"Localization and the Sullivan Fixed Point Conjecture"

Amir H. Assadi

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

University of Wisconsin
Dept. of Mathematics
Madison, WI

USA

MPI/90– 66

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Amir H. Assadi[(1)]

University of Wisconsin (Madison, WI USA),

Max–Planck–Institut für Mathematik (Bonn, FRG)

Introduction:

Let K be a finite dimensional G–space, where G is a p–elementary abelian group, i.e. $G \cong (\mathbb{Z}/p\mathbb{Z})^n$. The Borel–Quillen–Hsiang localization theorem states that $H_G^*(K;\mathbb{F}_p) \longrightarrow H_G^*(K^G;\mathbb{F}_p)$ is an isomorphism modulo $H_G^*(\{\text{point}\};\mathbb{F}_p)$–torsion, where H_G^* is Borel's equivariant cohomology ([B] [Q] [Hw]). The above theorem is not true for infinite dimensional spaces in general. As we shall see below, the Sullivan conjecture implies that such a localization holds for infinite dimensional G–spaces $\mathrm{Map}(E_G,K)$, where $\dim K < \infty$. Conversely, the main result of Section 2 proves that if the Borel–Quillen–Hsiang localization holds for $\mathrm{Map}(E_G,X)$, then $E_G \times X$ is G–homotopy equivalent to $E_G \times K$ with $\dim K < \infty$. Here E_G is the usual universal contractible free G–space. This provides an answer to a problem posed in [A2]. This question and other problems of this nature arise naturally in the geometric and differential topological aspects of transformation groups of manifolds. In particular, at present most methods of constructing group actions on a given manifold yield only infinite dimensional free

(1) Acknowledgement. Much stimulus for the question and the results of the present work was provided during my joint work [AB] and conversations with W. Browder, to whom I would like to express my gratitude.

It is a pleasure to thank G. Carlsson, H. Miller, and J. Lannes for conversations on their proofs of the Sullivan's fixed point conjecture.

The present version of this work was completed at the Max–Planck–Institut für Mathematik (Bonn). I would like to thank Professor F. Hirzebruch for his encouragement and support and the staff of the Institute for their hospitality during my stay at MPI.

Research partially supported by an NSF grant and the Max–Planck–Institut für Mathematik.

G–spaces. See [A1] [AB] [W] and their references.

While the localization theorem applies to p–elementary groups, and the Sullivan conjecture holds only for p–groups, we have formulated our results for all finite groups. The proof of the main topological results, (Theorem 2.4) is reduced to the case of cyclic groups of prime order using an inductive argument. The main tool which provides such a local–to–global passage is the algebraic result (Theorem 1.1) of Section 1 which is a projectivity criterion for integral and modular representations occuring as the cohomology of certain G–spaces.

The proof of our converse of the localization theorem for $G = \mathbb{Z}/p\mathbb{Z}$ does not use the proof of the Sullivan conjecture, but merely a statement of this kind. Therefore, it seems appropriate to present the statement and proofs in a sufficiently flexible manner to accommodate the possible improvements. Since the Borel–Quillen Localization theorem is essentially of homological nature, so are the proofs of our theorems. Thus, "the quasicompletion functors" which are modeled homologically after Bousfield–Kan's completion functors will also work in the context of Section 2. This approach emphasizes those homological properties of these functors which are relevant for our purposes and how they are used in the course of the proof. To apply the converse to the localization theorem, one needs to develop computation tools. At present, Lannes' results in [L] are the best available for $G = (\mathbb{Z}/p\mathbb{Z})^n$. Such results in conjunction with our theorems yield more general results for finite groups which are not necessarily p–elementary abelian. In non–technical terms, let us mention one corollary:

<u>Corollary</u>: Let G be a finite group and let X be a free G–space. Then there exists a finite dimensional G–space K such that $E_G \times K$ is G–homotopy equivalent to X if and only if for each prime $p \mid |G|$, and a representative p–Sylow subgroup $G_p \subseteq G$, there exists a finite dimensional G_p–space $K(p)$ such that $E_{G_p} \times K(p)$ is G_p–homotopy equivalent to

X .

An interesting feature of the localization theorem as pointed out by Quillen in [Q] is that it is valid for compact G–spaces even if they are infinite dimensional. This motivates the following.

Problem: Suppose $G = \mathbb{Z}/p\mathbb{Z}$ and X is a compact G–space. Does the Sullivan fixed point conjecture hold for X ?

Section 1. Algebraic Preliminaries

Let G be a finite group, and let k be an algebraic closure of $\mathbb{F}_p$ = the field with p–elements. All modules are assumed to be finitely generated. A classical result of Rim [R] states that a $\mathbb{Z}G$–module M is $\mathbb{Z}G$–projective if and only if its restrictions $M \mid \mathbb{Z}P$ are $\mathbb{Z}P$–projective for all Sylow subgroups $P \subseteq G$. Chouinard has refined this result [Ch] by replacing the p–Sylow subgroups in Rim's theorem by (maximal) p–elementary abelian subgroups. Thus the projectivity of M is detected by all its restrictions to $M \mid \mathbb{Z}A$ for all p–elementary abelian $A \subseteq G$, i.e. $A \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \dots \oplus \mathbb{Z}_p$. To decide the projectivity of $M \mid \mathbb{Z}A$, it suffices to consider the kA–module $M \otimes k$. Thus, let A be a p–elementary abelian group of rank n and with $\{e_1,\dots,e_n\}$ a set of generators, and let I be the augmentation ideal. It is possible to choose a k–subspace $L \subset I$ with $\dim_k L = r$ and such that $I \cong L \oplus I^2$ as k–vector spaces. Then L generates kA as a k–algebra and for each $\lambda \in L$, $(\lambda+1)^p = 1$. The elements $\sigma \in kA$ of the form $\sigma = \lambda+1$, $\lambda \in L$ (for such an L) are called "shifted units" and the cyclic subgroups $S \equiv \langle \sigma \rangle$ of order p are called "shifted cyclic subgroup". (See [Cj]). In [D] Dade has proved that a given kA–module M is kA–projective (hence kA–free since kA is local) if and only if $M \mid kS$ is kS–projective for all such shifted cyclic subgroups of kA . (Note that almost all shifted cyclic subgroups of kA do not come from cyclic subgroups of A .) We will fix L for the rest of the following discussion.

