POINTED ADMISSIBLE G-COVERS AND G-EQUIVARIANT COHOMOLOGICAL FIELD THEORIES

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ABSTRACT. For any finite group G we define the moduli space of pointed admissible G-covers and the concept of a G-equivariant cohomological field theory (G-CohFT), which, when G is the trivial group, reduce to the moduli space of stable curves and a cohomological field theory (CohFT), respectively. We prove that taking the "quotient" by G reduces a G-CohFT to a CohFT. We also prove that a G-CohFT contains a G-Frobenius algebra, a G-equivariant generalization of a Frobenius algebra, and that the "quotient" by G agrees with the obvious Frobenius algebra structure on the space of G-invariants, after rescaling the metric. We then introduce the moduli space of G-stable maps into a smooth, projective variety X with G action. Gromov-Witten-like invariants of these spaces provide the primary source of examples of G-CohFTs. Finally, we explain how these constructions generalize (and unify) the Chen-Ruan orbifold Gromov-Witten invariants of $[X/G]$ as well as the ring $H^\bullet(X, G)$ of Fantechi and Göttsche.

1. INTRODUCTION

The purpose of this paper is to introduce a generalization of Kontsevich and Manin's notion of a cohomological field theory (or CohFT) [KoMa], in the presence of a finite group G, which we call a G-equivariant cohomological field theory (or G-CohFT). Examples of (usual) CohFTs include the Gromov-Witten invariants of a smooth, projective variety (c.f., [Ma]) and the r-spin CohFT [JKV, PoVa, Po2]. A G-CohFT provides a framework for studying the physical procedure of orbifolding [Kau, Mo], as well as a structure for understanding both Chen-Ruan orbifold Gromov-Witten invariants of global quotients by a finite group [CR1, CR2, AGV] and the non-commutative ring structure of Fantechi and Göttsche [FG]. We now describe in some detail the motivation for studying G-CohFTs.

The first motivation comes from topological field theory. Recall that a Frobenius algebra $\mathcal{H}$ is a finite-dimensional, commutative, associative, unital algebra with an invariant metric. It can be regarded as a two-dimensional topological field theory, in the sense of Atiyah-Segal, associated to a cobordism category of two (real) dimensional, compact, oriented surfaces with boundary. A CohFT is a generalization of the above, but where the role of the cobordism category is replaced by $\{H_r(\overline{\mathcal{M}}_{g,n})\}$

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for all r, where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves of genus g with n marked points. By specializing to $r = 0$, one finds that the state space of the theory $\mathcal{H}$ recovers the structure of a Frobenius algebra.

For every finite group G Turaev [Tu] introduced a G-equivariant topological field theory (which he called a homotopy field theory) whose state space $\mathcal{H}$ is a (non-projective) G-Frobenius algebra associated to a cobordism category of principal G-bundles over two (real) dimensional, compact, oriented surfaces with boundary. A (non-projective) G-Frobenius algebra (borrowing terminology from [Kau]) is a finite-dimensional, G-graded G-module with a G-equivariant associative multiplication, metric, and unit, and whose multiplication is braided commutative, satisfying an additional genus-one compatibility condition (called the trace axiom). By *braided commutative* we mean that the multiplication commutes with the action of the generator of the braid group which acts on tensor products of G-graded G-modules. If G is the trivial group, then a G-Frobenius algebra is a Frobenius algebra. Furthermore, the space of G-invariants $\overline{\mathcal{H}}$ of a G-Frobenius algebra inherits the structure of a Frobenius algebra graded by $\overline{G}$, the set of conjugacy classes of G. Kaufmann [Kau] considered a generalization of the above construction which allowed for projective factors.

This procedure of restricting to the space of invariants can be interpreted as a kind of orbifolding procedure from physics [Kau, Mo] where the subspace of $\overline{\mathcal{H}}$ graded by 1 in $\overline{G}$ is called the *untwisted sector*, and the subspaces graded by nontrivial elements in $\overline{G}$ are called *twisted sectors.*

Question 1. *Is there a generalization of a CohFT, called a* G*-CohFT, where* $\overline{\mathcal{M}}_{g,n}$ *is replaced by another moduli space* $\overline{\mathcal{M}}^G_{g,n}$*, such that for all* r*, the collection* $\{H_r(\overline{\mathcal{M}}^G_{g,n})\}$ *endows the state space* $\mathcal{H}$ *of the theory with an algebraic structure whose specialization to* $r = 0$ *induces the structure of a* G*-Frobenius algebra on* $\mathcal{H}$*? A* G*-CohFT should also have the property that when* G *is the trivial group, a* G*-CohFT reduces to a CohFT. Furthermore, by performing the correct "quotient" by* G*, the space of* G*-invariants* $\overline{\mathcal{H}}$ *should inherit the structure of a CohFT graded by* $\overline{G}$.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{G-Frobenius algebra } \mathcal{H} \\ \text{From } H_0(\overline{\mathcal{M}}^G_{g,n})\end{array}} \arrow[d, mapsto, "/G"'] & \boxed{\begin{array}{c}\text{G-CohFT } \mathcal{H} \\ \text{From } H_\bullet(\overline{\mathcal{M}}^G_{g,n})\end{array}} \arrow[l, mapsto] \arrow[d, mapsto, "/G"'] \\
\boxed{\begin{array}{c}\overline{G}\text{-graded Frobenius algebra } \overline{\mathcal{H}} \\ \text{From } H_0(\overline{\mathcal{M}}_{g,n})\end{array}} & \boxed{\begin{array}{c}\overline{G}\text{-graded CohFT } \overline{\mathcal{H}} \\ \text{From } H_\bullet(\overline{\mathcal{M}}_{g,n})\end{array}} \arrow[l, mapsto]
\end{tikzcd}$$

FIGURE 1. Schematic of Question (1) where each box contains an algebraic structure and the responsible homology group, the horizontal arrows are restrictions, and the vertical arrows are "quotients" by G. When G is the trivial group, the two rows coincide.

The second motivation for studying G-CohFTs comes from orbifold Gromov-Witten invariants and is about how to construct certain examples of G-CohFTs associated to a smooth, projective variety X with an action of a finite group G.

Consider the G-graded G-module $\mathcal{H}(X) := \bigoplus_{m \in G} H^{\bullet}(X^m)$, where X^m denotes the fixed-point set in X of m, and let $\overline{\mathcal{H}}(X)$ denote its space of G-invariants. Chen and Ruan [CR1, CR2] introduced the notion of Gromov-Witten invariants for orbifolds, which, when applied to the global quotient $[X/G]$, has a state space isomorphic to $\overline{\mathcal{H}}(X)$. An algebro-geometric version of this theory was introduced by [AGV]. The key geometric object in these constructions was $\overline{\mathcal{M}}_{g,n}([X/G])$, the moduli space of orbifold stable maps into the quotient $[X/G]$. The state space $\overline{\mathcal{H}}(X)$ of this theory is graded by $\overline{G}$, and the Gromov-Witten invariants are expected to yield a CohFT associated to each $[X/G]$. An important special case arises by considering only those contributions from $\overline{\mathcal{M}}_{g,n}([X/G], 0)$, the moduli of orbifold stable maps which have degree zero. This endows $\overline{\mathcal{H}}(X)$ with the structure of a Frobenius algebra graded by $\overline{G}$, called variously *stringy orbifold cohomology*, *Chen-Ruan cohomology*, or just *orbifold cohomology* of $[X/G]$.

When G is a trivial group, $\overline{\mathcal{M}}_{g,n}([X/G])$ reduces to the usual moduli space $\overline{\mathcal{M}}_{g,n}(X)$ of stable maps into X, and the Gromov-Witten invariants of X make $H^{\bullet}(X)$ into a CohFT. Restricting to contributions from $\overline{\mathcal{M}}_{g,n}([X/G], 0)$ alone, one obtains the usual cohomology ring of X, which is a Frobenius algebra. "Forgetting" the stable map yields a morphism $\overline{\mathcal{M}}_{g,n}(X) \longrightarrow \overline{\mathcal{M}}_{g,n}$ for all stable pairs (g, n), which is an isomorphism when X is a point.

Fantechi and Göttsche [FG] were able to obtain the structure of the Chen-Ruan orbifold cohomology on $\overline{\mathcal{H}}(X)$ by first introducing a certain ring structure with metric on $\mathcal{H}(X)$ and then taking G-invariants. In fact, their ring satisfies all of the axioms of a G-Frobenius algebra except, possibly, the trace axiom. However, their construction is not obviously part of a larger structure and does not explicitly involve the moduli space of orbifold stable maps.

Question 2. *For any smooth, projective variety* X *with a* G*-action, does there exist a moduli space* $\overline{\mathcal{M}}^G_{g,n}(X)$ *of a* G*-equivariant version of stable maps such that "forgetting" the map yields a morphism* $\overline{\mathcal{M}}^G_{g,n}(X) \longrightarrow \overline{\mathcal{M}}^G_{g,n}$ *for stable pairs* (g, n)*? This map should be an isomorphism when* X *is a point.*

There should also exist G*-equivariant Gromov-Witten invariants associated to* $\overline{\mathcal{M}}^G_{g,n}(X)$ *which yield a* G*-CohFT with state space* $\mathcal{H}(X)$*, generalizing the usual construction when* G *is the trivial group. Furthermore, by taking the appropriate "quotient" by* G*, one should recover the orbifold Gromov-Witten invariants of* $[X/G]$ *as in* [CR1, CR2, AGV] *with associated state space* $\overline{\mathcal{H}}(X)$*.*

Finally, by considering only those contributions from the moduli $\overline{\mathcal{M}}^G_{g,n}(X, 0)$ *of stable maps of degree zero, one should be able to recover the* G*-Frobenius algebra structure in* [FG] *and prove that the trace axiom must hold.*

This paper provides affirmative answers to both of these questions.

