Generalized intermediate long wave hierachy in zero-curvature representation with noncommutative spectral parameter

(1) A. Degasperis (2) D. Lebedev (3) M. Olshanetsky (3) S. Pakuliak (3) A. Perelomov and (3) P. M. Santini

(1) Dipartimento di Fisica
Università di Roma I and INFN
Sezione di Roma, 00185 Roma
Italy

(2) Institute of Theoretical and
Experimental Physics
117259 Moscow
USSR

(3) Institute of Theoretical Physics
252130 Kiev
USSR

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI/91-63

GENERALIZED INTERMEDIATE LONG WAVE HIERARCHY IN ZERO-CURVATURE REPRESENTATION WITH NONCOMMUTATIVE SPECTRAL PARAMETER

A.Degasperis[1)], *D.Lebedev*[2)], *M.Olshanetsky*[2)], *S.Pakuliak*[3)],
A.Perelomov[2)] *and P.M.Santini*[1)]

[1)]Dipartimento di Fisica, Università di Roma I and INFN,
Sezione di Roma, 00185 Roma, Italy

[2)]Institute of Theoretical and Experimental Physics,
117259 Moscow, USSR

[3)]Institute of Theoretical Physics,
252130 Kiev, USSR

Abstract

We consider the simplest generalization of the Intermediate Long Wave hierarchy to show how to extend the Zakharov-Shabat dressing method to nonlocal, i.e. integro-partial differential, equations. The purpose is to give a procedure of constructing the zero-curvature representation of this class of equations. This result obtains by combining the Drinfeld-Sokolov formalism together with the introduction of an operator-valued spectral parameter, namely a spectral parameter which does not commute with the space variable x. This extension provides a connection between the ILW_k hierarchy and the Saveliev-Vershik continuum graded Lie-algebras. In the case of ILW_2 we find the Fairlie-Zachos *sinh*-algebra.

Typeset in TEX by A.Di Silvestro and M.Mastrofini

1. INTRODUCTION

The Intermediate Long Wave (ILW) equation appears first in physics as it models the propagation of long internal waves in a finite depth fluid [1-4]. From the mathematical side, this equation stays intermediate between the Korteweg-de Vries (KdV) equation [5,6] and the Benjamin-Ono (BO) equation [7-9] (see below), and, as such, it plays the same fundamental role in the theory of nonlocal integrable equations as the KdV equation does in the theory of integrable partial differential equations. Indeed, the solvability of the ILW equation by spectral analysis [10,11] originates from the spectral problem associated with the differential Riemann-Hilbert problem

$$\psi_x^+ + iu(x,t)\psi^+ = \lambda\psi^- \ , \quad \psi^\pm = \psi^\pm(x,\lambda,t) \ , \tag{1.1}$$

where $u(x,t)$ is a given function, λ a complex spectral parameter and the solution $\psi(z,\lambda,t)$ has to be analytic in the strip $-2\hbar < Imz < 0$ of the complex z-plane with the condition that its boundary values

$$\psi^+(x,\lambda,t) = \lim_{\varepsilon\to 0^+} \psi(x - i\varepsilon,\lambda,t) \ , \qquad x\epsilon\mathbf{R} \tag{1.2a}$$

$$\psi^-(x,\lambda,t) = \lim_{\varepsilon\to 0^+} \psi(x - 2i\hbar + i\varepsilon,\lambda,t) \ , \qquad x\epsilon\mathbf{R} \tag{1.2b}$$

satisfy the differential relation (1.1) (a subscript variable indicates partial differentiation, i.e. $\psi_x \equiv \partial\psi/\partial_x$).
Then the ILW equation

$$u_t = Tu_{xx} + 2uu_x \ , \qquad u = u(x,t) \tag{1.3}$$

obtains by requiring that the operator $L = \partial_x + iu$, which appears in the spectral problem (1.1), namely $L\psi^+ = \lambda\psi^-$, evolves in time according to the equation

$$L_t = M^- L - LM^+ \ , \tag{1.4}$$

where

$$M^\pm = i\partial_x^2 - (T \pm 1)u_x \ . \tag{1.5}$$

Here $\partial_x \equiv \partial/\partial_x$ and T is the convolution operator defined by the formula

$$Tf(x) = (1/2\hbar)P.V.\int_{-\infty}^{+\infty} dy\, coth[\pi(y-x)/2\hbar]f(y) \ , \tag{1.6}$$

$\hbar$ being a positive parameter (this notation merely suggests noncommutativity conditions as appearing in quantum mechanics). The important property of the ILW equation (1.3) of going into the KdV equation when $\hbar \to 0$ and the BO equation when $\hbar \to \infty$ follows from the asymptotic behaviour

$$T = -\frac{1}{\hbar}\partial_x^{-1} + \frac{1}{3}\hbar\partial_x + 0(\hbar^3) \ , \qquad \hbar \to 0 \tag{1.7a}$$

$$T = H + 0(\hbar^{-2}) \qquad , \qquad \hbar \to \infty \tag{1.7b}$$

of the operator (1.6), where H is the Hilbert operator

$$Hf(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} dy \, \frac{f(y)}{y-x} \; . \tag{1.8}$$

The ILW equation has proved to possess all beautiful properties of integrable systems [10-12], and, in a general algebraic setting, it has been shown to be the simplest equation of the hierarchy ILW_k of nonlocal integrable equations [13]. Various investigations in the direction of seeking other intermediate-type equations depending on a parameter $\hbar$ with the property of going, as $\hbar \to 0$, into other well-known integrable equations (such as, f.i., the modified KdV equation or the Nonlinear Schroedinger equation) and, as $\hbar \to \infty$, into the BO analogues, were made in [14-17].

In this paper we confine our attention to the ILW_2 hierarchy of evolution equations. In the same way as for the ILW equation (1.3), these equations are associated with a differential Riemann-Hilbert problem, i.e. $L\psi^+ = \lambda\psi^-$, where, however, L is the second order (rather than first order, see (1.1)) differential operator

$$L = \partial_x^2 + u_1(x,t)\partial_x + u_0(x,t) \; , \tag{1.9}$$

the coefficients u_1 and u_0 being two given functions. This class of evolution equations were investigated in [13] by extending the dressing method [18] based on Volterra operators, in the abstract form of pseudo differential operators, in order to deal with the nonlocality introduced by the Riemann-Hilbert problem. In this approach, the evolution equation for the operator L are derived in the form (1.4), where the operators $M^\pm$ are explicitly constructed by a recipe provided by the dressing technique. Our main goal here is to give a method to derive the zero-curvature representation of the ILW_2 equations, to say in the form of a vanishing commutator, $[\partial_x - U, \partial_t - V] = 0$. To this aim, we have first to generalize to nonlocal evolution equations the alternative version of the dressing method based on asymptotic expansions of dressing operators in inverse powers of the spectral parameter λ (for an introduction to this method, although in a different context, see [19]). The appropriate way to do so is to reformulate the second order scalar spectral problem $L\psi^+ = \lambda\psi^-$ in the equivalent form of a 2×2 matrix first order equation. Among several possibilities, we adopt the formalism introduced by Drinfeld and Sokolov [20] with the purpose of generalizing their approach to the present case. In this respect, we take advantage also of the results presented in [21], where our 2×2 matrix evolution equations are investigated in the general algebraic theory of hamiltonian reduction. In the process of building up the λ-dependent dressing technique we are naturally led to replace the ordinary spectral parameter λ with the operator-valued spectral variable

$$\hat{\lambda} \equiv \lambda \, exp(-2i\hbar \, \partial_x) \tag{1.10}$$

where λ is the usual complex variable and $exp(-2i\hbar \, \partial_x)$ is the shift-operator.
The appealing feature of the resulting theory is that the dressing operators are now expressed as formal inverse powers series in the new "spectral parameter" $\hat{\lambda}$, which does not commute with the space variable x.