In [A2], the author proved the following projectivity criterion which will be used in Section 2.

1.1 Theorem. Suppose X is a connected G–space such that for each maximal p–elementary abelian subgroup $A \subsetneq G$, the $H^*(-;k)$–spectral sequence $X \longrightarrow E_A \times_A X \longrightarrow BA$ collapses. Then $\bigoplus_{i>0} H^i(X;k)$ is a projective kG–module if and only if it is projective as a kC–module for every subgroup $C \subsetneq G$ of order p . Similarly, $\bigoplus_{i>0} H^i(X;\mathbb{Z})$ is a projective $\mathbb{Z}G$–module if and only if it is $\mathbb{Z}C$–projective for all cyclic subgroups C of prime order.

Note that if X is a Moore space with G–action and $X^G \neq \emptyset$, then the conditions of Theorem 1.1 are satisfied, and we get a projectivity criterion for the cohomology of Moore spaces with G–action.

Section 2. A Converse to the Localization Theorem

By a "converse" we mean the following. Given a finite dimensional G–space X , the Borel–Quillen–Hsiang theorem tells us how to get information about the cohomology of the fixed point sets for G = p–elementary abelian. Now suppose instead of the finite dimensional G–space X , we are given only the Borel construction $\pi : Y \longrightarrow BG$ (or equivalently the corresponding infinite dimensional free G–space $\tilde{Y} = \pi^*(E_G)$) and the localized equivariant cohomology information of the type in the conclusion of Borel–Quillen–Hsiang theorem. Then we can recover a finite dimensional G–space X whose Borel construction $E_G \times_G X \longrightarrow BG$ is "the same" as $Y \longrightarrow BG$ (in the sense of fibre homotopy equivalence). The key to such a construction is a statement of the type of Sullivan's fixed point conjecture.

In this section, we will use "completion functors", homologically modeled after Bousfield–Kan's completion functors [BK]. For simplicity of exposition, we will assume that our functors are defined for all topological spaces; however, such functors may have smaller domains of definition in the course of applications, in which case, the appropriate modification of the following properties is necessary. Recall that the Bousfield–Kan $\mathbb{F}_p$–completion functor satisfies the following:

(C0) $\mathcal{R}$ is a functor from the category of topological spaces to itself.

(C1) $\mathcal{R}$ commutes with arbitrary disjoint unions and finite products.

(C2) There is a coefficient ring R associated to $\mathcal{R}$ such that if $f : X \longrightarrow Y$ induces an isomorphism $F_* : H_*(X;R) \longrightarrow H_*(Y;R)$, then $\mathcal{R}(f)_* : H_*(\mathcal{R}(X);R) \longrightarrow H_*(\mathcal{R}(Y);R)$ is also an isomorphism.

(C3) There is a full subcategory of topological spaces $Top(\mathcal{R})$ (associated to $\mathcal{R}$) and there is a natural transformation τ : identity $\longrightarrow \mathcal{R}$ which satisfy:

- i) If $\pi_1(X) = 0$, then $X \in Top(\mathcal{R})$.
- ii) If $X \in Top(\mathcal{R})$ then $\mathcal{R}(X) \in Top(\mathcal{R})$.
- iii) For all $X \in Top(\mathcal{R})$, the map $\tau(X) : X \longrightarrow \mathcal{R}(X)$ induces an $H_*(-;R)$–isomorphism.

Definition (i) A functor R satisfying (C0)–(C3) above is called a "quasicompletion functor".

(ii) Let $\mathcal{G}$ be a category of groups and $\mathcal{R}$ be a quasicompletion functor. We say that $\mathcal{R}$ is adapted to $\mathcal{G}$ if the following is satisfied. Here E is a universal contractible G–space.

(C4) For all $G \in \mathcal{G}$ and all finite dimensional G–spaces X such that $H_*(X;R)$

and $H_*(X^G;R)$ are finitely generated, the map of constants

$X^G \longrightarrow \mathrm{Map}_G(E,X)$

induces an isomorphism $H_*(R(X^G);R) \longrightarrow H_*(\mathrm{Map}_G(E,R(X));R)$.

When $R = \mathbb{F}_p$ and R_p is the Bousfield–Kan [BK] $\mathbb{F}_p$–completion, then R_p is adapted to the category of all finite p–groups by the validity of the Sullivan's conjecture mentioned above. For the Bousfield–Kan $\mathrm{Top}(R)$ consists of $\mathbb{F}_p$–good spaces [BK].

Remarks. (a) The condition on $\pi_1(X)$ in (C3) may be weakened to $\overline{H}_*(\pi_1(X);R) = 0$, or even $H_1(X;R) = 0$ in applications.

(b) The condition (C4) is essentially the Sullivan fixed point conjecture which has been proved for p–group independently by G. Carlsson, J. Lannes, and H. Miller. The important special case where G acts trivially on X was done by H. Miller in [M]. See also [C] [L].