The first part of this paper is devoted to answering the first question. We introduce $\overline{\mathcal{M}}^G_{g,n}$, the moduli space of n-pointed admissible G-covers of genus g. Roughly speaking, it consists of a tuple $(E \xrightarrow{\pi} C; \widetilde{p}_1, \ldots, \widetilde{p}_n)$, where E and C are (at worst, nodal) curves, $(C, p_1, \ldots, p_n)$ is a stable curve of genus g, where $\widetilde{p}_i$ are points in E and $p_i := \pi(\widetilde{p}_i)$, and π maps nodes of E to nodes of C. Furthermore, we require that, away from $\pi^{-1}(p_i)$ and nodes, E is a principal G-bundle; however, E is allowed to have ramification over the marked points and nodes. Our construction

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{F-G } \mathsf{G}\text{-Frobenius Algebra } \mathcal{H}(\mathsf{X}) \\ \text{From } \overline{\mathcal{M}}^{\mathsf{G}}_{g,n}(\mathsf{X},0)\end{array}} \arrow[d, mapsto, "/\mathsf{G}"'] & \boxed{\begin{array}{c}\mathsf{G}\text{-Equivariant GW Theory } \mathcal{H}(\mathsf{X}) \\ \text{From } \overline{\mathcal{M}}^{\mathsf{G}}_{g,n}(\mathsf{X})\end{array}} \arrow[l, mapsto] \arrow[d, mapsto, "/\mathsf{G}"'] \\
\boxed{\begin{array}{c}\text{C-R Orbifold Cohomology } \overline{\mathcal{H}}(\mathsf{X}) \\ \text{From } \overline{\mathcal{M}}_{g,n}([\mathsf{X}/\mathsf{G}],0)\end{array}} & \boxed{\begin{array}{c}\text{C-R Orbifold GW Theory } \overline{\mathcal{H}}(\mathsf{X}) \\ \text{From } \overline{\mathcal{M}}_{g,n}([\mathsf{X}/\mathsf{G}])\end{array}} \arrow[l, mapsto]
\end{tikzcd}$$

FIGURE 2. Schematic of Question (2) where each box contains an algebraic structure and the relevant moduli space, F-G denotes Fantechi-Göttsche, C-R denotes Chen-Ruan, the horizontal arrows denote restriction, and the vertical arrows denote taking "quotients" by G.

differs from the stack of admissible covers in [ACV], as we require the additional data of $\tilde{p}_i$ in E over each marked point p_i in C.

By forgetting the data associated to the G-cover, one obtains a morphism st : $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$, where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable curves. We prove that $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}$ is a smooth, Deligne-Mumford stack, flat, proper, and quasi-finite (but not representable) over $\overline{\mathcal{M}}_{g,n}$. Furthermore, $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}$ has an action of the symmetric group S_n by permuting the ordering of the marked points, and it has an action of G^n by translation of the marked points. In fact, $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}$ admits an action of the braid group B_n, which factors through the S_n and G^n actions.

The collection $\{\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}\}$ possesses gluing morphisms, provided that the monodromies of the two marked points to be glued together are inverses of one another. These gluing morphisms are equivariant under the action of S_n and G^n. One may regard the collection $\{\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}\}$ as a G-equivariant colored modular operad, where the coloring is by elements of G. Furthermore, the morphism st respects the S_n actions and the gluing morphisms.

A G-CohFT is defined analogously to a CohFT, but where the role of $\overline{\mathcal{M}}_{g,n}$ is replaced by $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}$, and where G-equivariance is maintained throughout the construction. We prove that there is an external tensor product and a (usual) tensor product associated to equivariant CohFTs. We then define the correct notion of taking a "quotient" by G and prove that this procedure has the desired properties. The procedure of taking quotients involves an intermediate step on the stack $\overline{\mathcal{M}}_{g,n}(\mathcal{B}\mathsf{G})$ of stable maps into the classifying stack $\mathcal{B}\mathsf{G}$ (i.e., the stack of admissible covers without the additional points $\tilde{p}_i$). We show that in this intermediate step the stack $\overline{\mathcal{M}}_{g,n}(\mathcal{B}\mathsf{G})$ can be replaced by the quotient $[\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}/\mathsf{G}^n]$, but that the resulting "quotient" CohFTs are isomorphic.

The last part of this paper will treat the second question. We introduce the moduli space of G-stable maps $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}(\mathsf{X})$ and describe the G-equivariant Gromov-Witten invariants. By restricting to contributions from $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}(\mathsf{X},0)$ alone, we prove that the state space $\mathcal{H}(\mathsf{X})$ inherits a G-Frobenius algebra structure which agrees with that from [FG], and in particular that the trace axiom holds for their ring. The proof consists of relating the virtual fundamental class to an analogous cohomology class in their construction.

The details of the construction of a G-CohFT for general equivariant Gromov-Witten invariants, properties of potential functions, and applications to higher spin curves will be explored elsewhere [IP].

Finally, we observe that orbifolding plays an important role in mirror symmetry and in certain Landau-Ginzburg theories (see, for example, [CoKat, Ma]). We expect our construction to play a similar role in the theory of CohFTs. There are also related notions of orbifolding which appear in the context of vertex algebras (see, for example, [Kir]). It would be enlightening to better understand their relationship.

The outline of this paper is as follows. In Section 2 we describe the moduli spaces $\overline{\mathcal{M}}^{\mathsf{G}}_{g,n}$, their associated forgetful and gluing morphisms, group actions, and automorphism groups. In Section 3 we briefly review important facts from the category of G-graded G-modules, including the braid group action and tensor products. In Section 4 we define G-CohFTs and their tensor products. We prove that a (non-projective) G-CohFT always contains a G-Frobenius algebra. In Section 5 we define how to obtain a CohFT from a G-CohFT by taking the appropriate "quotient." We prove that this is consistent with the obvious notion of taking a quotient for a G-Frobenius algebra, after rescaling the metric, and then work out the example of the orbifold cohomology of $\mathcal{B}\mathsf{G}$. In Section 6 we introduce the moduli space of G-stable maps and equivariant Gromov-Witten invariants, reproduce the ring of [FG] as a special case, and prove that the trace axiom is satisfied.

Remark 1.1. Unless otherwise specified, we assume that all cohomology rings are over the ground ring $\mathbb{C}$, although all constructions here are also valid over the rationals $\mathbb{Q}$.

Also, unless otherwise specified, all groups which appear are finite and all group actions are right group actions.

Notation 1.2. The stack quotient of a variety X by G will be denoted $[\mathsf{X}/\mathsf{G}]$ and the coarse moduli space of this quotient will be denoted X/G.

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2. The moduli spaces

Let $(\mathsf{C} \xrightarrow{\varpi} \mathsf{T}, p_1, \ldots, p_n)$ be a stable curve over T of genus g, with marked points (sections) $p_1, \ldots, p_n$. We want to study a variant of the space of admissible G-covers of C, as defined in [ACV, Def 4.3.1]. We recall the definition here:

Definition 2.1. A finite morphism $\pi : E \longrightarrow C$ to an n-pointed, genus-g, stable curve $C \xrightarrow{\varpi} T, p_1, \ldots, p_n$ over T is an *admissible* G-*cover* if

(1) E/T is itself a nodal curve (not necessarily connected).
(2) Nodes of E map to nodes of C.
(3) There is a right action ρ_E of G on E preserving π, and such that
(4) the restriction of π to C_{gen}(the points of C which are neither marked points nor nodes) is a principal G-bundle.
(5) At points of E lying over nodes of C the structure of the maps $E \xrightarrow{\pi} C \xrightarrow{\varpi} T$ is locally the same as (analytically isomorphic to) that of
$$\operatorname{Spec} A[z,w]/(zw - t) \longrightarrow \operatorname{Spec} A[x,y]/(xy - t^r) \longrightarrow \operatorname{Spec} A,$$
where we have $t \in A$, $x = z^r$ and $y = w^r$, for some integer $r > 0$.
(6) At points of E lying over marked points of C the structure of the maps $E \xrightarrow{\pi} C \xrightarrow{\varpi} T$ is locally the same as (analytically isomorphic to) that of
$$\operatorname{Spec} A[z] \longrightarrow \operatorname{Spec} A[x] \longrightarrow \operatorname{Spec} A,$$
where $x = z^s$ for some integer $s > 0$.
(7) The action of the stabilizer $G_q \subseteq G$ at each node q of E is *balanced*; that is, the eigenvalues of the action on the tangent space at q are multiplicative inverses of each other.

Theorem 4.3.2 of [ACV] shows that the stack of admissible G-covers is isomorphic to the stack $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ of balanced twisted stable maps into the classifying stack of G.

2.1. Definition, construction, and basic properties of $\overline{\mathcal{M}}^G_{g,n}$.

Given an admissible G-cover $(E \xrightarrow{\pi} C, p_1, \ldots, p_n)$, let $\widetilde{p}_i \in \pi^{-1}(p_i)$ be a choice of a point in the fiber over p_i for all $i = 1 \ldots n$.

Definition 2.2. Let $\overline{\mathcal{M}}^G_{g,n}$ denote the stack of admissible G-covers
$$(\pi : E \longrightarrow C, p_1, \ldots, p_n, \widetilde{p}_1, \ldots, \widetilde{p}_n)$$
of n-pointed, genus-g, stable curves, together with a choice of n marked points $\widetilde{p}_i \in E$ such that $\pi(\widetilde{p}_i) = p_i$ for all $i = 1, \ldots, n$. We call such objects n-*pointed admissible* G-*covers*. A morphism of such objects is a G-equivariant fibered diagram; that is, a morphism of the underlying stable curves, together with a G-equivariant morphism of the induced admissible G-covers preserving the points $\widetilde{p}_i$.

Because the curve C is oriented, a pointed admissible G-cover $(E \xrightarrow{\pi} C, \widetilde{p}_1, \ldots, \widetilde{p}_n)$ has a well-defined monodromy m_i at each marked point $\widetilde{p}_i$; namely, E induces a principal G-bundle over $C - \{p_1, \ldots, p_n\}$, and the orientation gives a small loop in $C - \{p_1, \ldots, p_n\}$ around each p_i, with a lift to a path in a small neighborhood of $\widetilde{p}_i$ in $E - \{\widetilde{p}_1, \ldots, \widetilde{p}_n\}$. The lift is not uniquely determined, but the difference between the starting and ending sheets of the lifted path is given by a well-defined element $m_i \in G$.

Since the points $\widetilde{p}_i$ are determined up to a discrete choice by C, π, and the points p_i, the monodromy is invariant under deformation of the curve C, the cover E, and the points p_i. Also note that, while the action ρ_E acts on the points $\widetilde{p}_i$ by right multiplication, it acts on the holonomies by conjugation.

Let G_A denote the set G, considered as a right G-space under conjugation. Associated to any object $(E \xrightarrow{\pi} C, \widetilde{p}_1, \dots, \widetilde{p}_n)$ there exists an element $\mathbf{m} = (m_1, \dots, m_n) \in G_A^n$; namely, m_i is the monodromy of E at the point $\widetilde{p}_i$.

