# Variations on A. Kneser's theorem

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## 1 Introduction

At every point, a smooth plane curve can be approximated, to second order, by a circle; this circle is called osculating. One may think of the osculating circle as passing through three infinitesimally close points of the curve. A *vertex* of the curve is a point at which the osculating circle hyper-osculates: it approximates the curve to third order. Equivalently, a vertex is a critical point of the curvature function.

Consider a (necessarily non-closed) curve, free from vertices. The classical A. Kneser theorem [5] (see also [3, 10]), states that the osculating circles of the curve are pairwise disjoint, see Figure 1. This theorem is closely related to the four vertex theorem of S. Mukhopadhyaya [8] that a plane oval has at least 4 vertices (see again [3, 10]).

Figure 1 illustrates Kneser's theorem: it shows an annulus foliated by osculating circles of a curve.

**Remark 1.1** This foliation is not differentiable! Here is a proof. Let $f$ be a differentiable function in the annulus, constant on the leaves. We claim that $f$ is constant. Indeed, $df$ vanishes on the tangent vectors to the leaves. The curve is tangent to its osculating circle at every point, hence $df$ vanishes on the curve as well. Hence $f$ is constant on the curve. But the curve intersects all the circles that form the annulus, so $f$ is constant everywhere.

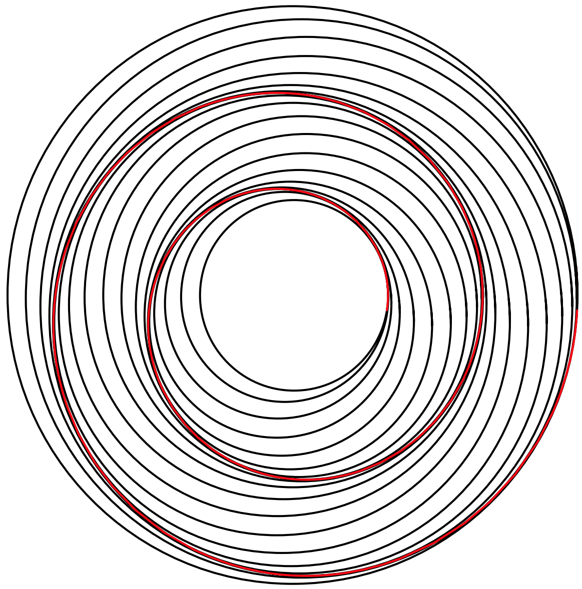

Figure 1: A spiral and its nested osculating circles

**Remark 1.2** Kneser's theorem has an analog in plane Minkowski geometry, see [11].

We will prove a number of analogs of Kneser's theorem; in each case, we will obtain a non-differentiable foliation with smooth leaves.

# 2 Osculating Taylor polynomials

Let $f$ be a smooth function of one real variable. Fix $n \geq 1$ and let $t \in \mathbf{R}$. The osculating (Taylor) polynomial $g_t$ of degree $n$ of the function $f$ at the point $t$ is the polynomial, whose value and the values of whose first $n$ derivatives at the point $t$ coincide with those of $f$:

$$g_t(x) = \sum_{i=0}^{n} \frac{f^{(i)}(t)}{i!}(x-t)^i. \tag{1}$$

The osculating polynomial $g_t$ is *hyper-osculating* if it approximates the function $f$ at the point $t$ up to $n+1$-st derivative, that is, if $f^{(n+1)}(t) = 0$.

Assume that $n$ is even and $f^{(n+1)}(t) \neq 0$ on some interval $I$ (possibly, infinite).