(c) Sullivan had stated his conjecture for p–groups. It is worth noticing that the Sullivan fixed point conjecture is not true for G–spaces where G is not a p–group. In [A3] the author has shown that for any finite group G which is not a p–group there exists a fixed–point free G–action on $\mathbb{R}^n$. These easily provide counterexamples. However, one may still ask the following:

Under which circumstances for a finite G–CW complex X the existence of an equivariant map $E_G \longrightarrow X$ implies that $X^G \neq \emptyset$?

We consider first the case $G = \mathbb{Z}/p\mathbb{Z}$. This case is sufficient for many applications.

2.1. Theorem. Let $G = \mathbb{Z}/p\mathbb{Z}$, and let X be a free G–space such that $\pi_1(X) = 0$ and $H_*(X)$ is finitely generated. Let R be the Bousfield–Kan $\mathbb{F}_p$–completion (or any quasicompletion functor adapted to $\mathcal{G} = \{\mathbb{Z}/p\mathbb{Z}\}$ whose coefficient is $\mathbb{F}_p$). The conditions (A0)–(A2) together are necessary and sufficient for the existence of a finite dimensional G–complex Y such that: (i) $Y^G \in \mathrm{Top}(R)$, (ii) $H_*(Y^G)$ is finitely generated, and (iii) $E \times Y$ and X are G–homotopy equivalent.

(A0) $\mathrm{Map}_G(E,R(X))$ belongs to the image of R up to $\mathbb{F}_p$–homology isomorphism.

(A1) There exists a finite dimensional complex $F \in \mathrm{Top}(R)$ with $H_*(F)$ finitely generated, and a map $\eta : F \longrightarrow \mathrm{Map}_G(E,X)$ such that "the induced map" $\hat{\eta} : R(F) \longrightarrow \mathrm{Map}_G(E,R(X))$ induces $H_*(-;\mathbb{F}_p)$–isomorphism. (See the remark below.)

(A2) The map $\lambda : \mathrm{Map}_G(E,R(X)) \longrightarrow \mathrm{Map}(E,R(X))$ induces a Borel–Quillen localized isomorphism in $H^*_G(-;\mathbb{F}_p)$–theory.

Remarks:

1. "The induced map" $\hat{\eta}$ is obtained as follows. The map η of (A1) has an adjoint map $\bar{\eta} : E \times F \longrightarrow X$. Then $\hat{\eta}$ is the adjoint map of the composition

$E \times R(F) \longrightarrow R(E) \times R(F) \cong R(E \times F) \xrightarrow{R(\bar{\eta})} R(X)$.

2. Let us observe that for $G = \mathbb{Z}/p\mathbb{Z}$ and $t \in H^2(G;\mathbb{Z}/p\mathbb{Z})$ nilpotent for p = odd or $t \in H^1(\mathbb{Z}/2\mathbb{Z};\mathbb{F}_2)$ for p = 2, the Tate cohomology $\hat{H}^*(G;\mathbb{F}_p)$ coincides with $H^*(G;\mathbb{F}_p)[\frac{1}{t}]$. When X = point, the localized equivariant cohomology reads: $H^*_G(\mathrm{point};\mathbb{F}_p)[\frac{1}{t}] \cong \hat{H}^*(G;\mathbb{F}_p)$. Thus, we may denote the functor $H^*_G(-;\mathbb{F}_p)[\frac{1}{t}]$ by $\hat{H}^*_G(-;\mathbb{F}_p)$ for short and suggest the properties of Tate cohomology as well.

Proof. Suppose such a Y exists, and let $F = Y^G$. Consider "The map of constants" $F \longrightarrow \mathrm{Map}_G(E,Y)$ which becomes a homology equivalence upon applying R (by virtue of condition (C4)): $H_*(R(F);\mathbb{F}_p) \cong H_*(\mathrm{Map}_G(E,R(Y);\mathbb{F}_p))$. For simplicity of notation, H_* denotes homology with $\mathbb{F}_p$–coefficients throughout this proof.

Since $\mathrm{Map}_G(E,E)$ has the homotopy type of a point, one has

$\mathrm{Map}_G(E,X) \simeq \mathrm{Map}_G(E,E\times Y) \simeq \mathrm{Map}_G(E,E) \times \mathrm{Map}_G(E,Y)$ on the level of path components. Thus one has the map $F \longrightarrow \mathrm{Map}_G(E,X)$ such that the composition below induces an H_*–equivalence:

$$\mathcal{R}(F) \longrightarrow \mathrm{Map}_G(E,\mathcal{R}(Y)) \longrightarrow \mathrm{Map}_G(E,E) \times \mathrm{Map}_G(E,\mathcal{R}(Y)) \longrightarrow$$

$$\longrightarrow \mathrm{Map}_G(E,E \times \mathcal{R}(Y) \longrightarrow \mathrm{Map}_G(E,\mathcal{R}(E \times Y)) \longrightarrow \mathrm{Map}_G(E,\mathcal{R}(X)) \ .$$

(Note that the homotopy fixed–point set is an invariant of G–maps which are non–equivariant homotopy equivalences.) Hence $\mathcal{R}(F) \longrightarrow \mathrm{Map}_G(E,\mathcal{R}(X))$ is also an H_*–equivalence and conditions (A0) and (A1) are seen to be necessary. To see the necessity of (A2), consider the diagram:

$$\begin{array}{ccccccccc}
\mathcal{R}(Y^G) & \xrightarrow{h_1} & Y^G & \longrightarrow & Y & \longrightarrow & \mathrm{Map}(E,E \times Y) & \longrightarrow & \mathrm{Map}(E,X) \\
\downarrow{\scriptstyle\sigma} & & & & & & & & \downarrow \\
\mathrm{Map}_G(E,\mathcal{R}(Y)) & & \longrightarrow & & \mathrm{Map}_G(E,\mathcal{R}(X)) & & \xrightarrow{\lambda} & & \mathrm{Map}(E,\mathcal{R}(X)) \\
& \searrow & & \nearrow & & & & & \\
& & \mathrm{Map}_G(E,\mathcal{R}(E \times Y)) & & & & & &
\end{array}$$

