:

Theorem (3.1). *(i) Let X be a projective homogeneous space over k, and let Y be a closed irreducible subvariety of X. If Y generates X then the natural map $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension.*

(ii) In the hypotheses of (i), let $\zeta \in K(\hat{X}_{/Y})$ be a formal function which is algebraic of degree d over $K(X)$. Then there exists a finite étale morphism $f : X' \to X$ of degree d from a projective irreducible variety X' such that the inclusion $Y \subset X$ lifts to an inclusion $Y \subset X'$ and such that $\tilde{f}^*(\zeta) \in K(X')$.

Note. Part (i) of theorem (3.1) is a fundamental result of Chow (see [Ch2]). Chow's proof of it (which makes use of non-archimedian function theory) is completely different from the proof below.

Proof. (i) We shall use the notations of section 1. The assumption that Y generates X implies - via lemma (1.6) - the existence of an integer $n \geq 1$, of $2n$ independent Ω-valued generic points $u_1, \dots, u_{2n}$ of H and of an Ω-valued generic point z of Y, such that $\epsilon := u_{2n}u_{2n-1}^{-1} \cdots u_t^{(-1)^t} \cdots u_1^{-1}$ is an Ω-valued generic point of X. This latter point is defined over the field $k(u_1, \dots, u_{2n}, z)$. Set $T := H^{2n}$ and $t_0 := (e, \dots, e)$ ($2n$ times). Then t_0 is a nonsingular point of T because e was a nonsingular point of H. Therefore restricting T along t_0 if necessary, we may assume that T is nonsingular. In the product $X \times T = X \times H^{2n}$ consider the closure Z over k of the point $(u_1, \dots, u_{2n}; u_{2n}u_{2n-1}^{-1} \cdots u_t^{(-1)^t} \cdots u_1^{-1}(z))$ (endowed with the reduced structure). Denote by $q : Z \to X$ and $p : Z \to T$ the restriction to Z of the canonical projections $X \times T \to X$ and $X \times T \to T$ respectively.

We want to apply, of course, theorem (2.6). The hypothesis (i) of (2.6) is fulfilled because Y generates X (via the above remarks). On the other hand, note that $p^{-1}(t_0) = Y \times \{t_0\}$. So, the hypothesis (ii) of (2.6) is verified by taking $Y' := Y \times \{t_0\}$. Finally, we need to check that (iii) is satisfied. But this is also easy because by the very definition of the morphism p it follows that the generic fibre F of p is isomorphic to $Y \otimes_k k(u_1, \dots, u_{2n})$, whence F is geometrically integral because Y is an integral scheme over the algebraically closed field k.

Then part (i) of theorem (3.1) follows by applying theorem (2.6).

For part (ii), by theorem (2.3), we get a finite surjective morphism $f : X' \to X$, with X' normal, a closed subvariety Y' of X' with the following properties: $f(Y') \subseteq Y$, the restriction $f|Y' : Y' \to Y$ is an isomorphism, f is étale in a neighbourhood of Y', and $\tilde{f}^*(\zeta) \in K(X')$. We will show that f is everywhere étale.

Let $\Delta' \subset X'$ be the ramification locus of f, and let $\Delta := f(\Delta')$ be the branch locus of f. Since f is étale along Y', $Y' \cap \Delta' = \emptyset$. Let u be the generic point of the group G, and let v be the generic point of Y. If $s : G \times X \to X$ is the action morphism of G on X, then by transitivity of this action we get that $s(u, v) = u \cdot v$ is the generic point of X. It follows that $u \cdot Y = s(\{u\} \times Y)$ is dense in X. Then it follows that $f^{-1}(u \cdot Y))$ is dense in X', and in particular, $f^{-1}(u \cdot Y)$ is an irreducible scheme over $k(u) = K(G)$ (because X' is irreducible). If $\overline{k(u)}$ is an algebraic closure of $k(u)$ we infer that the $\overline{k(u)}$-scheme $f^{-1}(u \cdot Y) \otimes_{k(u)} \overline{k(u)}$ consists of a number of irreducible components which are all conjugate over $k(u)$ under the natural action of the Galois group $\mathrm{Gal}(\overline{k(u)}|k(u))$. Since Y' is a specialization of one of these components (note that from the properties of the morphism f, Y' is a connected component of $f^{-1}(Y)$, which is irreducible because it is isomorphic to Y) and since $Y' \cap \Delta' = \emptyset$, it follows that for a sufficiently general element $g \in G$ we have

$$f^{-1}(g \cdot Y) \cap \Delta' = \emptyset.$$

In fact we get more, namely that for $g \in G$ sufficiently general, the irreducible components of $f^{-1}(g \cdot Y)$ are mutually disjoint and numerically equivalent to each other.

At this point it is easy to deduce that f is an étale morphism in the following way. The above equality implies that for $g \in G$ sufficiently general we have $g \cdot Y \cap \Delta = \emptyset$. Assuming that $\Delta \neq \emptyset$, then by the purity of the branch locus (recall that X is smooth) we obtain that Δ must be of pure codimension one in X. Then, recalling proposition (1.7), this contradicts the latter equality (because if Y generates X then $g \cdot Y$ also generates X). So $\Delta = \emptyset$, i.e. f is étale and theorem (3.1) is proved. $\square$

Note. The argument of proving part (ii) of theorem (3.1) is inspired by Chow's proof of theorem 3 of [Ch1] (see also [H-M]).

Corollary (3.2). *(Hironaka-Matsumura [H-M]) Let Y be an irreducible subvariety of an abelian variety X, let $i : Y \to \mathrm{Alb}(Y)$ the canonical morphism into the strict Albanese variety $\mathrm{Alb}(Y)$ of Y in the sense of [H-M], and let $f : \mathrm{Alb}(Y) \to X$ be the morphism induced by the inclusion $Y \to X$ arising from the universal property of strict Albanese varieties (f is a homomorphism of abelian varieties if we assume that Y passes through the origin 0_X of X). Then:*

(i) Y generates X if and only if f is surjective.