**Theorem 1** *For any distinct $a, b \in I$, the graphs of the osculating polynomials $g_a$ and $g_b$ are disjoint.*

**Proof.** To fix ideas, assume that $f^{(n+1)}(t) > 0$ on $I$. Let $a < b$ and suppose that $g_a(x) = g_b(x)$ for some $x \in \mathbf{R}$. It follows from (1) that

$$\frac{\partial g_t}{\partial t}(x) = \sum_{i=0}^{n} \frac{f^{(i+1)}(t)}{i!}(x-t)^i - \sum_{i=0}^{n} \frac{f^{(i)}(t)}{(i-1)!}(x-t)^{i-1} = \frac{f^{(n+1)}(t)}{n!}(x-t)^n,$$

and hence $(\partial g_t/\partial t)(x) > 0$ (except for $t = x$). It follows that $g_t(x)$ increases, as a function of $t$, therefore $g_a(x) < g_b(x)$. This is a contradiction. $\Box$

The same argument proves the following variant of Theorem 1. Let $n$ be odd. Assume that $f^{(n+1)}(t) \neq 0$ on an interval $I$. Consider two points $a < b$ from $I$.

**Theorem 2** *The graphs of the osculating polynomials $g_a$ and $g_b$ are disjoint over the segment $[b, \infty)$.*

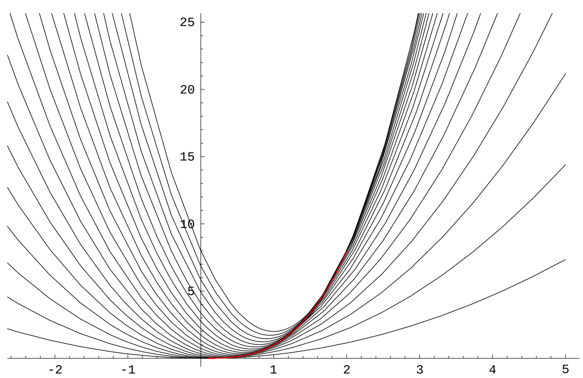


Figure 2: Osculating quadratic polynomials of the function $f(x) = x^3$

Figure 2 shows the graphs of the osculating quadratic polynomials of the function $f(x) = x^3$ and Figure 3 of the osculating cubic polynomials of the function $f(x) = x^4$.

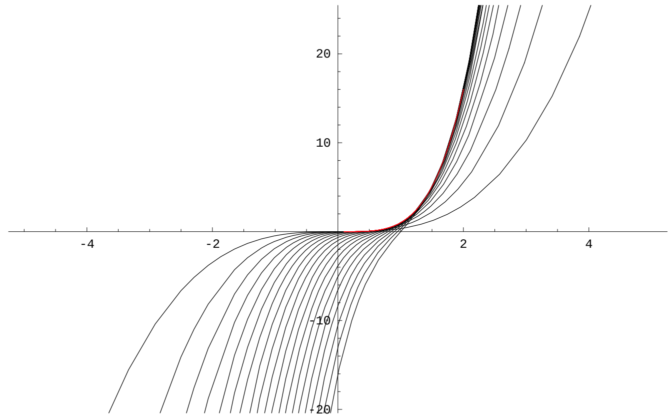


Figure 3: Osculating cubic polynomials of the function $f(x) = x^4$

# 3 Osculating trigonometric polynomials

Let $f$ be a $2\pi$-periodic smooth function, that is, a function on the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$. Fix $n \geq 1$ and let $t \in S^1$. A trigonometric polynomial of degree $n$

$$g_t(x) = c + \sum_{i=1}^{n}(a_i \cos ix + b_i \sin ix)$$

is the *osculating trigonometric polynomial* of the function $f(x)$ at the point $t$ if its value and the values of its first $2n$ derivatives at the point $t$ coincide with those of $f$.

**Remark 3.1** The osculating trigonometric polynomial always exists. Actually, the following more general fact is classical (one reference is [10]). Let $f_i$, $i = 1, \ldots, N$, be a system of functions on an interval $I$ such that the Wronski determinant of this system is nonzero everywhere on $I$. Then, for any sufficiently smooth function $g$ on $I$ and any $t_0 \in I$, there is a linear combination of functions $f_i$ that, at $t_0$, approximates $g$ up to the derivative of order $N-1$. This boils down to solving the linear system

$$g^{(j)}(t_0) = \sum c_i f_i^{(j)}(t_0), \quad j = 0, \ldots, N-1$$

with unknowns $c_i$, which has a solution due to non-zero determinant. The solution depends smoothly on $t_0$.