(Diagram 1)

In the above, all maps which are not labeled induce H_*–equivalence, since various spaces involved belong to $\mathrm{Top}(\mathcal{R})$ and $R = \mathbb{F}_p$ by the hypothesis. σ is an H_*–equivalence since $\mathcal{R}$ satisfies (C4). h_1 induces $\hat{H}_G(-;\mathbb{F}_p)$–isomorphism by the Borel–Quillen localization theorem ([Q] [Hw]). It follows that λ also induces such an isomorphism, and condition (A2) is also necessary.

We sketch now the proof that the above conditions are also sufficient. Consider the composition $f_0 \equiv \iota \cdot \eta$ where $\eta : F \longrightarrow Map_G(E,X)$, and $\iota : Map_G(E,X) \longrightarrow Map(E,X)$ are given by (A1) and the inclusion, respectively. The strategy is to add (finitely many) free G–cells to F so that the map f_0 is extended to a highly connected map. One obtains the diagram below:

$$\begin{array}{ccc} F & \xrightarrow{\ \eta\ } & Map_G(E,X) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} \\ Y_0 & \longrightarrow & Map(E,X) \end{array}$$

(Diagram 2)

Here, $Y_0^G = F$ and we may assume that the cofibre of f is a Moore space with finitely generated homology, since $H_*(X)$ is assumed to be finitely generated. At this point, it may be helpful for the reader to consider the special case where $\pi_1(F) = 0$, where (C4) is true without any completion (according to the validity of the Sullivan conjecture in this case.) In this case the proof is much simpler technically since f is readily seen to satisfy the following claim.

We claim that f induces an isomorphism for the functor $\hat{H}_G(-;\mathbb{F}_p)$ so that the reduced homology of the cofibre of f is cohomologically trivial as a G–module. From this claim, the proof of the theorem is completed as follows. Let C_f be the cofibre of f , and $\bar{H}^*(C_f)$ be its reduced homology. Then $\hat{H}(G;\bar{H}^*(C_f)) \cong \hat{H}_G(C_f,\{point\})$ vanishes ($\mathbb{F}_p$–coefficients), which is sufficient for the cohomological triviality of $H^*(C_f;\mathbb{F}_p)$ for $G = \mathbb{Z}/p\mathbb{Z}$. Since $H^*(C_f;\mathbb{Z})$ is finitely generated and we may assume it to be $\mathbb{Z}$–free as well, it follows that it is $\mathbb{Z}G$–projective ([R]). By standard arguments (e.g. [A3] Chapter I and II) we may add free G–cells to Y_0 and extend f to a homological equivalence, which we continue to call $f : Y \longrightarrow Map(E,X)$. Since X and Y are 1–connected, this yields a homotopy equivalence. The evaluation map $\epsilon : E \times \mathrm{Map}(E,X) \longrightarrow X$, $\epsilon(f,e) = f(e)$, $e \in E$, is equivariant and a homotopy equivalence. Hence it is a G–homotopy equivalence since both spaces are G–free. (Note that the action on $E \times \mathrm{Map}(E,X)$ is the diagonal action and the action on $\mathrm{Map}(E,X)$ is by conjugation, i.e. $f^g(\chi) \equiv gf(g^{-1}\chi)$.) This finishes the proof and it remains to establish the claim.

The proof of the claim is based on studying a number of commutative diagrams:

$$\begin{array}{ccccc}
F = Y_0^G & \xrightarrow{h_1} & Y_0 & \xrightarrow{f} & \mathrm{Map}(E,X) \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_1} \\
R(Y_0)^G & \longrightarrow & R(Y_0) & \longrightarrow & R\,\mathrm{Map}(E,X) \\
\downarrow & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_2} \\
\mathrm{Map}_G(E, R(Y_0)) & \longrightarrow & \mathrm{Map}(E, R(Y_0)) & \longrightarrow & \mathrm{Map}(E, R(\mathrm{Map}(E,X))) \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle T} & \swarrow{\scriptstyle \beta_3} & \\
\mathrm{Map}_G(E,R(X)) & \longrightarrow & \mathrm{Map}(E,R(X)) & &
\end{array}$$

(Diagram 3)

$$\begin{array}{ccc}
 & R(F) & \\
{\scriptstyle \sigma}\swarrow & & \searrow \\
\mathrm{Map}_G(E,R(Y_0)) & \xrightarrow{\varphi} & \mathrm{Map}_G(E,R(X))
\end{array}$$

(Diagram 4)

$$
\begin{array}{ccccc}
F & & \xrightarrow{\quad h_1 \quad} & & Y_0 \\
\Big\downarrow{\scriptstyle \gamma_0} & & R(Y_0)^G & & \Big\downarrow{\scriptstyle \alpha_1} \\
 & \nearrow & & \searrow{\scriptstyle h_2} & \\
R(F) & & \dashrightarrow{\;\hat{\theta}\;} & & (Y_0)
\end{array}
$$

(Diagram 5)