(ii) The morphism i is a closed embedding and $i(Y)$ is $G3$ in $\mathrm{Alb}(Y)$.

(iii) If Y generates X then Y is $G2$ in X. Moreover, Y is $G3$ in X if and only if Y generates X and the canonical morphism f has connected fibres.

Proof. Part (i) is trivial, as well as the fact that i is a closed embedding (because $f \circ i$ coincides with the inclusion $Y \subset X$). Since $i(Y)$ generates $A := \mathrm{Alb}(Y)$, by theorem (3.1), the map $K(A) \to K(\hat{A}_{/i(Y)})$ is an algebraic field extension. Assume that there is a formal function $\zeta \in K(\hat{A}_{/i(Y)}) \setminus K(A)$. Then by part (ii) of theorem (3.1) there is a finite étale morphism $g : A' \to A$ such that $\tilde{g}^*(\zeta) \in K(A')$ and such that the embedding $i : Y \to A$ lifts to an embedding $i' : Y \to A'$. Since g is étale, there is a unique structure of abelian variety on A' such that g becomes a homomorphism of abelian varieties. Then by the universal property of $\mathrm{Alb}(Y)$, there is a unique morphism $h : A \to A'$ such that $h \circ i = i'$. It follows that $g \circ h = \mathrm{id}_A$, which would disconnect A' if $\deg(g) > 1$. It follows that $\zeta \in K(A)$, i.e. $i(Y)$ is $G3$ in A. So, part (ii) is proved.

Part (iii) follows now easily from (ii). Indeed, look at the map $\tilde{f}^* : K(\hat{X}_Y) \to K(\hat{A}_{/i(Y)})$. By (ii), $K(\hat{A}_{/i(Y)}) = K(A)$, whence $\tilde{f}^*(K(\hat{X}_{/Y})) \subseteq K(A)$. But the extension $f^* : K(X) \to K(A)$ is finitely generated, so that the subextension $K(X) \to K(\hat{X}_{/Y})$ is also finitely generated. Being also algebraic, it follows that it is finite, i.e. Y is $G2$ in X. To prove the last statement of (iii), we need to apply a result of Hartshorne-Gieseker [Gi], according to which there exists a finite morphism $u : X' \to X$ with the following properties: X' is a normal projective variety, the inclusion $Y \subset X$ lifts to an inclusion $Y \subset X'$, u is étale along Y, and Y is $G3$ in X. Then the argument of the proof of part (ii) of theorem (3.1) applies also in this situation to yield the fact that u is everywhere étale. Since X is an abelian variety, X' is also an abelian variety and u is a homomorphism of abelian varieties. By the universal property of $\mathrm{Alb}(Y)$ the morphism f factors as $\mathrm{Alb}(Y) \to X' \to X$, and this yields immediately the conclusion. $\square$

Note. The above proof of the $G2$ condition in corollary (3.2) is completely different from Hironaka-Matsumura's proof [H-M].

Corollary (3.3). *Let X be a projective homogeneous space over k, and let Y be a closed irreducible subvariety of X that generates X. Assume furthermore that one of the following two conditions is satisfied:*

(i) X is a rational variety, or

(ii) $Y \cdot Y \neq 0$ (in the Chow ring of cycles modulo numerical equivalence).

Then the natural map $K(X) \to K(\hat{X}_{/Y})$ is an isomorphism, i.e. Y is $G3$ in X.

Note. Corollary (3.3) is just an algebraic analogue of an analytic result of Chow [Ch1].

Proof. By theorem (3.1) we have a finite surjective morphism $f : X' \to X$, with X' normal, a closed subvariety Y' of X' with the following properties: $f(Y') \subseteq Y$, the restriction $f|Y' : Y' \to Y$ is an isomorphism, f is étale, and the natural map $K(X') \to K(\hat{X'}_{/Y'})$ is an isomorphism, i.e. Y' is $G3$ in X'. Under hypothesis (i), X is algebraically simply connected (as any rational variety, see e.g. [SGA 1]), hence f is an isomorphism, and the proof is finished in this case.

Assume now that we are in the hypothesis (ii), and assume that f is not an isomorphism. Since f is a finite étale morphism, it follows that $n := \deg(f) \geq 2$. Then $f^{-1}(Y)$ consists of n irreducible components $Y_1 = Y', Y_2, \ldots, Y_n$. By what we said in the proof of theorem (3.1) Y_i is numerically equivalent to $Y_1 = Y'$, $i = 2, \ldots, n$. Moreover, these components do not meet each other. It follows that $Y_1 \cdot f^*(Y) = Y_1 \cdot (Y_1 + Y_2 + \cdots + Y_n) = nY_1 \cdot Y_2 = 0$. Then, by the projection formula, we get $0 = f_*(Y_1 \cdot f^*(Y)) = f_*(Y_1) \cdot Y = Y \cdot Y$, contradicting the hypothesis (ii). This completes the proof of corollary (3.3). $\square$

To state the next consequence of theorem (3.1) we need to recall the following:

Definition (3.4). (Ran-Debarre, see [R], and [De2]) Let Y be a d-dimensional irreducible subvariety ($d > 0$) of an n-dimensional abelian variety X, and let $s \geq 0$ be a non-negative integer. Then Y is said to be *s-geometrically nondegenerate* in X if for every abelian subvariety K of X the image under the canonical morphism $X \to X/K$ is either X/K, or has dimension $\geq d - s$. If Y is 0-geometrically nondegenerate in X one simply says that Y is geometrically nondegenerate ([R]). An irreducible subvariety Y is s-geometrically nondegenerate precisely if for every closed subvariety Z of X such that $\dim(Y) + \dim(Z) \geq \dim(X) + s$ one has $Y \cap Z \neq \emptyset$ (see [De2], (1.5)).

