WEAKLY NILPOTENT AND WEAKLY SEMISIMPLE POLYNOMIALS ON THE PLANE

YOSEF STEIN

ABSTRACT. In this paper we introduce weakly nilpotent and weakly semisimple polynomials in $K[x, y]$. We study the connection between these polynomials and the Jacobian Conjecture as well as the canonical form of the Newton polygon for these polynomials.

1. INTRODUCTION

Let K be an algebraically closed field of characteristic zero. For $f, g \in K[x, y]$ let:

$$[f, g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

For a fixed $A \in K[x, y]$ and $f \in K(x, y)$ let $D_A(f) = [A, f]$; D_A is a derivation on both $K[x, y]$ and $K(x, y)$. The operation $[\quad,\quad]$ imposes a Lie-algebra structure on both $K[x, y]$ and $K(x, y)$.

The main goal of this paper is to classify polynomials according to their Lie-algebra properties. The derivations D_A play a prominent role in studying seemingly purely algebraic properties of $K[x, y]$ (see [7]). They also play an important role in the analytic case $K = \mathbb{C}$, especially in studying problems related to the Jacobian Conjecture (see [1], [3], [5], [6], [8]). Some of the questions studied here were raised and some of the results obtained in the context of Weyl algebras (see [2]). There are also facts and results which are part of "folklore" or are scattered throughout various papers. This paper will, hopefully, serve two purposes: proving new results and arranging the old ones in some kind of order, making them accessible to mathematicians working on the Jacobian Conjecture and related topics.

We will start with some definitions. Let $S, v \in K[x, y]; S \neq 0, v \neq 0$ and let $\lambda \in K$ be such that $D_S(v) = \lambda v$. Such a number λ will be called an *eigenvalue* of S and v - a λ-*eigenfunction* of S.

Definition 1.1. *$S \in K[x, y]$ is called weakly semisimple (w.s.) if it has a non-zero eigenvalue.*

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Definition 1.2. *$A \in K[x,y]$ is called weakly nilpotent (w.n.) if there exists $B \in K[x,y]$ such that $D_A(B) \neq 0$ and $D_A^2(B) = 0$.*

For a w.s. polynomial S let $E(S)$ denote the set of all eigenvalues of S. For $\lambda \in E(S)$ we can consider the set of all λ-eigenfunctions of S. If we add 0 to this set, we obtain a vector space, which will be denoted by $V_\lambda(S)$.

Now we are able to state some of the more important results and to formulate several conjectures.

(1) If a w.s. polynomial S has a non-trivial rational λ-eigenfunction, then it has a non-trivial polynomial λ-eigenfunction (Corollary to Lemma 2.1).
(2) For a w.s. S, $E(S)$ is a group isomorphic to $\mathbb{Z}$ (Theorem 2.1).
(3) If a polynomial A is weakly nilpotent on the field $K(x,y)$, then it is w.n. on $K[x,y]$ (Proposition 2.4).
(4) A polynomial is w.n. iff it has a rational Jacobian mate (Proposition 3.1). If, in addition to this, all its fibers are irreducible, then it has a polynomial Jacobian mate (Corollary to Theorem 3.1).

So, if the Jacobian Conjecture is true, then every totally irreducible w.n. polynomial must be a coordinate. In other words, it would be possible to disprove the Jacobian Conjecture, if we were able to construct a totally irreducible w.n. polynomial which is not a coordinate. This provides, in my opinion, a good way to try to construct a counter-example, if such a thing exists, of course. Using the techniques developed in the end of Section 2, it is possible to construct a variety of more and more complicated w.n. polynomials as eigenfunctions of w.s. polynomials, and we can try to do it in such a way, that the end result is not a polynomial of a coordinate (which is easily done by applying the algorithm for reduction to the quasi-monic form, developed in Section 4), and is totally irreducible (which is much more difficult). The main advantage of such an approach is that it is necessary to construct only one polynomial in order to disprove the Jacobian Conjecture and not a pair of Jacobian mates. The degree of this polynomial does not have to be especially large, but the polynomial is guaranteed to have a Jacobian mate, which may have a very large degree. This situation is not covered by the famous result of Moh ([4]), which proves the Jacobian Conjecture in the case when the degrees of both mates are ≤ 100.

At this point I would like to formulate three conjectures:

Conjecture 1.

All irreducible components of a fiber of a w.n. or a w.s. polynomial are rational curves.

Proving this conjecture leads, in a standard way, to proving the Jacobian Conjecture.

Let us call both w.n. and w.s. polynomials *special*.

Conjecture 2.

All irreducible factors of a special polynomial are special.

Conjecture 3.

At least one of the irreducible factors of a w.n. polynomial is w.n.

Proving these two conjectures will, in my opinion, be a significant progress towards proving the Jacobian Conjecture.

Section 4 deals with a "canonical form" of the Newton polygon for w.s. and w.n. polynomials.

The motivation for the results comes from J. Dixmier [2], who obtained several similar results for the Weyl algebras. Our methods, however, are quite different. The main result of Section 4 is that the Newton polygon of a w.n. polynomial can be reduced by an automorphism of $K[x,y]$ to a special form (quasi-monic), unless the polynomial in question is a polynomial of a coordinate.

Remark. *If a polynomial P has a Jacobian mate, then P is w.n. It is possible, therefore, to reduce P to the quasi-monic form, unless P is a coordinate. This fact was known for a long time and proofs were published in various papers. Unfortunately, many unsuccessful attempts to prove the Jacobian Conjecture were based on nothing more than this single fact, without realizing that this property is shared by all w.n. polynomials, e.g. those which have* rational *and not necessarily polynomial Jacobian mates.*

2. Basic properties

We will, first of all, prove the following simple result:

Proposition 2.1. *Let $S \in K[x,y]$ be w.s. Then S is primitive.*

Proof. Assume, on the contrary, that $S = R(A)$; where $A \in K[x,y]$, $R(A) \in K[A]$ and $\deg R \geq 2$. Since S is w.s., there exist $v \in K[x,y]; v \neq 0$ and $\lambda \in K; \lambda \neq 0$ such that $D_S(v) = \lambda v$. Then: $\lambda v = D_S(v) = R^{'}(A)D_A(v)$. Applying D_S to v n times and using the fact that D_S and D_A commute we obtain by induction on n: $\lambda^n v = D_S^n v = (R^{'}(A))^n D_A^n(v)$. Thus $D_A^n(v) \neq 0$ and $\deg v \geq n \deg R^{'}(A)$ for all n. This is impossible since $\deg R \geq 2$. □

