Geometry of Möbius number systems

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Abstract

A Möbius iterative system is a system of real Möbius transformations indexed by a finite alphabet A. A Möbius number system is given by a subshift with alphabet A, such that each its word represents a real number, and this representation is continuous and surjective. We give some sufficient conditions on a subshift to form a Möbius number system. We show several examples based on continued fractions. We consider polygonal Möbius number systems whose transformations tesselate the hyperbolic space by regular polygons. We introduce the Biternary system which is based on a Fuchsian group whose fundamental domain is a rectangle with ideal vertices.

1 Introduction

The positional q-ary number system for the unit real interval $[0, 1]$ is the attractor of the iterative system of contractive linear mappings $x \mapsto (x + a)/q$, where $a \in \{0, 1, \ldots, q-1\}$. An iterative system $(F_a : X \to X)_{a \in A}$ consists of continuous self-maps of a compact metric space X indexed by a finite alphabet A. In contractive iterative systems, each infinite word $u \in A^{\mathbb{N}}$ determines a unique point $x = \Phi(u)$ which is contained in all images $F_{u_0} F_{u_1} \cdots F_{u_{n-1}}(X)$ of the state space X by the prefixes of u. The range of the symbolic representation Φ is a compact subset of X called the attractor of the system (see Barnsley [1]).

In Kůrka [7] and [8] we have studied number systems for the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ based on iterative systems of real Möbius transformations. Since Möbius transformations are not contractive on $\overline{\mathbb{R}}$, the Barnsley theorem does not work for them. Instead of convergence of sets, we use the convergence of measures. An infinite word of digits represents a real number x, if the images of the Cauchy measure by the prefixes of the word converge to the point measure concentrated on x. A Möbius number system is given by a subshift, on which the symbolic representation map is continuous and surjective. In [8] we have developed the theory of Möbius number systems with sofic subshifts. In the present paper we use subshifts which are obtained when we expand real numbers according to some interval cover. While these subshifts are in general not sofic, the arithmetical algorithms are simpler than in the sofic case.

We show that binary signed system and the continued fraction system are special cases of a Möbius number system and combine them into the Binary continued fractions system. We consider polygonal number systems whose transformations generate discrete Fuchsian groups, which tesselate the hyperbolic space by regular polygons. Finally we introduce the Biternary Möbius number system which is based on a Fuchsian group whose fundamental domain is a rectangle with ideal vertices.

2 Möbius transformations

The **extended real line** $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ can be regarded as a projective space, i.e., the space of one-dimensional subspaces of the two-dimensional vector space. On $\overline{\mathbb{R}}$ we have **homogenous coordinates** $x = (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ with equality $x = y$ iff $x_0 y_1 = x_1 y_0$. We regard $x \in \overline{\mathbb{R}}$ as a column vector, and write it usually as $x = x_0/x_1$, for example $\infty = 1/0$. For distinct $a, b \in \mathbb{R}$, the open interval (a, b) is the set $\{x \in \mathbb{R} : \ a < x < b\}$ if $a < b$, and $\{x \in \mathbb{R} : \ a < x \text{ or } x < b\} \cup \{\infty\}$

if $a > b$. We define closed intervals by $[a,b] := (a,b) \cup \{a,b\}$ if $a \neq b$, and $[a,b] = \overline{\mathbb{R}}$ if $a = b$. For $x \in \mathbb{R}$ we have $x \in (a,b)$ iff $(a-x)(x-b)(b-a) > 0$. In homogenous coordinates we get a formula which works for all $a, b \in \overline{\mathbb{R}}$:

$$(a,b) = \{x \in \overline{\mathbb{R}} : \ (a_0x_1 - a_1x_0)(x_0b_1 - x_1b_0)(b_0a_1 - b_1a_0) > 0\}$$

A real **orientation-preserving Möbius transformation** (MT) is a self-map of $\overline{\mathbb{R}}$ of the form

$$M_{(a,b,c,d)}(x) = \frac{ax+b}{cx+d} = \frac{ax_0 + bx_1}{cx_0 + dx_1}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. We can also regard $\overline{\mathbb{R}}$ as a subspace of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. MT act on the **upper half-plane** $\mathbb{U} = \{z \in \mathbb{C} : \ \Im(z) > 0\}$. If $z \in \mathbb{U}$, then $M(z) \in \mathbb{U}$ as well (see Katok [4]). The map $\mathbf{d}(z) = (iz+1)/(z+i)$ maps $\mathbb{U}$ conformally to the **unit disc** $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ and $\overline{\mathbb{R}}$ to the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : \ |z| = 1\}$. Define the **circle distance** on $\overline{\mathbb{R}}$ by

$$\varrho(x,y) = 2\arcsin\frac{|x-y|}{\sqrt{(x^2+1)(y^2+1)}} = 2\arcsin\frac{|x_0y_1 - x_1y_0|}{\sqrt{(x_0^2+x_1^2)(y_0^2+y_1^2)}}$$

which is the length of the shortest arc joining $\mathbf{d}(x)$ and $\mathbf{d}(y)$ in $\partial\mathbb{D}$. The length of a closed interval $B_r(a) = \{x \in \overline{\mathbb{R}} : \ \varrho(x,a) \leq r\}$ is $||B_r(a)|| = \min\{2r, 2\pi\}$. The length $||J||$ of a set $J \subseteq \overline{\mathbb{R}}$ is the length of the shortest interval which contains J. On $\overline{\mathbb{D}} := \mathbb{D} \cup \partial\mathbb{D}$ we get **disc Möbius transformations** $\widehat{M}$ defined by

$$\widehat{M}_{(a,b,c,d)}(z) = \mathbf{d} \circ M_{(a,b,c,d)} \circ \mathbf{d}^{-1}(z) = \frac{\alpha z + \beta}{\overline{\beta}z + \overline{\alpha}},$$

where $\alpha = (a+d) + (b-c)i$, $\beta = (b+c) + (a-d)i$. The disc MT preserve the hyperbolic metric $ds = |dz|/(1-|z|^2) = \sqrt{dx^2+dy^2}/(1-x^2-y^2$ and the hyperbolic distance

$$d(z,w) = \frac{1}{2}\ln\frac{|1-z\overline{w}| + |z-w|}{|1-z\overline{w}| - |z-w|}$$

Denote by $C_q(x) = x/q$, $R_\alpha(x) = (x\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})/(-x\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2})$ the contraction with the coefficient $q > 0$, and the rotation by the angle α. We have $\widehat{R}_\alpha(z) = \alpha z$. Define the **norm** of a Möbius transformation $M = M_{(a,b,c,d)}$ by $||M|| := (a^2+b^2+c^2+d^2)/(ad-bc)$. We have $||M|| \geq 2$ for each M, and $||M|| = 2$ iff M is a rotation, i.e., if $M = R_\alpha$ for some α. The **circle derivation**, the **expansion quotient** and the **expansion interval** of M are defined by

$$\begin{aligned} M^\bullet(x) &:= \lim_{y\to x}\frac{\varrho(M(y),M(x))}{\varrho(y,x)} = |\widehat{M}'(\mathbf{d}(x))| = \frac{(ad-bc)(x_0^2+x_1^2)}{(ax_0+bx_1)^2+(cx_0+dx_1)^2}, \\ \mathbf{q}(M) &:= \min\{M^\bullet(x) : \ x \in \overline{\mathbb{R}}\}. \end{aligned}$$

We have $(MN)^\bullet(x) = M^\bullet(N(x)) \cdot N^\bullet(x)$, $\mathbf{q}(M) \leq 1$, $\mathbf{q}(MN) \geq \mathbf{q}(M) \cdot \mathbf{q}(N)$, and (see Kůrka [8])

$$\begin{aligned} \mathbf{q}(M) &= \tfrac{1}{2}(||M|| - \sqrt{||M||^2 - 4}) = \frac{1 - |\widehat{M}(0)|}{1 + |\widehat{M}(0)|}, \\ 1/\mathbf{q}(M) &= \tfrac{1}{2}(||M|| + \sqrt{||M||^2 - 4}) = \max\{M^\bullet(x) : \ x \in \overline{\mathbb{R}}\}. \end{aligned}$$

3 Möbius number systems

For a finite alphabet A denote by $A^* := \bigcup_{m\geq 0} A^m$ the set of finite words and by $A^+ := A^* \setminus \{\lambda\}$ the set of finite non-empty words. The length of a word $u = u_0 \ldots u_{m-1} \in A^m$ is $|u| := m$. We denote by $A^{\mathbb{N}}$ the Cantor space of infinite words equipped with metric $d(u,v) := 2^{-k}$, where $k = \min\{i \geq 0 : u_i \neq v_i\}$. We denote by $u_{[i,j)} = u_i \ldots u_{j-1}$ and $u_{[i,j]} = u_i \ldots u_j$ subwords of u associated to intervals. We say that $v \in A^*$ is a subword of $u \in A^* \cup A^{\mathbb{N}}$ and write $v \sqsubseteq u$, if $v = u_{[i,j)}$ for some $0 \leq i \leq j \leq |u|$. Given $u \in A^n$, $v \in A^m$, denote by $u.v \in A^{\mathbb{N}}$ the **preperiodic**

word with preperiod u and period v defined by $(u.v)_i = u_i$ for $i < n$ and $(u.v)_{n+km+i} = v_i$ for $i < m$.

The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is defined by $\sigma(u)_i = u_{i+1}$. A **subshift** is a nonempty set $\Sigma \subseteq A^{\mathbb{N}}$ which is closed and σ-invariant, i.e., $\sigma(\Sigma) \subseteq \Sigma$. For a subshift Σ there exists a set $D \subseteq A^+$ of **forbidden words** such that $\Sigma = \mathcal{S}_D := \{x \in A^{\mathbb{N}} : \ \forall u \sqsubseteq x, u \notin D\}$. A subshift is uniquely determined by its **language** $\mathcal{L}(\Sigma) := \{u \in A^* : \ \exists x \in \Sigma, u \sqsubseteq x\}$. Denote by $[u] := \{v \in \Sigma : v_{[0,|u|)} = u\}$ the **cylinder** of $u \in \mathcal{L}(\Sigma)$. A subshift is of **finite type** (SFT), if the set D of forbidden words is finite. A subshift is **sofic**, if its language is regular. An **iterative system** is a continuous map $F : A^* \times X \to X$, or a family of continuous maps $(F_u : X \to X)_{u \in A^*}$ satisfying $F_{uv} = F_u \circ F_v$, and $F_\lambda = \mathrm{Id}$. It is determined by generators $(F_a : X \to X)_{a \in A}$.