Once we have obtained the zero-curvature representation with our noncommutative spectral parameter, we prove that the resulting ILW_2 evolution equations are indeed well defined (i.e. built out of the differential operator ∂_x and convolution operator T, see (1.6)), and coincide with those obtained in [13].

Finally, two by-products of our investigation should be pointed out. First, our observation that the ILW_k hierarchy can be obtained as a special reduction of a couple of KP hierarchies seems to be new, and it may be very useful in constructing explicit solutions. Second, the noncommutativity of the spectral parameter (1.10) naturally leads to replace the Kac-Moody algebra, which shows up in the local case, with the continuum graded Lie algebras. The case relevant to the ILW_2 hierarchy is briefly discussed in the last section.

2. PRELIMINARY OBSERVATIONS AND FORMULATION OF THE PROBLEM

In [13] the generalized ILW equations, termed ILW_k hierarchy, have been constructed. Here we briefly review this construction for the typical representative ILW_2. Let us first obtain the differential operator L (1.9) by dressing the bare operator ∂_x^2, namely assume that a solution $\psi(z,\lambda,t)$ of the spectral problem

$$L\psi^+ = \lambda\psi^- \ , \qquad \psi^\pm = \psi^\pm(x,\lambda,t) \ , \qquad x\epsilon\mathbf{R} \tag{2.1}$$

where $\psi^\pm$ are the boundary values (1.2), obtains from a solution $\psi^\circ(z,\lambda)$ of the bare problem ($\psi^\circ = exp(ikz)$, $\lambda = -k^2\, exp(-2k\hbar)$),

$$\psi(z,\lambda,t) = K(z,t,\partial_z)\psi^\circ(z,\lambda) \ , \tag{2.2}$$

where K is the dressing operator

$$K(z,t,\partial_z) = 1 + \sum_{j\geq 1} K_j(z,t)\partial_z^{-j} \tag{2.3}$$

and $K_j(z,t)$ are analytic functions in the strip $\Pi_{2\hbar} = \{z| -2\hbar < Imz < 0\}$. Then, it is easily seen that this assumption implies that the boundary-value operators

$$K^\pm = K^\pm(x,t,\partial_x) = 1 + \sum_{j\geq 1} K_j^\pm(x,t)\partial_x^{-j} \ , \qquad x\epsilon\mathbf{R}, \tag{2.4}$$

transform the operator ∂_x^2 into the operator L according to the equation

$$K^-\,\partial_x^2(K^+)^{-1} = L \ . \tag{2.5}$$

Of course, with obvious notation, $K_j^+(x,t) = K_j(x - i0^+,t)$ and $K_j^-(x,t) = K_j(x - 2i\hbar + i0^+,t)$, in agreement with definition (1.2). Inserting in the equation (2.5) the expressions (2.4) and (1.9) yields, for the coefficients K_j, the recursive equation

$$K_{j+2}^- - K_{j+2}^+ = (\partial_x^2 + u_1\partial_x + u_0)K_j^+ + (2\partial_x + u_1)K_{j+1}^+ \ , \ j \geq 1 \ , \tag{2.6}$$

with the initial conditions

$$K_1^- - K_1^+ = u_1 \ , \qquad K_2^- - K_2^+ = u_0 + (2\partial_x + u_1)K_1^+ \tag{2.7}$$

which allow to obtain the functions u_0 and u_1 from a given dressing operator K, or, viceversa, via a step-by-step calculation, all the coefficients $K_j(z,t)$ of the dressing operator K, see (2.3), from the knowledge of u_0 and u_1. Solving the latter problem requires the Sokhotsky-Plemely formulae, which allow to reconstruct a function $f(z)$, analytic in the strip $\Pi_{2\hbar}$, from the difference of its boundary values $f^\pm(x)$, namely

$$f(z) = (1/4i\hbar) \int_{-\infty}^{+\infty} dy \, coth[\pi(y-z)/2\hbar][f^-(y) - f^+(y)] \ , \ z \epsilon \Pi_{2\hbar} \ , \tag{2.8}$$

and, therefore, also the boundary values themselves, i.e.

$$f^{\pm}(x) = -\frac{1}{2}(iT \pm 1)[f^-(x) - f^+(x)] \ , \qquad x \epsilon \mathbf{R} \ , \tag{2.9}$$

where T is the integral operator defined by (1.6) (and, of course, assuming appropriate conditions on u_0 and u_1, such as smoothness and fast vanishing as $|x| \to \infty$). With respect to this construction, we should note that, if $f^-(x) - f^+(x) \epsilon L_1(\mathbf{R})$ and the solution $f(z)$ is assumed to be everywhere bounded in the strip $\Pi_{2\hbar}$ (including the point at ∞), then $f(z)$ is unique except for an additional constant. In the following, we set this arbitrary constant to zero since, in the solution of the dressing equations (2.6), it merely causes, via (2.2), a change of the normalization of the function $\psi(z, \lambda, t)$.

Let us consider now the time evolution; an interesting (and convenient) way of describing the class ILW_2 (or, more generally, ILW_k) of evolution equations is to consider it as a reduction of the KP hierarchy. More precisely, assume that each of the boundary values $K^{\pm}$ of $K(z, t, \partial_z)$ separately satisfy the KP hierarchy

$$K_t^{\pm} = -(P^{\pm})_- \, K^{\pm} \tag{2.10}$$

where

$$P^{\pm} \equiv K^{\pm} \, \partial_x^s (K^{\pm})^{-1} \ , \qquad s = 1, 2, 3 \ldots ; \tag{2.11}$$

here, and in the following, the operation $(A)_-((A)_+)$ projects a pseudo differential operator

$$A = \sum_{j=-N}^{\infty} A_j \, \partial_x^{-j} \ , \qquad N \geq 0 \ , \tag{2.12}$$

into its part containing only negative (nonnegative) powers of ∂_x, namely

$$(A)_- = \sum_{j \geq 1} A_j \, \partial_x^{-j} \ , \qquad (A)_+ = \sum_{j=-N}^{0} A_j \, \partial_x^{-j} \ . \tag{2.13}$$