Definition 2.3. Denote the canonical morphism we have just described by

$$\mathbf{e} : \overline{\mathcal{M}}^G_{g,n} \longrightarrow G_A^n, \tag{1}$$

and let

$$\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) := \mathbf{e}^{-1}(\mathbf{m}) \tag{2}$$

denote the substack of objects in $\overline{\mathcal{M}}^G_{g,n}$ that map to $\mathbf{m}$.

Since $\mathbf{e}$ is locally constant, we may write

$$\overline{\mathcal{M}}^G_{g,n} = \coprod_{\mathbf{m} \in G_A^n} \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}).$$

The stack $\overline{\mathcal{M}}^G_{g,n}$ and the substacks $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ can be explicitly constructed as follows.

Theorem 2.4. *The stack $\overline{\mathcal{M}}^G_{g,n}$ and the substacks $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ are smooth Deligne-Mumford stacks, flat, proper, and quasi-finite over $\overline{\mathcal{M}}_{g,n}$.*

Proof. Let $\mathsf{Adm}^G_{g,n}$ be the stack of admissible G-covers of n-pointed, genus-g curves, and let $E \xrightarrow{\pi} C \xrightarrow{\varpi} \mathsf{Adm}^G_{g,n}$ be the universal G-cover and stable curve, with universal sections p_i of ϖ. Let A_i be the image in C of the ith section p_i, and let

$$E_i := E \times_C A_i$$

be the restriction of E to A_i. Let

$$W := E_1 \times_{\mathsf{Adm}^G_{g,n}} E_2 \times_{\mathsf{Adm}^G_{g,n}} \cdots \times_{\mathsf{Adm}^G_{g,n}} E_n$$

be the fibered product of the E_i. It is straightforward to see that W is the stack of admissible G-covers, together with explicit choices of sections $\widetilde{p}_i \in E$ lying over the sections p_i; that is, $W = \overline{\mathcal{M}}^G_{g,n}$.

Theorems 3.0.2 and 4.3.2 of [ACV] show that the space $\mathsf{Adm}^G_{g,n}$ is isomorphic to $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$, the stack of twisted stable maps to the classifying stack $\mathcal{B}G$, and is a smooth DM stack, flat, proper, and quasi-finite over $\overline{\mathcal{M}}_{g,n}$. Since these properties are preserved by the above-listed fibered products, the theorem follows for $\overline{\mathcal{M}}^G_{g,n}$. The substacks $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ are finite disjoint unions of connected components of $\overline{\mathcal{M}}^G_{g,n}$, so the theorem also holds for them. $\square$

2.2. Morphisms and group actions on $\overline{\mathcal{M}}^G_{g,n}$.

There are several obvious morphisms on $\overline{\mathcal{M}}^G_{g,n}$. First, there are the forgetful morphisms

$$\overline{\mathcal{M}}^G_{g,n} \xrightarrow{\widetilde{\mathrm{st}}} \mathsf{Adm}^G_{g,n} \cong \overline{\mathcal{M}}_{g,n}(\mathcal{B}G) \xrightarrow{\widehat{\mathrm{st}}} \overline{\mathcal{M}}_{g,n},$$

which were shown in Theorem 2.4 to be proper, flat, and quasi-finite. We denote the composition by

$$\mathrm{st} := \widehat{\mathrm{st}} \circ \widetilde{\mathrm{st}}.$$

We also have the evaluation morphism (2):

$$\mathbf{e}: \overline{\mathcal{M}}^{G}_{g,n} \longrightarrow G^n_A.$$

Recall (see [JK]) that while $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ cannot be written as a disjoint union of substacks indexed by $\mathbf{m} \in G^n_A$, it does have a decomposition indexed by conjugacy classes of G.

Definition 2.5. We denote the set of conjugacy classes of G by $\overline{G}$ and the conjugacy class of $m \in G$ by $\overline{m}$. Similarly, we denote by $\overline{\mathbf{m}} \in \overline{G}^n$ the n-tuple of conjugacy classes determined by $\mathbf{m} \in G^n$.

As described in [JK], we have

$$\overline{\mathcal{M}}_{g,n}(\mathcal{B}G) = \coprod_{\overline{\mathbf{m}} \in \overline{G}^n} \overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}}),$$

where some of the substacks may be empty.

Definition 2.6. We let $\overline{\mathcal{M}}^{G}_{g,n}(\overline{\mathbf{m}})$ denote the preimage $\widetilde{\mathrm{st}}^{-1}(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}}))$, which is easily seen to be

$$\overline{\mathcal{M}}^{G}_{g,n}(\overline{\mathbf{m}}) = \coprod_{\mathbf{m}' \in \overline{\mathbf{m}}} \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m}').$$

The stack $\overline{\mathcal{M}}^{G}_{g,n}(\overline{\mathbf{m}})$ has a right G^n action

$$\rho(\gamma_1, \dots, \gamma_n) : \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m}) \longrightarrow \overline{\mathcal{M}}^{G}_{g,n}(\gamma_1^{-1} m_1 \gamma_1, \dots, \gamma_n^{-1} m_n \gamma_n), \tag{3}$$

which acts by right multiplication on the n marked points $(\tilde{p}_1, \dots, \tilde{p}_n) \mapsto (\tilde{p}_1 \cdot \gamma_1, \dots, \tilde{p}_n \cdot \gamma_n)$. We sometimes write ρ_i for the action on the ith factor: $\rho_i(\gamma) = \rho(1, \dots, \gamma, \dots 1)$.

Of course, there is the natural symmetric group action on $\overline{\mathcal{M}}^{G}_{g,n}$ obtained by relabeling points, but the stack $\overline{\mathcal{M}}^{G}_{g,n}$ also admits an action of the braid group B_n.

Definition 2.7. Let B_1 be the trivial group. If $n \geq 2$, let B_n be the group with generators $\{b_1, \dots, b_{n-1}\}$ subject to the relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \tag{4}$$

for all $i = 1, \dots, n-1$ and

$$b_i b_j = b_j b_i \tag{5}$$

if $|i - j| > 1$. B_n is called the *braid group on n-letters.*

For each generator b_i of the braid group B_n, there is an isomorphism

$$\overline{\mathcal{M}}^{G}_{g,n}(m_1, \dots, m_i, m_{i+1}, \dots, m_n) \xrightarrow{b_i} \overline{\mathcal{M}}^{G}_{g,n}(m_1, \dots, m_i m_{i+1} m_i^{-1}, m_i, \dots, m_n). \tag{6}$$

These are given by $b_i := \rho_i(m_i^{-1}) \circ s_i$, where s_i is the element $(i, i+1)$ in S_n which transposes i and $i+1$, and ρ_i is the group action on $\overline{\mathcal{M}}^{G}_{g,n}$ obtained by right translation of the i-th marked point. It is straightforward to check that the induced isomorphisms satisfy the braid relations (4) and (5), thus they induce an action of B_n on $\overline{\mathcal{M}}^{G}_{g,n}$.

Finally, there are the three fundamental morphisms: *forgetting tails*, *gluing trees*, and *gluing loops*.

Forgetting Tails: Let $\mathbf{m}$ be any n-tuple

$$\mathbf{m} := (m_1, \dots, m_n) \in G_A^n,$$

and let 1 be the identity in G. Whenever the pair (g, n) is stable (i.e., $2g - 2 + n > 0$) there is a natural *forgetting tails* morphism $\tilde{\tau} : \overline{\mathcal{M}}^G_{g,n+1}(\mathbf{m}, 1) \longrightarrow \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ defined as follows.

First, simply forgetting the data associated to the $(n+1)$st marked point usually yields an object of $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$, but if the resulting curve is unstable, then we need to contract the unstable component to a point p. In those cases it is true, but not immediately obvious, that we can produce a suitable G-cover E on the new curve, and where necessary, assign a point $\tilde{p}$ in E over p. We now describe how this works.

We have two cases: first, when the resulting unstable component D is a (genus-zero) -1-curve with one marked point p_i, and one node q; and second, when the unstable component D is a -2-curve with two nodes q and q' and no marked points.

In either case, the unstable component D is a genus-zero curve with two special points (call them q and q' for simplicity of notation). It is straightforward to see that for any $\tilde{q}' \in E$ over q' with monodromy m, the connected component $\widetilde{D}$ of E containing $\tilde{q}'$ is a finite cover of D with automorphism group $\mathrm{Aut}_D \widetilde{D}$ generated by m, which acts faithfully on all points but q and q'. In particular, it is fully ramified over q and q', and unramified at all other points. Thus there is a canonical G-equivariant isomorphism $\varphi : E|_q \xrightarrow{\sim} E|_{q'}$. This shows in the first case, where $p_i = q'$, that there is a canonical choice of $\tilde{q} \in E|_q$ (namely, $\tilde{q} = \varphi^{-1}(\tilde{p}_i)$), and thus a well-defined point of $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$.

In the second case, the isomorphism φ allows the construction of a principal G-bundle on the curve with the unstable component D contracted. In this case, we need no point $\tilde{q}$—the data we already have will give a point of $\overline{\mathcal{M}}^G_{g,n}$. Thus in every case the forgetting tails morphism exists.

Gluing Trees: Given any $\mathbf{m} \in G_A^{n_1}$ and $\mathbf{m}' \in G_A^{n_2}$, as well as an additional element $\mu \in G_A$, let $g := g_1 + g_2$ and $n := n_1 + n_2$. We have the *gluing trees* morphism:

$$\rho_{tree} : \overline{\mathcal{M}}^G_{g_1,n_1+1}(\mathbf{m}, \mu) \times \overline{\mathcal{M}}^G_{g_2,n_2+1}(\mu^{-1}, \mathbf{m}') \longrightarrow \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}, \mathbf{m}') \tag{7}$$

given by attaching the universal G-covers $E \xrightarrow{\pi} C' \xrightarrow{\varpi} \overline{\mathcal{M}}^G_{g_1,n_1+1}(\mathbf{m}, \mu)$ and $E' \xrightarrow{\pi'} C \xrightarrow{\varpi'} \overline{\mathcal{M}}^G_{g_2,n_2+1}(\mu^{-1}, \mathbf{m}')$ along the sections $\tilde{p}_{n_1+1} \in E$ and $\tilde{p}'_1 \in E'$, and attaching the universal curves C and C' along the sections p_{n_1+1} and p'_1. It is straightforward to see that, because the monodromies μ and μ^{-1} are inverses, the induced cover is indeed an admissible G-cover of the resulting stable curve, and thus gives an object in $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}, \mathbf{m}')$.