Definition 1 *We say that $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, is a* **Möbius iterative system***, if all $F_a : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ are orientation-preserving Möbius transformations. The* **convergence space** *$\mathbb{X}_F \subseteq A^{\mathbb{N}}$ and the* **symbolic representation** *$\Phi : \mathbb{X}_F \to \overline{\mathbb{R}}$ are defined by*

$$\begin{aligned} \mathbb{X}_F &:= \{u \in A^{\mathbb{N}} : \lim_{n\to\infty} F_{u_{[0,n)}}(i) \in \overline{\mathbb{R}}\}, \\ \Phi(u) &:= \lim_{n\to\infty} F_{u_{[0,n)}}(i), \end{aligned}$$

where $i \in \mathbb{U}$ is the imaginary unit. If $\Sigma \subseteq \mathbb{X}_F$ is a subshift such that $\Phi : \Sigma \to \overline{\mathbb{R}}$ is continuous and surjective, then we say that (F, Σ) is a **Möbius number system**. *We say that a Möbius number system is* **redundant***, if for every continuous map $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ there exists a continuous map $f : \Sigma \to \Sigma$ such that $\Phi f = g\Phi$.*

The continuity of function $f : \Sigma \to \Sigma$ is necessary for its computability (see e.g. Weihrauch [11]). We show that the system is redundant iff its cylinders overlap (Theorem 10(4)). The condition of convergence in Definition 1 has probabilistic meaning. Denote by μ the uniform measure on $\partial\mathbb{D}$ and by $\mu_n = \widehat{F}_{u_{[0,n)}}\mu$ its image. Define the mean of a measure by $\mathbb{E}(\mu_n) = \int_{\partial\mathbb{D}} z\, d\mu_n$. Then $\mathbb{E}(\mu_n) = \widehat{F}_{u_{[0,n)}}(0)$ (see Kůrka [7]). These means can be seen in Figure 1. The condition $\Phi(u) = x$ is equivalent to $\lim_{n\to\infty} \mu_n = \delta(\mathbf{d}(x))$, where $\delta(\mathbf{d}(x))$ is the point measure concentrated at $\mathbf{d}(x) \in \partial\mathbb{D}$. This is in turn equivalent to $\lim_{n\to\infty} \widehat{F}_{u_{[0,n)}}(0) = \mathbf{d}(x)$ and $\lim_{n\to\infty} F_{u_{[0,n)}}(i) = x$, where i is the imaginary unit. Another equivalent condition is established in Kůrka [8]:

Lemma 2 *Let $u \in A^{\mathbb{N}}$ and $x \in \overline{\mathbb{R}}$. Then $\Phi(u) = x$ iff there exists $c > 0$ and a sequence of intervals $I_m \ni x$ such that $\liminf_{n\to\infty} ||F^{-1}_{u_{[0,n)}}(I_m)|| > c$ for each m, and $\lim_{m\to\infty} ||I_m|| = 0$.*

Theorem 3 (Kůrka [8]) *Let $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and define the* **expanding intervals** *of $u \in A^*$ by*

$$\mathbf{V}_u := \{x \in \overline{\mathbb{R}} : \ (F_u^{-1})^\bullet(x) > 1\}$$

If $\{\mathbf{V}_u : \ u \in A^*\}$ is a cover of $\overline{\mathbb{R}}$ then $\Phi(\mathbb{X}_F) = \overline{\mathbb{R}}$ and there exists a subshift $\Sigma \subseteq A^{\mathbb{N}}$ such that (F, Σ) is a Möbius number system.

If F_u is a rotation, then $\mathbf{V}_u = \emptyset$, otherwise $\mathbf{V}_u$ is a nonempty interval.

Definition 4 *We say that $\mathcal{W} = \{W_a : \ a \in A\}$ is an* **interval cover** *for a Möbius iterative system $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, if each W_a is an open interval, and the union of all $\overline{W_a}$ is $\overline{\mathbb{R}}$.*

The **diameter** of $\mathcal{W}$ is $||\mathcal{W}|| := \max\{||W_a|| : \ a \in A\}$. The **Lebesgue number** $\ell(\mathcal{W})$ of $\mathcal{W}$ is the supremum of all $l \geq 0$ such that for each interval I of length at most l there exists $a \in A$ such that $I \subseteq W_a$ (this is the overlap of neighbouring intervals). For $u \in A^{n+1}$ set

$$\begin{aligned} W_u &:= W_{u_0} \cap F_{u_0}(W_{u_1}) \cap F_{u_{[0,2)}}(W_{u_2}) \cap \cdots \cap F_{u_{[0,n)}}(W_{u_n}) \\ \mathbf{q}(u) &:= \inf\{(F_u^{-1})^\bullet(x) : \ x \in W_u\} \end{aligned}$$

By definition, $W_\lambda := \overline{\mathbb{R}}$ and $\mathbf{q}(\lambda) := 1$. The sets W_u are not necessarily intervals, since the intersection of two intervals can be a union of two intervals. However, if $||F_a^{-1}(W_a)|| + ||W_b|| < 2\pi$ for each $a, b \in A$, then each W_u is an interval. If F_u is a rotation, then $\mathbf{q}(u) = 1$, otherwise $\mathbf{q}(u) < 1$. We have $W_u \subseteq \mathbf{V}_u$ iff $\mathbf{q}(u) \geq 1$.

Proposition 5 *If $\mathcal{W}$ is an interval cover for F, and $u, v \in A^*$, then $W_{uv} = W_u \cap F_u(W_v)$ and $\mathbf{q}(uv) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$.*

Proof: For $x \in W_{uv}$ we have $(F_{uv}^{-1})^\bullet(x) = (F_u^{-1})^\bullet(x) \cdot (F_v^{-1})^\bullet(F_u^{-1}(x)) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$, and therefore $\mathbf{q}(uv) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$. □

Definition 6 *Let $\mathcal{W} = \{W_a : a \in A\}$ be an interval cover for $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$. Define*

$$\begin{aligned}
\mathcal{L}_\mathcal{W} &:= \{u \in A^* : W_u \neq \emptyset\}, \\
\Sigma_\mathcal{W} &:= \{u \in A^\mathbb{N} : \forall n, W_{u_{[0,n)}} \neq \emptyset\}, \\
\mathcal{W}_n &:= \{W_u : u \in \mathcal{L}_\mathcal{W} \cap A^n\}, \\
\mathbf{Q}_n(\mathcal{W}) &:= \min\{\mathbf{q}(u) : u \in \mathcal{W}_n\}, \\
\mathbf{R}_n(\mathcal{W}) &:= ||\mathcal{W}_n||/2\pi
\end{aligned}$$

By definition $\mathcal{W}_0 = \{\overline{\mathbb{R}}\}$, $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{L}_\mathcal{W}$ is the language of $\Sigma_\mathcal{W}$. Denote by $\mathcal{L}_\mathcal{W}^n := \mathcal{L}_\mathcal{W} \cap A^n$.

Proposition 7 *Let $\mathcal{W}$ be an interval cover for a Möbius iterative system $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, and $n, m \geq 0$.*
(1) Each $\overline{\mathcal{W}}_n = \{\overline{W_u} : u \in \mathcal{L}_\mathcal{W}^n\}$ is a cover of $\overline{\mathbb{R}}$.
(2) $\mathbf{Q}_{n+m}(\mathcal{W}) \geq \mathbf{Q}_n(\mathcal{W}) \cdot \mathbf{Q}_m(\mathcal{W})$.
(3) $||\mathcal{W}_{n+m}|| \leq ||\mathcal{W}_m||/\mathbf{Q}_n(\mathcal{W})$.
(4) $\mathbf{R}_n(\mathcal{W}) \cdot \mathbf{Q}_n(\mathcal{W}) \leq 1$.

Proof: (1) Given $x \in \overline{\mathbb{R}}$ there exists $u \in A^n$ such that for each $k < n$ we have $(F_{u_{[0,k)}}^{-1}(x), y_k) \subseteq W_{u_k}$ for some $y_k \neq F_{u_{[0,k)}}^{-1}(x)$. It follows $(x, y) \subseteq W_u$ for some $y \neq x$, so $u \in \mathcal{L}_\mathcal{W}$, $x \in \overline{W_u}$, and $\overline{\mathcal{W}}_n$ is a cover.
(2) follows from $\mathbf{q}(uv) \geq \mathbf{q}(u) \cdot \mathbf{q}(v)$.
(3) For $u \in A^n$, $v \in A^m$ we have $W_{uv} \subseteq W_u$ and $F_u^{-1}(W_{uv}) \subseteq W_v$, so $\mathbf{q}(u) \cdot ||W_{uv}|| \leq ||F_u^{-1}(W_{uv})|| \leq ||W_v||$. It follows $\mathbf{Q}_n(\mathcal{W}) \cdot ||W_{uv}|| \leq ||W_v|| \leq ||\mathcal{W}_m||$ and therefore $||\mathcal{W}_{m+n}|| \leq ||\mathcal{W}_m||/\mathbf{Q}_n(\mathcal{W})$.
(4) By (3) we have $\mathbf{Q}_n(\mathcal{W}) \leq 2\pi/||\mathcal{W}_n|| = 1/\mathbf{R}_n(\mathcal{W})$. □

Proposition 8 *Assume that for each $u \in A^m$, $a \in A$ we have*

$$W_a \cap F_a(W_u) \neq \emptyset \;\Rightarrow\; F_a(W_u) \subseteq W_a$$

Then $\Sigma_\mathcal{W}$ is a SFT of order $m+1$.

Proof: Assume that $u \in A^n$, and for all $v \sqsubseteq u$ with $|v| = m+1$ we have $W_v \neq \emptyset$. Then

$$W_u = F_{u_0}(W_{u_{[1,n)}}) = \cdots = F_{u_{[0,n-m)}}(W_{u_{[n-m,n)}})$$

and $W_{u_{[n-m,n)}} \neq \emptyset$, so $W_u \neq \emptyset$. □

Definition 9 *The* **expansion quotient** *and the* **interval quotient** *of an interval cover $\mathcal{W}$ for a Möbius iterative system F are defined by*

$$\begin{aligned}
\mathbf{Q}(\mathcal{W}) &:= \lim_{n\to\infty} \sqrt[n]{\mathbf{Q}_n(\mathcal{W})}, \\
\mathbf{R}(\mathcal{W}) &:= \limsup_{n\to\infty} \sqrt[n]{\mathbf{R}_n(\mathcal{W})}.
\end{aligned}$$

Since $\mathbf{Q}_{n+m}(\mathcal{W}) \geq \mathbf{Q}_n(\mathcal{W}) \cdot \mathbf{Q}_m(\mathcal{W})$, the limit $\mathbf{Q}(\mathcal{W})$ exists and $\mathbf{Q}(\mathcal{W}) \geq \sqrt[n]{\mathbf{Q}_n(\mathcal{W})}$ for each n. Since $\mathbf{R}_n(\mathcal{W}) \cdot \mathbf{Q}_n(\mathcal{W}) \leq 1$, we have $\mathbf{R}(\mathcal{W}) \cdot \mathbf{Q}(\mathcal{W}) \leq 1$.

Theorem 10 *Let $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a Möbius iterative system and $\mathcal{W}$ an interval cover for F such that $\mathbf{Q}(\mathcal{W}) > 1$. Then*

(1) $\Sigma_{\mathcal{W}} \subseteq \mathbb{X}_F$.

(2) $\Phi([u]) = \overline{W_u}$ *for each* $u \in \mathcal{L}_{\mathcal{W}}$.

(3) $\Phi : \Sigma_{\mathcal{W}} \to \overline{\mathbb{R}}$ *is continuous and surjective.*

(4) $(F, \Sigma_{\mathcal{W}})$ *is redundant iff* $\ell(\mathcal{W}) > 0$.