In our case, the functions $f_i$ are 1, $\cos t$, $\sin t$, $\cos 2t$, $\sin 2t$, etc., and $N = 2n+1$. The Wronskian of these functions is constant, which can be seen by differentiating its columns. On the other hand, the functions are linearly independent solutions of a $N$-th order linear differential equation, hence the Wronskian is nonzero.

Geometrically, we consider $N$-dimensional projective space and the curve $[f_1 : ... : f_N : g]$. The osculating hyperplane of this curve at the point $t_0$ approximates the curve with $N-1$ derivatives. The equation of this hyperplane is $g = \sum c_i f_i$, and this gives the desired approximation.

The osculating trigonometric polynomial $g_t$ is *hyper-osculating* if it approximates the function $f$ at the point $t$ up to $2n+1$-st derivative, that is, if $f^{(2n+1)}(t) = g_t^{(2n+1)}(t)$. Trigonometric polynomials of degree $n$ are annihilated by the differential operator $\mathcal{D} := d(d^2+1)(d^2+4)\dots(d^2+n^2)$, where $d = d/dx$. Therefore $g_t$ hyper-osculates a function $f$ if and only if $(\mathcal{D}f)(t) = 0$.

Assume that the osculating trigonometric polynomials of degree $n$ for a function $f$ do not hyper-osculate on an interval $I \subset S^1$.

**Theorem 3** *For any distinct $a, b \in I$, the graphs of the osculating trigonometric polynomials $g_a$ and $g_b$ are disjoint.*

**Proof.** It is not hard to see that the real number $g_t^{(2n+1)}(t)$ depends continuously on $t$ (indeed, the function $g_t$ depends continuously on $t$ in the $C^{2n+1}$-metric).

To fix ideas, assume that $f^{(2n+1)}(t) > g_t^{(2n+1)}(t)$ for all $t \in I$. We will show that $\partial g_t(x)/\partial t > 0$ for all $t \in I$ and all $x \in S^1$ (except $t = x$), and this will imply the statement of the theorem as in the proof of Theorem 1.

Since $g_t$ is an osculating trigonometric polynomial, one has:

$$g_t^{(j)}(t) = f^{(j)}(t), \quad j = 0, \dots, 2n. \tag{2}$$

Differentiate:

$$\frac{\partial g_t}{\partial t}^{(j)}(t) + g_t^{(j+1)}(t) = f^{(j+1)}(t),$$

and combine with (2) to obtain:

$$\frac{\partial g_t}{\partial t}^{(j)}(t) = 0, \quad j = 0, \dots, 2n-1; \qquad \frac{\partial g_t}{\partial t}^{(2n)}(t) + g_t^{(2n+1)}(t) = f^{(2n+1)}(t). \tag{3}$$

The function $\partial g_t/\partial t$ is a trigonometric polynomial of degree $n$. If this trigonometric polynomial is not identically zero, then it has no more than $2n$ roots, counting with multiplicities. If $\partial g_t/\partial t \equiv 0$, then $(\partial g_t/\partial t)^{(2n)}(t) = 0$, and the last equality in (3) implies that $g_t$ hyper-osculates. Thus $\partial g_t/\partial t$ is not identically zero.

According to (3), the trigonometric polynomial $\partial g_t/\partial t$ already has a root at the point $t$ of multiplicity $2n$. Hence $(\partial g_t/\partial t)(x) \neq 0$ for $x \neq t$. By the assumption made at the beginning of the proof and the last equality in (3), we have $(\partial g_t/\partial t)^{(2n)}(t) > 0$. Hence $(\partial g_t/\partial t)(x) > 0$ for $x$ sufficiently close to $t$, and therefore $(\partial g_t/\partial t)(x) > 0$ for all $x \neq t$. $\square$