In order to set a connection between these two KP herarchies sitting on the two boundaries of the strip $\Pi_{2\hbar}$, we now ask that K^+ and K^- be the boundary values of an analytic operator related to each other by the dressing equation (2.5). This readly implies (as it is easily found by differentiating (2.5) with respect to t and using (2.10)) that the operator L evolves in time according to the equation (see (1.4))

$$L_t = M^- L - L M^+ \tag{2.14}$$

where we have defined

$$M^{\pm} \equiv (P^{\pm})_+ \tag{2.15}$$

and used the obvious formula $P^{\pm} = (P^{\pm})_- + (P^{\pm})_+$ together with the equation

$$P^- L - LP^+ = 0 \tag{2.16}$$

which follows from (2.5) and (2.11).
This class of evolution equations are precisely the ILW_2 hierarchy (the ILW_k class similarly obtains by replacing, in the dressing equation (2.5), ∂_x^2 with ∂_x^k). As a side remark, we observe that this connection with the KP hierarchy opens the way to construct solutions of the ILW_k equations by starting with Backer-Achiezer functions ψ (see (2.2)), or, equivalently, with the corresponding τ-functions, associated with the KP hierarchy. Indeed, solutions of the ILW_k equations obtain by imposing on the KP ψ-function the condition of satisfying the linear spectral problem (2.1), and this consequently implies special conditions on the general Riemann surface.
In this respect, we have been informed by I.Krichever that he has recently constructed a class of solution of the ILW_1 equations by using KP Backer-Achiezer functions, and, in this simple example, he has explicitly given the corresponding restriction on the Riemann surface. Our present derivation of the ILW_k equations via a reduction of the KP hierarchy provides, therefore, a natural way of understanding the result by I.Krichever [22].

For future reference, we now rewrite the evolution equation (2.14) in the following form. Note first that

$$P^+ = K^+ \partial_x^{s-2}(K^-)^{-1}\ K^-\partial_x^2(K^+)^{-1} = WL \tag{2.17a}$$

$$P^- = K^- \partial_x^2(K^+)^{-1}\ K^+\partial_x^{s-2}(K^-)^{-1} = LW \tag{2.17b}$$

where we have introduced the operator

$$W \equiv K^+ \partial_x^{s-2}(K^-)^{-1}\ ; \tag{2.18}$$

therefore, because of (1.9) (say $(L)_+ = L$),

$$M^+ = (WL)_+ = (W)_+ L + (XL)_+ \tag{2.19a}$$

$$M^- = (LW)_+ = L(W)_+ + (LX)_+ \tag{2.19b}$$

where we have set

$$X = (W)_-\ . \tag{2.20}$$

These formulae imply that the evolution equation (2.14) may take the alternative form

$$L_t = (LX)_+ L - L(XL)_+ \tag{2.21}$$

The generic equation of the ILW_2 class obtains by inserting in (2.21) the power expansion (see (2.20))

$$X = \sum_{j\geq 1} \partial_x^{-j} X_j(x,t) \tag{2.22}$$

where the coefficients X_j obtain through the definitions (2.20) and (2.18) together with the recursion relations (2.6) and (2.7). For instance, the simplest nontrivial equation in this class corresponds to $s = 2$ in (2.18) and reads

$$u_{1t} = (X_{1x} - 2X_2 - u_1 X_1)_x \ , \tag{2.23a}$$

$$u_{0t} = (X_{1x} - X_2 - u_1 X_1)_{xx} + u_1(X_{1x} - X_2 - u_1 X_1)_x + u_{0x} X_1 + 2u_0 X_{1x} \ , \tag{2.23b}$$

with

$$X_1 = -u_1 \ , \qquad X_2 = u_1^2 - u_0 + iTu_{1x} \tag{2.24}$$

or, more explicitly,

$$u_{1_t} = (2u_0 - u_{1x} - 2iTu_{1x} - u_1^2)_x \ , \tag{2.25a}$$

$$u_{0t} = (u_0 - u_{1x} - iTu_{1x})_{xx} + u_1(u_0 - u_{1x} - iTu_{1x})_x - u_1 u_{0x} - 2u_0 u_{1x} \ . \tag{2.25b}$$

Let us emphasize, at this point, that the evolution equation (2.21) has a universal form as it appears in different contexts; for instance, it has the same form of the generalized KdV equations associated to the $sl(2)$ algebra [20]. The distinction with respect to the present case comes only with the construction of the coefficients X_j through the equations (2.20), (2.18), (2.6) and (2.7) (in the KdV case, instead, $X = (K\,\partial_x^{s-2}\,K^{-1})_-$ where K satisfies the dressing equation $K\,\partial_x^2\,K^{-1} = \partial_x^2 + u_0$). Moreover, we note that the first equation of the ILW_2 hierarchy, namely (2.25), obtains with $s = 2$ and, therefore, having obviously no counterpart in the KdV class, is peculiar of the nonlocal feature introduced by the Riemann-Hilbert spectral problem (2.1).

Let us now turn our attention to the introduction of a λ-dependence in the evolution equations. This can obtain by means of the second version of the dressing technique, namely with dressing operators which are inverse power expansions in λ rather than in the differential operator ∂_x (see (2.3)). To this aim we the coefficients X_j through the equations (2.20), (2.18), (2.6) and (2.7) (in the KdV case, instead, $X = (K\partial_x^{S-2}K^{-1})_-$ where K satisfies the dressing equations $K\partial_x^2 K^{-1} = \partial_x^2 + u_o$). Moreover, we note that the first equation of the ILW_2 hierarchy, namely (2.25), obtains with $s = 2$ and, therefore, having obviously no counterpart in the KdV class, is peculiar of the nonlocal feature introduced by the Riemann-Hilbert spectral problem (2.1).

Let us now turn our attention to the introduction of a λ-dependence in the evolution equations. This can obtain by means of the second version of the dressing technique, namely with dressing operators which are inverse power expansions in λ rather than in the differential operator ∂_x (see (2.3)). To this aim we have first to transform the (scalar) spectral problem (2.1), with (1.9), into a matrix form, and we do so by rewriting the scalar second order differential operator L, see (1.9), as a first order 2x2 matrix operator according to the formulae [20,21]

$$\mathcal{L}^{can} \equiv \begin{pmatrix} \partial_x + u_1 & u_o \\ -1 & \partial_x \end{pmatrix} = \partial_x - I + q^{can} \ , \tag{2.26a}$$

$$I \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \qquad q^{can} \equiv \begin{pmatrix} u_1 & u_o \\ 0 & 0 \end{pmatrix} , \tag{2.26b}$$

$$\mathcal{L}^{can} \begin{pmatrix} \psi_x \\ \psi \end{pmatrix} = \begin{pmatrix} L\psi \\ 0 \end{pmatrix} . \tag{2.27}$$

These imply that the spectral problem (2.1), which formally reads (see (1.10))

$$L\psi = \hat{\lambda}\psi , \qquad \psi \equiv \psi^+(x, \lambda, t) , \tag{2.28}$$

takes the matrix form

$$\mathcal{L}^{can}(\hat{\Lambda})\Psi = 0 \tag{2.29}$$

where

$$\mathcal{L}^{can}(\hat{\Lambda}) = \mathcal{L}^{can} - \hat{\lambda} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \partial_x - \hat{\Lambda} + q^{can} \tag{2.30a}$$

$$\hat{\Lambda} = \begin{pmatrix} 0 & \hat{\lambda} \\ 1 & 0 \end{pmatrix} , \qquad \Psi = \begin{pmatrix} \psi_x \\ \psi \end{pmatrix} . \tag{2.30b}$$