In diagram 3 we have the following $\hat{H}_G$–isomorphisms: h_1, α_1, β_1, α_2, β_2 and β_3; and in diagram 4, we get σ and φ induce H_*–isomorphisms. In diagram 5, the dotted arrows exist by the functoriality of R and γ_0 induces an H_*–isomorphism. It follows from spectral sequence arguments that $\hat{\theta}$ induces a Borel–Quillen localized isomorphism. Combining these with a study of the diagram:

$$
\begin{array}{ccccc}
 & \overset{\alpha}{\nearrow} & \mathrm{Map}_G(E,R(Y_0)) & \xrightarrow{\quad h_3 \quad} & \mathrm{Map}(E,R(Y_0)) \\
R(F) & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \bar{f}} \\
 & \underset{\hat{\eta}}{\searrow} & \mathrm{Map}_G(E,R(X)) & \xrightarrow{\quad \lambda \quad} & \mathrm{Map}(E,R(X))
\end{array}
$$

we finally conclude that $\bar{f}$ induces a Borel–Quillen localized isomorphism. This is used in conjunction with a spectral sequence argument to show that in the diagram below f induces a Borel–Quillen localized isomorphism:

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{\quad f \quad} & \mathrm{Map}(E,X) \\
\Big\downarrow{\scriptstyle \alpha_3} & & \Big\downarrow{\scriptstyle \beta_4} \\
\mathrm{Map}(E,R(Y_0)) & \xrightarrow{\quad \bar{f} \quad} & \mathrm{Map}(E,R(X))
\end{array}
$$

Here $\beta_4 = \beta_3 \cdot \beta_2 \cdot \beta_1$. Thus, $\hat{H}_G(C_f, \{\text{Point}\}) = 0$ and the claim is established.

□

For the case of finite complexes, we find obstructions in $\tilde{K}_0(\mathbb{Z}G)$ which are algebraic in nature and may be treated separately from the homotopy–theoretic side of such problems.

2.2. Theorem. Given G,X , and R as in Theorem 2.1, suppose that (A0)–(A2) are satisfied, and in (A1) F is a finite complex with similar properties. Then there is an obstruction $w(X) \in \tilde{K}_0'(\mathbb{Z}G)$ such that $w(X) = 0$ if and only if there exists a finite G–complex Y such that $E \times Y$ and X are G–homotopy equivalent and $Y^G = F$. The obstruction w(X) does not depend on F as long as F satisfies (A1). ($\tilde{K}_0'(\mathbb{Z}G)$ is a certain subquotient of $\tilde{K}_0(\mathbb{Z}G)$ in general).

2.3. Remark. It is not always true that w(X) is independent of F for any finite group G . For $G = \mathbb{Z}/p\mathbb{Z}$, this is a consequence of the triviality of the Swan homomorphism $\sigma_G : (\mathbb{Z}/p\mathbb{Z})^{\times} \longrightarrow \tilde{K}_0(\mathbb{Z}G)$. Thus, in this case if such a finite Y exists, and if F′ is any finite complex which admits an $\mathbb{F}_p$–homotopy equivalence $F' \longrightarrow F$, then there exists a finite G–complex Y′ such that $(Y')^G = F'$ and $E \times Y'$ is G–homotopy equivalent to X as well (cf. [A3]).

2.4. Example. Let R be the Bousfield–Kan $\mathbb{F}_{23}$–completion functor, and let G be the cyclic group of order 23 acting on $M \cong \mathbb{Z}/47\mathbb{Z}$ via the inclusion of $G \subset \text{Aut}(\mathbb{Z}/47\mathbb{Z}) \cong \mathbb{Z}/46\mathbb{Z}$. The calculations of Swan shows that there is no finite G–complex X with $\bar{H}_*(X) \cong M$ as a $\mathbb{Z}G$–module. However, there are finite dimensional G–complexes Y such that $\bar{H}_*(Y) \cong M$ as $\mathbb{Z}G$–modules. For any such Y

$\overline{H}(Y;\mathbb{F}_{23}) = 0$ and $Y^G \in Top(R)$. However, it is not possible to find a finite G–complex K such that $E \times K$ and $E \times Y$ are G–homotopy equivalent.

Next, we briefly outline how we can generalize 2.1 from $\mathbb{Z}/p\mathbb{Z}$ to general finite groups. Let G be a finite group and p be any prime dividing order of G. We define the following sets of subgroups of G :

$$P_p(G) \equiv \{P \subseteq G \,\big|\, |P| \text{ is a p–power }\},\ P(G) \equiv \bigcup_{p\,|\,|G|} P_p(G)\ ;$$

$$A_p(G) \equiv \{(P_1,P_2)\,|\,P_i \in P_p(G),\ i = 1,2;\ P_2 \triangleleft P_1 \text{ and}$$

$$P_1/P_2 \cong (\mathbb{Z}/p\mathbb{Z})^r \text{ for some } r \geq 0\},\ A(G) \equiv \bigcup_{p\,|\,|G|} A_p(G)\ .$$

The following proposition provides us with the necessary conditions for "finiteness" of G–spaces in the appropriate context. A similar result with appropriate modifications hold for finitely dominated G–spaces in the equivariant sense. As pointed out earlier, the recent proofs of the equivariant Sullivan conjecture show that the quasicompletion functors which are used in the following proposition form a nonempty set !