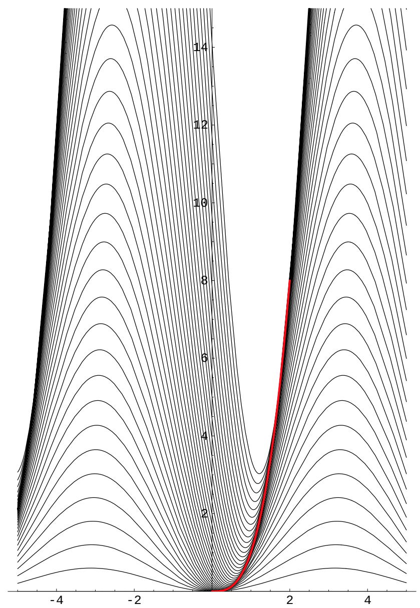


Figure 4: Osculating linear harmonics of the function $f(x) = x^3$

Theorem 3 is illustrated in Figure 4 depicting the graphs of osculating linear harmonics $c + a\cos x + b\sin x$ for the function $f(x) = x^3$.

**Remark 3.2** Theorem 3 extends from trigonometric polynomials to Chebyshev systems of functions; the proof remains the same.

**Remark 3.3** For $n = 1$, Theorem 3 implies Kneser's theorem: it suffices to consider the support function of the curve and use the fact that the support functions of circles are linear harmonics.

# 4 Osculating conics, cubics and fractional linear transformations

Fix $d \geq 1$ and consider the space of algebraic curves of degree $d$. This space has dimension $n(d) = d(d+3)/2$. At every point, a smooth plane curve $\gamma$ can be approximated, to order $n(d)-1$, by an algebraic curve of degree $d$; this algebraic curve is called the *osculating curve*. One may think of the osculating algebraic curve as passing through $n(d)$ infinitesimally close points of $\gamma$. A *$d$-extactic point* of the curve $\gamma$ is a point, at which the osculating algebraic curve *hyper-osculates*: it approximates $\gamma$ to order $n(d)$; see [1].

In this section, we extend Kneser's theorem to osculating conics and osculating cubic curves. We assume that the curve $\gamma$ is free from extactic points. We also assume that the osculating conics and cubic curves along $\gamma$ are non-degenerate.

Consider a smooth function $f$ with nowhere vanishing derivative. For every $t \in \mathbf{R}$, there exists a fractional-linear transformation $g_t$, whose value and the value of whose first two derivatives at the point $t$ coincide with those of $f$; this is the *osculating fractional-linear transformation*. As before, it *hyper-osculates* at the point $t$ if the third derivatives coincide as well. This happens if and only if the Schwarzian derivative of $f$ vanishes:

$$\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right)(t) = 0.$$

The graph of a fractional-linear transformation is a hyperbola with vertical and horizontal asymptotes (or a straight line); we refer to these graphs as the *osculating hyperbolas*. Assume that the osculating hyperbolas for the function $f$ do not hyper-osculate on an interval $I$. Let $\gamma$ be the graph of $f$ over $I$.

From the projective point of view, all non-degenerate conics are equivalent; since our results are projectively-invariant, we assume that the osculating conics of $\gamma$ are ellipses. In the case of cubic curves, we assume that the osculating cubics of $\gamma$ have two components, an oval and a branch going to infinity, and that the ovals, not the infinite branches, osculate $\gamma$. With these assumptions, we have the next theorem.

**Theorem 4** *1) The osculating ellipses along $\gamma$ are pairwise disjoint;*
*2) the ovals of the osculating cubic curves along $\gamma$ are pairwise disjoint;*
*3) the osculating hyperbolas of $\gamma$ are pairwise disjoint.*

Theorem 4, case 2) is illustrated by Figure 5.