Here is the place to recall some of the results obtained in [7]. The main object of study in the second section of [7] was the equation $D_A(f) = Tf$ with $A, T \in K[x,y]; f \in K(x,y), f \neq 0$. So, our present notion of eigenvalue is a particular case of a more general concept. The set of all polynomials T, satisfying the equation $D_A(f) = Tf$ for a given polynomial A and some none-0 rational function f, was denoted by $\tilde{\Gamma}(A)$ and the main result was that $\tilde{\Gamma}(A)$ is a free abelian group of finite rank $\rho(A)$ ([7], Theorem 2.4), where $\rho(A)$ is the total reducibility order of A ([S4], p.113). There is one additional result which can be easily deduced from the proofs of various results in [7], but we will prove it here to make it easier for the reader:

Lemma 2.1. *Let $A, T, B \in K[x,y]; B \neq 0$ satisfy the equation $D_A(B) = TB$. Then there exists $G \in K[x,y]; G \neq 0$ such that $GB = R(A) \in K[A]$.*

Proof. Let B be an irreducible polynomial satisfying the equation $D_A(B) = TB$ for a polynomial T. Then, by Lemma 2.1. of [7], B is a divisor of $A - \alpha$ for some $\alpha \in K$. Let now $B = B_0^n$, where n is a positive integer and B_0 is an irreducible polynomial and assume that B satisfies the equation $D_A(B) = TB$. Then $nB_0^{n-1}D_A(B_0) = TB_0^n$ or $nD_A(B_0) = TB_0$. It follows from the previous result that B is a divisor of $(A-\alpha)^n$. Let now $B = B_1B_2\dots B_m$, where the polynomials $B_1,\dots,B_m$ do not have common factors, and assume that $D_A(B) = TB$. Then

$$\sum_{i=1}^{m} B_1\dots B_{i-1}D_A(B_i)B_{i+1}\dots B_m = TB_1\dots B_m$$

This means that B_i divides $D_A(B_i)$. So, if we consider now the prime decomposition:

$$B = \prod_{i=1}^{m}(B_i-\alpha_i)^{k_i}$$

and assume that $D_A(B) = TB$, we can conclude that B divides some polynomial $R(A) \in K[A]$. □

Corollary. *Let $A, T \in K[x,y], f \in K(x,y); f \neq 0$ satisfy the equation $D_A(f) = Tf$. Then there exists $F \in K[x,y]$ satisfying the same equation.*

Proof. Let $f = \frac{U}{V}$, where $U, V \in K[x,y]$ and do not have common factors. Then:

$$D_A(U)V - D_A(V)U = TUV$$

This means that V divides $D_A(V)$. There exists,therefore, $G \in K[x,y]$ such that $GV = R(A) \in K[A]$. Thus $f = \frac{GU}{R(A)}$ and $D_A(f) = \frac{D_A(GU)}{R(A)} = \frac{TGU}{R(A)}$ or $D_A(GU) = TGU$. So the polynomial $F = GU$ satisfies the same equation as does the rational function f. □

Proposition 2.2. *Let $S \in K[x,y]$ be w.s. Then*

(1) *$E(S)$ is a free abelian group of finite rank.*
(2) *For $\lambda, \mu \in E(S)$ it follows that $V_\lambda V_\mu \subset V_{\lambda+\mu}$ and $[V_\lambda, V_\mu] \subset V_{\lambda+\mu}$.*
(3) *Let $\{\lambda_1,\dots,\lambda_n\} \subset E(S); \lambda_i \neq \lambda_j$ for $i \neq j$. Let $\{v_1,\dots,v_n\}$ be a collection of nontrivial eigenfunctions of S such that $v_i \in V_{\lambda_i}$. Then the collection $\{v_1,\dots,v_n\}$ is linearly independent.*

Proof. (1) Let $\lambda, \mu \in E(S); F_1 \in V_\lambda, F_1 \neq 0; F_2 \in V_\mu, F_2 \neq 0$. $D_S(F_1F_2) = (\lambda+\mu)F_1F_2$ and $D_S(\frac{F_1}{F_2}) = (\lambda-\mu)\frac{F_1}{F_2}$ since D_S is a derivation of $K(x,y)$. So, the sum of eigenvalues is an eigenvalue, and it follows from the Corollary to Lemma 2.1 that the difference of eigenvalues is an eigenvalue too. The result now follows from Theorem 2.4 of [7].
(2) Follows from the fact that D_S is a derivation of $K[x,y]$ as an algebra and as a Lie-algebra.
(3) Straightforward.

□

Corollary. *Set $V(S) = \sum_{\lambda\in E(S)} V_\lambda(S)$. Then $V(S) = \bigoplus_{\lambda\in E(S)} V_\lambda(S)$ and is a subalgebra of $K[x,y]$ as an algebra and as a Lie-algebra.*

Proof. Follows from (3) and (2). ☐

Lemma 2.2. *Let $S \in K[x,y]$ be w.s. and let $\lambda, \mu \in E(S); \lambda \neq 0, \mu \neq 0$. Then $\frac{\lambda}{\mu} \in \mathbb{Q}$.*

Proof. Let $v \in V_\lambda(S), u \in V_\mu(S); v \neq 0, u \neq 0$. Assume that $\frac{\lambda}{\mu} \notin \mathbb{Q}$. The polynomials S, v, u belong to $K[x,y]$ and are, therefore, algebraically dependent. This fact can be expressed as $\sum_{i,j} R_{ij}(S)v^i u^j = 0$, where $R_{ij}(S) \in K[S]$ and are not all trivial. Note that $R_{ij}(S)v^i u^j \in V_{i\lambda+j\mu}$ by Proposition 2.2.

If the pairs (i,j) and (i_1, j_1) are distinct, then the spaces $V_{i\lambda+j\mu}(S)$ and $V_{i_1\lambda+j_1\mu}(S)$ cannot coincide since, by our assumption, $\frac{\lambda}{\mu} \notin \mathbb{Q}$. Therefore those "monomials" $R_{ij}(S)v^i u^j$ which are nontrivial, are linearly independent - a contradiction.