Proof: (1) There exists $q > 1$ such that for all sufficiently large n we have $\mathbf{Q}_n(\mathcal{W}) > q^n$. Given $u \in \Sigma_{\mathcal{W}}$, we have $||W_{u_{[0,n)}}|| < 2\pi/q^n$, so the intersection $\bigcap_n \overline{W_{u_{[0,n)}}} = \{x\}$ is a singleton. Since $(F^{-1}_{u_{[0,n)}})^\bullet(x) > q^n$, we get $x = \Phi(u)$ by Lemma 2, so $\Sigma_{\mathcal{W}} \subseteq \mathbb{X}_F$.
(2) For $u \in \mathcal{L}_{\mathcal{W}}$ and $uv \in \Sigma_{\mathcal{W}}$ we have $\Phi(uv) \in \overline{W_u}$, so $\Phi([u]) \subseteq \overline{W_u}$. If $x \in \overline{W_u}$, then there exists v with $\Phi(uv) = x$, so $x \in \Phi([u])$.
(3) Since $\lim_{n\to\infty} ||\mathcal{W}_n|| = 0$, and $\Phi([u]) = \overline{W_u}$, Φ is continuous. Since each $\overline{\mathcal{W}_n}$ is a cover of $\overline{\mathbb{R}}$, Φ is surjective.
(4) If $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is continuous, then $g\Phi : \Sigma \to \overline{\mathbb{R}}$ is uniformly continuous. Given $u \in \Sigma$, we construct $v = f(u) \in \Sigma_{\mathcal{W}}$ by induction so that for each n there exists k_n such that $g\Phi([u_{[0,k_n)}]) \subseteq W_{v_{[0,n)}}$. If the condition holds for n, then there exists $k_{n+1} > k_n$ such that $||g\Phi([u_{[0,k_{n+1})}])|| \leq ||\mathcal{W}_{n+1}||$ so there exists v_n such that $g\Phi([u_{[0,k_{n+1})}]) \subseteq W_{v_{[0,n+1)}}$. Thus $f : \Sigma_{\mathcal{W}} \to \Sigma_{\mathcal{W}}$ is continuous and $\Phi f = g\Phi$. Conversely, if $\ell(\mathcal{W}) = 0$, there exists $y \in \overline{\mathbb{R}}$ and $a, b \in A$ such that $y \in \overline{W_a} \cap \overline{W_b}$ and $W_a \cap W_b = \emptyset$. Since the set of the endpoints of W_u is countable, there exists $x \in \overline{\mathbb{R}}$ such that whenever $x \in \overline{W_u}$ then $x \in W_u$. Let g be a Möbius transformation which maps x to $g(x) = y$. If $f : \Sigma_{\mathcal{W}} \to \Sigma_{\mathcal{W}}$ is such that $\Phi f = g\Phi$, and $\Phi(u) = x$, then f cannot be continuous at u. $\square$

Theorem 11 *Let* $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *be a Möbius iterative system and* $\mathcal{W}$ *an interval cover for* F *such that* $\mathbf{Q}_n(\mathcal{W}) = 1$ *for some* n, *and no* F_u *with* $u \in \mathcal{L}^n_{\mathcal{W}}$ *is a rotation. Then the claims of Theorem 10 hold.*

Proof: The assumptions imply that for each $u \in \mathcal{L}_{\mathcal{W}} \cap A^n$ we have $W_u \subseteq \mathbf{V}_u$. The claims then follow by a theorem of Kazda [5]. $\square$

The quantities $||\mathcal{W}_n||$ and $\mathbf{R}(\mathcal{W})$ express the speed of convergence in the system. On the other hand, high redundancy expressed by $\ell(\mathcal{W})$ means less delay in arithmetical algorithms. Thus optimal number systems have small interval quotient $\mathbf{R}(\mathcal{W})$ and large Lebesgue number $\ell(\mathcal{W})$. There is, however, a tradeoff between these two characteristics.

4 Arithmetical algorithms

In arithmetical algorithms we work with the extended rational numbers $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ with homogenous integer coordinates $x = x_0/x_1 \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Denote by $\mathcal{I}$ the set of open intervals $I = (a, b)$ with endpoints in $\overline{\mathbb{Q}}$, together with the full interval $\overline{\mathbb{R}}$. Denote by $\mathcal{M}_1$ the set of MT $M = M_{(a,b,c,d)}$ whose coefficients $a, b, c, d \in \mathbb{Z}$ are integers with $ad - bc > 0$. We assume that $F : A^* \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Möbius iterative system and $\mathcal{W} = \{W_a : a \in A\}$ is an interval cover such that $F_a \in \mathcal{M}_1$ and $W_a \in \mathcal{I}$ for each $a \in A$. We also assume that $\mathbf{Q}(\mathcal{W}) > 1$ and $\ell(\mathcal{W}) > 0$, so $(F, \Sigma_{\mathcal{W}})$ is a redundant Möbius number system. Denote by $\overline{A} := A \cup \{\lambda\}$ and $\overline{A^*} := A^* \cup A^{\mathbb{N}}$. Under these assumptions there exist algorithms for computing rational functions of one or more variables.

Definition 12 *A* (m, n)-**labelled graph** *over* A *(with* $n \geq 0$ *inputs and* $m \geq 0$ *outputs) is a structure* $G = (V, E, s, t, l)$, *where* V *is a countable set of vertices,* E *is a countable set of edges,* $s, t : E \to V$ *are computable source and target maps, and* $l : E \to \overline{A}^{m+n}$, *is a computable map such that for each* $q \in V$, *the set* $s^{-1}(q)$ *of edges with source* q *is finite, and the map* $q \mapsto s^{-1}(q)$ *is computable.*

A path in G is a word $u \in E^* \cup E^{\mathbb{N}}$ of edges such that $t(u_i) = s(u_{i+1})$. The label of a path is the concatenation of labels of its edges. The graph G determines a many-valued (nondeterministic) function $\Psi : V \times \overline{A^*}^n \to \overline{A^*}^m$ such that $w = \Psi(q, u)$ iff (w, u) is a label of a path with source q. The graph yields a machine consisting of a control unit (head) whose inner states are elements of V. The head is attached to n input tapes and m output tapes. At each time step, the head chooses one of the edges which leads from its state, updates its inner state, reads letters from input tapes and/or writes letters to output states.

Definition 13 *The* $(1,0)$-**number expansion graph** *(no inputs and* 1 *output) is a graph whose vertices are pairs* (x,d)*, where* $x \in \overline{\mathbb{Q}}$ *and* $d \in \{l,r\}$ *(left, right). We have a labelled edge* $(x,l) \xrightarrow{a} (F_a^{-1}(x), l)$ *if* $(x,x') \subseteq W_a$ *for some* $x' \neq x$ *and* $a \in A$*. We have a labelled edge* $(x,r) \xrightarrow{a} (F_a^{-1}(x), r)$ *if* $(x',x) \subseteq W_a$ *for some* $x' \neq x$ *and* $a \in A$*. The* $(1,0)$-**interval expansion graph** *(no inputs and* 1 *output) is a graph whose vertices are intervals* $I \in \mathcal{I}$*. There is an edge* $I \xrightarrow{a} F_a^{-1}(I)$ *whenever* $I \subseteq W_a$*.*

The condition $x \in \overline{W_a}$ alone in the number expansion graph is not sufficient to ensure the nonempty interior of W_u (see the proof of Proposition 7).

Proposition 14 *For each* $x \in \overline{\mathbb{Q}}$ *there exists an infinite path with source* (x,l)*, and an infinite path with source* (x,r)*. If* $u \in A^{\mathbb{N}}$ *is its label, then* $u \in \Sigma_{\mathcal{W}}$ *and* $\Phi(u) = x$*. If* $u \in A^*$ *is the label of a path with source* I*, then* $u \in \mathcal{L}_{\mathcal{W}}$*, and* $I \subseteq \Phi([u])$*.*

Proof: We have $(x,x') \subseteq W_{u_0}$, $(F_{u_0}^{-1}(x), x'') \subseteq W_{u_1}$, so $W_{u_{[0,1]}} \neq \emptyset$, and $x \in \overline{W_{u_{[0,1]}}}$. By induction $x \in \overline{W_{u_{[0,k)}}}$ and $W_{u_{[0,k)}} \neq \emptyset$ for each $k > 0$, so $x = \Phi(u)$. Similar argument works for the interval expansion graph. □

Definition 15 *The* $(0,1)$-**checking graph** *(*1 *input and no output) is a graph whose vertices are intervals* $I \in \mathcal{I}$*. We have a labelled edge* $I \xrightarrow{a} F_a^{-1}(I) \cap W_a$ *whenever* $F_a^{-1}(I) \cap W_a \neq \emptyset$ *is an interval.*

Proposition 16 *There exists a path with source* $\overline{\mathbb{R}}$ *and label* $u \in \overline{A^*}$ *iff* $u \in \mathcal{L}_{\mathcal{W}} \cup \Sigma_{\mathcal{W}}$*.*

Definition 17 *The* $(1,1)$-**linear graph** *(*1 *input and* 1 *output) has vertices* (M,a)*, where* $M \in \mathcal{M}_1$ *and* $a \in \overline{A}$*. The labelled edges are*

$$\begin{array}{lcll} (M,a) & \xrightarrow{(c,\lambda)} & (F_c^{-1}M, a) & \textit{if} \quad M(W_a) \subseteq W_c, \\ (M,a) & \xrightarrow{(\lambda,b)} & (MF_a, b) & \textit{if} \quad \neg\exists c, M(W_a) \subseteq W_c. \end{array}$$

Proposition 18 *If* (w,u) *is the label of a path with source* (M,λ) *and* $u \in \Sigma_{\mathcal{W}}$*, then* $w \in \Sigma_{\mathcal{W}}$ *and* $\Phi(w) = M(\Phi(u))$*. If* $u \in \mathcal{L}_{\mathcal{W}}$*, then* $w \in \mathcal{L}_{\mathcal{W}}$ *and* $M(\Phi([u])) \subseteq \Phi([w])$*.*

Proof: We show by induction that when there is a path with source (M,λ) and label $(w,u) \in A^* \times \mathcal{L}_{\mathcal{W}}$, then $M(W_u) \subseteq W_w$ and its target is $(F_w^{-1}MF_u, a)$, where $a = u_{|u|-1}$ is the last letter of u. Since $W_\lambda = \overline{\mathbb{R}}$, the first edge $(M,\lambda) \to (M,a)$ has label (λ,a), so $M(W_u) = M(W_a) \subseteq W_\lambda = W_w$ is satisfied. Suppose that the assumption holds for (w,u), and consider an edge $(F_w^{-1}MF_u, a) \to (F_w^{-1}MF_{ua}, b)$ with label (λ, b). Then $M(W_{ub}) \subseteq M(W_u) \subseteq W_w$, so the statement holds for the path label (w, ub). Consider an edge $(F_w^{-1}MF_u, a) \to (F_{wc}^{-1}MF_u, a)$, with label (c,λ), so $F_w^{-1}MF_u(W_a) \subseteq W_c$. Then $M(W_{ua}) \subseteq MF_u(W_a) \subseteq F_w(W_c)$. Since $M(W_{ua}) \subseteq M(W_u) \subseteq W_w$, we get $M(W_{ua}) \subseteq W_w \cap F_w(W_c) = W_{wc}$, so the statement holds for the path label (wc, u). □

Similar algorithms work for bilinear functions

$$P(x,y) = \frac{axy + bx + cy + d}{exy + fx + gy + h} = \frac{ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1}{ex_0y_0 + fx_0y_1 + gx_1y_0 + hx_1y_1}$$

These algorithms are based on the fact that for a bilinear function $P(x,y)$ and a MT M, the functions $M(P(x,y))$, $P(M(x),y)$, and $P(x, M(y))$ are bilinear. Similarly, if

$$P(x) = \frac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_nx^n} = \frac{a_0x_1^n + a_1x_0x_1^{n-1} + \cdots + a_nx_0^n}{b_0x_1^n + b_1x_0x_1^{n-1} + \cdots + b_nx_0^n}$$

is a rational function of degree n and M is a MT, then both $P \circ M$ and $M \circ P$ are rational functions of degree n. This yields algorithms for expansions of algebraic numbers and for evaluations of rational functions (see Gosper [3], Vuillemin [10], or Kornerup and Matula [6]).

