q-Supernomial Coefficients: From Riggings to Ribbons

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Dedicated to Barry M. McCoy on the occasion of his sixtieth birthday

ABSTRACT q-Supernomial coefficients are generalizations of the q-binomial coefficients. They can be defined as the coefficients of the Hall–Littlewood symmetric function in a product of the complete symmetric functions or the elementary symmetric functions. Hatayama et al. give an explicit expression for these q-supernomial coefficients. A combinatorial expression as the generating function of ribbon tableaux with (co)spin statistic follows from the work of Lascoux, Leclerc and Thibon. In this paper we interpret the formulas by Hatayama et al. in terms of rigged configurations and provide an explicit statistic preserving bijection between rigged configurations and ribbon tableaux thereby establishing a new direct link between these combinatorial objects.

1 Introduction

Lattice paths play an important rôle in combinatorics and exactly solvable lattice models. One distinguishes three types of paths: unrestricted, classically restricted and level-restricted paths. Amongst the easiest examples are Dyck paths consisting of up- and down-steps. For fixed $\lambda = (\lambda_1, \lambda_2)$, the set of unrestricted paths contains all paths with λ_1 up-steps and λ_2 down-steps. A path is classically restricted if the number of up-steps is greater or equal to the the number of down-steps in the first k steps for all $1 \leq k \leq \lambda_1 + \lambda_2$. A path is restricted of level ℓ if it is classically restricted and in addition the difference between the number of up- and down-steps in the first k steps does not exceed ℓ for all $1 \leq k \leq \lambda_1 + \lambda_2$. Examples of all three types of paths can be found in Figure 1.

The number of unrestricted paths with λ_1 up-steps and λ_2 down-steps is given by the binomial coefficient $\binom{\lambda_1+\lambda_2}{\lambda_1,\lambda_2} = (\lambda_1 + \lambda_2)!/\lambda_1!\lambda_2!$ which is the expansion coefficient of

$$(x_1 + x_2)^L = \sum_{\substack{\lambda_1,\lambda_2\geq 0\\ \lambda_1+\lambda_2=L}} x_1^{\lambda_1} x_2^{\lambda_2} \binom{L}{\lambda_1, \lambda_2}. \tag{1.1}$$

More generally, the steps of paths associated with the Lie algebra A_{n-1} are Young tableaux over the alphabet $\{1, 2, \ldots, n\}$ [7]. A Young tableau is

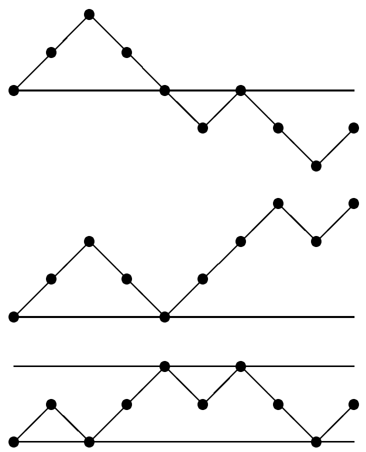


FIGURE 1. Examples of unrestricted, classically restricted and level-restricted paths.

a filling of a partition shape $\tau = (\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq 0)$ which is weakly increasing along rows and strictly increasing along columns. The content of a tableau $\lambda = (\lambda_1, \ldots, \lambda_n)$ records the number occurrences of the various letters in the tableau, i.e., λ_i specifies the numbers of i's in the tableau. In this language, Dyck paths are associated with the algebra A_1, and the up- and down-steps correspond to single-box Young tableaux filled with either 1 or 2.

The number of paths with single-row steps of widths $\mu_1, \mu_2, \ldots, \mu_L$ and total content $\lambda = (\lambda_1, \ldots, \lambda_n)$, denoted by $S_{\lambda\mu}$, is given by the coefficient of $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ in the expansion of $h_{\mu_1} \cdots h_{\mu_L}$ where h_r is the complete symmetric polynomial of degree r in n variables [14, Chap. I.2]. In analogy to (1.1) we have

$$h_{\mu_1} \cdots h_{\mu_L} = \sum_\lambda x^\lambda S_{\lambda\mu}.$$

Since the $S_{\lambda\mu}$ generalize the binomial and multinomial coefficients, they were coined supernomial coefficients (or more precisely completely symmetric supernomial coefficients) in [16, 17].

Similarly, the number of paths with single-column steps with heights $\mu_1, \mu_2, \ldots, \mu_L$ and total content $\lambda = (\lambda_1, \ldots, \lambda_n)$, denoted by $S'_{\lambda\mu}$, is given by the coefficient of $x^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ in the expansion of $e_{\mu_1} \cdots e_{\mu_L}$ where e_r is the elementary symmetric polynomial of degree r in n variables [14, Chap. I.2]. That is

$$e_{\mu_1} \cdots e_{\mu_L} = \sum_\lambda x^\lambda S'_{\lambda\mu}.$$

$S'_{\lambda\mu}$ is called the completely antisymmetric supernomial coefficient.

The one-dimensional configuration sums of exactly solvable lattice models, which are necessary for the calculation of order parameters of these models, require q-analogues of the supernomial coefficients. For example, the q-analogue of the binomial coefficient is the q-binomial coefficient

$$\begin{bmatrix} \lambda_1+\lambda_2 \\ \lambda_1, \lambda_2 \end{bmatrix} = \begin{cases} \frac{(q)_{\lambda_1+\lambda_2}}{(q)_{\lambda_1}(q)_{\lambda_2}} & \text{if } \lambda_1, \lambda_2 \text{ are nonnegative integers,} \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

where $(q)_m = \prod_{i=1}^m (1-q^i)$.

The q-analogues of $S_{\lambda\mu}$ and $S'_{\lambda\mu}$ are the coefficients of the Hall–Littlewood function $P_\mu(x;q)$ [14, Chap. III.2] in the expansion of $h_{\lambda_1}(x)\cdots h_{\lambda_n}(x)$ and $e_{\lambda_1}(x)\cdots e_{\lambda_n}(x)$, respectively. Here $h_r(x)$ and $e_r(x)$ are the complete and elementary symmetric functions in infinitely many variables. The q-supernomial coefficients can be expressed in terms of the Kostka polynomials $K_{\lambda\mu}(q)$ which are the entries of the transition matrix between the Schur function $s_\lambda(x)$ [14, Chaps. I.3] and the Hall–Littlewood function $P_\mu(x;q)$

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(q) P_\mu(x;q). \tag{1.3}$$

Combining (1.3) with the expansions $h_{\lambda_1}(x)\cdots h_{\lambda_n}(x) = \sum_\eta K_{\eta\lambda} s_\eta(x)$ and $e_{\lambda_1}(x)\cdots e_{\lambda_n}(x) = \sum_\eta K_{\eta^t\lambda} s_\eta(x)$ yields

$$S_{\lambda\mu}(q) = \sum_\eta K_{\eta\lambda} K_{\eta\mu}(q) \tag{1.4}$$

and

$$S'_{\lambda\mu}(q) = \sum_\eta K_{\eta^t\lambda} K_{\eta\mu}(q). \tag{1.5}$$

Here $K_{\lambda\mu} := K_{\lambda\mu}(1)$ are the Kostka numbers and η^t is the transpose partition of η obtained by interchanging the rôle of rows and columns.