☐

Theorem 2.1. *Let $S \in K[x,y]$ be w.s. Then the group $E(S)$ is isomorphic to $\mathbb{Z}$.*

Proof. $E(S)$ is a free abelian group of finite rank (by Proposition 2.2). Let $\{\lambda_1, \ldots, \lambda_n\}$ be a basis of $E(S)$ over $\mathbb{Z}$. Choose a non-zero $\lambda_0 \in E(S)$. Then, by Lemma 2.2, every $\lambda_k = \frac{p_k}{q_k}\lambda_0$ where $p_k \in \mathbb{Z}; q_k \in \mathbb{N}$. Let q be the least common denominator of $\{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\}$. Then every $\lambda \in E(S)$ can be written as $\lambda = \frac{p_\lambda}{q}\lambda_0; p_\lambda \in \mathbb{Z}$. The map $\lambda \to p_\lambda$ is, obviously, an isomorphism between $E(S)$ and a subgroup of $\mathbb{Z}$, but all subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$.

☐

Since $E(S)$ is isomorphic to $\mathbb{Z}$, it is always possible to multiply S by a constant c so that $E(cS) = \mathbb{Z}$. A w.s. polynomial S such that $E(S) = \mathbb{Z}$ will be called a *normalized* w.s. polynomial. Now we will establish some additional properties of w.s. polynomials.

Proposition 2.3. *Let S be w.s. Then $S - \alpha$ has no multiple factors for all $\alpha \in K$.*

Proof. Fix $\alpha \in K$ and assume $S - \alpha = A^k B; k \geq 2; A, B \in K[x,y]$. Let $v \in K[x,y], v \neq 0; \lambda \in K, \lambda \neq 0$ be such that $D_S(v) = \lambda v$. Let $v = A^n u; n \geq 0; u \in K[x,y]$ and A does not divide u. Then:

$$\lambda A^n u = [A^k B, A^n u] = A^{n+k-1}L$$

for some $L \in K[x,y]$. Therefore $\lambda u = A^{k-1}L$, which is impossible since $k \geq 2$ and A does not divide u.

☐

Theorem 2.2. *Let S be normalized w.s. Then there exist $v_1 \in V_1(S)$ and $v_{-1} \in V_{-1}(S)$ such that $V(S) = K[S][v_1] \oplus K[S] \oplus K[S][v_{-1}]$.*

Proof. First recall that S is primitive by Proposition 2.1. Let $\tilde{V_0}(S) = \{f \in K(x,y) : D_S(f) = 0\}$. It is well known (see, for example, Theorem 1.7. of [7]) that $V_0(S) = K[S]$ and $\tilde{V_0}(S) = K(S)$ if S is primitive. Let v_n be a polynomial of minimal degree in $V_n(S)$ and let $v \in V_n(S)$. Then $\frac{v}{v_n} \in \tilde{V_0}$. So there exist $F(S), G(S) \in K(S)$ such that $F(S)$ and $G(S)$ do not have common factors and $v = \frac{F(S)}{G(S)} v_n$. Or

$$G(S)v = F(S)v_n$$

Since $F(S)$ and $G(S)$ are coprime, $G(S)$ must divide v_n. But then the polynomial $\frac{v_n}{G(S)} \in V_n(S)$ and $\deg \frac{v_n}{G(S)} < \deg v_n$ unless $G(S) = const$. So $V_n(S) = K[S]v_n$. This also proves that v_n is unique up to a constant multiplier. Assume now $n > 0$. Then $v_1^n \in V_n(S)$ and, therefore, $v_1^n = F(S)v_n$ for some $F(S) \in K[S]$. It follows now from Proposition 2.3 that if $F(S)$ divides v_1^n, then it must divide v_1. This would contradict our choice of v_1, unless $F(S) = const$. Thus $V_n(S) = K[S]v_1^n$. Similarly for the negative values of n: $V_n(S) = K[S]v_{-1}^{-n}$ and the result follows from Proposition 2.2.

□

Corollary. *D_{v_1} and $D_{v_{-1}}$ are locally nilpotent on $V(S)$.*

Proof. Straightforward. □

These results hold (with obvious modifications) for all w.s. polynomials.

For a w.n. polynomial A let $N_0(A) = \ker D_A$ and let $N(A) = \{B \in K[x,y] : \exists n \in \mathbb{N} \;\; / \;\; D_A^n(B) = 0\}$. Both $N_0(A)$ and $N(A)$ are, obviously, subalgebras of $K[x,y]$ and D_A is locally nilpotent on $N(A)$.

Proposition 2.4. *Let $A \in K[x,y], f \in K(x,y)$ be such that $D_A(f) \neq 0; D_A^2(f) = 0$. Then A is w.n.*

Proof. It is well known that for any A, $\ker D_A = K[B]$ for some primitive B (see, for example, Corollary to Theorem 1.7 of [7]). So $N_0(A) = K[B]$ where B is primitive. $A = R(B) \in K[B]$ and, therefore, $D_A(f) = R'(B)D_B(f)$ and $D_A^2(f) = (R'(B))^2 D_B^2(f)$. Hence $D_B(f) \neq 0$ and $D_B^2(f) = 0$. Since B is primitive and $D_B(D_B(f)) = 0$, it follows that $D_B(f) = \frac{F(B)}{G(B)}$ where $F(B), G(B) \in K[B]$ and are coprime. Let $f = \frac{U}{V}$ where $U, V \in K[x,y]$ and have no common factors. Then $D_B(\frac{U}{V}) = \frac{F(B)}{G(B)}$ or $D_B(\frac{UG(B)}{V}) = F(B)$. Let $\frac{UG(B)}{V} = \frac{U_1}{V_1}$, where U_1, V_1 have no common factors. Then

$$D_B(U_1)V_1 - D_B(V_1)U_1 = F(B)V_1^2$$

This means that V_1 divides $D_B(V_1)$ and it follows from Lemma 2.1 that there exists $L \in K[x,y], L \neq 0$ such that $LV_1 = H(B) \in K[B]$. Then $D_B(\frac{LU_1}{H(B)}) = F(B)$ or $D_B(LU_1) = H(B)F(B)$. This proves that B is w.n. Then $A = R(B)$ is, obviously, also w.n.