Examples of s-geometrically nondegenerate subvarieties are smooth irreducible (positive dimensional) subvarieties Y of an abelian variety X whose normal bundle $N_{Y|X}$ is s-ample in the sense of Sommese [So], see [De2], (1.3). Every irreducible (positive-dimensional) subvariety Y of a simple abelian variety X is geometrically nondegenerate.

Corollary (3.5). *Let Y be an irreducible d-dimensional subvariety of an n-dimensional abelian variety X, and assume that $d \geq \frac{n}{2}$ and that Y is $(d - [\frac{n}{2}])$-geometrically nondegenerate in X. Then Y is $G3$ in X.*

In particular, every closed irreducible subvariety of dimension $d \geq \frac{n}{2}$ of a simple n-dimensional abelian variety X is $G3$ in X.

Proof. Since $d - [\frac{n}{2}] \leq d - 1$, the hypothesis implies that for every proper abelian subvariety K of X (i.e. $K \neq X$), the image of Y in X/K is at least one dimensional. It follows that Y cannot be contained in any proper abelian subvariety of X, hence Y generates X. Thus, by corollary (3.2) (or better, by [H-M]), Y is $G2$ in X.

To get the result, it will be sufficient to check that condition (ii) of corollary (3.3) is fulfilled. By [K], for general $x \in X$ the intersection of Y with the translate $t_x(Y) = Y + x$ is either empty, or smooth of dimension $2d - n$. In particular, Y and $t_x(Y)$ are in general position for $x \in X$ general. To show that $Y \cdot Y \neq 0$, it remains to show that for every $x \in X$, $Y \cap t_x(Y) \neq \emptyset$. This follows from the above mentioned intersection properties of s-geometrically nondegenerate subvarieties [De2], (1.5). $\square$

Note. The last part of corollary (3.5) was first proved by Hironaka-Matsumura [H-M], and in the analytic context, by Barth [Ba].

Corollary (3.6). *(Sommese [So]) Let Y be a positive dimensional connected subvariety of an abelian variety X. Assume that $\dim(Y) - \operatorname{codim}(Y) \geq m$, where m is the maximum dimension of a proper abelian subvariety of X. Then Y is $G3$ in X.*

Proof. The inequality $\dim(Y) - \operatorname{codim}(Y) \geq m$ implies that Y cannot be contained in any proper subvariety of X, whence Y generates X, and so, by [H-M] or by corollary (3.2), Y is $G2$ in X. As in the proof of corollary (3.5), it will then be sufficient to show that $Y \cap t_x(Y) \neq \emptyset$ $\forall x \in X$. Let K be an arbitrary proper abelian subvariety of X, and denote by $\varphi : X \to X/K$ the canonical morphism. Clerarly, $\dim(\varphi(Y)) \geq \dim(Y) - \dim(K) \geq \dim(Y) - m$. It follows that Y is m-geometrically nondegenerate, and since $\dim(Y) + \dim(t_x(Y)) = 2\dim(Y) \geq \dim(X) + m$ (by hypothesis), by [De2], (1.5) it follows that $Y \cap t_x(Y) \neq \emptyset$ $\forall x \in X$, as claimed. $\square$

(3.7) According to [De1], (2.2), a projective homogeneous space $X = G/P$, not necessarily rational, which is acted on transitively by a connected algebraic group G is said to be *simple* if one of the following three equivalent conditions is satisfied:

(3.7.1) Every closed irreducible positive-dimensional subvariety of X generates X.

(3.7.2) Every effective non-zero divisor on X is ample.

(3.7.3) Every closed subgroup of G containing P of dimension bigger than $\dim(P)$, coincides with G.

The proof of the equivalence of these three conditions uses proposition (1.7), a Bertini theorem for homogeneous spaces and the Nakai-Moishezon criterion (see [De1]). Examples of simple homogeneous spaces include all Grassmannians and the complex projective homogeneous spaces X with $b_2(X) = 1$ (with b_2 the second Betti number). Therefore theorem (3.1) implies:

Corollary (3.8). *Let Y be a closed irreducible subvariety of a projective homogeneous space X. Then the following conditions are equivalent:*

(i) Y generates X.

(ii) The extension $K(X) \to K(\hat{X})$ is algebraic.

(iii) Y is G1 in X, i.e. $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

Proof. The implication (i) implies (ii) is just theorem (3.1). The fact that (iii) implies (i) is immediate. Indeed, assume that (iii) holds and that Y does not generate X. Then the subgroup $P' := G_Y$ of G generated by $G_{Y,p}$ is closed in G and $\neq G$. Then it follows that $\pi(Y)$ is a point, where $\pi : X = G/P \to X' := G/P'$ is the canonical surjective proper morphism. Since $\dim(X') \geq 1$, this contradicts (iii) by [H-M], (2.10).

It remains therefore to prove that (ii) implies (iii). Assume therefore that (ii) holds and that $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) \neq k$. This means that there exists a non-constant formal holomorphic function $\zeta \in A := H^0(\hat{X}, \mathcal{O}_{\hat{X}})$. The argument is practically contained in [H-M], see also the proof of proposition 1.1 of [Ha1], pages 190-192. Indeed, in loc.cit one shows that A is a complete local ring with residue field k, and since $\zeta \in A$ is non-constant, ζ is trancendental over k. It follows that $k[[\zeta]] \subseteq A$, whence $k((\zeta)) \subseteq K(\hat{X})$, which implies that $K(\hat{X})$ is a field extension of k of infinite transcendence degree (because ζ is trancendental over k). It follows that the field extension $K(X) \subset K(\hat{X})$ is transcendental (because $K(X)$ is finitely generated over k, and hence of finite transcendence degree over k), which contradicts (ii). □

Corollary (3.9). *Let Y be a closed irreducible subvariety of a rational projective homogeneous space X. Then the following conditions are equivalent:*

(i) Y generates X.