More generally, let $I = \{i_1, \dots, i_{n_1}\}$ and $J = \{j_1, \dots, j_{n_2}\}$ be any disjoint subsets of $\{1, \dots, n\}$ such that $I \sqcup J = \{1, \dots, n\}$. For any integers s, t with $i \leq s \leq n_1$, $1 \leq t \leq n_2$ there is a morphism

$$\overline{\mathcal{M}}^G_{g_1,n_1+1}(m_{i_1}, \dots, m_{i_{s-1}}, \mu, m_{i_s}, \dots, m_{i_{n_1}}) \tag{8}$$

$$\times\overline{\mathcal{M}}^{G}_{g_2,n_2+1}(m_{j_1},\ldots,m_{j_{t-1}},\mu^{-1},m_{j_t},\ldots,m_{j_{n_2}}) \longrightarrow \overline{\mathcal{M}}^{G}_{g,n}(m_1,\ldots,m_n).$$

Gluing loops: Given any $\mathbf{m} \in G_A^n$ and $\mu \in G_A$ we have the *gluing loops* morphism:

$$\rho_{loop} : \overline{\mathcal{M}}^{G}_{g-1,n+2}(\mathbf{m},\mu,\mu^{-1}) \longrightarrow \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m}), \tag{9}$$

defined in a manner similar to the gluing trees morphism; namely, one attaches the universal G-cover E to itself along the two sections $\widetilde{p}_{n+1}$ and $\widetilde{p}_{n+2}$, and the universal curve C to itself along the sections p_{n+1} and p_{n+2}.

As with gluing trees, the gluing loops morphism can be defined more generally for any two sections $\widetilde{p}_i$, and $\widetilde{p}_j$, provided they have inverse monodromies.

Remark 2.8. Even more generally, if two points do not have inverse monodromies, the braid group action may still allow one to glue them. For example, for any $i_1 < i_2$ with $i_1, i_2 \in \{0,\ldots,n+1\}$ and $\sigma = m_{i_1+1}^{-1} m_{i_1+2}^{-1} \ldots m_{i_2}^{-1}\mu^{-1} m_{i_2} \ldots m_{i_1+1}$, we have a morphism

$$\begin{aligned}
&\overline{\mathcal{M}}^{G}_{g,n+2}(m_1,\ldots,m_{i_1},\mu,m_{i_1+1},\ldots,m_{i_2},\sigma,m_{i_2+1},\ldots,m_{n+2})\\
&\xrightarrow{b_{i_2+1}\circ b_{i_1+2}\circ\cdots\circ b_{i_2}} \overline{\mathcal{M}}^{G}_{g,n+2}(m_1,\ldots,m_{i_1},\mu,\mu^{-1},m_{i_1+1},\ldots,m_n)\\
&\xrightarrow{\rho_{loop}} \overline{\mathcal{M}}^{G}_{g+1,n}(m_1,\ldots,m_n).
\end{aligned}$$

Remark 2.9. Since the collection $\{\overline{\mathcal{M}}^{G}_{g,n}\}$ has gluing morphisms which are equivariant under the actions of G^n and S_n, one may regard $\{\overline{\mathcal{M}}^{G}_{g,n}\}$ as a G-equivariant colored modular operad where the set of colors is the G-set G_A. Since the action of the braid group B_n (see Equation (6)) on $\overline{\mathcal{M}}^{G}_{g,n}$ is constructed from the G^n and S_n actions, one may also regard $\{\overline{\mathcal{M}}^{G}_{g,n}\}$ as a colored modular operad, but where the role of the permutation group is replaced by the braid group.

Remark 2.10. It is worth pointing out that the stack $\overline{\mathcal{M}}^{G}_{g,n+1}(\mathbf{m},1)$ is not the universal curve over $\overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m})$, but rather the universal *orbicurve.*

2.3. Holonomy and other tools for studying G-covers.

Let G_R denote G considered as a right G-module. Note that the automorphism group $\mathrm{Aut}^G(G_R)$ of G_R is again G, acting by *left* multiplication. Given a pointed admissible cover $(E \xrightarrow{\pi} C, \widetilde{p}_1,\ldots,\widetilde{p}_n)$ and given any point $\widetilde{p}_0 \in E_{gen} := \pi^{-1}(C_{gen})$ lying over $p_0 \in C_{gen}$, we have an isomorphism of right G-modules $\nu_{\widetilde{p}_0} : E|_{p_0} \xrightarrow{\sim} G_R$, given by

$$\nu_{\widetilde{p}_0}(\widetilde{p}_0\gamma) := \gamma.$$

Changing the base point $\widetilde{p}_0$ to $\widetilde{p}_0\alpha$ changes the map $\nu_{\widetilde{p}_0}$ by left multiplication by α^{-1}.

$$\nu_{\widetilde{p}_0\alpha} = \alpha^{-1}\nu_{\widetilde{p}_0}.$$

Definition 2.11. The choice of $\widetilde{p}_0 \in E_{gen}$ gives a homomorphism from the fundamental group to G:

$$\chi_{\widetilde{p}_0} : \pi_1(C_{gen},p_0) \longrightarrow G,$$

which we call *holonomy.* One way to see this homomorphism explicitly is to pull E_{gen} back to the trivial admissible cover $\widetilde{E}$ of the universal cover U of C_{gen}. Automorphisms of U are precisely $\pi_1(C_{gen}, p_0)$, and they induce automorphisms of $\widetilde{E} \cong U \times G_R$, and therefore of G_R:

$$\pi_1(C_{gen}, p_0) \cong \mathrm{Aut}_{C_{gen}} U \xrightarrow{\chi} \mathrm{Aut}^G G_R = G.$$

Conversely, given any homomorphism $\chi : \pi_1(C_{gen}, p_0) \longrightarrow G$, it is easy to see that we get a uniquely determined admissible G-cover of $(C, p_1, \dots, p_n)$ and a distinguished point $\widetilde{p}_{0,\chi}$ over p_0. This G-cover is given by first taking the quotient of $U \times G_R$ by the action of $\pi_1(C_{gen}, p_0) \xrightarrow{\mathrm{id},\chi} \mathrm{Aut}_{C_{gen}} U \times \mathrm{Aut}^G G_R$ and then extending it to all of C. Such an extension is uniquely determined by the G-cover on C_{gen}. The point $\widetilde{p}_{0,\chi}$ is the image of $(p_0, 1) \in U \times G_R$ under this quotient. We call this cover *the admissible* G*-cover of* C *induced by* χ and p_0, and we denote it E_{χ,p_0}, or E_χ if p_0 is clear from context.

The following proposition is an immediate consequence of well-known corresponding results for principal G-bundles (see, for example, [Fu, Chapters 13–14]) and is straightforward to check.

Proposition 2.12. *Given a homomorphism* $\chi : \pi_1(C_{gen}, p_0) \longrightarrow G$, *the induced* E *and* $\widetilde{p}_{0,\chi}$ *have holonomy* $\chi_{\widetilde{p}_0}$ *equal to* χ, *and conversely, given an* E *and* $\widetilde{p}_0$ *the bundle* $E_{\chi_{\widetilde{p}_0}}$ *is canonically isomorphic to* E, *via an isomorphism identifying* $\widetilde{p}_{0,\chi_{\widetilde{p}_0}}$ *with the original* $\widetilde{p}_0$.

Thus the data of $E, \widetilde{p}_0$ *is equivalent to a choice of homomorphism* $\chi : \pi_1(C_{gen}, p_0) \longrightarrow G$.

A different choice of point $\widetilde{p}_0$, *say* $\widetilde{p}_0\alpha$, *changes* χ *by conjugation* $\chi_{\widetilde{p}_0\alpha} = \alpha^{-1}\chi\alpha$. *Furthermore, given a path* γ *from* p_0 *to* p_0' *in* C_{gen} *and the corresponding unique lift* $\widetilde{\gamma}$ *of* γ *from* $\widetilde{p}_0$ *to* $\widetilde{p}_0' \in E|_{p_0'}$, *the holonomy* $\chi_{\widetilde{p}_0'}$ *is induced from* $\chi_{\widetilde{p}_0}$ *by conjugation with* γ.

$$\begin{tikzcd}
\pi_1(C_{gen}, p_0) \arrow[r, "\chi_{\widetilde{p}_0}"] \arrow[d, "ad(\gamma)"'] & G \\
\pi_1(C_{gen}, p_0) \arrow[ur, "\chi_{\widetilde{p}_0'}"'] &
\end{tikzcd}$$

And conversely, given any $\chi' : \pi_1(C_{gen}, p_0) \longrightarrow G$ *determined from* χ *by conjugation by* γ, *the induced* G*-cover* $E_{\chi'}$ *is canonically isomorphic to* E_χ, *and the induced point* $\widetilde{p}_{0,\chi'}'$ *is that obtained by parallel transporting* $\widetilde{p}_0$ *along* γ *(i.e.,* $\widetilde{p}_{0,\chi}'$ *is the endpoint of* $\widetilde{\gamma}$*).*

Definition 2.13. For any path d from p_0 to p_i in C_{gen} (that is, a path in C such that the image of $(0, 1)$ lies in C_{gen} and $d(1) = p_i$ and $d(0) = p_0$), we have an induced element σ_d of $\pi_1(C_{gen}, p_0)$ defined by following d from p_0 to a little loop around p_i, tracing the loop out once counterclockwise, and then returning along d (or rather d^{-1}) to p_0.

Moreover, for any admissible G-cover with point $\widetilde{p}_0 \in E|_{p_0}$, the path d determines a point $\widetilde{p}(d) \in E|_{p_i}$, which is the endpoint of the unique lift $\widetilde{d}$ of d in E that begins at $\widetilde{p}_0$.

Finally, given $\widetilde{p}_0$ and a path d from p_0 to p_i, holonomy and the map $\nu_{\widetilde{p}_0}$ induce an isomorphism of right G-modules $\bar{\nu}_{d,\widetilde{p}_0} : E|_{p_i} \xrightarrow{\sim} \langle m_i \rangle \backslash G_R$, where $m_i := \chi_{\widetilde{p}_0}(\sigma_d)$, and $\bar{\nu}_{d,\widetilde{p}_0}$ maps the point $\widetilde{p}(d)$ to the coset $\langle m_i \rangle$.