In contrast with the scalar formalism, in this 2x2 matrix setting it has been shown in [21] that the ILW_2 hierarchy in the form (2.21) can be recast in the usual Lax form; this follows from the hamiltonian approach (for which we refer the reader to [20,21]), which leads to the equation

$$\mathcal{L}_t^{can} = [Grad_q \tilde{\ell}_X, \mathcal{L}^{can}] , \tag{2.31}$$

where $\mathcal{L}^{can}$ is defined by (2.26) and

$$Grad_q \tilde{\ell}_X = \begin{pmatrix} X_2 & X_{2x} - X_{1xx} + (u_1 X_1)_x - u_0 X_1 \\ X_1 & X_2 + u_1 X_1 - X_{1x} \end{pmatrix} . \tag{2.32}$$

In this last expression the functions X_1 and X_2 are the first two coefficients of the expansion (2.22), and the evolution equation (2.31) coincides with the system (2.23). Of course, these formulae apply as well to the local case, namely for $\hbar = 0$; on the other hand, in this case, it is well-known that the (KdV class of) equations (2.31), with (2.32) and $X = \sum_{j\geq 1} \partial_x^{-j} X_j = L^{(s-2)/2}$, can be lifted up to the so-called zero-curvature representation with dependence on the spectral parameter λ, which reads

$$\mathcal{L}_t^{can}(\Lambda) = [V(\Lambda), \mathcal{L}^{can}(\Lambda)] , \tag{2.33}$$

where, according to our notation (2.30),

$$\mathcal{L}^{can}(\Lambda) = \partial_x - \Lambda + q^{can} , \qquad \Lambda = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix} . \tag{2.34}$$

The main point here is that the matrix $V(\Lambda)$ is polynomial in λ of degree

$$\bar{s} = s/2 \; if \; s \; is \; even \; and \; \bar{s} = (s+1)/2 \; if \; s \; is \; odd , \tag{2.35a}$$

with s introduced by (2.11), i.e.

$$V(\Lambda) = \sum_{n=0}^{\bar{s}} V_n \, \lambda^n \tag{2.35b}$$

where the coefficient V_0 is (as it should, see (2.31)) the matrix $grad_q \, \tilde{\ell}_X$ given by the expression (2.32) (with $X = L^{(s-2)/2}$).

The main purpose of this paper is to extend to the nonlocal case, i.e. to the ILW_2 equations, the zero-curvature representation (2.33). We show that lifting up the equation (2.31) to this representation is indeed possible, but non trivial as it requires substituting the matrix Λ (see (2.34)) with the operator-valued matrix $\hat{\Lambda}$ (see (1.10) and (2.30)). Because of the noncommutativity of $\hat{\Lambda}$ with x, the polynomial expression $V(\hat{\Lambda})$ turns out to be rather formal, but we prove that the coefficient V_0 is again the matrix $Grad_q \, \tilde{\ell}_X$, as given by (2.32) with X constructed according to the formulae (2.20) and (2.18).

As a by-product, our investigation naturally leads to look at the algebra where the time evolution takes place. In the local (say KdV) case, the equation (2.33) shows that the lifting up to the λ-dependent zero-curvature representation takes into play the $\hat{g\ell}(2)$ Kac-Moody algebra [20]. In the present case, the lifting up to zero-curvature representation of the ILW_2 equations with a noncommutative spectral parameter leads us to find the cross-product algebras considered by Saveliev and Vershik [23], in particular the Fairlie-Zachos sinh-algebra [24].

3. DRESSING TECHNIQUE WITH NONCOMMUTATIVE SPECTRAL PARAMETER

Here we dress the "naked" spectral equation

$$\mathcal{L}_0 \, \Psi^0 = 0 \tag{3.1a}$$

$$\mathcal{L}_0 \equiv \partial_x - \hat{\Lambda} \ , \qquad \Psi^0 = \begin{pmatrix} ik \\ 1 \end{pmatrix} exp(ikx) \tag{3.1b}$$

wher $\hat{\Lambda}$ is defined by (2.30b) and (1.10) with λ and k related to each other by the equation $\lambda = -k^2 exp(-2\hbar k)$. We start by mimiking the local version of the dressing method, namely we look for an operator G which canonically takes the bare operator $\mathcal{L}_0$ into the dressed operator (2.30a), namely

$$G \, \mathcal{L}_0 \, G^{-1} = \mathcal{L}^{can}(\hat{\Lambda}) \ . \tag{3.2}$$

As a consequence, the vector

$$\Psi = G \, \Psi^0 \tag{3.3}$$

is a solution of the spectral equation (2.29). Rewriting the dressing equation (3.2) in the form

$$G_x + [G, \hat{\Lambda}] + q^{can} \, G = 0 \ , \tag{3.4}$$

clearly shows that, while in the local case ($\hat{\Lambda} = \Lambda$, see (2.34)) G is a 2x2 matrix satisfying an ordinary differential equation, in the present case, G is an operator valued 2x2 matrix, which, however, depends on the differential operator ∂_x only through the shift operator $exp(-2i\hbar\partial_x)$ (see (1.10)). As in the standard (local) case, a solution of the dressing equation (3.4) may be formally represented as an inverse power expansion, i.e. (in this section the dependence on time t is irrelevant, and, therefore, omitted)

$$G = \sum_{n=0}^{\infty} A_n(x)\hat{\lambda}^{-n} \ , \tag{3.5}$$

with respect to the operator-valued spectral parameter $\hat{\lambda}$, whose matrix coefficients $A_n(x)$ (as Ψ^0 and Ψ in (3.3)) are the boundary value on the real axis (from below, see (1.2a)) of a function analytic in the strip $\Pi_{2\hbar}$ (here, and wherever is convenient, we drop the upper sign "+" we have previously used to indicate such boundary value). Inserting the expansion (3.5) in (3.4), and noticing that the action of $\hat{\lambda}$ on the boundary value $A_n(x)$ is well-defined, as it reads (with obvious notation)

$$\hat{\lambda}\, A_n(x) = A_n^-(x)\hat{\lambda} \ , \tag{3.6}$$

easily yields a recursion equation for the coefficients A_n. Apart from unessential constants of integration, the solution of this equation turns out to be expressed in terms of the coefficients $K_n(x)(= K_n^+(x)$, see (2.4)) of the dressing operator K, namely (see the derivation in appendix A)

$$A_n = \begin{pmatrix} K_{2n} + K_{2n-1x} & K_{2n+1} + K_{2nx} \\ K_{2n-1} & K_{2n} \end{pmatrix} \ , \tag{3.7}$$

which, in particular, implies (see (2.7) and (2.9)) that the first coefficient is not the unit matrix, as it reads

$$A_0 = \begin{pmatrix} 1 & -\frac{1}{2}(u_1 + iTu_1) \\ 0 & 1 \end{pmatrix} \ . \tag{3.8}$$