(ii) Y is G2 in X.

(iii) The extension $K(X) \to K(\hat{X})$ is algebraic.

(iv) Y is G1 in X, i.e. $H^0(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

(v) Y is G3 in X.

(vi) $\mathrm{cd}(X \setminus Y) \leq \dim(X) - 2$ (where $\mathrm{cd}(V)$ denotes the cohomological dimension of an algebraic variety V, see [Ha1] or [Ha2]), i.e. $H^q(X \setminus Y, F) = 0$ for every $q \geq \dim(X) - 1$, and for every coherent sheaf F on $X \setminus Y$.

Proof. The equivalence of (i) and (v) is a direct consequence of corollary (3.8) and of corollary (3.3). Moreover, (v) together with proposition (1.7) imply (vi) - via a result of Speiser (see [Sp1] or [Ha1]). Conversely (vi) implies that Y meets every hypersurface of X, whence by proposition (1.7), Y generates X, i.e. (vi) implies (i). $\square$

(3.10) Before stating the last application let us recall Goldstein's notion of coampleness of a projective homogeneous space X, (see [Go]). For such a homogeneous space X the tangent bundle T_X is generated by its global sections. Consider the projective bundle $\mathbb{P}(T_X)$ associated to T_X. It follows that the tautological line bundle $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ is also generated by its global sections. Consider then the morphism $\varphi : \mathbb{P}(T_X) \to \mathbb{P}^N$ associated to the complete linear system $|\mathcal{O}_{\mathbb{P}(T_X)}(1)|$. By definition the ampleness of X, denoted $\operatorname{amp}(X)$, is the maximum fiber dimension of the morphism φ. By definition the coampleness of X, denoted $\operatorname{ca}(X)$, is given by the formula $\operatorname{ca}(X) := \dim(X) - \operatorname{amp}(X)$. Goldstein studied the coampleness of a rational projective homogeneous space $X = G/P$ over $\mathbb{C}$, showing that for G a semi-simple complex Lie group (which we can always assume) $\operatorname{ca}(X) \geq r$, where r is the minimum of ranks of the simple factors of G. In particular, for a complex rational projective homogeneous space X, $\operatorname{ca}(X) \geq 1$. More precisely, Goldstein computes $\operatorname{ca}(X)$ explicitly in each of the cases corresponding to the Dynkin diagrams (see [Go]). For example, in the case $G = SL_{n+1}(\mathbb{C})$ one has $\operatorname{ca}(X) = n$ for every parabolic subgroup P of G.

Next we will show that subvarieties of homogeneous spaces with small codimension generate the ambient variety.

Lemma (3.11). *Let X be a projective homogeneous space and let Y be a closed irreducible subvariety of X. If $\operatorname{codim}_X(Y) < \operatorname{ca}(X)$ then Y generates X.*

Proof. The argument is completely formal. Let $X = G/P$. Indeed, assume that Y does not generate X. Taking $p = e \bmod P$, let $Q := G_Y$ be the subgroup of G generated by $G_{Y,p}$. Q is a closed subgroup of G containing P. Moreover, Y is a closed subvariety of Q/P. Consider the canonical fibration $X = G/P \to G/Q$ with fiber Q/P. It follows that the normal bundle $N_{F|X}$ of $F := Q/P$ in $X = G/P$ is trivial, and since it is a quotient of $T_X|F$, we get

$$\operatorname{amp}(X) = \operatorname{amp}(T_X) \geq \operatorname{amp}(T_X|F) \geq \operatorname{amp}(N_{F|X}) = \dim(F).$$

It follows that $\operatorname{codim}_X(F) \geq \operatorname{ca}(X)$. Since Y is a subvariety of F, $\operatorname{codim}_X(Y) \geq \operatorname{codim}_X(F)$, whence $\operatorname{codim}_X(Y) \geq \operatorname{ca}(X)$, contradicting the hypothesis. $\square$

Corollary (3.12). *(Faltings [Fa1], [Fa3]) Let Y be a closed irreducible subvariety of a rational projective homogeneous space $X = G/P$ over $\mathbb{C}$. If $\operatorname{codim}_X(Y) < \operatorname{ca}(X)$ then Y is $G3$ in X.*

Proof. Use lemma (3.11) and corollary (3.9). $\square$

Remark. Using corollary (3.8) together with lemma (3.11) one gets the following. For every irreducible subvariety Y of a projective homogeneous space X such that

$\mathrm{codim}_X(Y) < \mathrm{ca}(X)$, the field extension $K(X) \subseteq K(\hat{X}_{/Y})$ is algebraic. If in addition $Y \cdot Y \neq 0$, by corollary (3.3), Y is $G3$ in X.