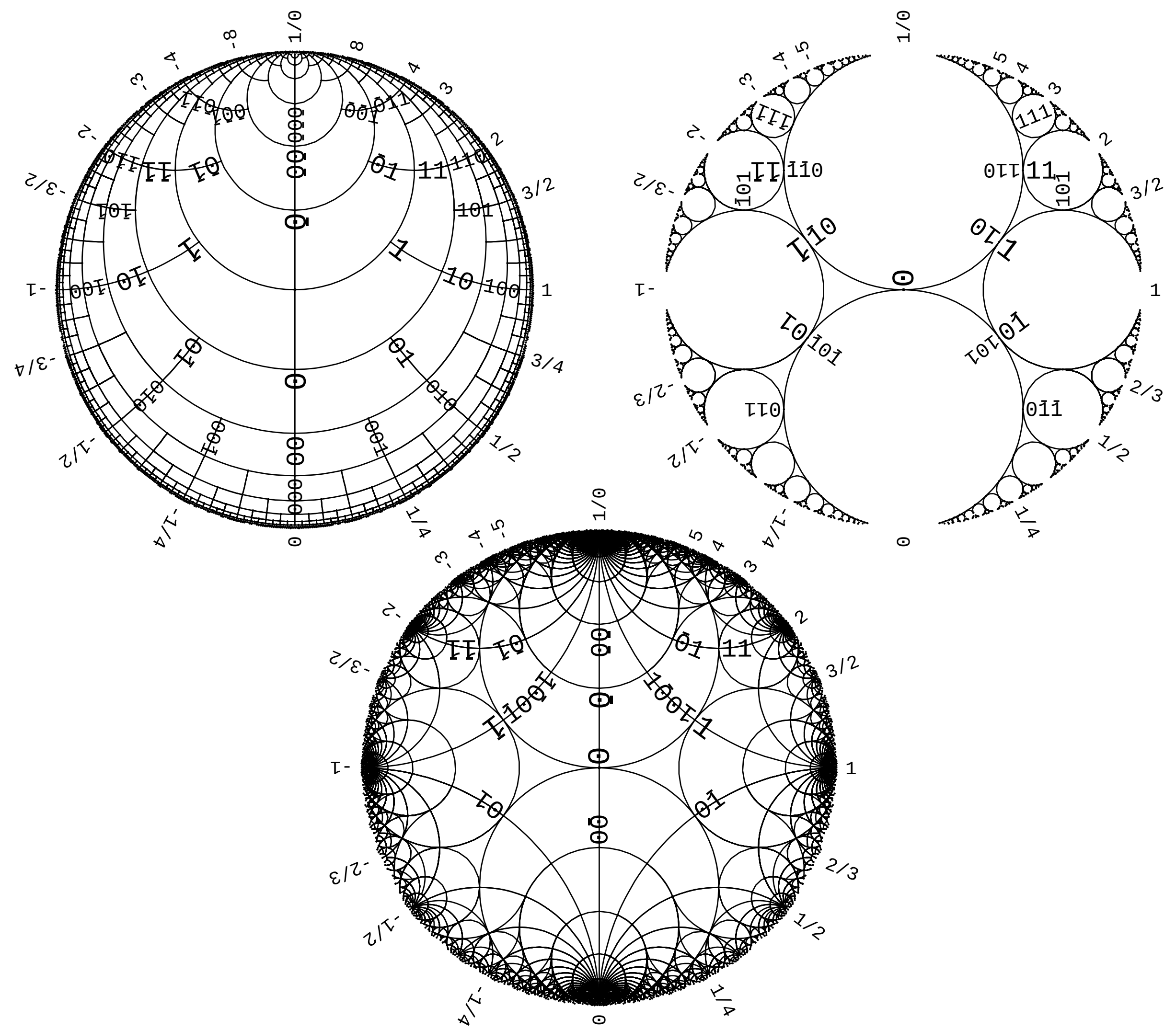


Figure 1: Means of the binary signed system (BSS, top left), semi-regular continued fractions (SRCF, top right) and binary continued fractions (BCF bottom)

5 Binary continued fractions

The binary signed number system for the interval $[-1,1]$ is based on iterations of mappings $(x-1)/2$, $x/2$, $(x+1)/2$. In fact $[-1,1]$ is the attractor of this system and $\Phi(u)=\sum_{n\geq 0} 2^{-i-1}u_i$ is its symbolic representation. We use simpler transformations $x-1$, $x/2$, $x+1$ and take also $2x$ to get the whole $\overline{\mathbb{R}}$. We use the alphabet $A=\{\overline{1},0,1,\overline{0}\}$ which represents numbers $-1,0,1,\infty$.

Example 1 *The* **Möbius binary signed system** *(BSS - Figure 1 top left) consists of the alphabet* $A=\{\overline{1},0,1,\overline{0}\}$*, transformations* $F_{\overline{1}}(x)=-1+x,\ F_0(x)=x/2,\ F_1(x)=1+x,\ F_{\overline{0}}(x)=2x$*, and the interval cover* $W_{\overline{1}}=(-2,-\frac{1}{2})$*,* $W_0=(-\frac{3}{4},\frac{3}{4})$*,* $W_1=(\frac{1}{2},2)$*,* $W_{\overline{0}}=(\frac{3}{2},-\frac{3}{2})$*.*

The intervals W_a are chosen with regard to the expansion intervals $\mathbf{V}_{\overline{1}}=(\infty,-\frac{1}{2})$, $\mathbf{V}_0=(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$, $\mathbf{V}_1=(\frac{1}{2},\infty)$, $\mathbf{V}_{\overline{0}}=(\sqrt{2},-\sqrt{2})$. We have $\ell(\mathcal{W})\doteq 0.249$, and $\mathbf{Q}(\mathcal{W})>1.36$, so $(F,\Sigma_{\mathcal{W}})$ is a Möbius number system. Since $\overline{1}0$, $0\overline{0}$, $1\overline{0}$ are forbidden words in $\Sigma_{\mathcal{W}}$, the letter $\overline{0}$ can occur only at the beginning of a word and each $u\in\mathcal{L}_{\mathcal{W}}$ can be written as $u=\overline{0}^n v$, where $v\in\{\overline{1},0,1\}^*$ and $n\geq 0$. Since $\overline{1}\overline{1}\overline{1}$, 111 are forbidden words, F_u can be written as $F_u(x)=2^n\left(s_0+\frac{1}{2}\left(s_1+\frac{1}{2}\left(s_2+\cdots+\frac{1}{2}\left(s_{k-1}+\frac{x}{2}\right)\cdots\right)\right)\right)$, where $k\geq 0$ and $s_i\in\{-2,-1,0,1,2\}$, $s_0\neq 0$.

The means $\widehat{F}_u(0)$ of words $u\in\mathcal{L}_{\mathcal{W}}$ can be seen in Figure 1. The curves between these means are constructed as follows. For each MT M there exists a family of MT $(M^t)_{t\in\mathbb{R}}$ such that $M^0=\mathrm{Id}$, $M^1=M$, and $M^{t+s}=M^tM^s$. In Figure 1, each mean $\widehat{F}_{ua}(0)$ is joined to $\widehat{F}_u(0)$ by the curve $(\widehat{F}_u\widehat{F}_a^t(0))_{0\leq t\leq 1}$. The labels $u\in A^+$ at $\widehat{F}_u(0)$ are written in the direction of the tangent vectors $\widehat{F}_u'(0)$. In fact the mean $\widehat{F}(0)$ and the unit tangent vextor $\widehat{F}'(0)/|\widehat{F}'(0)|$ determine the transformation F uniquely.

Regular continued fractions are based on iterations of transformations $1+x$ and $1/x$. Since $1/x$ is orientation-reversing, we use rather the orientation preserving transformation $F_0(x) = -1/x$ which corresponds to the rotation $\widehat{F}_0(z) = -z$ of the unit circle by π. It follows that $\widehat{F}_{u0}(0) = \widehat{F}_u(0)$, but the tangent vectors of u and ua differ by π (see Figure 1 top right).

Example 2 *The* **Möbius system of regular continued fraction** *(RCF see [8]) consists of the alphabet* $A = \{\overline{1}, 0, 1\}$*, transformations* $F_{\overline{1}}(x) = -1+x$*,* $F_0(x) = -1/x$*,* $F_1(x) = 1+x$*, and the interval cover* $W_{\overline{1}} = (\infty, -1)$*,* $W_0 = (-1, 1)$*,* $W_1 = (1, \infty)$*.*

The subshift $\Sigma_{\mathcal{W}} = \Sigma_{\{00,\overline{1}1,1\overline{1},\overline{1}0\overline{1},101\}}$ is of finite type. For each $u \in \mathcal{L}(\Sigma_D)$, the transformation F_u can be written as $F_u(x) = F_1^{a_0} F_0 F_1^{a_1} \cdots F_0 F_1^{a_n}(x)$ where $a_i \in \mathbb{Z}$, $a_i a_{i+1} \leq 0$ and $a_i \neq 0$ for $i > 0$. Thus we obtain a continued fraction whose partial quotients $(-1)^i a_i$ are either all positive or all negative and such continued fractions converge by the standard theory. Alternatively, we can use Theorem 11. We have $\mathbf{Q}_n(\mathcal{W}) = 1$ and $\mathbf{R}_n(\mathcal{W}) = ||(n,\infty)||/2\pi \approx 1/\pi n$, so $\mathbf{Q}(\mathcal{W}) = \mathbf{R}(\mathcal{W}) = 1$. Since $\ell(\mathcal{W}) = 0$, $(F, \Sigma_{\mathcal{W}})$ is a non-redundant Möbius number system (see Kůrka [8]). Each rational number has two preperiodic expansions with period length 1 of the form $u.1$ or $u.\overline{1}$.

Example 3 *The* **Möbius system of semi-regular continued fraction** *(SRCF - Figure 1 top right) consists of the alphabet* $A = \{\overline{1}, 0, 1\}$*, transformations* $F_{\overline{1}}(x) = -1+x$*,* $F_0(x) = -1/x$*,* $F_1(x) = 1+x$*, and the interval cover* $W_{\overline{1}} = (\infty, -\frac{1}{2})$*,* $W_0 = (-1, 1)$*,* $W_1 = (\frac{1}{2}, \infty)$*.*

Semi-regular continued fractions converge by a theory exposed in Perron [9]. The subshift

$$\Sigma_{\mathcal{W}} = \mathcal{S}_{\{00,\overline{1}1,1\overline{1},\overline{1}0\overline{1}0,1010,0\overline{1}0\overline{1},0101,\overline{1}0\overline{1}\overline{1}0\overline{1},101101\}} \subseteq \mathbb{X}_F$$

is of finite type. We have again $\ell(\mathcal{W}) = 0$ and $\mathbf{Q}(\mathcal{W}) = \mathbf{R}(\mathcal{W}) = 1$, so the system is not redundant and the convergence is slow. We add the transformation $F_2(x) = 2x$ to make it faster.