Definition 2.14. In genus zero, a choice of paths d_i from the point p_0 to the point p_i for each $i \in \{1, \dots, n\}$ induces loops σ_{d_i} that generate $\pi_1(C_{gen}, p_0)$. Not every choice of monodromy $\mathbf{m} \in G^n$ satisfies the same relations that the generators σ_i do, and thus not every choice of monodromy defines a holonomy χ, but for those $\mathbf{m}$ that do, there is a uniquely determined pointed admissible G-cover

$$\zeta(d_1, \dots, d_n; \mathbf{m}) := (E_\chi \longrightarrow \mathbb{CP}^1, \widetilde{p}_1, \dots, \widetilde{p}_n)$$

by defining the holonomy χ to be given by the monodromy

$$\chi(\sigma_{d_i}) = m_i, \quad \text{for } i \in \{1, \dots, n\}$$

and letting the points $\widetilde{p}_i := \widetilde{p}(d_i)$ be the points induced as in Definition 2.13. Since the loops σ_{d_i} generate the fundamental group of $\mathbb{CP}^1 - \{p_1, \dots, p_n\}$, this construction gives a well-defined pointed admissible G-cover.

It is clear from our discussion so far that every smooth, genus-zero, n-pointed, admissible G-cover $(E \longrightarrow \mathbb{CP}^1, \widetilde{p}_1, \dots, \widetilde{p}_n)$ that has all of its points $\widetilde{p}_i$ in the same connected component of E must be of the form $\zeta(d_1, \dots, d_n; \mathbf{m})$ for some choice of p_0, paths $(d_1, \dots, d_n)$, and $\mathbf{m} \in G^n$. Assume that the points $p_0, \dots, p_n \in C$ are given. Of special interest is the case where the induced generators of the fundamental group have product equal to 1. We denote the subset of such n-tuples of paths by

$$P_C := \{(d_1, \dots, d_n) | d_i \text{ a path from } p_0 \text{ to } p_i, \text{ and } \prod_{i=1}^{n} \sigma_{d_i} = 1\}, \tag{10}$$

and the corresponding pointed admissible G-covers of C by

$$\zeta_C := \{\zeta(\mathbf{d}; \mathbf{m}) | \mathbf{d} \in P_C,\ \mathbf{m} \in G^n,\ \prod_{i=1}^{n} m_i = 1\}. \tag{11}$$

Definition-Proposition 2.15. Given a choice of $p_0, \dots, p_n \in C$, with genus$(C) = 0$, there is a transitive action of the braid group on the set P_C, where

$$b_i d_i = d_{i+1} \tag{12}$$

$$b_i d_{i+1} = \sigma_{d_i+1} d_i \tag{13}$$

$$b_i d_j = d_j \text{ if } j \neq i, i+1. \tag{14}$$

This action of B_n on the set P_C is compatible with the usual braid action on $\pi_1(C_{gen}, p_0)$; that is, for each i we have $\sigma_{d_i} \in \pi_1(C_{gen}, p_0)$, and

$$\sigma_{b d_i} = b \sigma_{d_i}. \tag{15}$$

Consequently, the braid action on P_C induces an action of the braid group on ζ_C, distinct from the braid action on all of $\overline{\mathcal{M}}^G_{g,n}$ that we defined earlier. To distinguish the two, we will denote this new action by $\beta : B_n \longrightarrow \mathrm{Aut}(\zeta_C)$.

Proof. The fact that the given equation defines an action and that the action is compatible with the usual action on the fundamental group is a straightforward calculation. That the action is transitive follows from the classical fact that the braid group generates all automorphisms of the fundamental group that preserve the property of the product of generators being trivial.

Since the product of generators and the product of monodromies are both trivial, the induced holonomy $b\chi : \sigma_{(bd_i)} \mapsto m_i$ is still a well-defined homomorphism of groups. Thus for each admissible cover $\zeta(\mathbf{d};\mathbf{m}) \in \zeta_C$ and for each $b \in B_n$ we can define

$$\beta(b)\zeta(\mathbf{d};\mathbf{m}) := \zeta(b\mathbf{d};\mathbf{m}). \tag{16}$$

□

2.4. Automorphisms, isomorphisms, and fibers.

Definition 2.16. Let $\mathrm{Aut}_C^G E$ denote the group of G-equivariant automorphisms of E over C. Any $\varphi \in \mathrm{Aut}_C^G E$ must induce a G-equivariant automorphism $\varphi' : G_R \longrightarrow G_R$ of right G-modules by $\varphi' = \nu_{\widetilde{p}_0} \circ \varphi \circ \nu_{\widetilde{p}_0}^{-1}$. It is easy to see that if $\varphi(\widetilde{p}) = \widetilde{p}_0 g$, then φ' is simply left multiplication by g. This gives a homomorphism

$$\Psi_{\widetilde{p}_0} : \mathrm{Aut}_C^G E \longrightarrow G.$$

Proposition 2.17. *The homomorphism $\Psi_{\widetilde{p}_0} : \mathrm{Aut}_C^G E \longrightarrow G$ commutes with every element of* $\mathrm{im}(\chi_{\widetilde{p}_0})$*, and depends only upon the (path-)component of E_{gen} in which $\widetilde{p}_0$ lies. Moreover, if C is irreducible, then $\Psi_{\widetilde{p}_0}$ is an isomorphism to the centralizer of (i.e., the subgroup of G which commutes with every element of) the image of $\chi_{\widetilde{p}_0}$:*

$$\Psi_{\widetilde{p}_0} : \mathrm{Aut}_C^G E \xrightarrow{\sim} C(\mathrm{im}\,\chi_{\widetilde{p}_0}).$$

Proof. It is straightforward to check that a change of base point from $\widetilde{p}_0$ to $\widetilde{p}_0' = \widetilde{p}_0\gamma$ changes $\Psi_{\widetilde{p}_0}$ by conjugation.

$$\Psi_{\widetilde{p}_0\gamma} = \gamma^{-1}\Psi_{\widetilde{p}_0}\gamma.$$

On the other hand, given a path $\sigma : [0,1] \longrightarrow E_{gen}$ from $\widetilde{p}_0$ to another point $\widetilde{q}_0$ we may parallel transport any point $\widetilde{p}_0\gamma$ of the fiber $E|_{p_0}$ to the point $\widetilde{q}_0\gamma$ in the fiber $E|_{q_0}$, thus giving an isomorphism of right G-sets $\sigma_* : E|_{p_0} \xrightarrow{\sim} E|_{q_0}$, and one can check that the induced homomorphisms $\Psi_{\widetilde{p}_0}$ and $\Psi_{\widetilde{q}_0}$ are the same:

$$\Psi_{\widetilde{p}_0} = \Psi_{\widetilde{q}_0} : \mathrm{Aut}_C^G E \longrightarrow G.$$

The first two claims of the proposition follow.

It is straightforward to check that if C_{gen} is path connected, then $\Psi_{\widetilde{p}_0}$ is injective, and surjectivity can be seen by uniformizing C_{gen}, pulling E back to a trivial bundle on the uniformizer, and checking that left multiplication by any element of G which commutes with holonomy descends to a G-equivariant automorphism of E over C. □

We now turn our attention to automorphisms of *pointed* admissible G-covers. For a pointed admissible G-cover $(E \xrightarrow{\pi} C, \widetilde{p}_1, \dots, \widetilde{p}_n)$, we denote the group of G-equivariant automorphisms of E over C which fix the points $\widetilde{p}_i$ by $\mathrm{Aut}_C^G(E, \widetilde{p}_1, \dots, \widetilde{p}_n)$.

Proposition 2.18. *If C is an irreducible curve, and if $m_1, \dots, m_n \in G_A$ are the monodromies of the admissible G-cover E at $\widetilde{p}_1, \dots, \widetilde{p}_n$, respectively, then for any* *elements* $\gamma_1, \dots, \gamma_n$ *such that* $\widetilde{p}_0 \in E_{gen}$ *lies in the same connected component of* E_{gen} *as* $\widetilde{p}_1\gamma_1, \dots, \widetilde{p}_n\gamma_n$*, the map* $\Psi_{\widetilde{p}_0}$ *induces an isomorphism*

$$\Psi_{\widetilde{p}_0} : \mathrm{Aut}^G_C(E, \widetilde{p}_1, \dots, \widetilde{p}_n) \xrightarrow{\sim} \langle \gamma_1^{-1} m_1 \gamma_1 \rangle \cap \cdots \cap \langle \gamma_n^{-1} m_n \gamma_n \rangle \cap C(\mathrm{im}(\chi_{\widetilde{p}_0})),$$

where $C(\mathrm{im}(\chi_{\widetilde{p}_0}))$ *denotes the centralizer of the image of* $\chi_{\widetilde{p}_0}$.

Proof. If $\widetilde{p}_i\gamma_i$ is in the same component of E_{gen} as $\widetilde{p}_0$, then there is a path d in C_{gen} from p_0 to p_i which lifts to a path $\widetilde{d}$ from $\widetilde{p}_0$ to $\widetilde{p}_i$, and we have an isomorphism $\overline{\nu}_{d,\widetilde{p}_0} : E|_{p_i} \xrightarrow{\sim} \langle \gamma_i^{-1} m_i \gamma_i \rangle \backslash G_R$ of right G-sets taking $\widetilde{p}_i\gamma_i$ to the coset $\langle \gamma_i^{-1} m_i \gamma_i \rangle$. An automorphism $\varphi \in \mathrm{Aut}^G_C E$ with $\Psi_{\widetilde{p}_0}(\varphi) = g$ takes the coset $\langle \gamma_i^{-1} m_i \gamma_i \rangle$ to itself if and only if $g \in \langle \gamma_i^{-1} m_i \gamma_i \rangle$. Thus φ fixes the points $\widetilde{p}_i\gamma_i$ and also $\widetilde{p}_i$ if and only if $\Psi_{\widetilde{p}_0}(\varphi) \in \langle \gamma_i^{-1} m_i \gamma_i \rangle$ for every i. □

Of course, if $\widetilde{p}_i\gamma$ is in the same connected component of E_{gen} as $\widetilde{p}_i\alpha$, then, since $\Psi_{\widetilde{p}_0}(\varphi)$ commutes with holonomy, including $\gamma^{-1}\alpha$, the condition $\Psi_{\widetilde{p}_0}(\varphi) \in \langle \gamma^{-1} m_i \gamma \rangle$ is the same as the condition $\Psi_{\widetilde{p}_0}(\varphi) \in \langle {\alpha'}^{-1} m_i \alpha \rangle$.