Now let us recall the following problem raised by Hartshorne (see [Ha2], page 208, Problem 4.8):

Problem (3.13). *(Hartshorne) Let Y be a smooth closed subvariety of a smooth projective variety X of dimension n such that $\dim(Y) \geq \frac{n}{2}$ and the normal bundle $N_{Y|X}$ is ample. Is Y then $G3$ in X?*

In connection with this problem we have the following:

Proposition (3.14). *Let Y be a smooth closed subvariety of a rational homogeneous projective variety X of dimension n such that $\dim(Y) \geq \frac{n}{2}$ and the normal bundle $N_{Y|X}$ is ample. Then Y is $G3$ in X.*

Proof. Since $N_{Y|X}$ is ample, by a result of Hartshorne (see [Ha1], theorem (6.7)), Y is $G2$ in X. (Alternatively, the ampleness of $N_{Y|X}$ implies that Y generates X, so that, instead of using Hartshorne's result, one could use theorem (3.1) above to get that $K(X) \subseteq K(\hat{X}_{/Y})$ is an algebraic field extension.) So, to prove $G3$, in view of theorem 2 above it will be sufficient to check that $Y \cdot Y \neq 0$. To do this, we first apply a result of Kleiman [K] (valid if $\mathrm{char}(k) = 0$) which says that for general $g \in G$ (with G the connected algebraic group acting transitively on X) either $Y \cap gY = \emptyset$, or Y and gY have a non-empty proper intersection. Clearly, we have to rule out the first possibility. Since $N_{Y|X}$ is ample and $\dim(Y) + \dim(gY) = 2\dim(Y) \geq n$ (by hypothesis), we can conclude that $Y \cap gY \neq \emptyset$ by applying a result of Lübke (see [Lu], or [F-L3] for a more general result, with a different proof). $\square$

(3.15) Remarks. (3.15.1) Let Y be an irreducible subvariety of positive dimension of a simple projective homogeneous space X. Then by theorem (3.1) $K(X) \subseteq K(\hat{X}_{/Y})$ is an algebraic field extension. If in addition X is rational, Y is $G3$ in X (by corollary (3.9)). In particular we recover the result of [Bab-H] according to which every closed irreducible positive dimensional subvariety of a Grassmann variety X is G_3 in X.

(3.15.2) Corollary (3.9) shows in particular that in order to get the $G3$ condition (from the $G2$ one) in theorems 2 and 3 of [B-Sch] it is not necessary to appeal to Faltings' connectivity theorem proved in [Fa1]. In particular, it renders the proofs of theorems 2 and 3 of [B-Sch] even more elementary.

(3.15.3) The equivalent conditions of Corollary (3.9) are always satisfied for every irreducible positive dimensional subvariety Y of the projective space $\mathbb{P}^n$, whence corollary (3.9) generalizes Hartshorne's vanishing theorem for $\mathbb{P}^n \setminus Y$ (see [Ha2], theorem 7.5).

4. Further applications

(4.1) As a further direct application of theorem (2.6) we want to give a global proof of a result of Faltings, proved by local methods [Fa2].

Theorem (4.2). *(Faltings [Fa2]) Let X be a closed irreducible subvariety of a projective space P over k (of arbitrary characteristic). Let Y be the intersection of X with r*

hyperplanes $H_1, \dots, H_r$ of P, with $1 \le r < \dim(X)$. Here we do not assume that this intersection is proper. Then Y is $G3$ in X.

Proof. Let E be the image of the canonical map $H^0(P, \mathcal{O}_P(1)) \to H^0(X, \mathcal{O}_X(1))$, and set $T := \mathbb{P}(E)$ (the projective space associated to E). Let $t_0 \in T$ be the point corresponding to $(h_0, \dots, h_r) \in H^0(P, \mathcal{O}_P(1))$ (where h_i is the equation of the hyperplane H_i, $i = 1, \dots, r$). Consider the incidence variety

$$Z := \{(x; [s_1, \dots, s_r]) \in X \times T | s_1(x) = \dots = s_r(x) = 0\}.$$

It is trivial to verify that conditions (i) and (ii) of theorem (2.6) are satisfied. Condition (iii) follows from Bertini's theorem. So, by theorem (2.6), $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension. Moreover, by using the criterion of [H-M] (lemma (1.4)) together with connectivity results (see e.g. [SGA2], éxp. XIII, or [F-L1], theorem 2.1) we see that $K(\hat{X}_{/Y})$ is a field.

It remains to check that Y is $G3$ in X. Suppose Y is not $G3$ in X. Then by corollary (2.4) there exists a proper surjective morphism $f : X' \to X$ such that $f^{-1}(Y)$ is not connected. Considering the Stein decomposition of f, we may assume that f is finite. This however contradicts either [SGA2], XIII, or else, [F-L1] theorem 2.1. $\square$

By using a standard argument involving the join construction, one can show that theorem (4.2) implies the following:

Corollary (4.3). *(Faltings [Fa2]) Let V and W be two irreducible subvarieties of the projective space $\mathbb{P}^n$ such that $\dim(V) + \dim(W) > n$. Then $V \cap W$ is $G3$ in V and in W.*

Theorem (4.2) can be generalized (in characteristic zero) in the following way.

Theorem (4.4). *Let X be a projective irreducible variety over k, with $\mathrm{char}(k) = 0$, and let E be an ample vector bundle on X of rank r, with $1 \le r < \dim(X)$. Assume that E is generated by its global sections. Let $s \in H^0(X, E)$ be a non-zero global section and let us denote by $Y := Z(s)$ the zero locus of s. Then Y is $G3$ in X.*

The argument is similar to the proof of theorem (4.2). One has to use the connectivity of $Z(s)$ (which follows from [So] in case X is smooth, and from [F-L2] in general), as well as suitable Bertini theorems for this more general situation. We leave the details to the reader.

Note (4.5). In [B-Bal] a result similar to theorem (4.6) above has been proved. Specifically, in loc. cit. one didn't assume that E is generated by its global sections, but instead it was assumed that Y has the expected codimension r in X.