The q-analogues of the trinomial coefficients, which correspond to the case $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (2^L)$, were extensively studied by Andrews and Baxter [2, 3]. For $\lambda = (\lambda_1, \lambda_2)$ and general μ explicit formulas for $S_{\lambda\mu}(q)$ were given in [16] where they were also used to prove generalizations of Rogers–Ramanujan-type identities. Recently, an alternative definition as characters of coinvariants for one dimensional vertex operator algebras was given in [4]. Hatayama et al. [6, 8] provide an explicit expression for $S_{\lambda\mu}(q)$ and $S'_{\lambda\mu}(q)$ as given in equations (2.1) and (2.4) below. The motivation for these formulas comes from exactly solvable models, where $S_{\lambda\mu}(q)$ and $S'_{\lambda\mu}(q)$ can be interpreted as the generating function of unrestricted paths with an energy statistic which comes from crystal base theory.

Lascoux, Leclerc and Thibon [12] introduced q-supernomial coefficients as the generating function of ribbon tableaux with (co)spin statistic. As was shown in [13] these q-analogues are related to parabolic Kazhdan–Lusztig polynomials for affine symmetric groups. There exists a bijection between L-ribbon tableaux and L-tuples of ordinary tableaux [18]. Recently [15], the (co)spin statistic was translated into inversion statistic on tuples of tableaux under this bijection.

In this paper we show that the expressions (2.1) and (2.4) for the q-supernomial coefficients given below have a combinatorial interpretation in terms of "rigged configurations" similar to the ones introduced by Kerov, Kirillov and Reshetikhin [9, 10, 11] for generating functions of classically restricted paths. In addition we give a statistic preserving bijection between these rigged configurations and ribbon tableaux, thereby providing a new direct link between these combinatorial objects.

The paper is organized as follows. In Section 2 we state the explicit formulas for the q-supernomial coefficients as given by Hatayama et al. [6] and provide the combinatorial description in terms of rigged configurations. In Section 3 we review the generating functions of L-ribbon tableaux with (co)spin statistic and the analogous interpretation in terms of inversion statistic on L-tuples of tableaux. The bijection between ribbon tableaux and rigged configurations is presented in Section 4.

2 Rigged configurations

In this section we present the explicit expressions for the q-supernomial coefficients $S_{\lambda\mu}(q)$ and $S'_{\lambda\mu}(q)$ of Hatayama et al. [6] and provide a combinatorial interpretation of these formulas in terms of rigged configurations.

2.1 Explicit expressions for the q-supernomials

In the following it will be useful to consider a slight variant of the q-supernomial coefficients $S_{\lambda\mu}(q)$ and $S'_{\lambda\mu}(q)$ using a costatistic. Let $n(\mu) = \sum_{1\le i<j\le L} \min(\mu_i,\mu_j)$ for $\mu=(\mu_1,\dots,\mu_L)$. Then define

$$\tilde{S}_{\lambda\mu}(q) = q^{n(\mu)} S_{\lambda\mu}(q^{-1}) \qquad \text{and} \qquad \tilde{S}'_{\lambda\mu}(q) = q^{n(\mu)} S'_{\lambda\mu}(q^{-1}).$$

Let $\lambda=(\lambda_1,\dots,\lambda_n)$ and $\mu=(\mu_1,\dots,\mu_L)$ be partitions. According to [6, 8], the completely symmetric q-supernomial is given by

$$\tilde{S}_{\lambda\mu}(q) = \sum_{\{\nu\}} q^{\Phi(\nu)} \prod_{\substack{1\le a\le n-1\\ 1\le i\le \mu_1}} \begin{bmatrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{bmatrix}, \tag{2.1}$$

where the sum $\sum_{\{\nu\}}$ is over all sequences of partitions $\nu^{(1)},\dots,\nu^{(n-1)}$ such that

$$\begin{aligned} &\emptyset = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu^{(n)} = \mu^t \\ &|\nu^{(a)}| = \lambda_1 + \cdots + \lambda_a \quad \text{for all } 1\le a<n \end{aligned} \tag{2.2}$$

and

$$\Phi(\nu) = \sum_{\substack{1\le a<n \\ 1\le i\le \mu_1}} \nu_{i+1}^{(a)}(\nu_i^{(a+1)} - \nu_i^{(a)}). \tag{2.3}$$

Example 2.1. Take $\lambda = \mu = (2,2,1)$. Then the allowed sequences ν and their contributions to the sum in (2.1) are given by

$$\begin{array}{lllll}
\emptyset \subset & \ydiagram{2} & \subset \ydiagram{2,2} & \subset \ydiagram{3,2} & q^2 \\
\emptyset \subset & \ydiagram{2} & \subset \ydiagram{3,1} & \subset \ydiagram{3,2} & \begin{bmatrix}3\\2,1\end{bmatrix}\begin{bmatrix}2\\1,1\end{bmatrix} \\
\emptyset \subset & \ydiagram{1,1} & \subset \ydiagram{3,1} & \subset \ydiagram{3,2} & q^2\begin{bmatrix}2\\1,1\end{bmatrix} \\
\emptyset \subset & \ydiagram{1,1} & \subset \ydiagram{2,2} & \subset \ydiagram{3,2} & q^3\begin{bmatrix}2\\1,1\end{bmatrix}
\end{array}$$

Hence $\tilde{S}_{\lambda\mu}(q) = 1 + 2q + 4q^2 + 3q^3 + q^4$.

Similarly, the explicit expression for the completely antisymmetric supernomial is given by [6, 8]

$$\tilde{S}'_{\lambda\mu}(q) = \sum_{\{\nu\}} \prod_{\substack{1\le a\le n-1 \\ 1\le i\le \mu_1}} \begin{bmatrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a+1)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a+1)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{bmatrix}, \tag{2.4}$$

where the sum $\sum_{\{\nu\}}$ runs over all sequences of partitions $\nu^{(1)}, \dots, \nu^{(n-1)}$ such that

$$\begin{aligned} &\emptyset = \nu^{(0)} \subset \nu^{(1)} \subset \cdots \subset \nu^{(n-1)} \subset \nu^{(n)} = \mu^t \\ &\nu^{(a)}/\nu^{(a-1)} \quad \text{is a horizontal } \lambda_a\text{-strip.} \end{aligned} \tag{2.5}$$

Here $\nu^{(a)}/\nu^{(a-1)}$ is a skew shape obtained by considering all boxes in $\nu^{(a)}$ not in $\nu^{(a-1)}$, and a horizontal p-strip is a skew shape with p boxes such that every column contains at most one box.