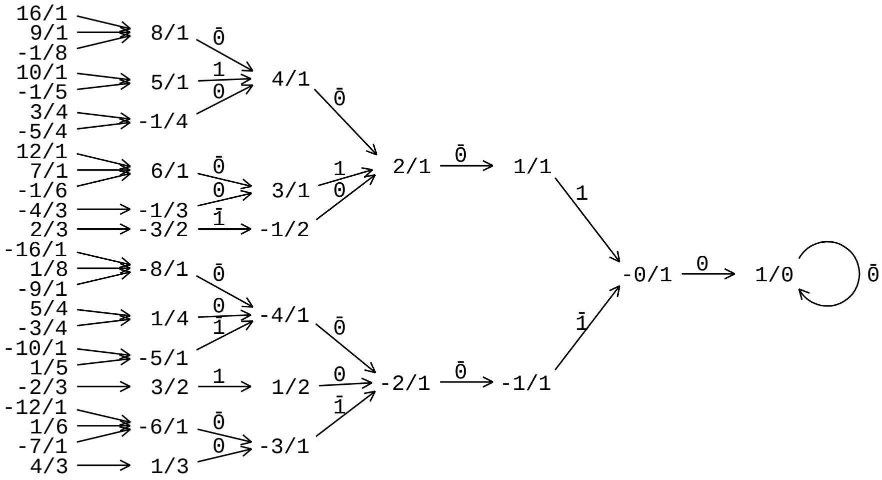
Example 4 *The* **Möbius system of binary continued fraction** *(BCF - Figure 1 bottom) consists of the alphabet* $A = \{\overline{1}, 0, 1, \overline{0}\}$*, transformations* $F_{\overline{1}}(x) = -1+x$*,* $F_0(x) = -1/x$*,* $F_1(x) = 1+x$*,* $F_{\overline{0}}(x) = 2x$*, and the interval cover* $W_{\overline{1}} = (\infty, -\frac{1}{2})$*,* $W_0 = (-1, 1)$*,* $W_1 = (\frac{1}{2}, \infty)$*,* $W_{\overline{0}} = (2, -2)$*.*

We get $\Sigma_{\mathcal{W}} = \mathcal{S}_{\{00,\overline{0}0,\overline{1}0\overline{1}0,1010,\overline{1}\overline{0}^*1,1\overline{0}^*\overline{1},0\overline{1}0\overline{0}^*\overline{1},010\overline{0}^*1,\overline{1}0\overline{1}\overline{1}0\overline{0}^*\overline{1},10110\overline{0}^*1,\overline{0}\overline{1}0\overline{0}^*\overline{1},\overline{0}102^*1\}}$, which is a sofic subshift. Since $F_1^{\bullet}(\infty) = 1$, we have $\mathbf{Q}_n(\Sigma_{\mathcal{W}}) = 1$ for each n, and the system converges by Theorem 11. The Lebesgue number is $\ell(\mathcal{W}) = \varrho(\frac{1}{2}, 1) \doteq 0.644$, so $(F, \Sigma_{\mathcal{W}})$ is a redundant system. Each rational number has an eventually periodic expansion with period length 1 of the form $u.\overline{0}$ (see Kůrka [7]). To obtain shorter expansions of rational numbers, we test the parity of the numerator and denominator:

Definition 19 *The arithmetical expansion graph (Figure 2 top) for the BCF system has vertices* $x = (x_0, x_1) \in \overline{\mathbb{Q}}$*, with* $x_1 \geq 0$ *and labeled edges*

$$\begin{array}{lcll}
x & \xrightarrow{\overline{0}} & (x_0/2, x_1) & \textit{if} \quad |x_0| \geq 2|x_1| \;\&\; 2|x_0 \\
x & \xrightarrow{\overline{0}} & (x_0, 2x_1) & \textit{if} \quad |x_0| \geq 2|x_1| \;\&\; 2|x_1 \\
x & \xrightarrow{\overline{1}} & (x_0 + x_1, x_1) & \textit{if} \quad x_0 \leq -x_1 \;\vee\; (2x_0 < -x_1 \;\&\; 2|x_1) \\
x & \xrightarrow{1} & (x_0 - x_1, x_1) & \textit{if} \quad x_0 \geq x_1 \;\vee\; (2x_0 > x_1 \;\&\; 2|x_1) \\
x & \xrightarrow{0} & (-x_1 \cdot \mathbf{sgn}(x_0), |x_0|) & \textit{otherwise}
\end{array}$$

In the expansion procedure of Definition 19, the first applicable rule is used, so each vertex has outdegree 1 and we get a deterministic expansion function $\mathcal{E} : \overline{\mathbb{Q}} \to \Sigma_{\mathcal{W}}$, such that $\mathcal{E}(x)$ is the label of the unique infinite path with source x. It follows $\Phi(\mathcal{E}(x)) = x$ for each $x \in \overline{\mathbb{Q}}$. Each rational number has expansion of the form $u.\overline{0}$ and integers have the same expansions as in the classical binary system. An integer can be written as $x = x_0 + 2x_1 + \cdots + 2^k x_k$, where $x_i \in \{-1, 0, 1\}$ are either all non-negative, or all non-positive. Then $\mathcal{E}(x) = s_0\overline{0}s_1\overline{0}\ldots\overline{0}s_{k-1}\overline{0}s_k0.\overline{0}$, where s_i is empty if $x_i = 0$, $s_i = \overline{1}$ if $x_i = -1$, and $s_i = 1$ if $x_i = 1$ (see Figure 2 bottom).



0/1	$0.\overline{0}$	1/5	$0\overline{1001}0.\overline{0}$	1/4	$0\overline{001}0.\overline{0}$	2/7	$0\overline{011}0\overline{001}0.\overline{0}$
1/3	$0\overline{101}0.\overline{0}$	3/8	$0\overline{01}01\overline{0}10.\overline{0}$	2/5	$0\overline{01}0\overline{00}10.\overline{0}$	3/7	$0\overline{11}01\overline{0}10.\overline{0}$
1/2	$0\overline{01}0.\overline{0}$	4/7	$0\overline{11}0\overline{001}0.\overline{0}$	3/5	$0\overline{1}010\overline{01}0.\overline{0}$	5/8	$10\overline{0}10\overline{101}0.\overline{0}$
2/3	$0\overline{1}0\overline{0}10.\overline{0}$	5/7	$0\overline{1}0\overline{0}10\overline{001}0.\overline{0}$	3/4	$10\overline{00}10.\overline{0}$	4/5	$0\overline{1}0\overline{00}10.\overline{0}$
1/1	$10.\overline{0}$	5/4	$10\overline{001}0.\overline{0}$	4/3	$10\overline{101}0.\overline{0}$	7/5	$10\overline{01}0\overline{00}10.\overline{0}$
3/2	$10\overline{01}0.\overline{0}$	8/5	$10\overline{1}010\overline{01}0.\overline{0}$	5/3	$10\overline{1}0\overline{0}10.\overline{0}$	7/4	$110\overline{00}10.\overline{0}$
2/1	$\overline{0}10.\overline{0}$	7/3	$110\overline{101}0.\overline{0}$	5/2	$\overline{0}10\overline{001}0.\overline{0}$	8/3	$\overline{0}10\overline{101}0.\overline{0}$
3/1	$1\overline{0}10.\overline{0}$	7/2	$\overline{0}110\overline{00}10.\overline{0}$	4/1	$\overline{00}10.\overline{0}$	5/1	$1\overline{00}10.\overline{0}$

Figure 2: Arithmetical expansions of rational numbers in BCF

6 Fuchsian groups

Given a Möbius iterative system F, denote by $\mathcal{G}(F)$ the group generated by the transformations $(F_a)_{a\in A}$. Discrete groups of MT are called Fuchsian groups (see Katok [4] or Beardon [2]). For example, the iterative system of the regular or semiregular continued fractions with transformations $F_{\overline{1}}(x) = -1 + x$, $F_0(x) = -1/x$, $F_1(x) = 1 + x$ generates the modular group $\mathcal{G}(F) = \{M_{(a,b,c,d)} : a, b, c, d \in \mathbb{Z},\ ad-bc = 1\}$. We consider Möbius number systems, whose groups generate tesselation of the hyperbolic space by regular polygons.

Definition 20 *The $(2n, 2m)$-polygonal system, where $\frac{1}{n} + \frac{1}{m} < 1$, has alphabet $A = \{0, 1, \ldots, 2n-1\}$ and transformations $F_j = R^j C_q R^{-j}$, where $R = R_{\pi/n}$ and*

$$q = q_{(2n,2m)} = \frac{1 + \sqrt{1 - \sin^2\frac{\pi}{2n} / \cos^2\frac{\pi}{2m}}}{1 - \sqrt{1 - \sin^2\frac{\pi}{2n} / \cos^2\frac{\pi}{2m}}}$$

Denote by $\mathcal{G}(2n, 2m)$ the group generated by $F_0, \ldots, F_{2n-1}$. Here $2m$ can be an odd integer, but $2n$ must be even.

Proposition 21 *$\mathcal{G}(2n, 2m)$ is a discrete group which satisfies identities $F_iF_{i+n} = \mathrm{Id}$, $R^iF_j = F_{i+j}R^i$, $F_0F_{(n-1)}F_{2(n-1)}\cdots F_{(2m-1)(n-1)} = R^{-2m}$, $F_0F_{(n+1)}F_{2(n+1)}\cdots F_{(2m-1)(n+1)} = R^{2m}$, (the addition is modulo $2n$).*

Proof: Denote by $A_i = \widehat{F}_0\widehat{F}_{n-1}\cdots\widehat{F}_{i(n-1)}(0)$. We search for the condition on q which implies $A_{2m-1} = 0$ and therefore $A_{2m} = A_0$. In this case the points $A_0, A_1, \ldots, A_{2m-1}$, form a regular $2m$-gon whose inner angles at vertices A_i are π/n. Denote by $a = \varrho(0, \widehat{F}_0(0))$ the hyperbolic length of the side of this polygon, by S its center and by B_0 the middle of the hyperbolic line A_0A_1. The hyperbolic triangle SA_0B_0 has angles $\pi/2m$, $\pi/2n$, $\pi/2$ and the side of length $a/2$ opposite to S. By the Cosine rule II we get

$$\frac{1}{\sqrt{1 - |\widehat{F}_0(0)|^2}} = \cosh\frac{a}{2} = \frac{\cos\frac{\pi}{2n}\cos\frac{\pi}{2} + \cos\frac{\pi}{2m}}{\sin\frac{\pi}{2n}\sin\frac{\pi}{2}} = \frac{\cos\frac{\pi}{2m}}{\sin\frac{\pi}{2n}}$$

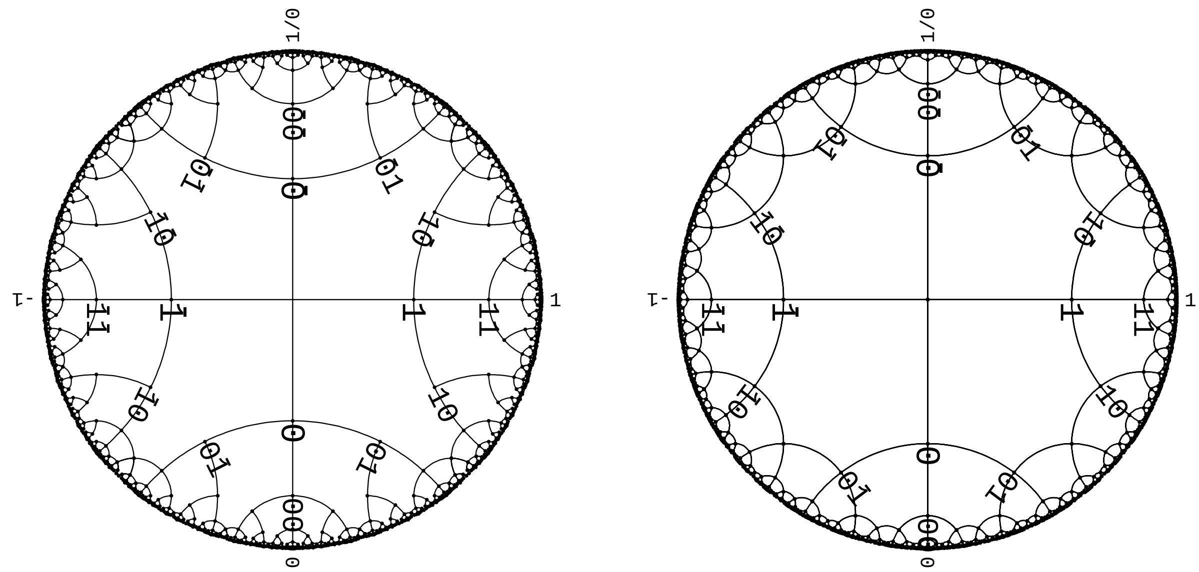


Figure 3: Polygonal $(4,5)$-system with $F_0F_1F_{\overline{0}}F_{\overline{1}}F_0 = R^{-1}$, and $(4,6)$-system with $F_0F_1F_{\overline{0}}F_{\overline{1}}F_0F_1 = R^2$