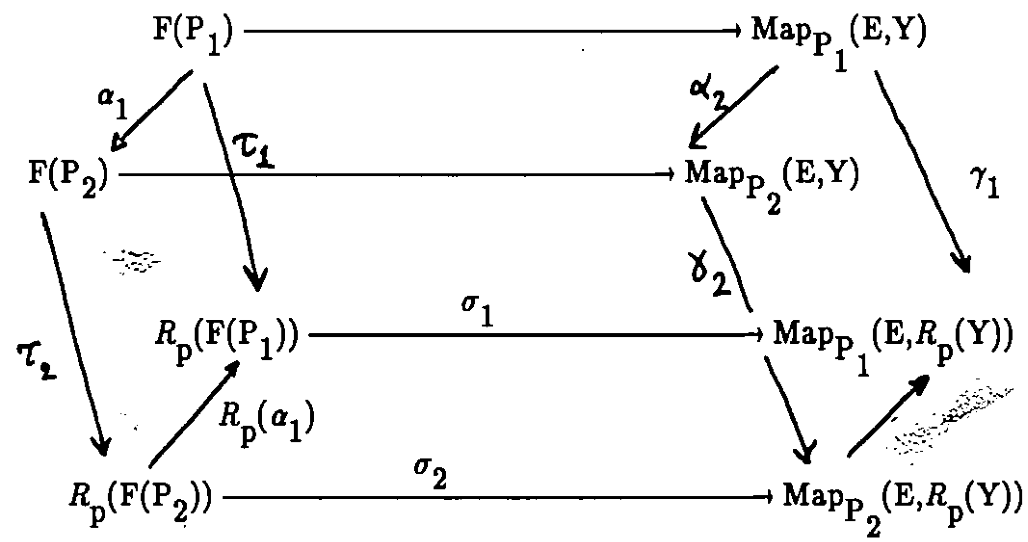
2.5. Proposition. Suppose that G is a finite group and p is any prime dividing $|G|$, and let R_p be the Bousfield–Kan completion or any quasi–completion functor whose associated coefficients is $\mathbb{F}_p$. Assume that Y is a finite dimensional G–space such that $H_*(Y^P;\mathbb{F}_p)$ is finitely generated for each $P \in P_p(G)$ and Y^P belong to $Top(R_p)$. Let X be a free G–space such that $E_G \times Y$ and X are G–homotopy equivalent. Then for each $P \in P(G)$ and each $(P_1,P_2) \in A_p(G)$ the following hold:

(B0) All spaces $\mathrm{Map}_P(E,R_p(X))$ are $H_*(-;\mathbb{F}_p)$–equivalent to spaces in the image of R_p.

(B1) There exist finite dimensional complexes $F(P) \in \mathrm{Top}(R_p)$ with finitely generated $H_*(F(P);\mathbb{F}_p)$ and maps $\eta(P) : F(P) \longrightarrow \mathrm{Map}_P(E,X)$ such that $\hat{\eta} : R_p(F(P)) \longrightarrow \mathrm{Map}_P(E,R_p(X))$ is an $H_*(-;\mathbb{F}_p)$–equivalence.

(B2) The map $\lambda(P_1,P_2) : \mathrm{Map}_{P_1}(E,R_p(X)) \longrightarrow \mathrm{Map}_{P_2}(E,R_p(X))$ induces a Borel–Quillen localized isomorphism for the group $A \equiv P_1/P_2$.

Proof. Let $F(P) = Y^P$ and $\eta(P)$ as in Theorem 2.1 (where $F(P)$ and $\eta(P)$ are denoted by F and η respectively). Since the first two conditions are consequences of the properties of quasi–completion functors as in Theorem 2.1, we will justify the last condition only. Consider the following commutative diagram.



The maps $F(P_i) \longrightarrow \mathrm{Map}_{P_i}(E,Y)$ are given by the maps of constants $Y^{P_i} \longrightarrow \mathrm{Map}_{P_i}(E,Y)$, and the maps τ_i and σ_i induce H_*–isomorphisms, where H_* denotes homology with $\mathbb{F}_p$–coefficients as in 1.1. Moreover, since $\dim F(P_2) < \infty$, α_1

induces a Borel–Quillen localized isomorphism in $\hat{H}_A$–theory, where $A \equiv P_1/P_2$. Thus $\hat{H}_A(F(P_2),F(P_1)) = 0$. Comparison of the Serre spectal sequences of the Borel constructions of various spaces involved show that the map

$$\mathrm{Map}_{P_1}(E,R_p(Y)) \longrightarrow \mathrm{Map}_{P_2}(E,R_p(Y))$$

induces also a Borel–Quillen localized isomorphism as well. Since $E \times Y$ and X are G–homotopy equivalent, the map

$$\lambda : \mathrm{Map}_{P_1}(E,R_p(X)) \longrightarrow \mathrm{Map}_{P_2}(E,R_p(X))$$

induces a Borel–Quillen localized isomorphism, as in Theorem 2.1.

□

2.6. Theorem. Let G, p, and R_p be as in Proposition 2.1 above. Let X be a free G–space such that the conditions B(0)–B(2) of Proposition 2.1 are satisfied. Then there exists a finite dimensional G–space Y such that $H_*(Y^P;\mathbb{F}_p)$ are finitely generated, $Y^P \in \mathrm{Top}(R_p)$ for each $P \in P_p(G)$, and $E \times Y$ and X are G–homotopy equivalent. If the complexes $F(P)$ are taken to be finite complexes, then there exists an obstruction $w(X) \in \tilde{K}_0'(\mathbb{Z}G)$ such that $w(X) = 0$ if and only if Y is G–homotopy equivalent to a finite G–complex.