Example 2.2. For the antisymmetric case also take $\lambda = \mu = (2,2,1)$. Then the allowed sequences ν and their contributions to the sum in (2.4) are given by

$$\begin{array}{lllll}
\emptyset \subset & \ydiagram{2} & \subset \ydiagram{2,2} & \subset \ydiagram{3,2} & 1 \\
\emptyset \subset & \ydiagram{2} & \subset \ydiagram{3,1} & \subset \ydiagram{3,2} & \begin{bmatrix}2\\1,1\end{bmatrix}^2
\end{array}$$

Hence $\tilde{S}'_{\lambda\mu}(q) = 2 + 2q + q^2$.

2.2 Combinatorial interpretation in terms of rigged configurations

The main tool for the combinatorial interpretation of (2.1) and (2.4) is the interpretation of the q-binomial coefficient $\begin{bmatrix} m+p \\ m,p \end{bmatrix}$ as the generating function of partitions in a box of width m and height p (see [1, Theorem 3.1]). That is

$$\begin{bmatrix} m+p \\ m,p \end{bmatrix} = \sum_{\lambda \subset (m^p)} q^{|\lambda|}$$

where $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$.

For the completely symmetric q-supernomials $\tilde{S}_{\lambda\mu}(q)$, set $m_i^{(a)} = \nu_i^{(a)} - \nu_{i+1}^{(a)}$ and $p_i^{(a)} = \nu_i^{(a+1)} - \nu_i^{(a)}$. The $p_i^{(a)}$ are called vacancy numbers. For fixed ν, each term in the summand of (2.1) corresponds to a labeling or rigging of ν in the following way. Label each column of $\nu^{(a)}$ with a quantum number j. If the column in $\nu^{(a)}$ has height i, the quantum number j has to be an integer satisfying $0 \le j \le p_i^{(a)}$. If $j = p_i^{(a)}$ it is called a singular quantum number. Riggings which differ only by reordering of quantum numbers corresponding to columns of the same height in a partition are identified. Hence for each partition $\nu^{(a)}$ and column height i we may view the riggings as a partition $J_i^{(a)}$ in a box of width $m_i^{(a)}$ and height $p_i^{(a)}$. A configuration together with an admissible rigging is called a rigged configuration.

Example 2.3. Continuing example 2.1,

$$\begin{array}{|c|c|l} \hline 1 & 0 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|l} \cline{1-1} 1 & \multicolumn{3}{l}{1} \\ \hline & 0 & 0 & 0 \\ \hline \end{array}$$

is a valid rigging of the second sequence, where the vacancy number $p_i^{(a)}$ is written on the right of row i in $\nu^{(a)}$ and the quantum numbers are put in the highest box of each column.

Let us denote the set of all rigged configurations corresponding to λ and μ by $\mathrm{RC}(\lambda, \mu)$. To each $(\nu, J) \in \mathrm{RC}(\lambda, \mu)$ associate a statistic

$$\tilde{c}(\nu, J) = \Phi(\nu) + \sum_{\substack{1 \le a \le n-1 \\ 1 \le i \le \mu_1}} |J_i^{(a)}|. \tag{2.6}$$

Then the combinatorial analogue of (2.1) in terms of rigged configurations is

$$\tilde{S}_{\lambda\mu}(q) = \sum_{(\nu,J) \in \mathrm{RC}(\lambda,\mu)} q^{\tilde{c}(\nu,J)}.$$

For the completely antisymmetric q-supernomials $\tilde{S}'_{\lambda\mu}(q)$, set $m_i^{(a)} = \nu_i^{(a)} - \nu_{i+1}^{(a+1)}$ and $p_i^{(a)} = \nu_i^{(a+1)} - \nu_i^{(a)}$. In this case attach labels to the last

$m_i^{(a)}$ boxes in the i-th row of $\nu^{(a)}$. A label j in row i of $\nu^{(a)}$ should satisfy $0 \le j \le p_i^{(a)}$. If $j = p_i^{(a)}$ the quantum number j is called singular. As in the symmetric case, riggings which differ by reordering of the labels within the same row of $\nu^{(a)}$ are identified. That is, to each $\nu^{(a)}$ and row i one may associate a partition $J_i^{(a)}$ which lies in a box of width $m_i^{(a)}$ and height $p_i^{(a)}$.

Example 2.4. The configurations of example 2.2 admit the following riggings, for example,

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|l}\cline{1-2} 0 & 0 & \,0 \\ \cline{1-2} & & \\ \cline{1-2} \end{array}$$

$$\begin{array}{|c|c|l}\cline{1-2} & 0 & \,1 \\ \cline{1-2} \end{array} \qquad\qquad \begin{array}{|c|c|c|l}\cline{1-1} 1 & \multicolumn{3}{l}{\,1} \\ \cline{1-3} & & 0 & \,0 \\ \cline{1-3} \end{array}$$

where the vacancy numbers are denoted to the right of each row.

Denote the set of all antisymmetric rigged configurations by $\mathrm{RC}'(\lambda, \mu)$. The statistic associated with $(\nu, J) \in \mathrm{RC}'(\lambda, \mu)$ is

$$\tilde{c}'(\nu, J) = \sum_{\substack{1 \le a \le n-1 \\ 1 \le i \le \mu_1}} |J_i^{(a)}|. \tag{2.7}$$

Then the combinatorial analogue of (2.4) in terms of rigged configurations is

$$\tilde{S}'_{\lambda\mu}(q) = \sum_{(\nu,J) \in \mathrm{RC}'(\lambda,\mu)} q^{\tilde{c}'(\nu,J)}.$$

3 Ribbon tableaux

Lascoux, Leclerc and Thibon [12] defined q-supernomial coefficients in terms of ribbon tableaux. Ribbon tableaux are the natural objects in the combinatorial description of the power-sum plethysm operators on symmetric functions. Here we give a brief review of the definition of the q-supernomial coefficients in terms of ribbon tableaux [12] and the reformulation in terms of inversion statistic on tuples of Young tableaux [15].

3.1 Cospin statistic

To define L-ribbon tableaux we will mimic the definition of Young tableaux as chains in Young's lattice. We begin with the review of the definition of Young tableaux.