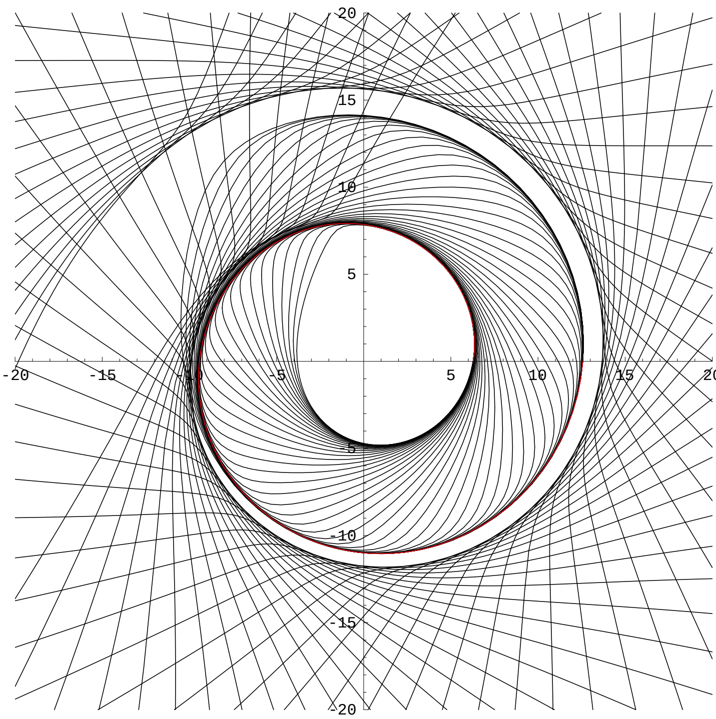


Figure 5: Osculating cubic curves of a spiral

**Proof of Theorem 4.** We will argue about cubic curves, indicating the difference with the case of conics and hyperbolas, when necessary.

Give the curve $\gamma$ a smooth parameterization, $\gamma(s)$. Let $\Gamma_s$ be the oval of the osculating cubic curve at the point $\gamma(s)$, and $f_s(x, y) = 0$ its cubic equation. It suffices to prove that the curves $\Gamma_a$ and $\Gamma_b$ are nested for distinct parameter values $a$ and $b$, sufficiently close to each other.

Give the ovals $\Gamma_s$ a smooth parameterization, $\Gamma_s(t)$, such that the tangency point with the curve $\gamma$ corresponds to $t = 0$, that is, $\Gamma_s(0) = \gamma(s)$. Let $F$ be the map $(s, t) \mapsto \Gamma_s(t)$. We claim that, for $t \neq 0$, this map is an immersion. This claim implies that $\Gamma_a$ and $\Gamma_b$ are nested for sufficiently close $a$ and $b$.

To prove the claim, we need the following lemma.

**Lemma 4.1** *Suppose that $F(s,t) = (x,y)$. The Jacobian of $F$ vanishes at point $(s,t)$ if and only if*

$$\frac{\partial f_s}{\partial s}(x,y) = 0, \quad f_s(x,y) = 0. \tag{4}$$

**Proof of Lemma.** The covector $df_s$ is nowhere zero since the curve $\Gamma_s$ is non-degenerate. This covector vanishes on $\partial F/\partial t$, the tangent vector to the curve $\Gamma_s$. Therefore the Jacobian of $F$ vanishes exactly when $df_s$ also vanishes on the vector $\partial F/\partial s$.

Differentiate the equation $f_s(F) = 0$ with respect to $s$:

$$\frac{\partial f_s}{\partial s} \circ F + df_s\left(\frac{\partial F}{\partial s}\right) = 0.$$

Thus $df_s$ vanishes on the vector $\partial F/\partial s$ if and only if $f_s = 0$ and $\partial f_s/\partial s = 0$. $\square$

Now we need to prove that the system of equations (4) has no solutions for $t \neq 0$ and point $(x,y)$ on the oval $\Gamma_s$. Both equations in (4) are cubic, and they are not proportional since $\gamma(s)$ is not an extactic point (in the case of osculating conics, the two equations are quadratic). By the Bezout theorem, the number of solutions is at most 9 (and 4, for conics). In the case of hyperbolas, $f_s(x,y) = (x-a)(y-b) - c$ where $a, b$ and $c$ depend on $s$; hence $\partial f_s/\partial s = 0$ is a linear equation in $x$ and $y$, and system (4) has at most 2 solutions.