Proposition 2.19. *For any smooth pointed curve* $(C, p_1, \dots, p_n)$ *having* no nontrivial automorphisms*, choose an admissible cover* $(E \xrightarrow{\pi} C, p_1, \dots, p_n) \in \mathrm{Adm}^G_{g,n}$. *For any* $\widetilde{p}_0 \in E_{gen}$ *and for any choice of paths* d_i *in* C_{gen} *from* $p_0 = \pi(\widetilde{p}_0)$ *to* p_i*, let* $\sigma_i = \sigma_{d_i}$ *be the corresponding element of* $\pi_1(C_{gen}, p_0)$*. We can describe the fiber* $(\widetilde{\mathrm{st}})^{-1}([E \xrightarrow{\pi} C, p_1, \dots, p_n])$ *of the forgetful map* $\widetilde{\mathrm{st}} : \overline{\mathcal{M}}^G_{g,n} \longrightarrow \mathrm{Adm}^G_{g,n}$ *as the quotient stack*

$$(\widetilde{\mathrm{st}})^{-1}(E \xrightarrow{\pi} C, p_1, \dots, p_n) = \left[\left(\prod_{i=1}^{n} \langle \chi_{\widetilde{p}_0}(\sigma_i) \rangle \backslash G_R\right) / C(\chi_{\widetilde{p}_0})\right] = \coprod_{I_{\widetilde{p}_0}} \mathcal{B}H_{\widetilde{p}_0},$$

where $C(\chi_{\widetilde{p}_0})$ *is the centralizer of the image of* $\chi_{\widetilde{p}_0}$*, acting diagonally on the product, the index set* $I_{\widetilde{p}_0}$ *is* $\left(\prod_{i=1}^{n} \langle \chi_{\widetilde{p}_0}(\sigma_i) \rangle \backslash G_R\right) / \left(C(\chi_{\widetilde{p}_0})/H_{\widetilde{p}_0}\right)$*, and the group* $H_{\widetilde{p}_0}$ *is the image under* $\Psi_{\widetilde{p}_0}$ *of the automorphism group of any pointing* $(\widetilde{p}_1, \dots, \widetilde{p}_n)$ *of* E*:*

$$H_{\widetilde{p}_0} = \Psi_{\widetilde{p}_0}(\mathrm{Aut}^G_C(E, \widetilde{p}_1, \dots, \widetilde{p}_n)) = C(\chi_{\widetilde{p}_0}) \cap \langle \chi_{\widetilde{p}_0}(\sigma_1) \rangle \cap \cdots \cap \langle \chi_{\widetilde{p}_0}(\sigma_n) \rangle.$$

Proof. A choice of pointing $\widetilde{p}_1, \dots, \widetilde{p}_n \in E$ is equivalent to a choice $\overline{\nu}_{\widetilde{p}_0}(\widetilde{p}_i) \in \langle \chi_{\widetilde{p}_0}(\sigma_i) \rangle \backslash G_R$ for each $i \in \{1, \dots, n\}$, and any isomorphism between two pointings $(E, \widetilde{p}_1, \dots, \widetilde{p}_n)$ and $(E, \widetilde{p}'_1, \dots, \widetilde{p}'_n)$ induces an automorphism of E. Conversely, the automorphisms of E act on the set of all pointings, thus Proposition 2.17 gives the first equality. For any pointing, the homomorphism $\Psi_{\widetilde{p}_0}$ takes the automorphism group $\mathrm{Aut}^G_C(E, \widetilde{p}_1, \dots, \widetilde{p}_n)$ to $H_{\widetilde{p}_0} := C(\chi_{\widetilde{p}_0}) \cap \langle \chi_{\widetilde{p}_0}(\sigma_1) \rangle \cap \dots, \cap \langle \chi_{\widetilde{p}_0}(\sigma_n) \rangle$ by Proposition 2.18. The second equality follows. □

Proposition 2.20. *Let* $C = C^1 \cup C^2$ *be the union of two irreducible curves joined at a single node* q*. Choose points* $\widetilde{p}_0^1, \widetilde{p}_0^2 \in E_{gen}$ *lying over* C^1_{gen} *and* C^2_{gen}*, respectively, and such that* $\widetilde{p}_0^1$ *and* $\widetilde{p}_0^2$ *lie in the same connected component of* E*. Also, choose a point* $\widetilde{q} \in E|_q$ *of the fiber over* q *which lies in the same connected component of* E *as* $\widetilde{p}_0^1$ *and* $\widetilde{p}_0^2$*. Let* μ *and* μ^{-1} *be the monodromy of* E *at* q *with respect to the orientations of* C^1 *and* C^2 *respectively. We have an injective homomorphism*

$$\Psi := (\Psi_{\widetilde{p}_0^1}, \Psi_{\widetilde{p}_0^2}) : \mathrm{Aut}^G_C(E, \widetilde{p}_1, \dots, \widetilde{p}_n) \hookrightarrow G \times G,$$

which depends only on the connected component of E *in which* $\widetilde{p}_0^1$ *and* $\widetilde{p}_0^2$ *lie, and*

$$\operatorname{im}\Psi = \{(g_1, g_2) \in \operatorname{im}\Psi_{\widetilde{p}_0^1} \times \operatorname{im}\Psi_{\widetilde{p}_0^2} | g_1 g_2^{-1} \in \langle\mu_1\rangle\}.$$

Proof. The injectivity follows from arguments similar to the irreducible case. The condition on the elements $(g_1, g_2) \in \operatorname{im}\Psi_{\widetilde{p}_0^1} \times \operatorname{im}\Psi_{\widetilde{p}_0^2}$ comes from the fact that any automorphism of E must take both "sides" of the node $\widetilde{q}$ to the same point: $\widetilde{q}g_1 = \widetilde{q}g_2$, but $\widetilde{q}g_i$ is only determined up to a (left) coset of $\langle\mu\rangle$. $\square$

Let C be an irreducible curve with one node q obtained by attaching 2 points q_+ and q_- of the normalized curve C^ν. An admissible G-cover E of C is obtained by attaching two points $\widetilde{q}_+ \in E^\nu|_{q_+}$ and $\widetilde{q}_- \in E^\nu|_{q_-}$ of an admissible G-cover E^ν on C^ν which have monodromy μ and μ^{-1}, respectively, for some $\mu \in G$. Let $\widetilde{p}_0 \in E^\nu_{gen} = E_{gen}$ be in the same connected component of E^ν as $\widetilde{q}_+$ is, and let $\gamma \in G$ be chosen so that $\widetilde{q}_-\gamma^{-1}$ is in that same component of E^ν.

Proposition 2.21. *Any automorphism* $\varphi \in \operatorname{Aut}^G_C(E, \widetilde{p}_1, \ldots, \widetilde{p}_n)$ *induces an automorphism* $N(\varphi) \in \operatorname{Aut}^G_{C^\nu}(E^\nu, \widetilde{p}_1, \ldots, \widetilde{p}_n)$ *by pullback to the normalization. For any* $\widetilde{p}_0 \in E_{gen}$ *the homomorphism* N *is injective and is compatible with* $\Psi_{\widetilde{p}_0}$*; that is, the following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Aut}^G_C(E, \widetilde{p}_1, \ldots, \widetilde{p}_n) \arrow[r, "\Psi_{\widetilde{p}_0}"] \arrow[d, hook, "N"'] & G \arrow[d, equal] \\
\operatorname{Aut}^G_{C^\nu}(E^\nu, \widetilde{p}_1, \ldots, \widetilde{p}_n) \arrow[r, "\Psi_{\widetilde{p}_0}"] & G
\end{tikzcd}$$

Moreover, we have

$$\operatorname{im}(\Psi_{\widetilde{p}_0} \circ N) = \{g \in \Psi_{\widetilde{p}_0}(\operatorname{Aut}^G_{C^\nu} E^\nu) | g \in C(\gamma)\}.$$

Proof. Commutativity of the diagram is straightforward to check and injectivity of N follows from the fact that $\Psi_{\widetilde{p}_0}$ is injective. The fact that the image commutes with γ follows from an argument similar to that for holonomy in Proposition 2.17. $\square$

2.5. **Distinguished components of $\overline{\mathcal{M}}^G_{g,n}$.**

Several distinguished components of $\overline{\mathcal{M}}^G_{g,n}$ will be useful for our construction of G-CohFTs. We describe them and their basic properties in this subsection.

2.5.1. *The substack* $\xi(\mathbf{m})$ *of* $\overline{\mathcal{M}}^G_{0,3}(\mathbf{m})$.

Definition 2.22. For any $\mathbf{m} := (m_1, m_2, m_3) \in G^3_A$, if the product $\prod_{i=1}^3 m_i$ is not 1, then we define the stack $\xi(\mathbf{m})$ to be the empty stack. Otherwise, let C denote the sphere $\mathbb{CP}^1$ with special points $p_0 := 0$ and $p_j = \exp(2\pi j\sqrt{-1}/3)$ for $j \in \{1, 2, 3\}$. For each $j \in \{1, 2, 3\}$ let d_i be the path in C determined by following a straight line from p_0 to p_i. These paths induce elements $\sigma_i := \sigma_{d_i} \in \pi_1(\mathbb{CP}^1 - \{p_1, p_2, p_3\}, p_0)$ (see Definition 2.13), which generate the fundamental group and have trivial product: $\sigma_1\sigma_2\sigma_3 = 1$. Thus the triple $\mathbf{d} = (d_1, d_2, d_3)$ is in P_C.

We define $\xi(\mathbf{m})$ to be the connected component of $\overline{\mathcal{M}}^G_{0,3}(\mathbf{m})$ containing the geometric point $\zeta(\mathbf{d}; \mathbf{m})$, as defined in Definition 2.14.

Remark 2.23. For any $m \in G$, it is clear that the component $\xi(m, m^{-1}, 1)$ is the unique component of $\overline{\mathcal{M}}^G_{0,3}(m, m^{-1}, 1)$ such that all the points $\widetilde{p}_i$ lie in the same connected component of the admissible G-cover E.

Lemma 2.24. *For any* $\mathbf{m} \in G^3_A$ *we have the following identities for the* $\xi(\mathbf{m})$*.*

(1) $\rho(\gamma, \gamma, \gamma)(\xi(\mathbf{m})) = \xi(\gamma m_1 \gamma^{-1}, \gamma m_2 \gamma^{-1}, \gamma m_3 \gamma^{-1})$ *for any* $\gamma \in G$*.*
(2) $\rho(m_1, 1, 1)(\xi(\mathbf{m})) = \rho(1, m_2, 1)(\xi(\mathbf{m})) = \rho(1, 1, m_3)(\xi(\mathbf{m})) = \xi(\mathbf{m})$*.*
(3) *For the generators* b_1, b_2 *of the braid group* B_3

$$b_1 \xi(\mathbf{m}) = \xi(m_1 m_2 m_1^{-1}, m_1, m_3)$$
$$b_2 \xi(\mathbf{m}) = \xi(m_1, m_2 m_3 m_2^{-1}, m_2).$$

Thus for any element $b \in B_3$*, we have*

$$b\xi(\mathbf{m}) = \xi(b\mathbf{m}),$$

where b *acts on the triple* $\mathbf{m}$ *via the Hurwitz action (i.e., the obvious action where, for example,* $b_1(m_1, m_2, m_3) := (m_1 m_2 m_1^{-1}, m_1, m_3)$*).*
(4) *Let* s *be an isomorphism induced from a cyclic permutation (also denoted* s*) in* S_3*, then*

$$s\xi(\mathbf{m}) = \xi(s\mathbf{m}).$$

Proof. The first identity follows from the fact that the global right action translates all points in the admissible G-cover in ξ by γ. Under this action, the i-th monodromy m_i changes to $\gamma^{-1} m_i \gamma$ for all $i = 1, \ldots, n$.