Containment defines a partial order on partitions. If ν is contained in λ we write $\nu \subset \lambda$. Young's lattice $\mathcal{Y}$ is the set of partitions under the partial order $\subset$. A horizontal strip is a skew shape that has at most one cell in each column. A Young tableau of shape λ/ν and weight $\mu = (\mu_1, \dots, \mu_r)$ is a chain of partitions

$$\nu = \alpha^0 \subset \alpha^1 \subset \cdots \subset \alpha^r = \lambda$$

such that α^i/α^{i-1} is a horizontal strip with μ_i cells. As before, a Young tableau is represented graphically by the diagram of λ/ν where the cells in α^i/α^{i-1} are numbered with i. The set of all Young tableaux of shape λ/ν and weight μ is denoted by $\mathrm{Tab}(\lambda/\nu, \mu)$.

An L-ribbon is a connected skew shape consisting of L cells, which does not contain any 2×2 squares. The rightmost and lowermost cell is called the origin of the ribbon. Define the spin of a ribbon R by $\mathrm{spin}(R) = h(R) - 1$ where $h(R)$ is the height of R.

The relation $\nu \lessdot_L \mu$ on $\mathcal{Y}$ means that $\nu \subset \mu$ and the skew shape μ/ν is an L-ribbon; it is the covering relation for a partial order $\leq_L$ on $\mathcal{Y}$. Each component of the poset $(\mathcal{Y}, \leq_L)$ has a unique minimum. The $\leq_L$-minima are called L-cores. The L-ribbon lattice $\mathcal{R}_L$ is the component of the empty partition $\emptyset$ in $(\mathcal{Y}, \leq_L)$.

The skew shape μ/ν is a horizontal L-ribbon strip of weight m if there is a saturated chain $\nu = \alpha^0 \lessdot_L \alpha^1 \lessdot_L \cdots \lessdot_L \alpha^m = \mu$ such that the origin of each ribbon $R_i = \alpha^i/\alpha^{i-1}$ is in the lowermost cell of its column in μ/ν. If such a saturated chain exists it is unique if we require in addition that the origin of R_i is to the right of R_{i-1}.

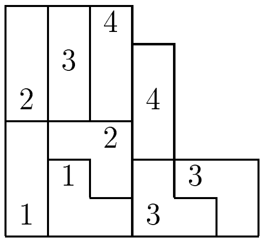
An L-ribbon tableau T of shape μ/ν and weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is a chain of partitions

$$\nu = \alpha^0 \leq_L \alpha^1 \leq_L \cdots \leq_L \alpha^n = \mu$$

such that α^i/α^{i-1} is a horizontal L-ribbon strip of weight λ_i. Graphically, T may be represented by its L-ribbons where each L-ribbon in α^i/α^{i-1} is numbered by i. The spin of an L-ribbon tableau T is the sum of the spins of its ribbons. Let $\mathrm{Tab}_L(\mu/\nu, \lambda)$ be the set of all L-ribbon tableaux of shape μ/ν and weight λ. The cospin of $T \in \mathrm{Tab}_L(\mu/\nu, \lambda)$ is defined as

$$\mathrm{maxspin}(\mu/\nu) = \max\{\mathrm{spin}(S) \mid S \in \mathrm{Tab}_L(\mu/\nu, \cdot)\}$$
$$\mathrm{cospin}(T) = \frac{1}{2}(\mathrm{maxspin}(\mu/\nu) - \mathrm{spin}(T)).$$

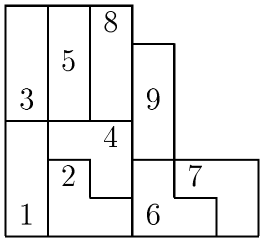
Example 3.1. The 3-ribbon tableau T



of shape $\mu = (6,6,4,4,4,3)$ and weight $\lambda = (2,2,3,2)$ has $\mathrm{spin}(T) = 14$ and $\mathrm{cospin}(T) = (16-14)/2 = 1$.

A standard L-ribbon tableau of shape μ/ν is a saturated $\le_L$-chain from ν to μ. Equivalently, it is an L-ribbon tableau of weight $(1^{|\mu/\nu|/L})$. The standardization $\mathrm{st}(T)$ of an L-ribbon tableau T is the standard L-ribbon tableau obtained by joining together the saturated chains of all its horizontal L-ribbon strips.

Example 3.2. The standardization $\mathrm{st}(T)$ of the tableau in example 3.1 is



The generating function of L-ribbon tableaux with spin and cospin statistics are defined as follows

$$\begin{aligned} R_L(\mu/\nu,\lambda) &= \sum_{T\in \mathrm{Tab}_L(\mu/\nu,\lambda)} q^{\mathrm{spin}(T)} \\ \tilde{R}_L(\mu/\nu,\lambda) &= \sum_{T\in \mathrm{Tab}_L(\mu/\nu,\lambda)} q^{\mathrm{cospin}(T)}. \end{aligned} \tag{3.1}$$

3.2 Inversion statistic

As was mentioned earlier, the set of L-ribbon tableaux can be identified with the set of L-tuples of Young tableaux. We follow mainly ref. [15] in this exposition.

Denote by $\mathcal{Y}^L$ the L-fold direct product of the poset $\mathcal{Y}$, which is by definition the set of L-tuples of partitions $\mu^\bullet = (\mu^0, \mu^1, \dots, \mu^{L-1})$, also called L-multipartitions. The set $\mathcal{Y}^L$ has a partial order $\nu^\bullet \subset \mu^\bullet$ given by $\nu^i \subset \mu^i$ for all $0 \le i \le L-1$. Say that $\mu^\bullet/\nu^\bullet$ is a horizontal L-multistrip of weight m if it is an L-tuple of horizontal strips with a total number of m cells. Then an L-multitableau of shape $\mu^\bullet/\nu^\bullet$ and weight $\lambda = (\lambda_1, \dots, \lambda_n)$ is a chain

$$\nu^\bullet = \alpha^{\bullet 0} \subset \alpha^{\bullet 1} \subset \cdots \subset \alpha^{\bullet n} = \mu^\bullet \tag{3.2}$$

such that $\alpha^{\bullet i}/\alpha^{\bullet i-1}$ is a horizontal L-multistrip of weight λ_i. Equivalently, an L-multitableau T may be viewed as an L-tuple of Young tableau $T = (T^0, T^1, \dots, T^{L-1})$. The set of all L-multitableaux of shape $\mu^\bullet/\nu^\bullet$ and weight λ is denoted by $\mathrm{Tab}^L(\mu^\bullet/\nu^\bullet, \lambda)$.