Outline of proof: In order to prove that such a Y exists, we actually proceed to construct the p–singular set of Y, i.e. $S_p(Y) = \bigcup_{1 \neq P \in P_p(G)} Y^P$ for each $p \mid |G|$, in order to obtain maps $h_p : S_p(Y) \longrightarrow \mathrm{Map}(E,X)$ which are equivariant and such that the induced maps $\hat{h}_p^P : R_p(S_p(Y)^P) \longrightarrow \mathrm{Map}_P(E,R_p(X))$ induce H_*–isomorphisms for each $P \in \mathcal{P}_p(G)$, where $H_* = H_*(-;\mathbb{F}_p)$ as before. By adding free G–cells to $\bigcup_p \mathcal{S}_p(Y)$ we make the map $\bigcup_p h_p : \bigcup_p \mathcal{S}_p(Y) \longrightarrow \mathrm{Map}(E,X)$ highly connected and we obtain $f : Y_0 \longrightarrow \mathrm{Map}(E,X)$ so that the cofibre C_f of f is a Moore space, and $\mathcal{S}_p(Y_0) \equiv \mathcal{S}_p(Y)$. Then we try to show that $\bar{H}_*(C_f;\mathbb{Z})$ is $\mathbb{Z}G$–projective. In cases where we deal with finite complexes, the class $[\bar{H}_*(C_f)] \in \tilde{K}_0(\mathbb{Z}G)$ will represent the finiteness obstruction $w(X)$ which will be only well–defined up to ambiguity arising from different choices of $\mathcal{S}_p(Y)$ in the course of this construction. This leads, then, to a well–defined obstruction, denoted again by $w(X)$ (by abuse of notation) in a subquotient of $\tilde{K}_0(\mathbb{Z}G)$.

In order to show that $\bar{H}_*(C_f;\mathbb{Z})$ is $\mathbb{Z}G$–projective, we use the projectivity criterion Theorem 1.1 to reduce the problem to showing that $H_*(C_f;\mathbb{Z}) \mid \mathbb{Z}C$ is $\mathbb{Z}C$–projective for each $C \subseteq G$, $|C| = p$. But in this case, we are in the situation of Theorem 2.1, since by construction $R_p(Y_0{}^C) \longrightarrow \mathrm{Map}_C(E,R_p(X))$ induces a homology isomorphism, and other conditions are also satisfied, as one can check from the hypothesis. Hence the proof of Theorem 2.1 shows that $H_*(C_f) \mid \mathbb{Z}C$ is $\mathbb{Z}C$–projective for any such C.

Fix a $K \in \mathcal{P}_p(G)$. It remains to show how to construct $\mathcal{S}_p(Y)^K$. We proceed by induction on the lattice of p–subgroups $\mathcal{P}_p(G)$. Suppose that $h^P : \mathcal{S}_p(Y)^P \longrightarrow \mathrm{Map}(E,X)$ is constructed for all subgroups P such that $K \subsetneq P$, $h_p{}^P : R_p(\mathcal{S}_p(Y)^P) \longrightarrow \mathrm{Map}_P(E,R_p(X))$ induces an H_*–isomorphism. Let L denote $\mathcal{S}_p(Y)$ for short. We add free $W(K) \equiv N(K)/K$ cells to L^K and extend it to G–orbits (which are added to L in the usual fashion) so that the map $\alpha : L_0 \longrightarrow \mathrm{Map}(E,X)$ in this way satisfies the following: the cofibre of $\alpha(K) : L_0{}^K \longrightarrow \mathrm{Map}_K(E,K)$, call it $C(\alpha(K))$ has homology (i.e. $\bar{H}_*(-;\mathbb{F}_p)$) only in one dimension, i.e. it is a $H_*(-;\mathbb{F}_p)$–Moore space. Now $\bar{H}_*(C(\alpha(K)))$ is an $\mathbb{F}_p(W)$–module and we claim that it is $\mathbb{F}_p(W)$–free. Using the modular version of the projectivity criterion (Theorem 1.1), we need to check this for each cyclic subgroup of order p, say $C \subset W$, $|C| = p$. We have the exact sequence: $1 \longrightarrow K \longrightarrow K' \longrightarrow C \longrightarrow 1$ where $|K'| = p \cdot |K|$. Hence, by the induction hypothesis $\mathcal{R}_p(L_0{}^{K'}) \longrightarrow \mathrm{Map}_{K'}(E,\mathcal{R})_p(X))$ induces an H_*–isomorphism. Translating this into W–actions, we have $(L_0{}^K)^{C'} = L_0{}^{K'}$ and $\mathcal{R}_p((L_0{}^K)^C \longrightarrow \mathrm{Map}_C(E,\mathcal{R}_p(X)^K)$ is a homology isomorphism. On the other hand, by studying the diagram

$$\begin{array}{ccc} L_0{}^K & \xrightarrow{\quad\alpha(K)\quad} & \mathrm{Map}_K(E,X) \\ \downarrow & & \downarrow \\ (L_0{}^K)^C & \xrightarrow{\quad\ell\quad} & \mathrm{Map}_K(E,X)^C \\ \downarrow = & & \downarrow = \\ L_0{}^{K'} & \xrightarrow{\quad\ell\quad} & \mathrm{Map}_{K'}(E,X) \\ \downarrow & & \downarrow \\ \mathcal{R}_p(L_0{}^{K'}) & \xrightarrow{\quad\ell'\quad} & \mathrm{Map}_{K'}(E,\mathcal{R}_p(X)) \end{array}$$

as in Theorem 2.1, we conclude that $H_*(C(\alpha(K)))$ is cohomologically trivial, hence $\mathbb{F}_pG$–projective. This $\mathbb{F}_pG$–projective module can be killed and the map $\alpha(K)$ will be made more connected so we achieve the inductive step. □

We have the following interesting application:

<u>2.7. Theorem</u>. Let G, p , and $\mathcal{R}_p$ be as in Proposition 2.5. Let X be a G–space such that X and $\mathrm{Map}_P(E,X)$ belong to $\mathrm{Top}(\mathcal{R}_p)$ for each $P \in \mathcal{P}_p(G)$. Then there exists a finite dimensional G–complex K such that $E \times X$ and $E \times K$ are G–homotopy equivalent, if and only if for each cyclic subgroup C_i of order p_i there exists a finite dimensional C_i–complex K_i such that $E \times X$ and $E \times K_i$ are C_i–homotopy equivalent.