Since $\widehat{F}_0(0) = -i(q-1)/(q+1)$, we get

$$q = \frac{1+|\widehat{F}_0(0)|}{1-|\widehat{F}_0(0)|} = \frac{1+\sqrt{1-1/\cosh^2(a/2)}}{1-\sqrt{1-1/\cosh^2(a/2)}} = e^a$$

and the formula for q follows. The angle between the hyperbolic geodetic A_0A_1 and the euclidean geodetic (straight line) A_0A_1 is $\frac{\pi}{2} - \frac{\pi}{2n} - \frac{\pi}{2m}$, therefore the rotation angles of $\widehat{F}_0 \cdots \widehat{F}_{i(n-1)}$ and $\widehat{F}_0 \cdots \widehat{F}_{i(n-1)}\widehat{F}_{(i+1)(n-1)}$ differ by $2(\frac{\pi}{2} - \frac{\pi}{2n} - \frac{\pi}{2m})$. Since $R^{2n} = \mathrm{Id}$, we get

$$F_0 \cdots F_{(2m-1)(n+1)} = R_{4m(\frac{\pi}{2} - \frac{\pi}{2n} - \frac{\pi}{2m})} = R_{-2\pi m/n} = R^{-2m} \qquad \square$$

For a $(2n,2m)$ polygonal system and $a > 0$ consider an interval cover $\mathcal{W}_a = \{W_k : k \in A\}$, where $W_k = (R_{k\pi/n}(-a), R_{k\pi/n}(a))$. We can find $a > 0$ such that $\mathcal{W}_a$ satisfies Proposition 8 and $\Sigma_{(2n,2m)} := \Sigma_{\mathcal{W}_a}$ is a Möbius number system. For $2n = 4$ and $2m = 5,6$, we get

$$\begin{aligned} \Sigma_{(4,5)} &= \mathcal{S}_{\{\overline{1}1,0\overline{0},1\overline{1},\overline{0}0,\overline{1}01,\overline{10}1,01\overline{0},0\overline{10},1\overline{01},10\overline{1},\overline{01}0,\overline{0}10\}} \\ \Sigma_{(4,6)} &= \mathcal{S}_{\{\overline{1}1,0\overline{0},1\overline{1},\overline{0}0,\overline{1}01\overline{0},\overline{10}10,01\overline{01},0\overline{10}1,1\overline{01}0,10\overline{10},\overline{01}01,\overline{0}10\overline{1}\}} \end{aligned}$$

The means of these systems can be seen in Figure 3. We use again the alphabet $\{0,1,\overline{0},\overline{1}\}$ instead of $\{0,1,2,3\}$. The quotients of polygonal systems are not rational, but they are algebraic. The algorithms of Section 4 work if we use the countable field $\overline{\mathbb{Q}}[q]$ instead of $\overline{\mathbb{Q}}$. However, the arithmetics in $\overline{\mathbb{Q}}[q]$ is slower and needs more memory. Moreover, rational numbers do not have preperiodic expansions in these systems. In the next section we construct another system based on a Fuchsian group in which rational numbers do have preperiodic expansions.

7 Biternary system

Consider rectangle systems with $2n = 4$, $R(x) = R_{\pi/2}(x) = (x_0 + x_1)/(x_0 - x_1)$ but different quotients $q_0, q_1 > 1$ in vertical and horizontal directions. With the alphabet $A = \{\overline{1}, 0, 1, \overline{0}\}$ we get transformations

$$F_{\overline{1}}(x) = \frac{(q_1+1)x + (1-q_1)}{(1-q_1)x + (q_1+1)},\ F_0(x) = \frac{x}{q_0},\ F_1(x) = \frac{(q_1+1)x + (q_1-1)}{(q_1-1)x + (q_1+1)},\ F_{\overline{0}}(x) = q_0 x$$

For $q_0 = 4$, $q_1 = 9$ the group $\mathcal{G}(F)$ is Fuchsian. Its tessellation is in Figure 4 left. Here C is the Ford fundamental region bounded by geodesics joining ideal points $\mathbf{d}(\frac{1}{2})$, $\mathbf{d}(2)$, $\mathbf{d}(-2)$, $\mathbf{d}(-\frac{1}{2})$ at

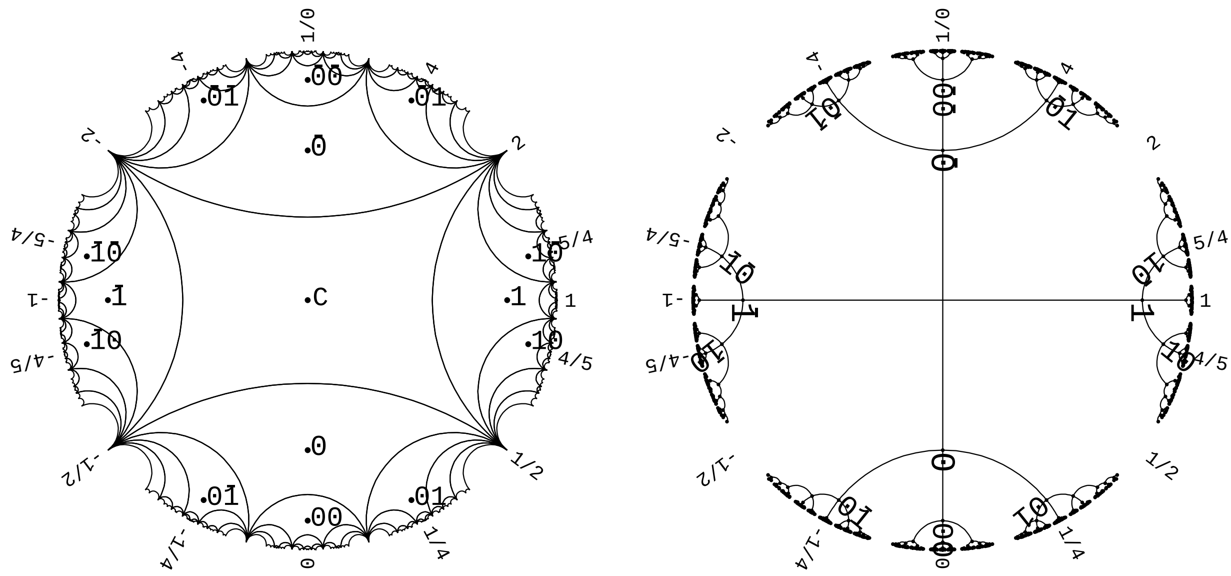


Figure 4: The tesselation and means of the quadrononary system

the boundary $\partial\mathbb{D}$. The images $F_u(C)$ tesselate the hyperbolic plane. The expanding intervals are $\mathbf{V}_{\overline{1}} = (-2, -\frac{1}{2})$, $\mathbf{V}_0 = (-\frac{1}{2}, \frac{1}{2})$, $\mathbf{V}_1 = (\frac{1}{2}, 2)$, $\mathbf{V}_{\overline{0}} = (2, -2)$. The Möbius quadrononary number system with interval cover $W_a = \mathbf{V}_a$ has forbidden words $\overline{1}1, 0\overline{0}, 1\overline{1}, \overline{0}0$. It is convergent but not redundant (Figure 4 right). The biternary system with quotients $q_0 = 2$, $q_1 = 3$ is the "square root" of the quadrononary system. Its transformations are

$$F_{\overline{1}}(x) = \frac{2x_0 - x_1}{2x_1 - x_0},\ F_0(x) = \frac{x_0}{2x_1},\ F_1(x) = \frac{2x_0 + x_1}{x_0 + 2x_1},\ F_{\overline{0}}(x) = \frac{2x_0}{x_1} \tag{1}$$

The expansion intervals are $\mathbf{V}_{\overline{1}} = (-2-\sqrt{3}, -2+\sqrt{3})$, $\mathbf{V}_0 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\mathbf{V}_1 = (2-\sqrt{3}, 2+\sqrt{3})$, $\mathbf{V}_{\overline{0}} = (\sqrt{2}, -\sqrt{2})$. Consider interval covers

$$\begin{array}{lllll} \mathcal{W}_0: & W_{\overline{1}} = (-2, -\frac{1}{2}), & W_0 = (-\frac{1}{2}, \frac{1}{2}), & W_1 = (\frac{1}{2}, 2), & W_{\overline{0}} = (2, -2) \\ \mathcal{W}_1: & W_{\overline{1}} = (\infty, 0), & W_0 = (-1, 1), & W_1 = (0, \infty), & W_{\overline{0}} = (1, -1) \end{array}$$

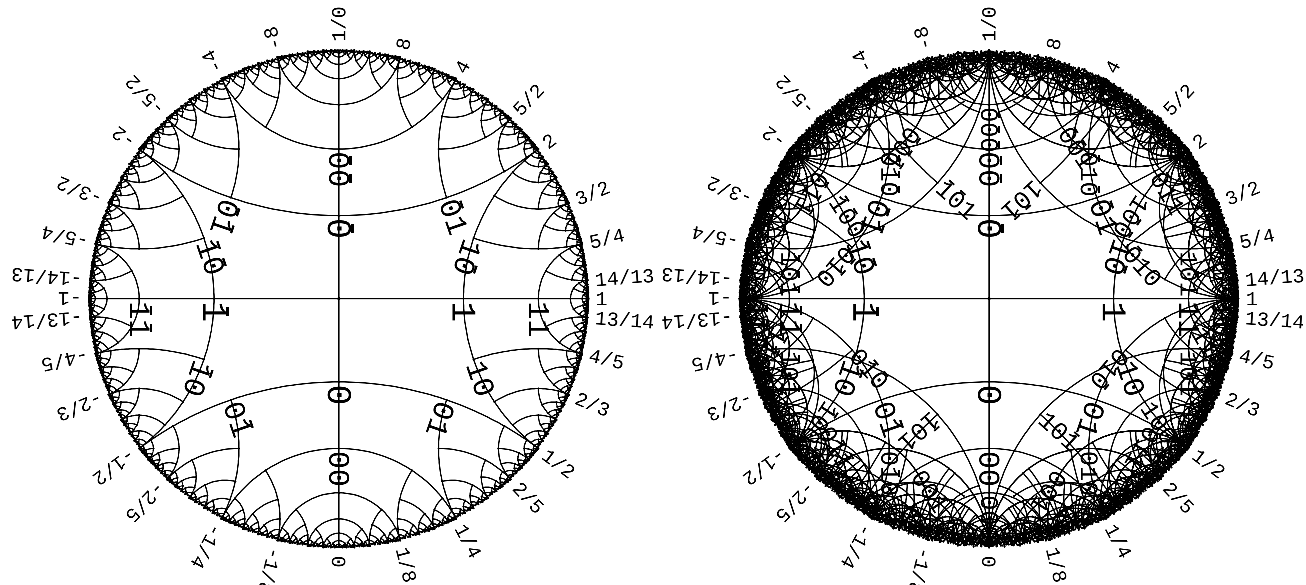
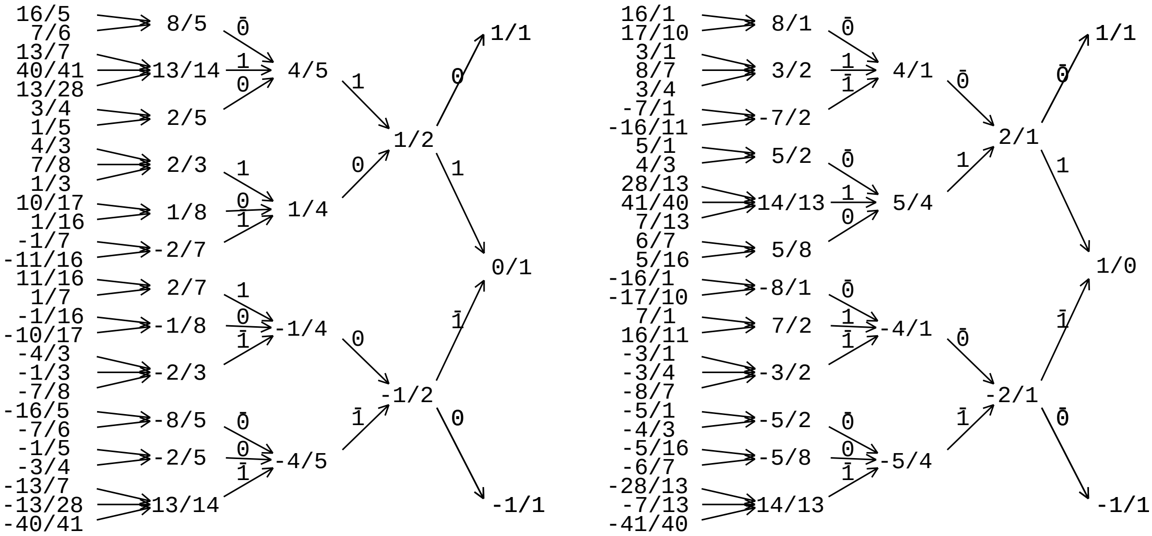