Littlewood's L-quotient map gives a poset isomorphism $\mathrm{quot}_L : \mathcal{R}_L \to \mathcal{Y}^L$ (see [5]). We use the following normalization of the L-quotient. Let λ be a partition. By appending zeros it may be assumed that the number of parts of λ is a multiple mL of L. Define the staircase partition $\rho^{(r)} = (r-1, r-2, \ldots, 1, 0) \in \mathbb{Z}^r$. It can be shown that λ has empty L-core (that is, $\lambda \in \mathcal{R}_L$) if and only if $\lambda + \rho^{(mL)}$ has exactly m parts that are congruent to i mod L for each $0 \le i \le L-1$. In that case, define the L-quotient $\mathrm{quot}_L(\lambda)$ of λ to be the L-tuple of partitions $(\lambda^0, \lambda^1, \ldots, \lambda^{L-1})$ where the partition $L(\lambda^i + \rho^{(m)}) + i$ coincides with the partition obtained by selecting the parts of $\lambda + \rho^{(mL)}$ that are congruent to i modulo L. This definition is independent of m.

The map quot_L induces a weight-preserving bijection

$$\mathrm{quot}_L : \mathrm{Tab}_L(\mu, \lambda) \to \mathrm{Tab}^L(\mu^\bullet, \lambda) \tag{3.3}$$

from L-ribbon tableaux of shape μ to L-multitableaux of shape $\mu^\bullet = \mathrm{quot}_L(\mu)$. This map is induced by observing that $\nu \le_L \mu$ if and only if $\mathrm{quot}_L(\nu) \subset \mathrm{quot}_L(\mu)$, and λ/ν is a horizontal L-ribbon strip if and only if $\mathrm{quot}_L(\mu)/\mathrm{quot}_L(\nu)$ is a horizontal L-multistrip. The bijection in (3.3) is called the Stanton–White bijection [18].

Example 3.3. Under quot_3 the 3-ribbon tableau T of example 3.1 becomes

$$\mathrm{quot}_3(T) = \left(\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c}\hline 4 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array} \right).$$

A standard L-multitableau of shape $\mu^\bullet/\nu^\bullet$ is a saturated chain in $\mathcal{Y}^L$ from $\nu^\bullet$ to $\mu^\bullet$. A cell in an L-multipartition $\mu^\bullet$ is a triple $s = (i, j, p) \in \mathbb{Z}^2 \times \{0, 1, \ldots, L-1\}$ such that $(i, j) \in \mu^p$. Write $\mathrm{pos}(s)$ for the index p and set $\mathrm{row}(s) = i$, $\mathrm{col}(s) = j$ and $\mathrm{diag}(s) = j - i$. Define the partial order $\preceq$ on elements of $\mathbb{Z}^2 \times \{0, 1, \ldots, L-1\}$ by $s \prec s'$ if $\mathrm{diag}(s) < \mathrm{diag}(s')$, or if $\mathrm{diag}(s) = \mathrm{diag}(s')$ and $\mathrm{pos}(s) < \mathrm{pos}(s')$. The standardization of an L-multitableau T of shape $\mu^\bullet/\nu^\bullet$ is the saturated chain in $\mathcal{Y}^L$ compatible with (3.2) such that the cells within each horizontal L-multistrip are added in $\preceq$-increasing order.

Example 3.4. The standardization of the 3-multitableau in example 3.3 is

$$\left(\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 5 & 9 \\ \hline 4 & 6 \\ \hline \end{array}, \begin{array}{|c|c}\hline 8 & \\ \hline 2 & \multicolumn{1}{c|}{7} \\ \hline \end{array} \right).$$

The set of L-ribbon tableaux is endowed with the spin and cospin statistic. Under the bijection quot_L the cospin statistic on $\mathrm{Tab}_L(\mu, \cdot)$ becomes an inversion statistic on $\prod_{i=0}^{L-1} \mathrm{Tab}(\mu^i, \cdot)$ as shown in [15]. We define the inversion statistic on standard L-multitableaux; the inversion statistic of a general L-multitableau is the inversion of its standardization.

Let T be a standard L-multitableau of shape $\mu^\bullet$. For $s \in \mu^\bullet$ write $T(s)$ for the value of the cell s in T, which is the index i such that s is in the i-th multipartition in the chain T but not in the $(i-1)$-st.

Given cells $s, t \in \mu^\bullet$, say that (s, t) is an inversion of T if the following conditions hold:

1. $\mathrm{diag}(s) = \mathrm{diag}(t)$ and $\mathrm{pos}(s) < \mathrm{pos}(t)$, or $\mathrm{diag}(s) = \mathrm{diag}(t) - 1$ and $\mathrm{pos}(s) > \mathrm{pos}(t)$. Note that in either case $s \prec t$.
2. $\mathrm{row}(s) \leq \mathrm{row}(t)$.
3. $T(t) < T(s) < T(t\uparrow)$ (where $t\uparrow$ is the cell directly above t and $T(t\uparrow) = \infty$ if $t\uparrow \notin \mu^\bullet$).

Write $\mathrm{inv}(T)$ for the number of inversions of T. A standard multitableau T of shape $\mu^\bullet = ((1), \ldots, (1))$ can be identified with a permutation of the set $\{1, 2, \ldots, L\}$ and in this case $\mathrm{inv}(T)$ is the usual inversion number.

Theorem 3.1 ([15]). *For $T \in \mathrm{Tab}_L(\mu, \cdot)$,*

$$\mathrm{cospin}(T) = \mathrm{inv}(\mathrm{quot}_L(T)).$$

Example 3.5. The only inversion of the tableau T of example 3.4 is (s, t) where $T(s) = 4$ and $T(t) = 2$ so that $\mathrm{inv}(T) = 1$ which agrees with the cospin of the 3-ribbon tableau of example 3.1.

4 Statistic preserving bijection

In this section we give a bijection from L-multitableaux to rigged configurations in the symmetric (all tableaux in the multitableaux are single rows) and the antisymmetric (all tableaux in the multitableaux are single columns) case. This bijection preserves the statistics, which means that the inversion statistic on multitableaux equals the statistic on rigged configurations under this bijection. In conjunction with the Stanton–White bijection of the previous section, this defines a bijection from certain ribbon tableaux to rigged configurations.