A simpler expression of the expansion coefficients of the dressing operator G obtains if the expansion is performed in inverse powers of the matrix $\hat{\Lambda}$ (see (2.30b)), rather than of the operator $\hat{\lambda}$. These new matrix coefficients G_n are therefore introduced by the formula

$$G = \sum_{n=0}^{\infty} G_n(x)\hat{\Lambda}^{-n} \ , \tag{3.9}$$

which, however, does not uniquely define the matrices $G_n(x)$. Indeed, because of the following property

$$\hat{\Lambda}^2 = \hat{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.10}$$

of the off-diagonal matrix $\hat{\Lambda}$, these exists an expansion, with non vanishing matrix coefficients, which is identically zero. As is easily verified, this is explicitly displayed by the expression

$$\sum_{n=0}^{\infty} Z_n(x)\hat{\Lambda}^{-n} = 0 \tag{3.11a}$$

$$Z_n(x) = \begin{pmatrix} \alpha_n(x) & -\alpha_{n+1}(x) \\ \beta_n(x) & -\beta_{n+1}(x) \end{pmatrix} \quad , \qquad n = 0, 1, 2, \ldots \tag{3.11b}$$

where $\{\alpha_n(x)\}$ and $\{\beta_n(x)\}$ are two sequences of functions which, except for the condition $\alpha_0 = \beta_0 = 0$, are arbitrary (and note that this property of the expansions in powers of $\hat{\Lambda}$ holds true also in the local case since the operator character of the element $\hat{\lambda}$ of the matrix $\hat{\Lambda}$ is completely irrelevant). As a consequence of this observation, the general solution of the dressing equation (3.2) (or, equivalently, (3.4)), in the form (3.9), reads

$$G_{2n} = A_n + Z_{2n} \;; \qquad G_{2n+1} = Z_{2n+1} \;, \qquad n = 0, 1, 2, \ldots \tag{3.12}$$

where the matrices Z_n are given by (3.11b) in terms of the arbitrary functions $\alpha_n(x)$ and $\beta_n(x)$.

We now display few expressions, which the coefficients $G_n(x)$ can take by appropriately choosing the arbitrary functions $\alpha_n(x)$ and $\beta_n(x)$. The simplest form of G_n is the upper-triangular matrix

$$G_n = \begin{pmatrix} K_n & K_{nx} \\ 0 & K_n \end{pmatrix} \;, \qquad G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \tag{3.13}$$

which obtains from the general expression (3.12) by setting

$$\alpha_{2n} = -K_{2n-1x} \;, \qquad \alpha_{2n+1} = K_{2n+1} \;, \tag{3.14a}$$

$$\beta_{2n} = -K_{2n-1} \;, \qquad \beta_{2n+1} = 0 \;. \tag{3.14b}$$

An alternative form is the lower-triangular matrix

$$G_n = \begin{pmatrix} K_n + K_{n-1x} & 0 \\ -K_{n-2x} & K_n + K_{n-1x} \end{pmatrix} \;, \qquad G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \tag{3.15}$$

which corresponds to the choice

$$\alpha_{2n} = 0 \;, \qquad \alpha_{2n+1} = K_{2n+1} + K_{2nx} \;, \tag{3.16a}$$

$$\beta_{2n} = -K_{2n-1} - K_{2n-2x} \;, \qquad \beta_{2n+1} = -K_{2n-1x} \;. \tag{3.16b}$$

A third expression may be the diagonal matrix

$$G_n = \begin{pmatrix} K_n + K_{n-1x} & 0 \\ 0 & K_n \end{pmatrix} \;, \qquad G_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;, \tag{3.17}$$

which obtains with

$$\alpha_{2n} = 0 \;, \qquad \alpha_{2n+1} = K_{2n+1} + K_{2nx} \tag{3.18a}$$

$$\beta_{2n} = -K_{2n-1} \;, \qquad \beta_{2n+1} = 0 \tag{3.18b}$$

In the following, if an explicit expression of G_n had to be used, we will adopt that one given by (3.13).

These findings can be summarized by the following *Lemma 1*: if the functions $K_n(x)(= K_n^+(x))$ are the solution of the Riemann-Hilbert recursive problem (2.6), with (2.7), or, equivalently, the coefficients of the operator (2.4) which solves the dressing equation (2.5), then the solution of the dressing equation (3.2) with the noncommutative spectral parameter $\hat{\lambda}$ is provided by (3.5), with (3.7), or, alternatively, by (3.9) and (3.13) (with the observation that alternative expressions are possible, such as, f.i., (3.15) and (3.17)).

4. CONSTRUCTION OF THE HIERARCHY OF EVOLUTION EQUATIONS

In section 2 the ILW_2 hierarchy of evolution equations has been given in the form (2.14) by using the scalar dressing formalism. Here we derive the hierarchy of evolution equations in the $\hat{\lambda}$-dependent version of the dressing method, and prove that these equations coincide with the ILW_2 class. To this aim, we consider first the class of operators which i) commute with $\mathcal{L}_0 = \partial_x - \hat{\Lambda}$ (see (3.1b)), ii) depend on the operator ∂_x only through the shift operator, i.e. the operator $\hat{\lambda}$ (see (1.10)), and iii) are polynomial in $\hat{\lambda}$. This class is obviously given by $\hat{\Lambda}^s, s = 1, 2, 3, \ldots$, and, therefore, by standard arguments, we dress, by means of the operator G introduced in the previous section, the (trivial) equation $[\partial_t - \hat{\Lambda}^s, \partial_x - \hat{\Lambda}] = 0$, which then becomes

$$[\partial_t - G_t G^{-1} - G\hat{\Lambda}^s G^{-1}\, , \mathcal{L}^{can}(\hat{\Lambda})] = 0 \tag{4.1}$$

where, of course, we have used (3.2). At this point, we observe that the operators $G_t G^{-1}$ and $G\hat{\Lambda}^s G^{-1}$, because of the representation (3.5), can be formally expanded in powers of $\hat{\Lambda}$. This expansion is indeed formal since, if $s > 1$, it requires computing the action of $\hat{\lambda}^n$ on a function $f(x)$, for $n > 1$, which can be done only if $f(x)$ satisfies the strong condition of being the boundary value on the real axis of a function analytic in the domain $x\epsilon\mathbf{R}, -2n\hbar \le y \le 0$. Nevertheless, we proceed in a formal way, and introduce the projections

$$(F)_+ = \sum_{n=-N}^{0} F_n\, \hat{\lambda}^{-n}\ , \qquad (F)_- = \sum_{n=1}^{\infty} F_n\, \hat{\lambda}^{-n} \tag{4.2}$$

of the operator

$$F = \sum_{n=-N}^{\infty} F_n\, \hat{\lambda}^{-n} \tag{4.3}$$

in its nonnegative and negative power expansions. Of course, a similar definition of the projections $(\cdot)_+$ and $(\cdot)_-$, but with respect to the powers of the matrix $\hat{\Lambda}$ rather than to the spectral parameter $\hat{\lambda}$, would be meaningless because of the equations (3.11).