□

Lemma 2.3. *Let S be w.s. and let A be an eigenfunction of S. Then:*

(1) *A is w.n.*
(2) *If $B \in N_0(A)$, then $S+B$ is w.s.*

Proof. (1) $D_A(S) = -D_S(A) = -\lambda A$ and $D_A^2(S) = 0$.
(2) Straightforward.

□

So, if A is an eigenfunction of a w.s. S, then A is w.n. Then, of course, every non-constant polynomial in $N_0(A)$ is also w.n. Can all w.n. polynomials be obtained in this way? The answer is no.

Example 2.1. *Let $A = x^3y + x^2 + x$. Then A is w.n. and primitive and no element of $K[A]$ is an eigenfunction of a w.s. polynomial.*

Proof. Let $P = x(xy+1)$. Then $A = (P+1)x$ and it is easy to check that $D_A((P+1)^2) = 2A^2$. A is, obviously, primitive. We will prove now that $A+c$ cannot be an eigenfunction of a w.s. polynomial for any $c \in K$. Assume, on the contrary, that $D_S(A) = A + c$ for some $S \in K[x,y]; c \in K$. Now: $[A, \frac{(P+1)^2}{2A}] = A$ and $[A, \frac{c(P+1)^2}{2A^2}] = c$. Therefore

$$[A, S + \frac{(P+1)^2}{2A} + \frac{c(P+1)^2}{2A^2}] = 0$$

Since $A = (P+1)x$, we obtain that $[A, S + \frac{P+1}{2x} + \frac{c}{2x^2}] = 0$ Since A is primitive, it follows that

$$S + \frac{P+1}{2x} + \frac{c}{2x^2} = \frac{F(A)}{G(A)}$$

where $F(A), G(A) \in K[A]$ and are coprime. This implies that $G(A)$ divides x^2. Hence $G(A) = const$. So:

$$2x^2S + x(P+1) + c = 2x^2F(A)$$

Then $c = 0$ and $2xS + P + 1 = 2xF(A)$. Thus x divides $P+1$, which is impossible. If we assume now that a polynomial of A is an eigenfuction of a w.s. polynomial, then it is very easy prove that some $A+c$ must also be an eigenfunction. Thus no element of $N_0(A)$ is an eigenfunction of a w.s. polynomial.

□

Remark. *This example was suggested to me by* Pierrette Cassou-Nogues*.*

Definition 2.1. *A w.n. polynomial A is called natural if an element of $N_0(A)$ is an eigenfunction of a w.s. polynomial.*

The previous example shows that not all w.n. polynomials are natural. So, is it possible to find a good description of the set of all natural w.n. polynomials? I don't know how to do it.

Definition 2.2. *A w.s. polynomial S is called semisimple if $V(S) = K[x,y]$.*

Definition 2.3. *A w.n. polynomial A is called locally nilpotent if $N(A) = K[x,y]$.*

Now, using Lemma 2.3, we can construct increasingly more complicated w.s. and w.n. polynomials. We start with a very basic semisimple polynomial like $S = xy$ and one of its eigenfunctions and apply the two steps, described in Lemma 2.3, in succession.

Let $S_1 = xy$. Then $A_1 = xy^3 \in V_2(S_1)$. Now let $S_2 = \frac{xy+xy^3}{2}$. Then, by Lemma 2.3, S_2 is w.s. (it is not semisimple anymore!) and $A_1 \in V_1(S_2)$. Let $A_2 = S_2^2 A_1$. Then $A_2 \in V_1(S_2)$. Let $S_3 = S_2 + A_2$ for instance. Then S_3 is w.s. and so on. This machinery enables us to produce more and more complicated examples of w.s. and w.n. polynomials and this is important in the context of the Jacobian Conjecture.

3. w.s. and w.n. polynomials in the context of the Jacobian Conjecture

Let us recall first what the Jacobian Conjecture is.

Definition 3.1. *Two rational functions $f, g \in K(x,y)$ are called Jacobian mates if $[f,g] = const \neq 0$.*

Definition 3.2. *$A \in K[x,y]$ is called a coordinate if there exists $B \in K[x,y]$ such that $K[A,B] = K[x,y]$.*

The Jacobian Conjecture can be formulated as follows:
A polynomial has a polynomial Jacobian mate iff it is a coordinate.

Proposition 3.1. *$A \in K[x,y]$ is w.n. iff it has a rational Jacobian mate.*

Proof. Assume first that A is w.n. Then there exists $B \in K[x,y]$ such that $D_A(B) \neq 0$ and $D_A(B) \in N_0(A)$. But then $D_A(\frac{B}{D_A(B)}) = 1$.

Assume now that $D_A(f) = 1$ for some $f \in K(x,y)$. Let $f = \frac{F}{G}; F, G \in K[x,y]$ and do not have common factors. Then

$$D_A(F)G - D_A(G)F = G^2$$

This implies that G divides $D_A(G)$ and by Lemma 2.1, there exists $B \in K[x,y]; B \neq 0$ such that $BG = R(A) \in K[A]$. Then $D_A(FB) = R(A)$ and A is w.n. □

Definition 3.3. *$A \in K[x,y]$ is called totally irreducible if $A - \lambda$ is irreducible for each $\lambda \in K$.*

Theorem 3.1. *Let P be totally irreducible and let $f \in K(x,y)$ be such that $D_P(f) = U \in K[x,y]; U \neq 0$. Then there exists a polynomial F such that $D_P(F) = U$.*