4.1 Symmetric case

Let $\mu^\bullet = (\mu^0, \mu^1, \ldots, \mu^{L-1})$ be a multipartition where the μ^i are single rows. The bijection

$$\Psi : \mathrm{Tab}^L(\mu^\bullet, \lambda) \to \mathrm{RC}(\lambda, \mu),$$

where μ is the partition with parts $|\mu^i|$, is defined recursively. Let $T = (T^0, \ldots, T^{L-1}) \in \mathrm{Tab}^L(\mu^\bullet, \lambda)$ where the entries of each tableau in this sequence are denoted by $T^k = t_1^k t_2^k \ldots t_{|\mu^k|}^k$. The L-multitableau T is built up by successively adding the letters t_i^k for $k = L-1, L-2, \ldots, 0$ and $i = 1, 2, \ldots, |\mu^k|$. The addition of the letter t_i^k to T corresponds to an analogous operation on rigged configurations given by the following algorithm:

1. Define $(\nu, J)_{0,L-1}$ to be the empty rigged configuration in $\mathrm{RC}(\lambda, \mu)$.

2. Suppose the rigged configuration $(\nu, J)_{i-1,k}$ which corresponds to the L-multitableau built up to letter t^k_{i-1} is already known. In this notation we identify $(\nu, J)_{0,k}$ with $(\nu, J)_{|\mu^{k+1}|,k+1}$ and set $t^k_0 = 1$. Denote by $(\nu, J)^{(a)}_{i-1,k}$ the a-th rigged partition in $(\nu, J)_{i-1,k}$. Adding the entry t^k_i to the k-th tableau in T with $t^k_i \geq t^k_{i-1}$ corresponds to the following operation on the rigged configuration $(\nu, J)_{i-1,k}$. For all $t^k_i \leq a < n$, add a box to row i in $(\nu, J)^{(a)}_{i-1,k}$ such that a singular box is removed from row $i-1$ and the new box in row i is singular. The resulting rigged configuration is $(\nu, J)_{i,k}$.

3. The rigged configuration $(\nu, J) = \Psi(T)$ is obtained from $(\nu, J)_{|\mu^0|,0}$ by inverting all quantum numbers. More precisely, a quantum number j in row i of the a-th rigged partition is replaced by $p^{(a)}_i - j$.

Example 4.1. Take the 3-multitableau $T = (\boxed{2\,|\,3}\,, \boxed{1\,|\,1}\,, \boxed{1\,|\,3\,|\,4})$ which has inversion statistic $\mathrm{inv}(T) = 3$. The corresponding rigged configuration $\Psi(T)$ is

$$
\begin{array}{|c|l} \hline 0 & 0 \\ \hline \end{array}\!\!\!\!\begin{array}{c}\end{array}
\qquad
\begin{array}{c}
\begin{array}{|c|c} \hline 0 & 0 \\ \hline \end{array} \\[-1pt]
\begin{array}{|c|c|l} \hline \ \ & 1 & 1 \\ \hline \end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{|c|c} \hline 1 & 2 \\ \hline \end{array} \\[-1pt]
\begin{array}{|c|c|c|l} \hline \ \ & 0 & 0 & 0 \\ \hline \end{array}
\end{array}
\qquad
\begin{array}{c}
\begin{array}{|c|c|c|l} \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \\[-1pt]
\begin{array}{|c|c|c|} \hline \ \ & \ \ & \ \ \\ \hline \end{array}
\end{array}
$$

with $\tilde{c}(\Psi(T)) = 3$. All intermediate steps in the algorithm for the calculation of $\Psi(T)$ are listed in Figure 2. $\Psi(T)$ is obtained from the last rigged configuration by inverting all quantum numbers.

Theorem 4.1. *The map* $\Psi : \mathrm{Tab}^L(\mu^\bullet, \lambda) \to \mathrm{RC}(\lambda, \mu)$ *defined by the above algorithm is a statistic preserving bijection. That is,* $\mathrm{inv}(T) = \tilde{c}(\Psi(T))$ *for* $T \in \mathrm{Tab}^L(\mu^\bullet, \lambda)$.

Proof. First we prove that Ψ defines a bijection. To this end it needs to be shown that Ψ is well-defined and that the algorithm is invertible.

Adding the letter t to the i-th column of a tableau in T adds a box in row i to the a-th rigged partition for all $t \leq a < n$. This operation leaves all vacancy numbers unchanged, except $p^{(t-1)}_i$ which changes by $+1$. Since $t^k_i \geq t^k_{i-1}$ there hence exists a singular box in row $i-1$ in $(\nu, J)^{(a)}_{i-1,k}$ for all $t^k_i \leq a < n$. This shows that Ψ is well-defined.

It is easy to see that the algorithm for Ψ is invertible. Given $(\nu, J) \in \mathrm{RC}(\lambda, \mu)$ one constructs $T = (T^0, \ldots, T^{L-1}) \in \mathrm{Tab}^L(\mu^\bullet, \lambda)$ with $T^k = t^k_1 t^k_2 \cdots t^k_{|\mu^k|}$ by successively determining t^k_i for $k = 0, 1, \ldots, L-1$ and $i = |\mu^k|, |\mu^k - 1|, \ldots, 1$. Set $(\nu, J)_{|\mu^0|,0}$ to be (ν, J) with inverted quantum numbers and let $(\nu, J)_{i+1,k}$ be the rigged configuration after determining t^k_{i+1}. To determine t^k_i, one finds the smallest index t such that there is a singular string in row i of $(\nu, J)^{(a)}_{i+1,k}$ for all $t \leq a < n$. Set $t^k_i = t$. The

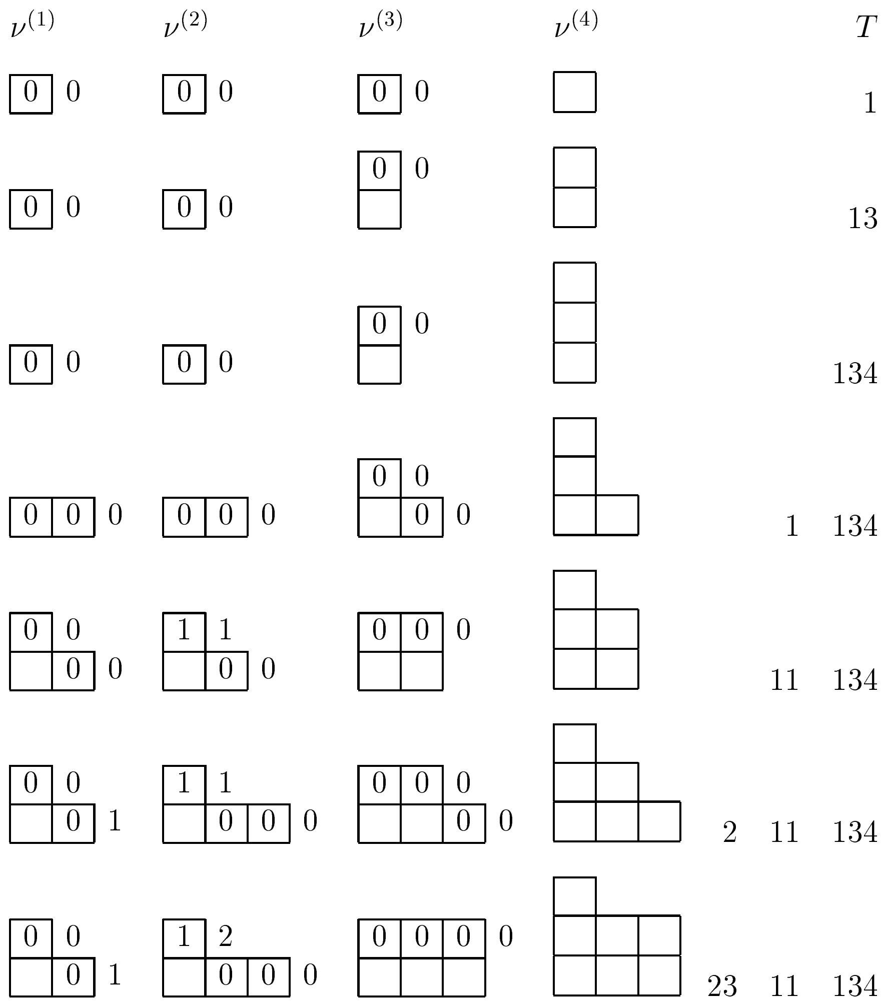


FIGURE 2. The intermediate steps in the algorithm for $\Psi(T)$ with T of example 4.1