Let us now ask that the evolution in time be such that

$$[G_t G^{-1} + G\hat{\Lambda}^s G^{-1}]_- = 0\ ; \tag{4.4}$$

this, together with (4.1) and the definition (see (2.35a))

$$V(x,t,\hat{\lambda}) = \sum_{n=0}^{s} V_n(x,t)\hat{\lambda}^n = [G_t G^{-1} + G\hat{\Lambda}^s G^{-1}]_+ \ , \tag{4.5}$$

imply the following zero-curvature representation of the class of evolution equations associated with the equation (4.4)

$$\mathcal{L}_t^{can}(\hat{\Lambda}) = [V, \mathcal{L}^{can}(\hat{\Lambda})] \ , \tag{4.6}$$

or, equivalently (see (2.30)),

$$[\partial_t - V, \partial_x - U] = 0 \ , \qquad U \equiv \hat{\Lambda} - q^{can} \ . \tag{4.7}$$

In order to show that, indeed, this construction leads to lifting up the ILW_2 class of equations to the zero-curvature representation with the spectral parameter, we state the following *Lemma 2*: if the dressing operator G satisfies the evolution equation (4.4), then the dressing operator $K(=K^+$, see (2.10) and (2.11)) solves the equation

$$K_t K^{-1} + (K\partial_x^s K^{-1})_- = 0 \tag{4.8}$$

and viceversa, i.e. (4.4) is equivalent to the KP hierarchy (4.8). The proof of this result is given in appendix B. As a simple consequence of (2.17a) and (2.20), the following corollary holds true: the equation (4.4) is equivalent to the evolution equation

$$K_t K^{-1} + (XL)_- = 0 \ ; \tag{4.9}$$

for instance, the evolution of the first coefficient of the expansion (2.4) reads

$$K_{1t} + X_{1xx} + u_0 X_1 - (u_1 X_1)_x - 2X_{2x} + u_1 X_2 + X_3 = 0 \ . \tag{4.10}$$

At this point, it remains to prove that the equation (4.6), for $\lambda = 0$, namely (see (4.5))

$$\mathcal{L}_t^{can} = [V_0, \mathcal{L}^{can}] \ , \tag{4.11}$$

does indeed coincide with the already known equation (2.31), with (2.26a); or, more precisely, the following *Theorem* holds: if G satisfies the evolution equation (4.4) and $V(x,t,\hat{\lambda})$ is defined by (4.5), then

$$V_0(x,t) = Grad_q \tilde{\ell}_X \tag{4.12}$$

where the r.h.s. of this equality is explicitly given by (2.32). In contrast with the local case, because of the noncommutativity of the operator $\hat{\lambda}$ with x, the equivalence of the evolution equation (4.11) with (2.31) is not evident. The proof of this theorem is provided in appendix B.

5. CONCLUSIONS

In this paper we have confined our attention to the class ILW_2 of evolution equations with the purpose of introducing the novel dressing technique to deal with a nonlocal (or intermediate) generalization of the KdV hierarchy. The appropriate formalism, based on expansions in powers of the operator-valued spectral parameter $\hat{\lambda}$, see (1.10), can be applied, of course, to the more general class $ILW_k, k \epsilon \mathbf{Z}$. In the paper [25], we apply this approach to the generalized MKdV and two-dimensional Toda lattice to construct their nonlocal partners.

Finally, we note that the lifting up the ILW_k hierarchy to zero curvature representation with a noncommutative spectral parameter clearly changes the algebra where the evolution takes place in the local case. In order to point out the type of algebras which therefore appear in the nonlocal case, it may be convenient to consider the simplest class, i.e. the ILW_1 equations. In this particular case $\mathcal{L} = \partial_x + u - \hat{\lambda}$ and $V = \hat{\lambda}^s + \sum_{n=0}^{s-1} v_n(x,t)\hat{\lambda}^n$. Thus, the relevant Lie algebra is generated by operators of the form $v_n(x)\hat{\lambda}^n$. More precisely, if the functions $v_n(x)$ are supposed to be periodic in x, then it is natural to define the generators

$$T(\vec{n}) = \frac{1}{2} exp(\hbar n_1 n_2) exp(i n_2 x)\hat{\lambda}^{n_1} \ , \tag{5.1}$$

where $\vec{n} = (n_1, n_2)$ is a two-dimensional vector with $n_1, n_2 \epsilon \mathbf{Z}$. As it is easily verified, this expression of $T(\vec{n})$ satisfies the standard $sinh$-algebra commutation relations

$$[T(\vec{n}), T(\vec{m})] = sinh[\hbar(\vec{n} \times \vec{m})] T(\vec{n} + \vec{m}) \ ,$$

with $\vec{n} \times \vec{m} = n_1 m_2 - n_2 m_1$. Combining this observation with the formulae reported in [23], it is apparent that the ILW_k hierarchy leads to evolutions in the so-called cross product Lie algebras introduced by Saveliev and Vershik. This direction of the present investigation certainly deserves a detailed investigation.

ACKNOWLEDGMENTS

Two of us (A.D. and D.L) whish to thank F.Hirzebruch for his kind hospitality at the Max Planck Institut für Mathematik in Bonn, where this work has been partially done.

Appendix A

Here we show that the power expansion (3.5) of the dressing operator G, with the coefficients (3.7), satisfies the equation (3.4). To this aim, we first note that the equation (3.3), together with the expression (2.30b) of the vector Ψ in terms of the solution ψ, implies that

$$G^{11}\psi^o_x + G^{12}\psi^o = (G^{21}\psi^o_x + G^{22}\psi^o)_x \tag{A.1}$$

where we have introduced the operator-valued matrix elements of the dressing operator G, i.e.

$$G = \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix} \ , \tag{A.2}$$

and, of course, $\psi^o = exp(ikx)$. Since ψ^o is a solution of the undressed equation,

$$\psi^o_{xx} = \hat{\lambda}\psi^o \tag{A.3}$$

the relation (A.1) is equivalent to the two conditions

$$G^{11} = G^{21}_x + G^{22} \ , \qquad G^{12} = G^{22}_x + G^{21}\hat{\lambda} \ , \tag{A.4}$$

which give the matrix (A.2) the general structure

$$G = \begin{pmatrix} G^{21}_x + G^{22} & G^{22}_x + G^{21}\hat{\lambda} \\ G^{21} & G^{22} \end{pmatrix} \ ; \tag{A.5}$$

as the reader may easily verify, this condition on G is consistent with the dressing equation (3.4).