(4.6) In [Fa1], Faltings proved the following connectivity result (which generalizes the connectivity theorems proved in [F-H] and in [Han] when $k = \mathbb{C}$):

Theorem (4.7). *(Faltings [Fa1], [Fa3]) Let X be a rational projective homogeneous space over $\mathbb{C}$, and let $f : V \to X \times X$ be a proper morphism from an irreducible variety V.*

(i) If $\operatorname{codim}_{X\times X}(f(V)) \leq \operatorname{ca}(X)$ *then* $f^{-1}(\Delta) \neq \emptyset$*, where* Δ *is the diagonal of* $X \times X$*.*
(ii) If $\operatorname{codim}_{X\times X}(f(V)) < \operatorname{ca}(X)$ *then* $f^{-1}(\Delta)$ *is connected.*

On the other hand, in [B] the connectedness theorem of Fulton and Hansen [F-L] has been generalized and strengthened in the following way:

Theorem (4.8). *(Bădescu [B]) Let* $f : V \to \mathbb{P}^n(e) \times \mathbb{P}^n(e)$ *be a proper morphism from an irreducible variety* V *such that* $\dim(f(V)) > n$*, where* $\mathbb{P}^n(e)$ *denotes the* n*-dimensional weighted projective space over* k *of weights* $e = (e_0, e_1, \dots, e_n)$*. Then* $f^{-1}(\Delta)$ *is* $G3$ *in* V*, where* Δ *is the diagonal of* $\mathbb{P}^n(e) \times \mathbb{P}^n(e)$*.*

We want to show that theorem (2.6) above allows one to strengthen theorem (4.8) in a similar way to get:

Theorem (4.9). *Let* X *be a rational projective homogeneous space over* $\mathbb{C}$*, and let* $f : V \to X \times X$ *be a proper morphism from an irreducible variety* V*. If* $\operatorname{codim}_{X\times X}(f(V)) < \operatorname{ca}(X)$ *then* $f^{-1}(\Delta)$ *is* $G3$ *in* V*.*

Proof. Set $W := f(V)$. By theorem (2.2) it will be sufficient to prove that $W \cap \Delta$ is $G3$ in W (the $G3$ condition is stable by taking inverse images under proper surjective morphisms). By hypothesis, $\operatorname{codim}_{X\times X}(W) < \operatorname{ca}(X)$.

In order to apply theorem (2.6), consider the followings subvariety of $W \times T$, with $T = G$:

$$Z := \{((x, y), g) \in W \times T | g \cdot x = y\}.$$

Denote by q and p the restrictions to Z of the projections $W \times G \to X$ and $W \times G \to G$ respectively. By the transitivity of the action of G on X, q is surjective, so condition (i) of theorem (2.6) is fulfilled. If we set $Y' := \{((x, x), e) | (x, x) \in W \cap \Delta\}$ and $t_0 = e$, then $p^{-1}(t_0)_{\mathrm{red}} = Y'$, whence condition (ii) of theorem (2.6) is also fulfilled because by Faltings' connectivity theorem, p is a proper surjective morphism with connected fibres.

To check the condition (iii), let $u : W' \to W$ be a resolution of singularities of W. Such a resolution exists by Hironaka since we are in characteristic zero. Since W' and Δ are nonsigular and $X \times X$ is a homogeneous space under the action of $G \times G$ on $X \times X$ on each component, we can apply a transversality result due to Kleiman (see [K], theorem 2) to deduce that $u^{-1}(\Delta^g)$ is smooth for every g belonging to a non-empty Zariski open subset U of G, where $\Delta^g := \{(x, gx) | x \in X\}$. On the other hand, by Faltings' connectivity, $u^{-1}(\Delta^g)$ is also connected for every $g \in G$. It follows that $u^{-1}(\Delta^g)$ is irreducible for every $g \in U$, whence $W \cap \Delta^g = p^{-1}(g)$ is also irreducible for every $g \in U$. In this way we checked the condition (iii) of theorem (2.6).

Therefore, by theorem (2.6), $K(W) \to K(\hat{W}_{/W\cap\Delta})$ is an algebraic field extension. Finally, by using Faltings' connectivity theorem again via an argument completely similar to that used in the last part of the proof of theorem (4.2), we infer that $W \cap \Delta$ is $G3$ in W. $\square$

Remarks (4.10). (4.10.1) In view of corollary (2.4), Faltings' connectivity theorem (4.7), (ii) is equivalent to the following statement: for every closed irreducible subvariety W of $X \times X$ (with X a rational projective homogeneous space over a field of characteristic zero) with $\operatorname{codim}(W) < \operatorname{ca}(X)$ the ring $K(\hat{W}_{/W\cap\Delta})$ is a field and $K(W)$ is algebraically closed in $K(\hat{W}_{/W\cap\Delta})$. On the other hand, theorem (4.9) asserts that the natural map $K(W) \to K(\hat{W}_{/W\cap\Delta})$ is actually an isomorphism.

(4.10.2) If we apply theorem (4.9) to $V := Y \times Y$ and to $f := i \times i$, where $i : Y \to X$ is any closed irreducible subvariety of the rational projective homogeneous space X such that $\operatorname{codim}_X(Y) < \frac{1}{2}\operatorname{ca}(X)$, we get that the diagonal embedding Δ_Y is $G3$ in $Y \times Y$. Then by a general fact, this implies the surjectivity of the map

$$\pi_1(\Delta_Y) = \pi_1(Y) \to \pi_1(Y \times Y) = \pi_1(Y) \times \pi_1(Y)$$

induced by the inclusion $\Delta_Y \subset Y \times Y$ of algebraic fundamental groups. Therefore $\pi_1(Y) = 0$, i.e. Y is algebraically simply connected. However this fact could be deduced from Faltings' connectivity theorem in the same way as in [F-L1] for the case $X = \mathbb{P}^n$.