rigged configuration $(\nu, J)_{i,k}$ is obtained from $(\nu, J)_{i+1,k}$ by removing the selected singular strings in row i and making the new boxes in row $i-1$ singular. Then it is clear that $t_i^k \le t_{i+1}^k$ (where $t_{i+1}^k = \infty$ if $i = |\mu^k|$) so that T will indeed be an L-multipartition of shape $\mu^\bullet$ and content λ.

It remains to prove that Ψ preserves the statistic. Since we are dealing with L-multitableaux of single rows the inversion statistic of Section 3.2 simplifies as follows. The pair (s,t) is an inversion of T if $T(t) < T(s)$ and either $\mathrm{diag}(s) = \mathrm{diag}(t)$ and $\mathrm{pos}(s) < \mathrm{pos}(t)$ or $\mathrm{diag}(s) = \mathrm{diag}(t) - 1$ and $\mathrm{pos}(s) > \mathrm{pos}(t)$. Furthermore, the following relation between a multitableau T and the corresponding configuration ν will be needed

$$\nu_i^{(a)} - \nu_i^{(a-1)} = \mathrm{card}\{s \in \mu^\bullet \mid T(s) = a,\, \mathrm{diag}(s) = i-1\}. \tag{4.1}$$

Building T up successively, the addition of the letter t_i^k in box $t \in \mu^\bullet$ changes the inversion statistic by

$$\begin{aligned}\Delta\mathrm{inv} = {} & \mathrm{card}\{s \in \mu^\bullet \mid \mathrm{pos}(s) > \mathrm{pos}(t),\, T(s) < T(t),\, \mathrm{diag}(s) = \mathrm{diag}(t)\}\\ & + \mathrm{card}\{s \in \mu^\bullet \mid \mathrm{pos}(s) > \mathrm{pos}(t),\, T(s) > T(t),\, \mathrm{diag}(s) = \mathrm{diag}(t) - 1\}.\end{aligned}$$

Denote by ν the configuration corresponding to T built up to letter t_i^k in the algorithm for Ψ. Then, setting $a = T(t)$ and inserting (4.1) yields

$$\begin{aligned}\Delta\mathrm{inv} &= \sum_{\alpha=1}^{a-1}(\nu_i^{(\alpha)} - \nu_i^{(\alpha-1)}) + \sum_{\alpha=a+1}^{n}(\nu_{i-1}^{(\alpha)} - \nu_{i-1}^{(\alpha-1)})\\ &= \nu_i^{(a-1)} + \nu_{i-1}^{(n)} - \nu_{i-1}^{(a)}.\end{aligned} \tag{4.2}$$

On the other hand, the change of the rigged configuration statistic $\tilde{c}$, taking into account that we have to invert the quantum numbers, is given by

$$\begin{aligned}\Delta\tilde{c} &= \nu_{i+1}^{(a-1)} + (\nu_{i-1}^{(a+1)} - \nu_{i-1}^{(a)}) + (\nu_{i-1}^{(a+2)} - \nu_{i-1}^{(a+1)}) + \cdots + (\nu_{i-1}^{(n)} - \nu_{i-1}^{(n-1)})\\ &\quad + (\nu_i^{(a-1)} - \nu_{i+1}^{(a-1)})\\ &= \nu_i^{(a-1)} + \nu_{i-1}^{(n)} - \nu_{i-1}^{(a)},\end{aligned} \tag{4.3}$$

where the first line comes from the change in $\Phi(\nu)$ and the second line is induced by the change of the vacancy number. Since (4.2) and (4.3) agree at each step in the algorithm and $\mathrm{inv}(\emptyset) = \tilde{c}(\emptyset) = 0$, Ψ is statistic preserving. □

4.2 Antisymmetric case

Let $\mu^\bullet = (\mu^0, \mu^1, \ldots, \mu^{L-1})$ be a multipartition where the μ^i are single columns. As in the symmetric case, the bijection

$$\Psi' : \mathrm{Tab}^L(\mu^\bullet, \lambda) \to \mathrm{RC}'(\lambda, \mu),$$

is defined recursively where μ is the partition with parts $|\mu^i|$. Let $T = (T^0, \ldots, T^{L-1}) \in \mathrm{Tab}^L(\mu^\bullet, \lambda)$. The letters in the single column tableau T^k are denoted by $t^k_{|\mu^k|} t^k_{|\mu^k|-1} \cdots t^k_1$ in strictly decreasing order. The L-multitableau T is built up by successively adding the letters t^k_i for $k = L-1, L-2, \ldots, 0$ and $i = 1, 2, \ldots, |\mu^k|$. The addition of the letter t^k_i to T corresponds to an analogous operation on rigged configurations given by the following algorithm:

1. Define $(\nu, J)_{0,L-1}$ to be the empty rigged configuration in $\mathrm{RC}'(\lambda, \mu)$.

2. Suppose the rigged configuration $(\nu, J)_{i-1,k}$ which corresponds to the L-multitableau built up to letter t^k_{i-1} is already known. Here we identify $(\nu, J)_{0,k}$ with $(\nu, J)_{|\mu^{k+1}|,k+1}$ and set $t^k_0 = 0$. Denote by $(\nu, J)^{(a)}_{i-1,k}$ the a-th rigged partition in $(\nu, J)_{i-1,k}$. Adding the entry t^k_i to the k-th tableau in T with $t^k_i > t^k_{i-1}$ corresponds to the following operation on the rigged configuration $(\nu, J)_{i-1,k}$. For all $t^k_i \leq a < n$, add a box to row i in $(\nu, J)^{(a)}_{i-1,k}$ and make the new box singular. Remove a singular quantum number from row $i-1$ in all $(\nu, J)^{(a)}_{i-1,k}$ for $t^k_i - 1 \leq a < n$. Call the resulting rigged configuration $(\nu, J)_{i,k}$.