We then ask that the solution ψ of the linear problem (2.28), which is the second component of the vector (see (3.3) and (2.30b))

$$\Psi = G\Psi^o = \begin{pmatrix} \psi_x \\ \psi \end{pmatrix} \ , \tag{A.6}$$

be the solution given by the dressing operator K (see (2.2)), say

$$G\Psi^o = \begin{pmatrix} (K\psi^o)_x \\ K\psi^o \end{pmatrix} \ . \tag{A.7}$$

Therefore the operators G and K, when applied to the bare solution, are simply related to each other, and, in particular, combining (A.5) with (A.7), we have

$$(G^{21}\partial_x + G^{22})exp(ikx) = K\,exp(ikx) \ . \tag{A.8}$$

It now remains to use the expansions (2.4) and (3.5), and to observe that $\hat{\lambda}\,exp(ikx) = (ik)^2 exp(ikx)$, to obtain from (A.8) the equality

$$\sum_{n\geq 0}[A^{21}_n(ik)^{-2n+1} + A^{22}_n(ik)^{-2n}] = 1 + \sum_{n\geq 1} K_n(ik)^{-n} \ , \tag{A.9}$$

which obviously implies

$$A^{21}_n = K_{2n-1} \ , \qquad A^{22}_n = K_{2n} \ , \tag{A.10}$$

with the notation (see (3.5))

$$A_n = \begin{pmatrix} A^{11}_n & A^{12}_n \\ A^{21}_n & A^{22}_n \end{pmatrix} \ . \tag{A.11}$$

The expression (3.7) of the matrix coefficients A_n finally follows from (3.5), (A.5) and (A.10). To complete our task, one has to substitute (3.7) into the dresssing equation (3.4) to merely verify that this is satisfied because of the recursion equation (2.6).

Appendix B

Here we prove *Lemma 2*, and the *Theorem*, stated in section 4. In the following, computations involving powers of the operator $\hat{\lambda}$ are formal, and we assume that, if need be, a function $f(x)$ is analytic off the real x-axis in a strip large enough to give meaning to expressions such as $\hat{\lambda}^n f(x) = f(x - 2in\hbar)\hat{\lambda}^n$. We further observe that, in order to compare a power expansion in $\hat{\lambda}$ with a power expansion in ∂_x, we have to consider their action on $\psi^o = exp(ikx)$, or on the dressed solution $\psi = K\psi^o$ (see (2.2)). In fact, the connection formulae are the linear problems (see (A.3) and (2.28) with (1.9))

$$\hat{\lambda}\psi^o = \partial_x^2\psi^o \ , \qquad \psi^o = exp(ikx) \ , \tag{B.1}$$

$$\hat{\lambda}\psi = (\partial_x^2 + u_1\partial_x + u_0)\psi \ . \tag{B.2}$$

As a consequence, we have that, for any integer n,

$$\hat{\lambda}^n\psi^o = \partial_x^{2n}\psi^o = (ik)^{2n}\psi^o \ , \tag{B.3}$$

while, if n is a nonnegative integer, $n \geq 0$, then

$$\hat{\lambda}^n\psi = Y^{(n)}\psi \ , \tag{B.4}$$

where $Y^{(n)}$ is a purely differential oeprator of order $2n$, whose expression is easily found to be

$$Y^{(n)} = L^{(n-1)}L^{(n-2)}\dots L^{(1)}L \tag{B.5}$$

where L is the second order differential operator (1.9), and $L^{(n)}$ is the second order differential operator which obtains by shifting $L\,n$ times, namely

$$L^{(n)} = exp(-2in\hbar\partial_x)L\,exp(2in\hbar\partial_x) = \partial_x^2 + u_1(x - 2in\hbar)\partial_x + u_0(x - 2in\hbar) \ , \tag{B.6}$$

On the other hand, for $n \geq 0$, the action of the differential operator ∂_x^n on ψ is equivalent to an operator polynomial of $\hat{\lambda}$ according to the formula

$$\partial_x^n\psi = [Z_1^{(n)}(\hat{\lambda})\partial_x + Z_0^{(n)}(\hat{\lambda})]\psi \ , \tag{B.7}$$

where the polynomials $Z_1^{(n)}$ and $Z_0^{(n)}$ can be computed from the recursion equations (use (B.2) to eliminate ∂_x^2)

$$Z_1^{(n+1)} = Z_{1x}^{(n)} - Z_1^{(n)}u_1 + Z_0^{(n)} \ , \tag{B.8a}$$

$$Z_0^{(n+1)} = Z_{0x}^{(n)} + Z_1^{(n)}(\hat{\lambda} - u_0) \ , \tag{B.8b}$$

together with the obvious initial condition

$$Z_1^{(0)} = 0 \ , \qquad Z_0^{(0)} = 1 \ . \tag{B.9}$$

One more technical observation is the following *Lemma 3*: if the (possibly x-dependent) coefficients a_n, b_n and c_n do not depend on k, then the equations

$$\sum_n (a_n \partial_x + b_n) \hat{\lambda}^{-n} \psi^o = 0 \ , \tag{B.10}$$

$$\sum_n c_n \partial_x^{-n} \psi^o = 0 \ , \tag{B.11}$$

where ψ^o is given by (B.1) and the range of values of n in the sum needs not to be specified, respectively imply, if satisfied for all values of k, the vanishing of the coefficients, i.e.

$$a_n = b_n = 0 \ , \tag{B.12}$$

and

$$c_n = 0 \ . \tag{B.13}$$

The proof is a simple consequence of (B.3).

As for the proof of *Lemma 2*, our starting point is the relation

$$(G_t G^{-1} + G \hat{\Lambda}^s G^{-1}) \Psi = \begin{pmatrix} [(K_t K^{-1} + K \partial_x^s K^{-1}) \psi]_x \\ (K_t K^{-1} + K \partial_x^s K^{-1}) \psi \end{pmatrix} \tag{B.14}$$

whose derivation easily goes via (A.6), (A.7) and (3.1). Assume now that the dressing operator G satisfy the evolution equation (4.4), then, using the notation (4.5) with

$$V_n = \begin{pmatrix} V_n^{11} & V_n^{12} \\ V_n^{21} & V_n^{22} \end{pmatrix} \ , \tag{B.15}$$

the second component of the vector equation (B.14), together with (A.6), implies the scalar equation

$$(K_t K^{-1} + K \partial_x^s K^{-1}) \psi = \sum_{n=0}^{\bar{s}} (V_n^{21} \partial_x + V_n^{22}) \hat{\lambda}^n \psi \ , \tag{B.16}$$

where $\bar{s}$ is defined by (2.35a). Since K is the pseudo-differential operator defined by the expansion (2.3), and since the differential operators $Y^{(n)}$ introduced by (B.4) allow to write the operator acting on ψ in the r.h.s. of (B.15) as a differential operator, we can define the coefficients $f_n(x,t)$ through the equation

$$K_t K^{-1} + K \partial_x^s K^{-1} - \sum_{n=0}^{\bar{s}} (V_n^{21} \partial_x + V_n^{22}) Y^{(n)} \equiv \Big(\sum_{n=-s}^{\infty} f_n \partial_x^{-n} \Big) K^{-1} \ . \tag{B.17}$$

In this way, the equation (B.16), with (2.2), reduces to

$$\sum_{n=-s}^{\infty} f_n \partial_x^{-n} \psi^o = 0 \ , \tag{B.18}$$

which, because of *Lemma 3* (see (B.11) and (B.13)), implies $f_n = 0$, and, finally, the KP hierarchy

$$(K_t K^{-1} + K \partial_x^s K^{-1})_- = 0 \ , \tag{B.19}$$

and the relation ($M = M^+$, see (2.15) and (2.11))

$$M = \sum_{n=0}^{\bar{s}} (V_n^{21}\partial_x + V_n^{22}) Y^{(n)} \ . \tag{B.20}$$