Corollary (4.11). *Let Y be a closed irreducible subvariety of the rational projective homogeneous space X over $\mathbb{C}$ such that* $\operatorname{codim}_X(Y) < \frac{1}{2} \cdot \operatorname{ca}(X)$. *Then the diagonal Δ_Y is $G3$ in $Y \times Y$ and Δ_Y meets every hypersurface of $Y \times Y$. If furthermore Y has (locally) the $\mathbf{S}_2$ property of Serre then the weak Grothendieck-Lefschetz property* $\operatorname{Lef}(Y \times Y, \Delta_Y)$ *holds, i.e. for every vector bundle E on $Y \times Y$ the natural map $H^0(Y \times Y, E) \to H^0(\widehat{Y \times Y}_{/\Delta_Y}, \hat{E})$ is an isomorphism, where $\hat{E}$ denotes the completion of E along Δ_Y.*

Proof. The proof is completely similar to the proof of theorem (0.2) of [B] as soon as we know that Δ_Y meets every hypersurface V of $Y \times Y$. To check this latter fact, consider the embedding $f : V \to X \times X$ obtained as the composition of the embeddings

$$V \subset Y \times Y \subset X \times X.$$

Since $\operatorname{codim}_{X\times X}(V) = \operatorname{codim}_{X\times X}(Y \times Y) + 1 = 2\operatorname{codim}_X(Y) + 1 \leq \operatorname{ca}(X)$, from theorem (4.7), (i) of Faltings it follows that $\Delta \cap V \neq \emptyset$, or else $\Delta_Y \cap V \neq \emptyset$. □

Remark (4.12). In the hypotheses of corollary (4.11) one could ask to what extent the effective Grothendieck-Lefschetz condition $\operatorname{Leff}(Y \times Y, \Delta_Y)$ (which is, at least when Y is smooth, stronger than $G3$) holds (see [SGA2], or [Ha2] for the definition of Leff). The answer is negative in general. Take for example $Y = X$ a projective space. Then $\operatorname{Pic}(X) = \mathbb{Z}[\mathcal{O}_X(1)]$, and if $L := p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_X(1))$, then L is not divisible in $\operatorname{Pic}(X \times X)$, while the restriction of L to Δ_X is isomorphic to $L \otimes L$, whence divisible by 2 in $\operatorname{Pic}(\Delta_X) = \operatorname{Pic}(X)$. By a general elementary well known property (see e.g. [Fa4]) one cannot have $\operatorname{Leff}(X \times X, \Delta_X)$.

Corollary (4.13). *Let Y be a closed irreducible subvariety of the rational projective homogeneous space X, which is locally $\mathbf{S}_2$ and satisfies* $\operatorname{codim}(Y) < \frac{1}{2} \cdot \operatorname{ca}(X)$. *Then every stratified vector bundle on Y is trivial (see [Gr], [Sp2], or [B] for the definition of stratified vector bundles).*

Proof. The result is a consequence of corollary (4.11) via an argument which can be found in [Sp2] (see also [B], proof of theorem (0.3)). □

Remark. A stratified vector bundle on a complex projective variety Y amounts to a vector bundle E on Y endowed with an integrable connection. Corollary (4.14) considerably extends a result of Ogus [Og].

5. Formal-rational functions and meromorphic functions

(5.1) Let X be a complex projective algebraic variety and let Y be a closed irreducible subvariety of X. For an arbitrary complex algebraic variety Z we shall denote by Z^{an} the corresponding analytic space. Then in the analytic category it makes sense to consider the formal completion $\widehat{X^{\mathrm{an}}}_{/Y^{\mathrm{an}}}$. By projectivity assumptions we can apply GAGA to get a canonical isomorphism

$$K(\hat{X}_{/Y}) \cong K(\widehat{X^{\mathrm{an}}}_{/Y^{\mathrm{an}}}),$$

where the second ring is the ring of formal functions in the analytic category (defined in a completely analogous way as in the algebraic setting). Let U be a connected neighbourhood of Y^{an} in X^{an}. Clearly $\mathcal{M}(U) \subseteq K(\widehat{X^{\mathrm{an}}}_{/Y^{\mathrm{an}}})$ and therefore we get a natural inclusion $\mathcal{M}(U) \subseteq K(\hat{X}_{/Y})$.

The Gi conditions ($i = 1, 2, 3$) have the following analytic counterparts:

Definition (5.2). (i) Y is said to be *analytically* $G1$ if for any connected complex neighbourhood U of Y in X we have $H^0(U, \mathcal{O}_U^{\mathrm{an}}) = \mathbb{C}$.

(ii) Y is said to be *analytically* $G2$ if for any connected complex neighbourhood U of Y in X the restriction map $\mathcal{M}(X) \to \mathcal{M}(U)$ is a finite field extension.

(iii) Y is said to be *analytically* $G3$ if for any connected complex neighbourhood U of Y in X the restriction map $\mathcal{M}(X) \to \mathcal{M}(U)$ is an isomorphism.

As in the algebraic case "analytic Gi" implies "analytic $G(i-1)$" for $i = 2, 3$. On the other hand by our remarks above "algebraic Gi" implies "analytic Gi" for $i = 1, 2, 3$. In general the converse is not true. We illustrate this by the following classical famous example due to Serre.

Example (5.3). There is a complex projective geometrically ruled surface $p : X \to B$ over an elliptic curve B and a curve Y in X, which is a section of p, with the following properties (see [Ha1], page 231, example 3.2):

(i) $Y^2 = 0$ (and in particular, $X \setminus Y$ is not affine).

(ii) $(X \setminus Y)^{\mathrm{an}}$ is analytically isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$, and therefore is a Stein manifold.

By condition (ii) Y cannot be $G2$, see [H-M] or also [Ha1]. We are going to show however that Y is analytically $G3$ in X. In particular $K(X) = \mathcal{M}(U) \subsetneq K(\hat{X}_{/Y})$.