Proof. Let $f = \frac{A}{B}; A, B \in K[x,y]$ and do not have common factors. Then

$$D_P(A)B - D_P(B)A = UB^2$$

This implies that B divides $D_P(B)$ and, by Lemma 2.1, there exists a polynomial $H \neq 0$ such that $HB = R(P) \in K[P]$. Then $D_P(HA) = UR(P)$. Let $W(P,U)$ denote the set of all $G \in K[x,y]$ such that $D_P(G) \neq 0$ and $\frac{D_P(G)}{U} \in K[P]$. We have just proved that $W(P,U) \neq \emptyset$. Let $G_0 \in W(P,U)$ be such that

$$\deg G_0 = \min_{G \in W(P,U)} \deg G$$

So, $D_P(G_0) = UR_0(P)$ for some $R_0(P) \in K[P]$. If $R_0(P) \in K$, then we are done. Assume that $\deg R_0 > 0$ and let $\alpha \in K$ be a root of the polynomial R_0, $R_0(P) = R_1(P)(P-\alpha)$. Then $P-\alpha$ divides $D_{G_0}(P)$ and, by Lemma 2.1, $P-\alpha$ must divide a polynomial of G_0. $P-\alpha$ is irreducible, since P is totally irreducible, and this implies that $P-\alpha$ divides $G_0 - c$ for some $c \in K$. Thus $\frac{G_0-c}{P-\alpha} = G_1 \in K[x,y]$ and

$$D_P(G_0) = D_P(G_1)(P-\alpha) = UR_1(P)(P-\alpha)$$

Therefore $D_P(G_1) = UR_1(P)$, but $\deg G_1 < \deg G_0$, contradicting our choice of G_0. Thus $R_0(P) = \mathit{const} \neq 0$ and this concludes the proof.

□

Corollary. *Let P be w.n. and totally irreducible. Then P has a polynomial Jacobian mate.*

Proof. Obvious.

□

4. The Newton polygon

In this section we will construct the "canonical" form of the Newton polygons of w.s. and w.n. polynomials. Similar results can be found in [2] in the context of Weyl algebras. Our methods are, however, somewhat different and based on the results obtained in the previous sections.

Definition 4.1. *Let $r, s \in \mathbb{N}$. Let*

$$A = \sum_{i,j} \alpha_{ij} x^i y^j \in K[x,y]$$

The (r,s)-degree is defined as follows:

$$\deg_{r,s} A = \max_{\alpha_{ij} \neq 0} (ri + sj)$$

Remark. *The same definition was used in* [2] *with $r, s \in \mathbb{R}$. In this paper we will need only $r, s \in \mathbb{N}$. Polynomials homogeneous with respect to this degree are sometimes called quasi-homogeneous. We will call them (r,s)-homogeneous.*

We will now list some well known properties of the (r,s)-degree without proof (which is a very easy one) in:

Proposition 4.1. (1) $\deg_{r,s}(AB) = \deg_{r,s} A + \deg_{r,s} B$

(2) $\deg_{r,s}(A+B) \leq \max(\deg_{r,s} A, \quad \deg_{r,s} B)$

(3) *Every* $A \in K[x,y]$ *can be decomposed into a sum of* (r,s)*-homogeneous polynomials:*

$$A = \sum_{i=1}^{k} A_i$$

where each A_i *is* (r,s)*-homogeneous,* $\deg_{r,s} A_i = n_i$ *and* $n_1 < \cdots < n_k$*.* A_k *is called the* (r,s)*-leading term of* A *and is denoted by* $\bar{A}_{r,s}$*.*

(4) $\deg_{r,s}[A,B] \leq \deg_{r,s} A + \deg_{r,s} B - (r+s)$ *and* $\deg_{r,s}[A,B] < \deg_{r,s} A + \deg_{r,s} B - (r+s)$ *iff* $[\bar{A}_{r,s}, \bar{B}_{r,s}] = 0$.

(5) *If* A *is an* (r,s)*-homogeneous polynomial with* $\deg_{r,s} A = n$*, then* A *satisfies the Euler's equation*

$$rx\frac{\partial A}{\partial x} + sy\frac{\partial A}{\partial y} = nA$$

(6) *Let* A *be a* (r,s)*-homogeneous polynomial with* $\deg_{r,s} A = n$ *and let* $m = \min_{B \in N_0(A), B \notin K} \deg_{r,s}(B)$*. Then* $\frac{n}{m} = k \in \mathbb{N}$ *and* $N_0(A) = K[A_0]$*, where* A_0 *is primitive* (r,s)*-homogeneous,* $\deg_{r,s} A_0 = m$ *and* $A = const \cdot A_0^k$*.* (This follows easily from (5)).

Lemma 4.1. *Let* $r,s \in \mathbb{N}$ *and let* $W \subset K[x,y]$ *be a subalgebra satisfying the following condition: for each pair of polynomials* $A, B \in W$ $[\bar{A}_{r,s}, \bar{B}_{r,s}] = 0$*. Then* $W \subset \ker D_A$ *for all* $A \in W$*.*

Proof. We can assume, of course, that $W \neq K$. Let $W_A = \{B \in W : D_A(B) \neq 0\}$. Suppose that $W_A \neq \emptyset$ for some $A \in W$. We will define now a function $d(B)$ on the set W_A as follows:

$$d(B) = \deg_{r,s} D_A(B) - \deg_{r,s} B$$

$d(B) \neq -\infty$, is integer-valued, and it follows from (4) of Proposition 4.1 that $d(B) \leq \deg_{r,s} A - (r+s)$, so that $d(B)$ is bounded from above. There exists, therefore, $M \in W_A$ such that

$$d(M) = \max_{B \in W_A} d(B)$$

It is easy to see that $d(B^n) = d(B)$ for $n \in \mathbb{N}$. Indeed:

$$d(B^n) = (n-1)\deg_{r,s} B + \deg_{r,s} D_A(B) - n \deg_{r,s} B = d(B)$$

Since $[\bar{A}_{r,s}, \bar{M}_{r,s}] = 0$ by our assumption, it follows from (6) of Proposition 4.1 that there exist $n, m \in \mathbb{N}$ and $c \in K$ such that $\bar{M}_{r,s}^n - c\bar{A}_{r,s}^m = 0$. Let $H = M^n - cA^m$. Then, obviously, $H \in W_A$, $\deg_{r,s} H < \deg_{r,s} M^n$ and

$$d(H) = \deg_{r,s} D_A(M^n) - \deg_{r,s} H > d(M^n) = d(M)$$

This contradicts our choice of M and, therefore, $W_A = \emptyset$.

□

Remark. *This kind of proof, using the defect function, was, as far as I know, first suggested be* Leonid Makar-Limanov *about 30 years ago.*

Lemma 4.2. *Let $F \in K[x,y]$ be irreducible and let a non-constant rational function $h = \frac{A}{B} \in K(x,y)$ be such that A and B do not have common factors and $D_F(h) = 0$ on the curve $\{F = 0\}$. Then there exist $a, b \in K$ such that at least one of them $\neq 0$ and F divides $aA - bB$.*

Proof. If $D_F(h) = D_F(\frac{A}{B}) = 0$ on the curve $F = 0$, then F divides $BD_F(A) - AD_F(B)$.