For any parameter value $s$, the point $\gamma(s)$ is a multiple solution of system (4). Since the curve $\{f_s = 0\}$ is the osculating curve of degree $d$ for the curve $\gamma$ at the point $\gamma(s)$, the function $s' \mapsto f_s(\gamma(s'))$ has zero of order $n(d)$ at point $s' = s$. We can view $f_s(\gamma(s'))$ as a smooth function of two variables $s$ and $s'$. This function vanishes on the line $s' = s$. According to a version of the preparation theorem for differentiable functions [4, 6] (see also [7]), there exists a smooth function $\phi$ of two variables such that

$$f_s(\gamma(s')) = (s-s')^m \phi(s,s')$$

and $\phi(s,s) \neq 0$ locally near a given value of $s$. Restricting this equation to a line $s = const$, we obtain $m = n(d)$. Differentiating with respect to $s$, we see that $\frac{\partial f_s}{\partial s}(\gamma(s'))$ starts with terms of order $n(d) - 1$ in $s - s'$. Then $\frac{\partial f_s}{\partial s}(\Gamma_s(t))$

vanishes for $t = 0$ with order $n(d) - 1$, because $\Gamma_s$ approximates $\gamma$ up to order $n(d)$ at $\gamma(s)$. Hence the multiplicity of the solution $\gamma(s)$ of system (4) is $n(d) - 1$.

For $d = 2$ (the case of osculating ellipses), this multiplicity is 4, and hence there are no other solutions. For $d = 3$ (the case of osculating cubics), the multiplicity is 8, and there may be one other solution. However, the number of intersection points of an oval with any curve is even, and therefore the 9-th point (if it exists) lies on the other branch. Therefore system (4) has no solutions for $t \neq 0$.

Finally, in the case of hyperbolas, the multiplicity of the solution of system (4) at the point $\gamma(s)$ is 2, therefore there are no other solutions again. This completes the proof. □

**Remark 4.2** It is interesting to compare Theorem 4 with three results on the existence of "vertices": a plane oval has at least six sextactic (i.e., 2-extactic) points [8]; a closed plane curve, sufficiently close to an oval of a cubic curve, has at least ten 3-extactic points [1]; and the Schwarzian derivative of a diffeomorphism of $\mathbf{RP}^1$ has at least four zeros [2] (see also [9, 10]).

**Remark 4.3** In fact, the osculating hyperbolas *are* the osculating circles in Lorentz metric [2, 12].

**Remark 4.4** Theorem 4 does not generalize to osculating quartics. This can be seen on Figure 6, where several osculating quartics for the curve $x^{2/3} + y^{2/3} = 1$ are drawn. Each quartic in the picture splits into two ovals, one being below and one above the curve. One can see that nearby ovals below the curve intersect.

# 5 Infinitesimal intersection indices

In this section, we give some more general results that may highlight the proof of Theorem 4.

Consider a smooth map $F$ of a region in $\mathbf{R}^2$ to a region in $\mathbf{R}^2$. The map $F$ gives rise to a family of curves. Namely, for any $s \in \mathbf{R}$, we have the parameterized curve $\Gamma_s : t \mapsto F(s, t)$, where the parameter $t$ runs through all real numbers such that $(s, t)$ is in the domain of $F$. Suppose that the curve

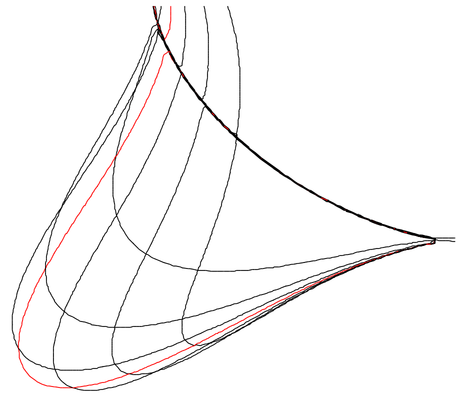

Figure 6: Osculating quartics of the curve $x^{2/3} + y^{2/3} = 1$.

$\Gamma_s$ is given locally by an equation $f_s = 0$, which depends smoothly on $s$. We will assume that $df_s$ never vanishes (e.g., if $f_s$ are polynomials, then we are talking about nonsingular algebraic curves $\Gamma_s$).