Figure 5: The small and large biternary systems

Definition 22 *The small biternary system (BTS0 - Figure 5 left) and large biternary system (BTS1 Figure 5 right) have alphabet $A = \{\overline{1}, 0, 1, \overline{0}\}$, transformations (1) and sofic subshifts*

$$\Sigma_0 \quad = \quad \Sigma_{\mathcal{W}_0} = \{\overline{1}, 0\}^{\mathbb{N}} \cup \{0, 1\}^{\mathbb{N}} \cup \{1, \overline{0}\}^{\mathbb{N}} \cup \{\overline{0}, \overline{1}\}^{\mathbb{N}}$$

$$\Sigma_1 \quad = \quad \Sigma_{\mathcal{W}_1} \cap \mathcal{S}_{\{\overline{1}1,0\overline{0},1\overline{1},0\overline{0}\}} = \Sigma_{\{\overline{1}1,0\overline{0},1\overline{1},\overline{0}0,\overline{1}01^*\overline{0},\overline{10}1^*0,01\overline{0}^*\overline{1},0\overline{10}^*1,1\overline{01}^*0,10\overline{1}^*\overline{0},\overline{01}0^*1,\overline{0}10^*\overline{1}\}}$$



0/1	$.0$	1/5	0010.1	1/4	00.1	2/7	0100010.1
1/3	0100.1	3/8	01010.1	2/5	010.1	3/7	011001010100.1
1/2	0.1	4/7	100010.1	3/5	100100010.1	5/8	1001010100.1
2/3	100.1	5/7	10100.1	3/4	1010.1	4/5	10.1
1/1	$.1$	5/4	$11.\overline{0}$	4/3	$1\overline{0}11.\overline{0}$	7/5	$1\overline{0}1\overline{0}1.\overline{0}$
3/2	$1\overline{0}1.\overline{0}$	8/5	$1\overline{00}1\overline{0}1\overline{0}1\overline{0}1.\overline{0}$	5/3	$1\overline{00}1\overline{000}11.\overline{0}$	7/4	$1\overline{000}11.\overline{0}$
2/1	$1.\overline{0}$	7/3	$\overline{0}11\overline{00}1\overline{0}1\overline{0}1\overline{0}1.\overline{0}$	5/2	$\overline{0}11.\overline{0}$	8/3	$\overline{0}1\overline{0}11.\overline{0}$
3/1	$\overline{0}1\overline{0}1.\overline{0}$	7/2	$\overline{0}1\overline{000}11.\overline{0}$	4/1	$\overline{0}1.\overline{0}$	5/1	$\overline{00}11.\overline{0}$

0/1	$.0$	1/5	$10\overline{1}0\overline{1}.0$	1/4	01.0	2/7	$10\overline{1}.0$
1/3	0101.0	3/8	$001\overline{0}1.\overline{0}$	2/5	$100\overline{1}.0$	3/7	$01011.\overline{0}$
1/2	1.0	4/7	$011\overline{0}1.\overline{0}$	3/5	$1010\overline{1}.0$	5/8	$011.\overline{0}$
2/3	101.0	5/7	10101.0	3/4	$01\overline{0}1.\overline{0}$	4/5	11.0
1/1	$.1$	5/4	$11.\overline{0}$	4/3	$\overline{0}101.0$	7/5	$1\overline{0}1\overline{0}1.\overline{0}$
3/2	$1\overline{0}1.\overline{0}$	8/5	$\overline{0}11.0$	5/3	$1\overline{0}1\overline{01}.\overline{0}$	7/4	$\overline{0}1101.0$
2/1	$1.\overline{0}$	7/3	$\overline{0}1\overline{0}11.0$	5/2	$1\overline{001}.\overline{0}$	8/3	$\overline{00}101.0$
3/1	$\overline{0}1\overline{0}1.\overline{0}$	7/2	$1\overline{01}.\overline{0}$	4/1	$\overline{0}1.\overline{0}$	5/1	$1\overline{0101}.\overline{0}$

Figure 6: Expansion graph of BTS1 (top), expansions of rationals in BTS0(center) and arithmetical expansions in BTS1 (bottom)

The quotient of BTS0 is greater than one, so the system converges, but it is not redundant. In BTS1, $\mathbf{Q}_n(\Sigma) = 1$ for each n divisible by four, so (F, Σ_1) is a redundant Möbius number system by Theorem 11. Alternatively we can use the sofic subshift

$$\Sigma_2 = \mathcal{S}_{\{\overline{1}1,0\overline{0},1\overline{1},0\overline{0}\}} \cap (\{\overline{1},0,1\}^{\mathbb{N}} \cup \{0,1,\overline{0}\}^{\mathbb{N}} \cup \{1,\overline{0},\overline{1}\}^{\mathbb{N}} \cup \{\overline{0},\overline{1},0\}^{\mathbb{N}})$$

which satisfies $\Sigma_0 \subset \Sigma_2 \subset \Sigma_1$ and is redundant as well.

We conjecture that rational numbers have in BTS0 preperiodic expansions with period length 1 of the form $u.a$ (see Figure 6). In BTS1 we get short preperiodic expansions if we test divisibility by 2 and 3. Define the extended rationals and rationals modulo $n > 0$ by

$$\begin{aligned}
\overline{\mathbb{Q}} &= \{\tfrac{p}{q} :\ p, q \in \mathbb{Z},\ q \geq 0,\ |p| + |q| > 0\} \\
\mathbb{Q}_0 &= \{\tfrac{p}{q} \in \overline{\mathbb{Q}} :\ q \geq 0, \gcd(p, q) = 1\}, \\
\mathbb{Q}_n &= \{\tfrac{p}{q} :\ p, q \in \mathbb{Z}_n,\ \gcd(p, q) = 1\}, \\
\overline{\mathbb{Q}}_n &= \bigcup_{m|n} \mathbb{Q}_m \ \cup\ \{\tfrac{0}{0}\},
\end{aligned}$$

where we write $\frac{p}{q}$ for the pair (p,q), and $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$. For $x \in \overline{\mathbb{Q}}$ we write $p|x$ if $p|\gcd(x_0,x_1)$. We have a homomorphism $\mathbf{m} : \overline{\mathbb{Q}} \to \mathbb{Q}_0$ defined by $\mathbf{m}(\frac{p}{q}) = \frac{p/\gcd(p,q)}{q/\gcd(p,q)}$. For each $n > 0$ we have a homomorphism $\mathbf{m}_n : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_n$ defined by $\mathbf{m}_n(\frac{x_0}{x_1}) = \frac{\mathbf{mod}_n(x_0)}{\mathbf{mod}_n(x_1)}$. An integer MT acts on $\overline{\mathbb{Q}}$ and its composition with $\mathbf{m}$ acts on $\mathbb{Q}_0$. We write $x \xrightarrow{a} y$ if $y = F_a^{-1}(x)$. In $\mathbb{Q}_2 = \{\frac{0}{1},\frac{1}{0},\frac{1}{1}\}$ we have $\frac{1}{0} \xrightarrow{1} \frac{0}{1} \xrightarrow{1} \frac{1}{0}$, $\frac{1}{1} \xrightarrow{1} \frac{1}{1}$, $\frac{1}{0} \xrightarrow{\overline{1}} \frac{0}{1} \xrightarrow{\overline{1}} \frac{1}{0}$, $\frac{1}{1} \xrightarrow{\overline{1}} \frac{1}{1}$. In $\mathbb{Q}_3 = \{\frac{0}{1},\frac{0}{2},\frac{1}{0},\frac{1}{1},\frac{1}{2},\frac{2}{0},\frac{2}{1},\frac{2}{2}\}$ we have $\{\frac{0}{1},\frac{1}{0}\} \xrightarrow{1} \frac{2}{2} \xrightarrow{1} \frac{2}{2}$, $\{\frac{0}{2},\frac{2}{0}\} \xrightarrow{1} \frac{1}{1} \xrightarrow{1} \frac{1}{1}$, $\{\frac{1}{2},\frac{2}{1}\} \xrightarrow{1} \frac{0}{0}$, $\{\frac{0}{1},\frac{2}{0}\} \xrightarrow{\overline{1}} \frac{1}{2} \xrightarrow{\overline{1}} \frac{1}{2}$, $\{\frac{0}{2},\frac{1}{0}\} \xrightarrow{\overline{1}} \frac{2}{1} \xrightarrow{\overline{1}} \frac{2}{1}$, $\{\frac{1}{1},\frac{2}{2}\} \xrightarrow{\overline{1}} \frac{0}{0}$.

Definition 23 *The arithmetical expansion algorithm for BTS1 is defined by the following rules. For each $x \in \mathbb{Q}_0$, the first applicable rule is chosen.*

$$\begin{array}{llll}
\overline{1}a: & x \xrightarrow{\overline{1}} \mathbf{m}F_1(x) & \textit{if} \;\; x \in [-2,-\frac{1}{2}] & \& \;\; 3|F_1(x) \\
\overline{1}b: & x \xrightarrow{\overline{0}} \mathbf{m}F_0(x) & \textit{if} \;\; x \in [-2,-1) & \& \;\; (2|F_0(x) \vee 3|F_{\overline{1}}(x)) \\
\overline{1}c: & x \xrightarrow{0} \mathbf{m}F_{\overline{0}}(x) & \textit{if} \;\; x \in (-1,-\frac{1}{2}] & \& \;\; (2|F_{\overline{0}}(x) \vee 3|F_{\overline{1}}(x)) \\
\overline{1}d: & x \xrightarrow{\overline{1}} \mathbf{m}F_1(x) & \textit{if} \;\; x \in [-2,-\frac{1}{2}] & \\
0a: & x \xrightarrow{0} \mathbf{m}F_{\overline{0}}(x) & \textit{if} \;\; x \in (-\frac{1}{2},\frac{1}{2}) & \& \;\; 2|F_{\overline{0}}(x) \\
0b: & x \xrightarrow{\overline{1}} \mathbf{m}F_1(x) & \textit{if} \;\; x \in (-\frac{1}{2},0) & \& \;\; (3|F_1(x) \vee 2|F_0(x)) \\
0c: & x \xrightarrow{1} \mathbf{m}F_{\overline{1}}(x) & \textit{if} \;\; x \in (0,\frac{1}{2}) & \& \;\; (3|F_{\overline{1}}(x) \vee 2|F_0(x)) \\
0d: & x \xrightarrow{0} \mathbf{m}F_{\overline{0}}(x) & \textit{if} \;\; x \in (-\frac{1}{2},\frac{1}{2}) & \\
1a: & x \xrightarrow{1} \mathbf{m}F_{\overline{1}}(x) & \textit{if} \;\; x \in [\frac{1}{2},2] & \& \;\; 3|F_{\overline{1}}(x) \\
1b: & x \xrightarrow{0} \mathbf{m}F_{\overline{0}}(x) & \textit{if} \;\; x \in [\frac{1}{2},1) & \& \;\; (2|F_{\overline{0}}(x) \vee 3|F_1(x)) \\
1c: & x \xrightarrow{\overline{0}} \mathbf{m}F_0(x) & \textit{if} \;\; x \in (1,2] & \& \;\; (2|F_0(x) \vee 3|F_1(x)) \\
1d: & x \xrightarrow{1} \mathbf{m}F_{\overline{1}}(x) & \textit{if} \;\; x \in [\frac{1}{2},2] & \\
\overline{0}a: & x \xrightarrow{\overline{0}} \mathbf{m}F_0(x) & \textit{if} \;\; x \in (2,-2) & \& \;\; 2|F_0(x) \\
\overline{0}b: & x \xrightarrow{1} \mathbf{m}F_{\overline{1}}(x) & \textit{if} \;\; x \in (2,\infty) & \& \;\; (3|F_{\overline{1}}(x) \vee 2|F_{\overline{0}}(x)) \\
\overline{0}c: & x \xrightarrow{\overline{1}} \mathbf{m}F_1(x) & \textit{if} \;\; x \in (\infty,-2) & \& \;\; (3|F_1(x) \vee 2|F_{\overline{0}}(x)) \\
\overline{0}d: & x \xrightarrow{\overline{0}} \mathbf{m}F_0(x) & \textit{if} \;\; x \in (2,-2) &
\end{array}$$