□

Section 3. Some Applications and Problems

To show that the theorems of Section 2 are useful, we need to verify the hypotheses in some geometrically interesting situations. This involves, in particular, cohomology computations of some equivariant function spaces, or equivalently, the space of sections of fibrations over $B(\mathbb{Z}/p\mathbb{Z})^n$ arising from Borel constructions. In this respect, J. Lannes'work [L] is quite relevant. Combined with some cohomology calculations of certain classifying spaces, Lannes' theorem leads to finiteness results, from which we derive the validity of the hypotheses of the main theorem 2.1 for $G = \mathbb{Z}/p\mathbb{Z}$. Then Theorem 2.7 allows us to derive the finiteness conclusions for a general finite group.

We recall below the following theorem of Lannes (conjectured by H. Miller in [Mm]). Let π be a p–elementary abelian group, and let K be the category of unstable algebras over the mod p Steenrod algebra. For any space X a homotopy class of maps $B\pi \longrightarrow X$ induces a homomorphism $H^*(X;\mathbb{F}_p) \longrightarrow H^*(B\pi;\mathbb{F}_p)$ in K.

3.1. Theorem (J. Lannes [L]). Let X be a simply–connected space such that $\dim H^i(X;\mathbb{F}_p) < \infty$ for all $i \geq 0$. Then the natural map

$$[B\pi,X] \longrightarrow \mathrm{Hom}_K(H^*(X;\mathbb{F}_p)\,,\ H^*(B\pi;\mathbb{F}_p))$$

is bijective.

The first interesting case that we consider is a classical problem. Let X be a free G–space which is (non–equivariantly homotopy equivalent to the n–sphere S^n.

3.2. Problem. When does there exist a G–action on S^n such that $E_G \times S^n$ is G–homotopic to X ?

In homotopy theory, this is a problem about spherical fibrations. Let $\mathscr{H}_+(S^n)$ be the monoid of self–maps of degree one of S^n. Then the spherical fibration $X \longrightarrow X/G \longrightarrow BG$ is classified by a map $\lambda : BG \longrightarrow B\,\mathscr{H}_+(S^n)$ provided that G acts on X by degree one homeomorphisms. Problem 3.2 now translates into a lifting problem for the fibration $B\,Top_+(S^n) \longrightarrow B\,\mathscr{H}_+(S^n)$ for the map λ. A more refined question is the following:

3.3 Problem. When is a spherical fibration over BG fibre homotopy equivalent to an ortthogonal fibration ?

This problem involves a similar lifting problem for the fibration $B0(n+1) \longrightarrow B\,\mathscr{H}_+(S^n)$ for λ.

According to Theorem 3.1 this is reduced to a lifting problem on the level of cohomology over the Steenrod algebra (which is not an easy problem in general either !). Now let us recall that according to Theorem 2.7, it suffices to solve the lifting problem of 3.2 for $\mathbb{Z}/p\mathbb{Z}$. (Note that Bousfield–Kan's completion [BK] suffices in this case). The lifting problem of 3.3 for $G = \mathbb{Z}/p\mathbb{Z}$ in fact can be solved on the level of cohomology due to deep calculations of the structure of $H^*(B\,\mathscr{H}_+(S^n);\mathbb{F}_p)$ over the Steenrod algebra due to F. Cohen [CLM] and related computations of J. Milgram and Madsen–Milgarm (Cf. [MJ], [Mj] and [MM] for example).

Positive solutions to Problem 3.3 for $G = \mathbb{Z}/p\mathbb{Z}$ and Theorem 2.7 give a partial answer to Problem 3.2. Namely, let X be a free G–space such that $X \simeq S^n$. Then there exists a finite dimensional G–complex K such that $E_G \times K$ is G–homotopy equivalent to X. In fact K may be taken equivariantly finitely dominated in the appropriate context. This result is the first step towards a complete solution of Problems 3.2 and 3.3 via methods of equivariant surgery, and it suggests that there are interesting relationships between Problem 3.3 and Atiyah's theorem on the K–theory of BG (to the effect that K(BG) is the I–adic completion of the representation ring R(G) CF. [At]).

Another interesting case is to consider G–actions on simply–connected Moore spaces. Let X be a Moore space on which a finite group of square–free order acts freely. Suppose that $\bar{H}_*(X)$ has the following property with respect to the induced $\mathbb{Z}G$–module structure: For each prime order subgroup $C \subset G$, $\bar{H}_*(X) | \mathbb{Z}C \cong P \oplus Q$, where P is $\mathbb{Z}C$–projective and Q is indecomposable. (P and Q depend on C). Then there exists a finite dimensional G–space K such that $E_G \times K$ is G–homotopy equivalent to X . The proof of the existence of the G–space K is reduced to the special case $G = \mathbb{Z}/p\mathbb{Z}$, thanks to Theorem 2.7 above. In this case, $\hat{H}^*(C;Q)$ is isomorphic to either $\hat{H}^*(C;\mathbb{Z})$ or $\hat{H}^*(C;I)$, where I is the augmentation ideal. This allows one to modify the arguments (involving the Sullivan fixed point conjecture and Lannes' Theorem 3.1) for the above special case $X = S^n$ in order to construct the desired K .

Finally, the above discussion leads us to the following conjecture which has intersting implications for the topological realizability of homotopy actions and the Steenrod problem, cf. [A2] for related discussions.

<u>3.4. Conjecture</u>. Let M be a finitely generated $\mathbb{Z}$–torsion free $\mathbb{Z}G$–module, where G is a finite group. Suppose that there exists a Moore space X with G–action such that $\bar{H}_*(X)$ is isomorphic to M as $\mathbb{Z}G$–modules. Then there exists a finite dimensional Moore G–space K with the same property.

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