Let $(x, y)$ be a point $\Gamma_s(t)$ on a curve $\Gamma_s$ so that $F(s, t) = (x, y)$. Define the *infinitesimal intersection multiplicity* of $\Gamma_s$ at point $(x, y)$ as the order of vanishing of the function

$$t \mapsto Jacobian[F](s, t)$$

at point $t$. In particular, if the infinitesimal intersection multiplicity is zero, then the family $F$ looks like a foliation locally near the point $(x, y)$ and for parameter values near $s$ (however, the curves from the family $F$ corresponding to far-away parameter values may also pass through $(x, y)$). The *infinitesimal intersection index* of a curve $\Gamma_s$ (in the family $F$) is the sum of local intersection multiplicities at all points of this curve. The following theorem is an infinitesimal version of the classical Bezout theorem:

**Theorem 5** *Suppose that all curves $\Gamma_s$ are algebraic of degree $d$. Then the infinitesimal intersection index of each curve $\Gamma_s$ is at most $d^2$.*

The proof of this theorem is based on the following lemma:

**Lemma 5.1** *The infinitesimal intersection multiplicity of* $\Gamma_s$ *at a point* $(x, y)$ *is equal to the intersection multiplicity of the curves* $\Gamma_s = \{f_s = 0\}$ *and* $\{\frac{\partial f_s}{\partial s} = 0\}$ *at the same point.*

This is a direct generalization of Lemma 4.1.

**Proof.** The Jacobian of $F$ is, by definition, $\det(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t})$. We have $df_s(\frac{\partial F}{\partial t}) = 0$, and neither the 1-form $df_s$ nor the vector field $\frac{\partial F}{\partial t}$ ever vanish. It follows that the Jacobian of $F$ is $df_s(\frac{\partial F}{\partial s})$ times a nowhere vanishing differentiable function. In particular, the order of vanishing of the Jacobian coincides with the order of vanishing of the function $df_s(\frac{\partial F}{\partial s})$ at the same point, and, by the equality

$$\frac{\partial f_s}{\partial s} \circ F + df_s \left( \frac{\partial F}{\partial s} \right) = 0,$$

with the order of vanishing of $\frac{\partial f_s}{\partial s} \circ F$ at the same point. Restrict all functions considered to a curve $\Gamma_s$ and express them it terms of the local parameter $t$. Then the order of vanishing of the function $\frac{\partial f_s}{\partial s}$ is, by definition, the intersection multiplicity of the curves $\Gamma_s$ and $\{\frac{\partial f_s}{\partial s} = 0\}$. $\Box$

Theorem 5 now follows.

The following statement provides a description of families $F$ that consist of osculating algebraic curves to a given plane curve:

**Theorem 6** *Under the assumptions of Theorem 5, suppose also that there is a smooth plane curve* $\gamma$ *parameterized by* $s$ *and such that* $\gamma(s) \in \Gamma_s$ *for each* $s$*, and each curve* $\Gamma_s$ *has infinitesimal intersection multiplicity* $n(d) - 1$ *at the point* $\gamma(s)$*. Then* $\Gamma_s$ *are osculating algebraic curves of degree* $d$ *for the curve* $\gamma$*.*

This theorem generalizes the well-known algorithm of finding the envelope of a family of lines: the envelope coincides with the locus of points, where two infinitesimally close lines intersect.

**Proof.** Since $\frac{\partial f_s}{\partial s} = 0$ on $\gamma$, the curve $\gamma$ is the envelope of curves $\Gamma_s$ (this follows from the classical description of the envelope).

Then the function $s' \mapsto f_s(\gamma(s'))$ has a multiple zero at point $s' = s$. By the preparation theorem for differentiable functions [6, 4, 7], we have

$$f_s(\gamma(s')) = (s - s')^m \phi(s, s'),$$

where $m > 1$ is an integer and $\phi$ is a smooth function of two variables such that $\phi(s,s) \neq 0$ locally near a given value of $s$. In particular, the curves $\Gamma_s$ approximate the curve $\gamma$ up to order $m$ for $s$ in the chosen neighborhood. Reparameterize curves $\Gamma_s$ to make $\Gamma_s(t)$ coincide with $\gamma(s)$ for $t = 0$. Then $\frac{\partial f_s}{\partial s}(\Gamma_s(t))$ vanishes at point $t = 0$ with order $m - 1$. On the other hand, the order of vanishing is $n(d) - 1$, hence $m = n(d)$. $\Box$

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