The conditions in the arithmetical expansion algorithm can be tested by simple rules. We have

$$\begin{array}{lclclcl}
3|F_{\overline{1}}(x) & \Leftrightarrow & 3|R(x)_0 \Leftrightarrow 3|(x_0+x_1), & 2|F_0(x) & \Leftrightarrow & 2|x_0, \\
3|F_1(x) & \Leftrightarrow & 3|R(x)_1 \Leftrightarrow 3|(x_0-x_1), & 2|F_{\overline{0}}(x) & \Leftrightarrow & 2|x_1.
\end{array}$$

Proposition 24 *Each rational number has a preperiodic arithmetical expansion with period length 1.*

Proof: We show that the norm of $x \in \mathbb{Q}_0$ defined by $||x|| := |x_0| + |x_1|$ is a Lyapunov function for the arithmetical expansion algorithm in the following sense. If $x \in \mathbb{Q}_0 \setminus \{-1,0,1,\infty\}$, then $||y|| < ||x||$ for some y on the path with source x. Note that if both $x, F_{\overline{1}}(x) \in (0,\infty)$ are positive, then $||F_{\overline{1}}(x)|| = ||x||$ and if moreover $3|(x_0+x_1)$, then $||\mathbf{m}F_{\overline{1}}(x)|| = ||x||/3$. Similarly, if both $x, F_1(x) \in (\infty,0)$ are negative, then $||F_1(x)|| = ||x||$ and if moreover $3|(x_0-x_1)$, then $||\mathbf{m}F_1(x)|| = ||x||/3$. The proof of the claim distinguishes 45 cases which are summarized in Table 1. These cases depend on modulo classes $\mathbf{m}_6(x)$. For example, the first item means that if $x \in (\infty,-6)$ and $\mathbf{m}_6(x) \in \{\frac{1}{3},\frac{3}{1},\frac{3}{5},\frac{5}{3}\}$, then $x \xrightarrow{\overline{0}1001} \mathbf{m}F_{\overline{1}0010}(x)$ and

$$||\mathbf{m}F_{\overline{1}0010}(x)|| = \left|\left|\frac{12x_1}{-6x_0-30x_1}\right|\right|/6 = 2x_1 - x_0 - 5x_1 < -x_0 + x_1 = ||x|| \qquad \square$$

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$\mathrm{mod}_6(x)$	interval	norm	$\mathrm{mod}_6(x)$	interval	norm
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{1}{0}, -\frac{6}{1})$	$\|\|F_{\bar{1}0010}\|\|/6$	$\frac{1}{0}, \frac{1}{2}, \frac{3}{2}, \frac{3}{4}, \frac{5}{0}, \frac{5}{4}$	$(-\frac{1}{0}, -\frac{7}{2})$	$\|\|F_{\bar{1}001}\|\|/6$
$\frac{2}{5}, \frac{4}{1}$	$(-\frac{1}{0}, -\frac{2}{1})$	$\|\|F_{0}\|\|/2$	$\frac{1}{5}, \frac{5}{1}$	$(-\frac{1}{0}, -\frac{1}{1})$	$\|\|F_{10}\|\|/3$
$\frac{0}{1}, \frac{0}{5}, \frac{2}{1}, \frac{2}{3}, \frac{4}{3}, \frac{4}{5}$	$(-\frac{1}{0}, -\frac{1}{1})$	$\|\|F_{0}\|\|/2$	$\frac{1}{4}, \frac{5}{2}$	$(-\frac{1}{0}, -\frac{1}{2})$	$\|\|F_{1}\|\|/3$
$\frac{1}{1}, \frac{5}{5}$	$(-\frac{1}{0}, \frac{0}{1})$	$\|\|F_{1}\|\|/3$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{6}{1}, -\frac{2}{1})$	$\|\|F_{010}\|\|/2$
$\frac{1}{2}, \frac{5}{4}$	$(-\frac{7}{2}, -\frac{2}{1})$	$\|\|F_{01}\|\|/2$	$\frac{1}{0}, \frac{3}{2}, \frac{3}{4}, \frac{5}{0}$	$(-\frac{7}{2}, -\frac{1}{1})$	$\|\|F_{01}\|\|/2$
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{2}{1}, -\frac{1}{1})$	$\|\|F_{101}\|\|/3$	$\frac{1}{2}, \frac{5}{4}$	$(-\frac{2}{1}, -\frac{1}{1})$	$\|\|F_{10}\|\|/3$
$\frac{2}{5}, \frac{4}{1}$	$(-\frac{2}{1}, \frac{0}{1})$	$\|\|F_{1}\|\|/3$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{1}{1}, -\frac{1}{2})$	$\|\|F_{1\bar{0}1}\|\|/3$
$\frac{2}{1}, \frac{4}{5}$	$(-\frac{1}{1}, -\frac{1}{2})$	$\|\|F_{1\bar{0}}\|\|/3$	$\frac{0}{1}, \frac{0}{5}, \frac{2}{3}, \frac{4}{3}$	$(-\frac{1}{1}, -\frac{2}{7})$	$\|\|F_{\bar{0}1}\|\|/2$
$\frac{1}{5}, \frac{5}{1}$	$(-\frac{1}{1}, \frac{0}{1})$	$\|\|F_{1\bar{0}}\|\|/3$	$\frac{1}{2}, \frac{5}{4}$	$(-\frac{1}{1}, \frac{1}{2})$	$\|\|F_{\bar{0}}\|\|/2$
$\frac{1}{0}, \frac{3}{2}, \frac{3}{4}, \frac{5}{0}$	$(-\frac{1}{1}, \frac{1}{1})$	$\|\|F_{\bar{0}}\|\|/2$	$\frac{2}{1}, \frac{4}{5}$	$(-\frac{1}{2}, -\frac{2}{7})$	$\|\|F_{\bar{0}1}\|\|/2$
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{1}{2}, -\frac{1}{6})$	$\|\|F_{\bar{0}1\bar{0}}\|\|/2$	$\frac{1}{4}, \frac{5}{2}$	$(-\frac{1}{2}, \frac{1}{1})$	$\|\|F_{\bar{0}}\|\|/2$
$\frac{0}{1}, \frac{0}{5}, \frac{2}{1}, \frac{2}{3}, \frac{4}{3}, \frac{4}{5}$	$(-\frac{2}{7}, \frac{0}{1})$	$\|\|F_{\overline{1001}}\|\|/6$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(-\frac{1}{6}, \frac{0}{1})$	$\|\|F_{\overline{1001}\bar{0}}\|\|/6$
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{0}{1}, \frac{1}{6})$	$\|\|F_{1\overline{0010}}\|\|/6$	$\frac{0}{1}, \frac{0}{5}, \frac{2}{3}, \frac{2}{5}, \frac{4}{1}, \frac{4}{3}$	$(\frac{0}{1}, \frac{2}{7})$	$\|\|F_{1\overline{001}}\|\|/6$
$\frac{1}{1}, \frac{5}{5}$	$(\frac{0}{1}, \frac{1}{1})$	$\|\|F_{\overline{10}}\|\|/3$	$\frac{2}{1}, \frac{4}{5}$	$(\frac{0}{1}, \frac{2}{1})$	$\|\|F_{\bar{1}}\|\|/3$
$\frac{1}{5}, \frac{5}{1}$	$(\frac{0}{1}, \frac{1}{0})$	$\|\|F_{\bar{1}}\|\|/3$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{1}{6}, \frac{1}{2})$	$\|\|F_{\overline{010}}\|\|/2$
$\frac{2}{5}, \frac{4}{1}$	$(\frac{2}{7}, \frac{1}{2})$	$\|\|F_{\overline{01}}\|\|/2$	$\frac{0}{1}, \frac{0}{5}, \frac{2}{3}, \frac{4}{3}$	$(\frac{2}{7}, \frac{1}{1})$	$\|\|F_{\overline{01}}\|\|/2$
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{1}{2}, \frac{1}{1})$	$\|\|F_{\overline{101}}\|\|/3$	$\frac{2}{5}, \frac{4}{1}$	$(\frac{1}{2}, \frac{1}{1})$	$\|\|F_{\overline{10}}\|\|/3$
$\frac{1}{2}, \frac{5}{4}$	$(\frac{1}{2}, \frac{1}{0})$	$\|\|F_{\bar{1}}\|\|/3$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{1}{1}, \frac{2}{1})$	$\|\|F_{\bar{1}0\bar{1}}\|\|/3$
$\frac{1}{4}, \frac{5}{2}$	$(\frac{1}{1}, \frac{2}{1})$	$\|\|F_{\bar{1}0}\|\|/3$	$\frac{1}{0}, \frac{3}{2}, \frac{3}{4}, \frac{5}{0}$	$(\frac{1}{1}, \frac{7}{2})$	$\|\|F_{0\bar{1}}\|\|/2$
$\frac{1}{1}, \frac{5}{5}$	$(\frac{1}{1}, \frac{1}{0})$	$\|\|F_{\bar{1}0}\|\|/3$	$\frac{0}{1}, \frac{0}{5}, \frac{2}{3}, \frac{2}{5}, \frac{4}{1}, \frac{4}{3}$	$(\frac{1}{1}, \frac{1}{0})$	$\|\|F_{0}\|\|/2$
$\frac{1}{4}, \frac{5}{2}$	$(\frac{2}{1}, \frac{7}{2})$	$\|\|F_{0\bar{1}}\|\|/2$	$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{2}{1}, \frac{6}{1})$	$\|\|F_{0\bar{1}0}\|\|/2$
$\frac{2}{1}, \frac{4}{5}$	$(\frac{2}{1}, \frac{1}{0})$	$\|\|F_{0}\|\|/2$	$\frac{1}{0}, \frac{1}{4}, \frac{3}{2}, \frac{3}{4}, \frac{5}{0}, \frac{5}{2}$	$(\frac{7}{2}, \frac{1}{0})$	$\|\|F_{100\bar{1}}\|\|/6$
$\frac{1}{3}, \frac{3}{1}, \frac{3}{5}, \frac{5}{3}$	$(\frac{6}{1}, \frac{1}{0})$	$\|\|F_{100\bar{1}0}\|\|/6$			

Table 1: Norm in arithmetical expansion algorithm

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