Case 1. A and B are algebraically dependent.

In this case there exists a primitive $P \in K[x,y]$ such that $A = R(P) \in K[x,y]; B = G(P) \in K[x,y]$. Then F divides $(G(P)R^{'}(P) - R(P)G^{'}(P))D_F(P)$. Since F is irreducible, it must divide either $D_F(P)$ or $G(P)R^{'}(P) - R(P)G^{'}(P)$ or both. If F divides $D_F(P)$, then, by Lemma 2.1, it must divide a polynomial of P. Since F is irreducible, this implies that F divides $P - c$ for some constant c. Then F divides $G(c)R(P) - R(c)G(P)$. Note that $G(c)$ and $R(c)$ cannot both be 0, since $R(P)$ and $G(P)$ do not have common factors. Assume now that F divides $G(P)R^{'}(P) - R(P)G^{'}(P)$. This polynomial cannot be 0, since in this case $h = \frac{R(P)}{G(P)}$ would be a constant. Neither can it be a non-0 constant, since F cannot divide a non-0 constant. Therefore again F must divide $P - c$ for some constant c and this concludes the proof for this case.

Case 2. A and B are algebraically independent.

Since $D_F(h) = 0$ on the curve $F = 0$, we obtain that F divides $BD_F(A) - AD_F(B)$. Since $A, B, F \in K[x,y]$ and A and B are algebraically independent, there must exist a nontrivial polynomial $R(A,B) \in K[A,B]$ such that F divides $R(A,B)$. We can assume that this polynomial is of minimal degree in (A,B). Let's define a derivation δ on $K(x,y)$ in the following way:

$$\delta(g) = B[g,A] - A[g,B]$$

Then F divides $\delta(F)$, $\delta(A) = -A[A,B]$ and $\delta(B) = B[B,A]$. Applying δ to $R(A,B)$, we obtain:

$$(A\frac{\partial R}{\partial A} + B\frac{\partial R}{\partial B})[A,B] = UF^n$$

where $n > 0$ and F does not divide U.

Let $L(A,B) = A\frac{\partial R}{\partial A} + B\frac{\partial R}{\partial B}$. Assume first that F divides $L(A,B)$. Since R is minimal, it follows that $L(A,B) = \lambda R(A,B); \lambda \in K$. This implies that $L(A,B)$ is homogeneous in A, B and is a product of linear factors. Then F must divide one of them and we are done. Assume now that F does not divide $L(A,B)$. Then F^n must divide $[A,B]$. Applying D_A to $R(A,B)$, we obtain

$$\frac{\partial R}{\partial B}[A,B] = D_A(U)F^n + nUD_A(F)F^{n-1}$$

Therefore F must divide $D_A(F)$. Then, by Lemma 2.1, F divides a polynomial of A and, being irreducible, it must divide $A - c_1$ for some constant c_1. In other words, $A - c_1 = A_0F$ and similarly $B - c_2 = B_0F$. c_1 and c_2

cannot both be 0, since A and B do not have common factors. It is clear that F divides $c_2A - c_1B$. This concludes the proof.

$\square$

Theorem 4.1. *Let $A \in K[x,y]$ be w.n. and let $r,s \in \mathbb{N}$ be coprime. Then there are three possible cases:*

(1) $r = 1, s > 1$ *and the (r,s)-leading term of A has the form:*
$$\bar{A}_{1,s} = const \cdot x^{n_1}(y + ax^s)^{n_2}$$
$a \in K, a \neq 0, n_1 \neq n_2$.

(2) $s = 1, r > 1$ *and the (r,s)-leading term of A has the form:*
$$\bar{A}_{r,1} = const \cdot y^{n_1}(x + by^r)^{n_2}$$
$b \in K, b \neq 0, n_1 \neq n_2$.

(3) $s = 1, r = 1$ *and the (r,s)-leading term of A has the form:*
$$\bar{A}_{1,1} = const \cdot (a_1x + b_1y)^{n_1}(a_2x + b_2y)^{n_2}$$
$a_1, a_2, b_1, b_2 \in K$, $n_1 \neq n_2$ *and* $a_1b_2 - a_2b_1 \neq 0$.

Proof. $N(A)$ is a subalgebra of $K[x,y]$ and, since A is w.n., $N(A) \neq N_0(A)$. There exists, therefore, by Lemma 4.1, $B \in N(A)$ such that $[\bar{A}_{r,s}, \bar{B}_{r,s}] \neq 0$. It is obvious, since A is w.n., that $\bar{B}_{r,s} \in N(\bar{A}_{r,s})$. Thus $D^n_{\bar{A}_{r,s}}(\bar{B}_{r,s}) = 0$ and $D^{n-1}_{\bar{A}_{r,s}}(\bar{B}_{r,s}) \neq 0$ for some $n > 1$. Let $D^{n-2}_{\bar{A}_{r,s}}(\bar{B}_{r,s}) = H$ and let G be a primitive (r,s)-homogeneous polynomial such that $N_0(\bar{A}_{r,s}) = K[G]$. Such a polynomial exists by (6) of Proposition 4.1. We now have the following situation: $D_G(H) \in K[G]; D_G(H) \neq 0$. Since both G and H are (r,s)-homogeneous, it follows that $D_G(H) = const \cdot G^m$ for some $m \geq 0$.