3. The rigged configuration $(\nu, J) = \Psi'(T)$ is obtained from $(\nu, J)_{|\mu^0|,0}$ by inverting all quantum numbers. More precisely, a quantum number j in row i of the a-th rigged partition is replaced by $p^{(a)}_i - j$.

Example 4.2. Consider the 3-multitableau

$$T = \left(\begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline \end{array}\ , \ \begin{array}{|c|} \hline 2 \\ \hline \end{array}\ , \ \begin{array}{|c|} \hline 4 \\ \hline 3 \\ \hline 1 \\ \hline \end{array} \right)$$

with inversion statistic $\mathrm{inv}(T) = 2$. The corresponding rigged configuration $\Psi'(T)$ is

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array}\ 1 \qquad \begin{array}{|c|c|} \hline & 1 \\ \hline \end{array}\ 1 \qquad \begin{array}{l} \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|c|c|} & & 0 \\ \hline \end{array}\ 0 \end{array}$$

with $\tilde{c}'(\Psi'(T)) = 2$. The intermediate steps of the algorithm are listed in Figure 3 and $\Psi'(T)$ is obtained from the last line by inversion of the quantum numbers.

Theorem 4.2. *The map* $\Psi' : \mathrm{Tab}^L(\mu^\bullet, \lambda) \to \mathrm{RC}'(\lambda, \mu)$ *defined by the above algorithm is a statistic preserving bijection. That is,* $\mathrm{inv}(T) = \tilde{c}'(\Psi'(T))$ *for* $T \in \mathrm{Tab}^L(\mu^\bullet, \lambda)$.

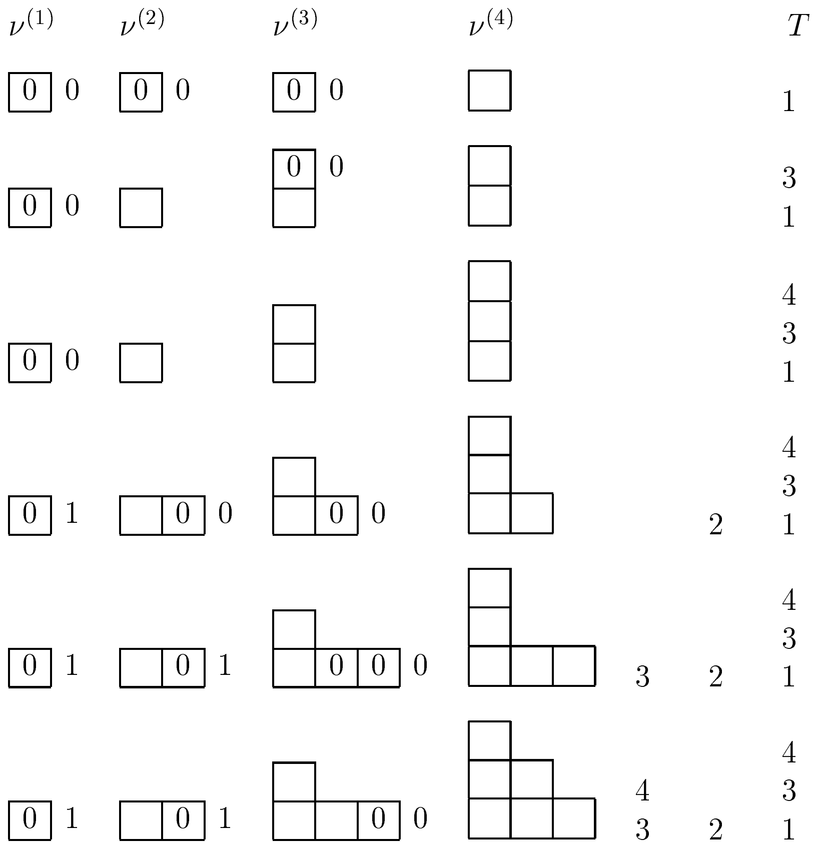


FIGURE 3. The intermediate steps in the algorithm for $\Psi'(T)$ with T of example 4.2

Proof. We begin by proving that Ψ' defines a bijection. Adding the letter t to the i-th row of a tableau in T adds a box in row i to the a-th rigged partition for all $t \le a < n$. This operation leaves all vacancy numbers unchanged, except $p_i^{(t-1)}$ which changes by $+1$. Since $t_i^k > t_{i-1}^k$ there exists a singular box in row $i-1$ in $(\nu, J)_{i-1,k}^{(a)}$ for all $t_i^k - 1 \le a < n$. This shows that Ψ' is well-defined. It is easy to see that the algorithm for Ψ' is invertible.

It remains to show that the statistic is preserved. In the single column case (s,t) is an inversion if $\mathrm{diag}(s) = \mathrm{diag}(t)$, $\mathrm{pos}(s) < \mathrm{pos}(t)$ and $T(t) < T(s) < T(t\uparrow)$. Furthermore, the analog of (4.1) becomes

$$\nu_i^{(a)} - \nu_i^{(a-1)} = \mathrm{card}\{s \in \mu^\bullet \mid T(s) = a, \mathrm{diag}(s) = 1 - i\}. \tag{4.4}$$

Adding the letter t_i^k to T in box $t \in \mu^\bullet$ changes the inversion number by

$$\begin{aligned}\Delta\mathrm{inv} = {} & \mathrm{card}\{s \in \mu^\bullet \mid \mathrm{pos}(s) > \mathrm{pos}(t),\, T(s) < T(t),\, \mathrm{diag}(s) = \mathrm{diag}(t)\} \\ & - \mathrm{card}\{s \in \mu^\bullet \mid \mathrm{pos}(s) > \mathrm{pos}(t),\, T(s) \le T(t),\, \mathrm{diag}(s) = \mathrm{diag}(t) - 1\}\end{aligned}$$

where the last line subtracts all contributions which do not satisfy $T(t) < T(s\uparrow)$. Denote by ν the configuration corresponding to T built up to letter t_i^k in the algorithm Ψ'. Then, setting $a = T(t)$ and inserting (4.4) yields

$$\begin{aligned}\Delta\mathrm{inv} &= \sum_{\alpha=1}^{a-1} (\nu_i^{(\alpha)} - \nu_i^{(\alpha-1)}) - \sum_{\alpha=1}^{a} (\nu_{i+1}^{(\alpha)} - \nu_{i+1}^{(\alpha-1)}) \\ &= \nu_i^{(a-1)} - \nu_{i+1}^{(a)}.\end{aligned}$$

On the other hand, taking into account that the quantum numbers need to be inverted for the correct statistic, we have

$$\Delta\tilde{c}' = \nu_i^{(a-1)} - \nu_{i+1}^{(a)}$$

from the change in vacancy numbers. Since $\Delta\mathrm{inv}$ and $\Delta\tilde{c}'$ agree at each step of the algorithm, Ψ' is statistic preserving. $\square$

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