The opposite implication can be proved in a similar way. Assume that the dressing operator K satisfy the evolution equation (B.19), then

$$K_t K^{-1} + K\partial_x^s K^{-1} = M = \sum_{n=0}^{s} M_n \partial_x^n \ . \tag{B.21}$$

In this case, the equation (B.14) reads

$$(G_t G^{-1} + G\hat{\Lambda}^s G^{-1})\Psi = \sum_{n=0}^{s} \begin{pmatrix} M_n & M_{nx} \\ 0 & M_n \end{pmatrix} \partial_x^n \Psi \ ; \tag{B.22}$$

observe now that the relation (B.7) and the expression (A.6) of Ψ imply that

$$\partial_x^n \Psi = \begin{pmatrix} Z_1^{(n+1)} & Z_0^{(n+1)} \\ Z_1^{(n)} & Z_0^{(n)} \end{pmatrix} \Psi \ , \tag{B.23}$$

where $Z_1^{(n)}$ and $Z_0^{(n)}$ are polynomials in $\hat{\lambda}$. This naturally leads to define the matrix coefficients $F_n(x,t)$ via the relation

$$(G_t G^{-1} + G\hat{\Lambda}^s G^{-1}) - \sum_{n=0}^{s} \begin{pmatrix} M_n & M_{nx} \\ 0 & M_n \end{pmatrix} \begin{pmatrix} Z_1^{(n+1)} & Z_0^{(n+1)} \\ Z_1^{(n)} & Z_0^{(n)} \end{pmatrix} \equiv \Big(\sum_{n=-\bar{s}}^{\infty} F_n \hat{\lambda}^{-n} \Big) G^{-1} \ , \tag{B.24}$$

so that the equation (B.22) also reads

$$\sum_{n=-\bar{s}}^{\infty} F_n \hat{\lambda}^n \begin{pmatrix} \psi_{0x} \\ \psi_0 \end{pmatrix} = 0 \ . \tag{B.25}$$

Since the two components of this vector equation takes the form (B.10), the *Lemma 3* readly implies $F_n = 0$, and, therefore, the evolution equation (4.4).

Let us now prove the theorem of section 4, namely the equality (4.12). We first observe that the expression (2.19a) can be rewritten as

$$M = (W)_+ L + X_1 \partial_x - X_{1x} + u_1 X_1 + X_2 \ , \tag{B.26}$$

this resulting from computing $(XL)_+$ with (2.22). Moreover, for the ILW_2 class of evolution equations, because of *Lemma 2*, the equation (B.14) becomes

$$V\Psi = \begin{pmatrix} (M\psi)_x \\ M\psi \end{pmatrix} \ . \tag{B.27}$$

This equation has two consequences; the first is

$$V_{11}\psi_x + V_{12}\psi = (V_{21}\psi_x + V_{22}\psi)_x \ , \tag{B.28}$$

and the second is

$$(V_{21}\partial_x + V_{22})\psi = M\psi \ . \tag{B.29}$$

By using the linear problem (B.2), the equation (B.28) takes the form

$$(V_{11}\partial_x + V_{12})\psi = [(V_{21x} - V_{21}u_1 + V_{22})\partial_x + V_{22x} + V_{21}(\hat{\lambda} - u_0)]\psi \ , \tag{B.30}$$

while the equation (B.29), with (B.26), becomes

$$(V_{21}\partial_x + V_{22})\psi = \{[W_1(\hat{\lambda})\partial_x + W_0(\hat{\lambda})]\hat{\lambda} + X_1\partial_x - X_{1x} + u_1X_1 + X_2\}\psi \ , \tag{B.31}$$

where the action of the differential operator $(W)_+$ on ψ has been appropriately rewritten in powers of $\hat{\lambda}$.

Arguments similar to those used in proving *Lemma 2* easily take the equation (B.30) and (B.31) into operator equations, which are

$$V_{11} = V_{21x} - V_{21}u_1 + V_{22} \ , \qquad V_{12} = V_{22x} + V_{21}(\hat{\lambda} - u_0) \ , \tag{B.32}$$

$$V_{21} = W_1(\hat{\lambda})\hat{\lambda} + X_1 \ , \qquad V_{22} = W_0(\hat{\lambda})\hat{\lambda} - X_{1x} + u_1X_1 + X_2 \ . \tag{B.33}$$

It now remains to set $\lambda = 0$ in these equations to obtain the coefficient V_0 of the polynomial (4.5), whose expression turns out to be precisely that given by (2.32).

REFERENCES

1. Joseph R.I., *J.Phys.A.* **101**, 1225 (1977).
2. Kubota T., Ko D.R.S. and Dobbs D., *Hydronaut.* **12**, 157 (1978).
3. Joseph R.I. and Egri R., *J.Phys.A.* **11**, L97 (1978).
4. Chen H.H. and Lee Y.C., *Phys.Rev.Lett.* **43**, 264 (1979).
5. Korteweg D.J. and de Vries G., *Phil.Mag.* **39**, 422 (1895).
6. Gardner C.S., Green J.M., Kruskal M.D. and Miura R.M., *Phys.Rev.Lett.* **19**, 1095 (1967).
7. Benjamin T.B., *J.Fluid.Mech.* **29**, 559 (1967).
8. Ono H., *J.Phys.Soc. Japan* **39**, 1082 (1975).
9. Fokas A.S. and Ablowitz M.J., *Stud.Appl.Math.* **68**, 1 (1983).
10. Satsuma J., Ablowitz M.J. and Kodama Y., *Phys.Lett. A* **73**, 283 (1979).
11. Kodama Y., Ablowitz M.J. and Satsuma J., *J.Math.Phys.* **23**, 564 (1982).
12. Radul A.O., *Dokl.Acad.Nauk.* SSSR **283**, 303 (1985).
13. Lebedev D.R. and Radul A.O., *Commun.Math.Phys.* **91**, 543 (1983).
14. Satsuma J., Taha T.R. and Ablowitz M.J., *J.Math.Phys.* **25**, 900 (1984).
15. Gibbons J. and Kuppershmidt B., *Phys.Lett.* **79A**, 31 (1980).
16. Degasperis A. and Santini P.M., *Phys.Lett.* **98A**, 240 (1983).
17. Degasperis A., Santini P.M. and Ablowitz M.J., *J.Math.Phys.* **26**, 2469 (1985).
18. Zakharov V.E. and Shabat A.B., *Funct.Anal.* **8**, 43 (1974).
19. Zakharov V.E. and Shabat A.B., *Funct.Anal.* **13**, 13 (1979).
20. Drinfeld V. and Sokolov V., *Journal of Soviet Math.* **30**, 1975 (1984).

21. Lebedev D. and Pakuliak S., Preprint GEF-Th-9/1990.
22. Krichever I., to be published.
23. Saveliev M.V. and Vershik A.M., *Phys.Lett.* **143A**, 121 (1990).
24. Fairlie D.B. and Zachos C.K., *Phys.Lett.* **224B**, 101 (1989).
25. Degasperis A., Lebedev D., Olshanetsky M., Pakuliak S., Perelomov A. and Santini P.M., *Commun.Math.Phys.* to be published.