We can assume without loss of generality that $m > 0$, since if $m = 0$, then we can multiply H by G. Our goal is to prove that G^{m-1} divides H. Assume, on the contrary, that G^{m-1} does not divide H. Then there exists an (r,s)-homogeneous, irreducible F such that:
$G = G_0F^k; H = H_0F^l$, where $k > 0, l \geq 0$, F does not divide G_0H_0 and $k(m-1) > l$. Then:
$$D_G(H) = const \cdot G_0^m F^{km} = F^{k+l-1}(kG_0[F,H_0] + F[G_0,H_0] + lH_0[G_0,F])$$
Therefore $kG_0[F,H_0] + lH_0[G_0,F] = UF$ for some polynomial U. This is equivalent to:
$$[\frac{G_0^l}{H_0^k}, F] = U\frac{G_0^{l-1}}{H_0^{k+1}}F$$
It follows now from Lemma 4.2 that there exists a nontrivial pair $c_1, c_2 \in K$ such that $c_1G_0^l - c_2H_0^k = MF; M \in K[x,y]$. Moreover, neither c_1 nor c_2 can be 0. Indeed, if $c_1 = 0$, then $c_2 = 0$ since $k > 0$ and F does not divide H_0. If $c_2 = 0$, then either $c_1 = 0$, which is impossible, or $c_1 \neq 0$ and $l = 0$, which would imply that F divides a nonzero constant. Assume for a moment that $l\deg_{r,s} G_0 \neq k\deg_{r,s} H_0$. Then the decomposition of M into

the sum of (r,s)-homogeneous polynomials must contain exactly two terms: M_1 and M_2, such that $M_1F = c_1G_0^l$ and $M_2F = c_2H_0^k$ which is obviously impossibble. Thus G_0^l and H_0^k must have the same (r,s)-degree.

Since $[G,H] = const \cdot G^m \neq 0$ and both G and H are (r,s)-homogeneous, it follows from (4) of Proposition 4.1 that

$$\deg_{r,s}[G,H] = \deg_{r,s} G + \deg_{r,s} H - (r+s)$$

Therefore, since $[G,H] = const \cdot G^m$,

$$\deg_{r,s} H_0 = r + s + (m-1)\deg_{r,s} G_0 + (k(m-1)-l)\deg_{r,s} F$$

Now we can use the fact that $\deg_{r,s} G_0^l = \deg_{r,s} H_0^k$ and we obtain:

$$(k(m-1)-l)(\deg_{r,s} G_0 + k\deg_{r,s} F) + k(r+s) = 0$$

This is impossible if $k(m-1) > l$. Therefore $k(m-1) \leq l$, which contradicts our earlier assumption. Thus G^{m-1} divides H.

Hence $L = \frac{H}{G^{m-1}}$ is an (r,s)-homogeneous polynomial and

$$[G,L] = const \cdot G \neq 0$$

Thus L is w.s. and G is an eigenfunction of L.

$\deg_{r,s}[G,L] = \deg_{r,s} G + \deg_{r,s} L - (r+s)$ by (4) of Proposition 4.1 and, therefore, $\deg_{r,s} L = r+s$. Assume now that L contains a monomial $const \cdot x^iy^j$ with $const \neq 0$. Then $r(i-1) + s(j-1) = 0$. Since r and s are coprime, there are only three alternatives:

(1) $i = 0, s = 1, r = j-1$
(2) $j = 0, r = 1, s = i-1$
(3) $i = 1, j = 1$

If L contains only one nontrivial monomial, then alternatives (1) and (2) are obviously impossible since in this case L cannot be w.s. So we can assume $L = xy$. Thus G is an eigenfunction of xy. Assume for a moment that G contains at least two nontrivial monomials: $ax^{i_1}y^{j_1}$ and $bx^{i_2}y^{j_2}$. Then: $r(i_1 - i_2) = s(j_2 - j_1)$ since G is (r,s)-homogeneous and $i_1 - j_1 = i_2 - j_2$ since G is an eigenfunction of xy. $i_1 \neq i_2$ because the equality would imply that $j_1 = j_2$ and the monomials are distinct. Therefore $r = s$ and, since they are coprime, $r = s = 1$. This immediately implies that $i_1 = i_2$ and $j_1 = j_2$. Thus G must be a single monomial: $G = const \cdot x^{n_1}y^{n_2}$. $n_1 \neq n_2$, since otherwise $D_G(L)$ would be 0, and we obtain Case 3 of the theorem since $\bar{A}_{r,s} = G^n$.

If L contains three nontrivial monomials, then the only possible form for L is $L = ax^i + by^j + cxy; abc \neq 0$. In this case both $r = 1$ and $s = 1$ and it follows immediately that $i = j = 2$. Then $L = (a_1x + b_1y)(a_2x + b_2y)$ and $a_1b_2 - a_2b_1 \neq 0$, since otherwise $L = const \cdot (ax+by)^2$ and cannot be w.s. Then there exists an automorphism τ of $K[x,y]$ such that
$\tau(x) = a_1x + b_1y; \tau(y) = a_2x + b_2y$. Thus $L = \tau(xy)$ and
$G = const \cdot (a_1x + b_1y)^{n_1}(a_2x + b_2y)^{n_2}$ with $n_1 \neq n_2$ and we obtain Case 3 again.

It is obvious from the previous argument, that L cannot contain more then three nontrivial monomials. So, we have either alternative (1) or alternative (2).

Assume first $L = ay^{r+1} + bxy; s = 1$. Then $L = y(ay^r + bx)$ and there exists an automorphism τ of $K[x, y]$ such that $\tau(y) = y; \tau(x) = bx + ay^r$. Thus $L = \tau(xy)$ and $G = (bx + ay^r)^{n_1} y_2^n$ with $n_1 \neq n_2$. This leads to Case 2 of the theorem.

Similarly, $L = ax^{s+1} + bxy$ leads to Case 1.

□

Definition 4.2. *$A \in K[x, y]$ is called quasi-monic if*

$$A = cx^{n_1}y^{n_2} + \sum_{i \leq n_1, j \leq n_2, i+j < n_1+n_2} a_{ij}x^iy^j$$

where $n_1 > 0$, $n_2 > 0$, and c is a nonzero constant.

Theorem 4.2. *Let A be w.n. Then either A is locally nilpotent and there exists an automorphism τ of $K[x, y]$ such that $\tau(A) \in K[x]$, or there exists τ such that $\tau(A)$ is quasi-monic with* $\deg_x \tau(A) \neq \deg_y \tau(A)$.

Proof. Let $d(A) = \deg_x A + \deg_y A$. Let

$$d_0 = \min_{\tau \in Aut(K[x,y])} d(\tau(A))$$

We can assume, without loss of generality, that $d(A) = d_0$. Assume that A is not quasi-monic. Then either all the points of the Newton polygon of A are located on one of the coordinate axes, or there exist coprime $r, s \in \mathbb{N}$ such that $\bar{A}_{r,s}$ is not a monomial. In the first case $A \in K[x]$ or $A \in K[y]$ and is, therefore, locally nilpotent. Thus if A is not locally nilpotent, then the Newton polygon of A must contain an edge with negative slope corresponding to $\bar{A}_{r,s}$.