More generally, let Y be a closed irreducible subvariety of a smooth complex projective variety X of dimension ≥ 2 such that$(X \setminus Y)^{\mathrm{an}}$ is a Stein manifold. Then it is well known that there exists a strictly plurisubharmonic exhaustion function $\varphi : (X \setminus Y)^{\mathrm{an}} \to \mathbb{R}$, whence $\forall a \in \mathbb{R}$ the open subsets $\{\varphi < a\}$ are relatively compact in $(X \setminus Y)^{\mathrm{an}}$. The unions of Y^{an} and the complements of the adherence (in $(X \setminus Y)^{\mathrm{an}}$) of $\{\varphi < a\}$, $\forall a \in \mathbb{R}$, form a fundamental system of strictly pseudoconcave neighbourhoods of Y^{an} in X^{an}, Then one can apply the well known results of Andreotti to deduce that for every such connected pseudoconcave neighbourhood U of Y^{an} in X^{an}, $\mathcal{M}(X) = K(X) \to \mathcal{M}(U)$ is a finite algebraic field extension, i.e. Y is analytically $G2$ in X.

In fact one can say more. Since every $\zeta \in \mathcal{M}(U)$ can be extended to a meromorphic function to the whole X^{an} because on a Stein manifold Z of dimension ≥ 2 every meromorphic function defined in the complement of a compact set K such that $Z \setminus K$ is connected, can be extended to a unique meromorphic function on Z (see [Ka-Sa]).

Thus, in the above situation one deduces that the canonical map $\mathcal{M}(X) = K(X) \to \mathcal{M}(U)$ is an isomorphism for every connected complex neighbourhood U of Y in X, i.e. Y is even analytically $G3$ in X.

On the other hand we have the following criterion which relates $\mathcal{M}(U)$ to $K(\hat{X}_{/Y})$:

Proposition (5.4). *Let Y be a closed irreducible subvariety of the complex projective variety X such that $K(X) \to K(\hat{X}_{/Y})$ is an algebraic field extension. Then the natural map*

$$\operatorname{dir}\lim_{U} \mathcal{M}(U) \to K(\hat{X}_{/Y}),$$

induced by the inclusions $\mathcal{M}(U) \subseteq K(\hat{X}_{/Y})$, is an isomorphism, where U runs over all connected complex open neighbourhoods of Y in X.

Proof. Let $\zeta \in K(\hat{X}_{/Y})$ be an arbitrary formal-rational function. By hypothesis ζ is algebraic over $K(X)$, whence by theorem (2.3), there exists a finite surjective morphism $f : X' \to X$ and a closed subvariety Y' of X' with the following properties: $f(Y') \subseteq Y$, $f|Y' : Y' \to Y$ is an isomorphism, f is étale along Y', and $\tilde{f}^*(\zeta) \in K(X')$. In particular, for every connected complex open neighbourhood U' of Y' in X', $\tilde{f}^*(\zeta) \in \mathcal{M}(U')$. Since f is étale along Y', for sufficiently small U' the restriction $f|U' : U' \to U := f(U')$ is an isomorphism. This implies that $\zeta \in \mathcal{M}(U)$. Since U' can be chosen to be an arbitrarily small neighbourhood of Y', this yields the conclusion. □

Corollary (5.5). *If (X, Y) is the geometrically ruled surface over B,and Y is the curve on X given in Serre's example (see (5.3)) then the map $K(X) \to K(\hat{X}_{/Y})$ is a transcendental field extension, and in particular, $\mathcal{M}(U) \neq K(\hat{X}_{/Y})$ for every connected complex open neighbourhood of Y in X.*

Proof. Indeed, we saw above that $\mathcal{M}(U) = K(X)$ for every such U. Therefore, if the conclusion of the corollary were not true then by proposition (5.4) it would follow that $K(X) \cong K(\hat{X}_{/Y})$, i.e. Y would be $G3$ in X. However this is not possible because by [H-M] (see also [Ha1]), this latter fact could happen only if $Y^2 > 0$. □

Corollary (5.6). *Let Y be a closed irreducible subvariety of a complex projective homogeneous space X which generates X. Then*

$$K(\hat{X}_{/Y}) \cong \operatorname{dir}\lim_{U} \mathcal{M}(U),$$

where U runs over all connected complex open neighbourhoods U of Y in X.

Proof. Combine theorem (3.1) with proposition (5.4). □

Proposition (5.7). *Let Y be a closed irreducible subvariety of a complex projective homogeneous space X. Then Y is analytically Gi in X if and only if Y is algebraically Gi in X, for $i = 1$ or for $i = 3$.*

Proof. The case $i = 3$ follows from corollary (5.6). Assume that Y is analytically $G1$ in X. By corollary (3.8) it will be sufficient to prove that Y generates X. Assume it does not. In the notations of the proof of corollary (3.8) ((iii) implies (i)), this means that $P' \neq G$, i.e. $\dim(X') > 0$. Then since $\pi(Y)$ is a point of X', one can obviously produce a non-constant holomorphic funcion ζ' in a small connected complex open neighbourhood U' of $\pi(Y)$ in X'. Then $\zeta := \zeta' \circ \pi$ is a noncontant holomorphic function in $U := \pi^{-1}(U')$, which contradicts the fact that Y is analytically $G1$ in X. $\square$

Remark (5.8). The corollary (3.9) implies in particular that every irreducible subvariety Y of positive dimension of a complex projective homogeneous space X with $b_2(X) = 1$ is analytically $G3$ in X. Indeed, such a homogeneous space is necessarily rational. In particular, Y is analytically $G1$ in X. This last fact is a result proved first by Severi-Remmert-Van de Ven (see [Se] if $X = \mathbb{CP}^n$, and [R-VdV] in general). On the other hand, corollaries (3.9) and (3.5) imply the main results of Barth's paper [Ba].

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