The second statement follows from the fact that the action of ρ_i on the i-th point $\widetilde{p}_i$ is the same (via the map $\bar{\nu}_{\widetilde{p}_0}$) as right multiplication acting on the right G-coset $\langle m_i \rangle$; that is, the action $\varphi_i(m_i)$ is trivial.

The third statement follows from studying the results of sliding points p_j around p_i, which we now describe in the case of b_2. The case of b_1 is essentially the same.

The transformation $T : z \mapsto 1/z$ takes $p_0 = 0$ to ∞, fixes p_1, and interchanges p_2 and p_3. Let $E' = T_* E := (T^{-1})^* E$, $\widetilde{p}'_2 := T_*(\widetilde{p}_3)$, $\widetilde{p}'_3 := T_*(\widetilde{p}_2)$, and $\widetilde{p}'_1 := T_*(\widetilde{p}_1)$. The pointed cover $(E', \widetilde{p}'_1, \widetilde{p}'_2, \widetilde{p}'_3)$ corresponds to the geometric point representing the image of $\xi(\mathbf{m})$ under the action of the transposition $s_{(2,3)}$. Let γ be a straight path from p_0 to ∞ that passes between p_3 and p_1, e.g., the path $\gamma(t) = -i/(1-t)$. Note that via γ we have an isomorphism of (un-pointed) admissible G-covers $E' \cong E_{\chi, p_0}$, where χ is the homomorphism $\pi_1(C_{gen}, p_0) \longrightarrow G$, given by taking $\gamma T_* \sigma_i \gamma^{-1}$ to m_i and with the induced $\widetilde{p}_{0,\chi}$ being the "parallel transport" of $T^*(\widetilde{p}_0)$ along γ. The loop $\gamma T_* \sigma_3 \gamma^{-1}$ (around $T(p_3) = p_2$) and the loop $\gamma T_* \sigma_1 \gamma^{-1}$ (around $T(p_1) = p_1$) are homotopic to the loops σ_2 and σ_1, respectively. But the loop $\gamma T_* \sigma_2 \gamma^{-1}$ is homotopic to $\sigma_2 \sigma_3 \sigma_2^{-1}$. Thus the (un-pointed) G-cover E' is isomorphic to $E'' := E_{\chi, p_0}$, where χ is the homomorphism taking σ_1 to m_1, σ_2 to $m_2 m_3 m_2^{-1}$, and σ_3 to m_2, that is, to the G-cover E'' associated to $\xi(b_2 \mathbf{m})$. And the points $\widetilde{p}'_1$ and $\widetilde{p}'_3$ are the same as those that are induced on $\xi(b_2 \mathbf{m})$. However, the point $\widetilde{p}'_2$ is not the same as the point $\widetilde{p}''_2$ induced on $\xi(b_2 \mathbf{m})$; indeed, $\widetilde{p}''_2$ is induced by parallel transport from $\widetilde{p}_0$ along the path d_2, whereas $\widetilde{p}'_2$ is induced by parallel transport from $\widetilde{p}_0$ along $\gamma T_* d_3 \gamma^{-1} = (\sigma_3)^{-1} d_2$. Thus they differ by the holonomy m_2 of the loop σ_3; i.e., the claim of the third statement holds.

Finally, the last statement of the lemma follows from the fact that a rotation (multiplication by $\pm \exp(2\pi i/3)$) of $\mathbb{CP}^1$ will induce the permutation s on ξ. $\square$

The construction of $\xi(\mathbf{m})$ depends *a priori* on the choices of p_i and d_i, but we will see in Proposition 2.26 that it is independent of these choices.

Before we give that proposition, we need to understand better how the different braid actions interact. The fact that the braid action on paths agrees with the usual braid action on loops shows that for any $b \in B_3$ we may write

$$b\mathbf{d} = (\omega_1 d_{\psi(b(1))}, \omega_2 d_{\psi(b(2))}, \omega_3 d_{\psi(b(3))})$$

for some choice of $\omega_i \in \pi_1(C_{gen}, p_0)$ (and $\psi(b(i))$ is the action on i of the permutation induced by the standard surjection $B_3 \xrightarrow{\psi} S_3$). Let $b\chi$ denote the induced holonomy $\sigma_{bd_i} \mapsto m_i$, and let γ_i be the image of ω_i in G via $b\chi$. It is clear that $b\chi$ is the same homomorphism as that induced by taking $d_i \mapsto bm_i$, but the point $\widetilde{p}(bd_i)$ differs from that defined by $\widetilde{p}(d_{\psi(b(i))}$ by $\rho(\gamma_i)$. That is, we have

$$\begin{aligned}\beta(b)\xi(\mathbf{m}) &= \beta(b)\zeta(\mathbf{d};\mathbf{m})\\ &= \zeta(b\mathbf{d};\mathbf{m})\\ &= \rho(\boldsymbol{\gamma})\zeta(\mathbf{d};b\mathbf{m})\\ &= \rho(\boldsymbol{\gamma})\xi(b\mathbf{m})\\ &= \rho(\boldsymbol{\gamma})b\xi(\mathbf{m}).\end{aligned} \tag{17}$$

Lemma 2.25. *The braid action β on $\zeta_C \subset \overline{\mathcal{M}}^G_{0,3}$ factors through the standard symmetric group action on $\overline{\mathcal{M}}^G_{0,3}$ via the usual surjection $\psi : B_3 \longrightarrow S_3$ to the symmetric group. That is, for any $b \in B_3$, $\mathbf{d} \in P_C$, and $\mathbf{m}$, such that $\prod m_i = 1$, we have*

$$\beta(b)\zeta(\mathbf{d};\mathbf{m}) = \psi(b)\zeta(\mathbf{d};\mathbf{m}).$$

Proof. By transitivity of the B_n action on P_C, for every $\zeta(\mathbf{d}';\mathbf{m})$ there exists a $b' \in B_n$ such that $\mathbf{d}' = b'\mathbf{d}$, where $\mathbf{d}$ is the set of paths used to define ξ. So it suffices to check this only in the case of $\xi(\mathbf{m})$; i.e., where the paths are the standard $\mathbf{d}$. Checking the generators of B_3 is now quite easy. For example, in the case of $b = b_1$ the shift $\boldsymbol{\gamma}$ is simply $(m_1, 1, 1)$ and so equation (17) and Lemma 2.24 item (3) gives

$$\begin{aligned}\beta(b_1)\xi(\mathbf{m}) &= \rho(m_1,1,1)\xi(b\mathbf{m})\\ &= \rho(m_1,1,1)b\xi(\mathbf{m})\\ &= s_{1,2}\xi,\end{aligned}$$

as desired. □

Proposition 2.26. *For any $\mathbf{m} \in G^3$ with $\prod_{i=1}^3 m_i = 1$, any choice of points $p'_0, \dots, p'_3 \in \mathbb{CP}^1$, and any choice of paths d'_i from p'_0 to p'_i for each $i \in \{1,2,3\}$ with trivial product (i.e., $\mathbf{d}' = (d'_1, d'_2, d'_3) \in P_C$), the geometric point of $\overline{\mathcal{M}}^G_{0,3}(\mathbf{m})$ defined by $\zeta(\mathbf{d}', \mathbf{m})$ lies in the component $\xi(\mathbf{m})$.*

Proof. Using the action of $PGL(2, \mathbb{C})$ we may assume that $p'_1 = p_1, p'_2 = p_2$, and $p'_3 = p_3$.

Moreover, given any path δ from p_0 to p'_0, we may replace the paths d'_i by $d'_i\delta$. This gives an isomorphism between the n-pointed admissible G cover defined by the d'_i and that defined by the $d'_i\delta$. Thus we may assume that $p'_0 = p_0$.

Since both sets of paths $\mathbf{d} = (d_1, d_2, d_3)$ (from the definition of $\xi(\mathbf{m})$) and $\mathbf{d}' = (d'_1, d'_2, d'_3)$ lie in the set P_C, and since the braid action on P_C is transitive (Definition-Proposition 2.15), there is an element $b' \in B_3$ such that

$$\zeta(\mathbf{d}'; \mathbf{m}) = \zeta(b'\mathbf{d}; \mathbf{m}) = \beta(b')\xi(\mathbf{m}).$$

Moreover, since the endpoint of each d'_i is p_i, we must have

$$b' \in \ker(\psi : B_3 \longrightarrow S_3),$$

that is, b' lies in the *pure braid group*.

The proposition now follows from Lemma 2.25.

□

2.5.2. *Distinguished components of* $\overline{\mathcal{M}}^G_{0,4}$.