Therefore, by Theorem 4.1, there are three possible cases:

(1) $s = 1, r > 1$ and

$$A = c(x + ay^r)^{n_1}y^{n_2} + \sum_{ri+j < rn_1+n_2} a_{ij}x^iy^j$$

with $c \neq 0, a \neq 0, n_1 > 0$.

(2) $r = 1, s > 1$ and

$$A = c(y + bx^s)^{n_1}x^{n_2} + \sum_{i+sj < sn_1+n_2} a_{ij}x^iy^j$$

with $c \neq 0, b \neq 0, n_1 > 0$.

(3) $r = 1, s = 1$ and

$$A = c(a_1x + b_1y)^{n_1}(a_2x + b_2y)^{n_2} + \sum_{i+j < n_1+n_2} a_{ij}x^iy^j$$

where $c \neq 0, a_1b_2 - a_2b_1 \neq 0$ and at most one of a_1, b_1, a_2, b_2 is 0.

In Cases 1 and 2 there exists a triangular and in Case 3 a linear automorphism τ such that $\tau(\bar{A}_{r,s})$ is a monomial. It is easy to verify that the action of τ on A decreases either $\deg_x A$ or $\deg_y A$ or both. This contradicts our choice of d_0. Thus A must be quasi-monic and, since A is w.n., it follows from Theorem 4.1 that $\deg_x A \neq \deg_y A$.

On the other hand, if A is quasi-monic, then its (r,s)-leading term for any $r>0, s>0$ is $const \cdot x^{n_1}y^{n_2}$ with $n_1>0, n_2>0$. If $n_1 \geq n_2$, then, as can be easily verified, $x \notin N(A)$ and if $n_1 < n_2$, then $y \notin N(A)$, so that a quasi-monic polynomial can never be locally nilpotent.

□

Corollary. *Let S be w.s. Then either S is semisimple and there exists an automorphism τ of $K[x,y]$ such that $\tau(S) = axy+b; a,b \in K, a \neq 0$, or there exists τ such that $\tau(A)$ is quasi-monic and* $\deg \tau(A) > 2$.

Proof. We can assume, without loss of generality, that S is normalized (e.g. $E(S) = \mathbb{Z}$). Then, by Theorem 2.2 and its Corollary, there exist two eigenfunctions $v_1 \in V_1(S)$ and $v_{-1} \in V_{-1}(S)$ such that $V(S) = K[S][v_1] \oplus K[S] \oplus K[S][v_{-1}]$ and the derivations D_{v_1} and D_{v-1} are locally nilpotent on $V(S)$. Applying a suitable automorphism τ, we obtain that either $\tau(v_1) \in K[x]$ or $\tau(v_1)$ is quasi-monic. Assume first that S is semisimple. Then, of course, $\tau(S)$ is also semisimple, $V(\tau(S)) = K[x,y]$ and $\tau(v_1)$ is locally nilpotent. Therefore it cannot be quasi-monic and $\tau(v_1) = R(S) \in K[x]$. Then

$$[\tau(S), R(x)] = -R'(x)\frac{\partial \tau(S)}{\partial y} = \gamma R(x)$$

where $\gamma \in K, \gamma \neq 0$.

This implies that $R'(x)$ divides $R(x)$, which is possible only if $R(x) = const \cdot (x-c)^n; c \in K$. A straightforward calculation shows that, by applying a triangular automorphism if necessary, $\tau(S)$ can be reduced to the form $axy+b$.

If S is not semisimple, then $\tau(v_1)$ must be quasi-monic, because otherwise $\tau(S)$ can be reduced to the form $axy+b$, as we have just shown, and this polynomial is semisimple. Thus the (r,s)-leading term of $\tau(v_1)$ is the same for all $r>0, s>0$ and is $const \cdot x^{n_1}y^{n_2}$ with $n_1>0, n_2>0$ and $n_1 \neq n_2$. Let V denote this leading term. Let us fix a pair (r,s) of coprime positive integers. Let U denote the (r,s)-leading term of $\tau(S)$. Assume first that $[U,V] \neq 0$. Then , since $[\tau(S), \tau(v_1)] = const \cdot \tau(v_1)$, we obtain that $[U,V] = const \cdot V, const \neq 0$. thus U is w.s., V is its eigenfunction, and

$$\deg_{r,s}[U,V] = \deg_{r,s} U + \deg_{r,s} V - (r+s) = \deg_{r,s} V$$

Therefore $\deg_{r,s} U = r+s$. Let U contain a nontrivial monomial $const \cdot x^i y^j$. Then $r(i-1)+s(j-1) = 0$, so either $i=0$ or $j=0$ or $i=j=1$. Assume first $i=0$. Then $r = s(j-1)$ and, since r and s are coprime, it follows that $s=1, j=r+1$. Since V is a monomial and an eigenfunction of U, it must be an eigenfunction of each nontrivial monomial contained in U. V obviously cannot be an eigenfunction of y^{r+1}, so the case $i=0$ is impossible, Similarly, $j \neq 0$. This leaves $i=1, j=1$ as the only alternative, but in this case $\tau(S)$ can be reduced to $axy+b$, which is impossible since S is not semisimple.

Thus $[U,V]=0$. Then there exists a monomial $W=x^i y^j$ such that $V = const \cdot W^k$ and $U = const \cdot W^m$. Since $n_1>0, n_2>0, n_1 \neq n_2$, it follows that $i>0, j>0, i\neq j$ and, therefore, U is the (r,s)-leading term for all pairs (r,s) and $\tau(S)$ is quasi-monic.

□

The Newton polygon of a w.n. or a w.s. polynomial may contain vertical or horizontal edges. In fact, it may consist of just one such edge. In such a case the polynomials cannot be "simplified" by an automorphism.

Example 4.1. *Let $A = y^3(y-1)^3x^2$ and let $S = y(y-1)(2y-1)x$. Then it is easy to verify that $D_S(A)=A$ and, therefore, S is w.s. and A is w.n. The Newton polygons of both S and A consist of a single vertical edge.*

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: ystein@mpim-bonn.mpg.de