Definition 2.27. Let $\mathbf{m} = (m_1, \ldots, m_4)$ be chosen so that $\prod_{i=1}^4 m_i = 1$, and let

$$m_+ := (m_1 m_2)^{-1}, \quad m_- = m_+^{-1}.$$

We let $\xi_{0,4}(\mathbf{m})$ denote the component of $\overline{\mathcal{M}}^G_{0,4}(\mathbf{m})$ which contains the image of $\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4)$ under the gluing map

$$\rho : \overline{\mathcal{M}}^G_{0,3}(m_1, m_2, m_+) \times \overline{\mathcal{M}}^G_{0,3}(m_-, m_3, m_4) \longrightarrow \overline{\mathcal{M}}^G_{0,4}(\mathbf{m}).$$

Definition 2.28. For any closed substack $Q \subseteq \overline{\mathcal{M}}^G_{g,n}$, consider the homology class $[Q]$ in $H_\bullet(\overline{\mathcal{M}}^G_{g,n})$. We define

$$[\![Q]\!] := 0$$

when Q is empty, otherwise,

$$[\![Q]\!] := \frac{1}{\deg(st_Q)}[Q],$$

where $\deg(st_Q)$ is the degree of the forgetful morphism restricted to Q:

$$st_Q : Q \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Lemma 2.29. *Using the notation of Definition 2.27, let*

$$m'_+ := (m_4 m_1)^{-1}, \quad m'_- := (m'_+)^{-1}.$$

We further let ρ' *denote the gluing map composed with the cyclic permutation* $s = (4,3,2,1) \in S_4$*, that is,* $\rho' = s \circ \rho_{tree}$*:*

$$\overline{\mathcal{M}}^G_{0,3}(m_4, m_1, m'_+) \times \overline{\mathcal{M}}^G_{0,3}(m'_-, m_2, m_3) \xrightarrow{\rho_{tree}} \overline{\mathcal{M}}^G_{0,4}(m_4, m_1, m_2, m_3) \xrightarrow{s} \overline{\mathcal{M}}^G_{0,4}(\mathbf{m}),$$

and we let ρ'' *denote the gluing map*

$$\rho'' : \overline{\mathcal{M}}^G_{0,3}(m_1, m'_+, m_4) \times \overline{\mathcal{M}}^G_{0,3}(m_2, m'_-, m_3) \longrightarrow \overline{\mathcal{M}}^G_{0,4}(\mathbf{m}).$$

(1) *The component* $\xi_{0,4}(\mathbf{m})$ *contains the image of* $\xi(m_4, m_1, m'_+) \times \xi(m'_-, m_2, m_3)$ *under the map* ρ' *and the image of* $\xi(m_1, m'_+, m_4) \times \xi(m_2, m'_-, m_3)$ *under the map* ρ''.

(2) *We have the following equalities in* $H_2(\xi_{0,4}(\mathbf{m}))$*:*

$$[\![\rho(\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4))]\!] = [\![\rho'(\xi(m_4, m_1, m'_+) \times \xi(m'_-, m_2, m_3))]\!],$$

and

(3)

$$\rho_*([\![\xi(m_1, m_2, m_+)]\!] \otimes [\![\xi(m_-, m_3, m_4)]\!]) = \rho'_*([\![\xi(m_4, m_1, m'_+)]\!] \otimes [\![\xi(m'_-, m_2, m_3)]\!]).$$

Proof. For any choice $\mathbf{m} \in G_A^4$ with $\prod_{i=1}^4 m_i = 1$, a construction similar to that of $\xi(m_1, m_2, m_3)$ on $\mathbb{CP}^1 - \{p_1, p_2, p_3, p_4\}$, with, say, $p_i := (\sqrt{-1})^i$ and $p_0 := 0$, and with straight-line paths d_i to each p_i, gives a pointed admissible G-cover of $\mathbb{CP}^1 - \{p_1, p_2, p_3, p_4\}$ which has two obvious degenerations. The first degeneration is given by contracting the great circle defined by $\{z = t(1+\sqrt{-1}) | t \in \mathbb{R} \cup \infty\}$. This can easily be seen to be the image of $\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4)$ under the gluing map $\rho_{tree} : \overline{\mathcal{M}}^G_{0,3}(m_1, m_2, m_+) \times \overline{\mathcal{M}}^G_{0,3}(m_-, m_3, m_4) \longrightarrow \overline{\mathcal{M}}^G_{0,4}(\mathbf{m})$. Similarly, the second degeneration, given by contracting the great circle $\{z = t(1 - \sqrt{-1}) | t \in \mathbb{R} \cup \infty\}$, is the image of $\xi(m_1, m'_+, m_4) \times \xi(m_2, m'_-, m_3)$ under the gluing map $\overline{\mathcal{M}}^G_{0,3}(m_1, m'_+, m_4) \times \overline{\mathcal{M}}^G_{0,3}(m_2, m'_-, m_3) \longrightarrow \overline{\mathcal{M}}^G_{0,4}(\mathbf{m})$. The first claim follows from Lemma 2.24 item 4 and the fact that all these gluing morphisms are well-behaved under cyclic permutations.

To see the second claim, consider the forgetful morphism

$$\mathrm{st} : \xi_{0,4}(\mathbf{m}) \longrightarrow \overline{\mathcal{M}}_{0,4}.$$

By pulling back the corresponding boundary divisors on $\overline{\mathcal{M}}_{0,4}$, one obtains the equality

$$[\rho(\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4))]\frac{A}{B} = [\rho'(\xi(m_4, m_1, m'_+) \times \xi(m'_-, m_2, m_3))]\frac{A'}{B},$$

where A is the order of the automorphism group of $\rho(\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4))$, A' is the order of the automorphism group of $\rho'(\xi(m_4, m_1, m'_+) \times \xi(m'_-, m_2, m_3))$, and B is the order of the automorphism group of a generic point in $\overline{\mathcal{M}}^G_{0,4}(\mathbf{m})$.

Finally, we observe that

$$\rho_*([\xi(m_1, m_2, m_+)] \otimes [\xi(m_-, m_3, m_4)]) = [\rho(\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4))]\frac{C}{D_+ D_-},$$

where C is the order of the automorphism group of a generic point in $\rho(\xi(m_1, m_2, m_+) \times \xi(m_-, m_3, m_4))$, D_+ is the order of the automorphism group of $\xi(m_1, m_2, m_+)$, and D_- is the order of the automorphism group of $\xi(m_-, m_3, m_4)$. This equation, together with its counterpart from $\rho_*([\xi(m_1, m_2, m_+)] \otimes [\xi(m_-, m_3, m_4)])$ and the previously derived equation, yields the desired result. □

2.5.3. *Distinguished components of* $\overline{\mathcal{M}}^G_{1,1}$.

Definition 2.30. Choose elements $a, b, m_1 \in G$ such that $m_1 = [a, b]$. Let ρ_b be the composition of the morphisms

$$\xi(m_1, b, ab^{-1}a^{-1}) \xrightarrow{\rho_3(a)} \overline{\mathcal{M}}^G_{0,3}(m_1, b, b^{-1}) \xrightarrow{\rho'_b} \overline{\mathcal{M}}^G_{1,1}(m_1), \tag{18}$$

where the first morphism is right action by a in the third factor, and the second morphism in the gluing morphism identifying the 2nd and 3rd marked points.

Similarly, let ρ_a be the composition of the morphisms

$$\xi(m_1, bab^{-1}, a^{-1}) \xrightarrow{\rho_2(b)} \overline{\mathcal{M}}^G_{0,3}(m_1, a, a^{-1}) \xrightarrow{\rho'_a} \overline{\mathcal{M}}^G_{1,1}(m_1), \tag{19}$$

where the first morphism is right action by b in the second factor, and the second is again the gluing morphism identifying the 2nd and 3rd marked points.

We define $\xi_{1,1}(m_1, a, b)$ to be the component of $\overline{\mathcal{M}}^G_{1,1}(m_1)$ containing the image of ρ_b.

Lemma 2.31. *The images of ρ_a and ρ_b lie in the same connected component $\xi_{1,1}(m_1, a, b)$ of $\overline{\mathcal{M}}^G_{1,1}(m_1)$. Moreover, the following equation holds in $H_2(\overline{\mathcal{M}}^G_{1,1}(m_1))$:*

$$\rho'_{b*}([\![\rho_3(a)\xi(m_1, b, ab^{-1}a^{-1})]\!]) = \rho'_{a*}([\![\rho_2(b)\xi(m_1, bab^{-1}, a^{-1})]\!]). \tag{20}$$

Proof. The images of ρ_a and ρ_b are degenerations of the same smooth admissible G-cover over a smooth torus. In particular, consider a smooth, one-pointed torus (T, p_1) with generators α, β, and γ of $\pi_1(T, p_0)$ for some point p_0, with γ corresponding to the loop σ_d induced by a path d from p_0 to p_1 (as in Definition 2.13), and $[\alpha, \beta] = \gamma$. The homomorphism $\chi : \pi_1(T, p_0) \longrightarrow G$ that takes α, β , and γ to a, b, and m_1, respectively, defines an admissible G-cover E_χ, and a point $\widetilde{p}_{0,\chi}$. Parallel transport along d induces a point $\widetilde{p}_1$ with monodromy m_1, giving us a pointed admissible G-cover $E_\chi, \widetilde{p}_1$.

It is straightforward to see that the image of ρ_a corresponds to the α-cycle shrinking to become a node, while the image of ρ_b corresponds to the β-cycle shrinking to become a node. Thus both images lie in the same connected component $\xi_{1,1}(m_1)$ of $\overline{\mathcal{M}}^G_{1,1}(m_1)$.

Equation (20) follows from the identity

$$\rho'_{b*}([\rho_3(a)\xi(m_1, b, ab^{-1}a^{-1})])\frac{A}{B} = \rho'_{a*}([\rho_2(b)\xi(m_1, bab^{-1}, a^{-1})])\frac{A'}{B'},$$

where A is the order of the automorphism group of $\rho_3(a)\xi(m_1, b, ab^{-1}a^{-1})$, A' is the order of the automorphism group of $[\rho_2(b)\xi(m_1, bab^{-1}, a^{-1})]$, B is the order of the automorphism group of $\rho'_b(\rho_3(a)\xi(m_1, b, ab^{-1}a^{-1}))$, and B' is the order of the automorphism group of $\rho'_a(\rho_2(b)\xi(m_1, bab^{-1}, a^{-1}))$.

However, $B = B'$, as their corresponding automorphism groups are both isomorphic to $C(a, b) \subseteq G$ (see Proposition 2.21). □

3. The category of G-graded G-modules

In this section, we briefly review some well-known facts from the category of G-graded G-modules (see [Kas, BaKi]) which will be useful in the sequel.

3.1. G-graded G-modules and their G-coinvariants.

Definition 3.1. Let $\mathcal{H} := \bigoplus_{m\in G} \mathcal{H}_m$ be a finite-dimensional G_A-graded vector space which is endowed with the structure of a right G-module $\rho(\gamma) : \mathcal{H} \xrightarrow{\sim} \mathcal{H}$ for all γ in G, with $\rho(\gamma)$ taking $\mathcal{H}_m$ to $\mathcal{H}_{\gamma^{-1}m\gamma}$ for all m in G. $(\mathcal{H}, \rho)$ *is said to be a* G*-graded* G*-module.*

A G*-invariant metric* η *on a* G*-graded* G*-module* $\mathcal{H}$ is a symmetric, nondegenerate, bilinear form η on $\mathcal{H}$ which is G-invariant (under the diagonal G action) and which respects the grading, i.e., for all v_{m_+} in $\mathcal{H}_{m_+}$ and v_{m_-} in $\mathcal{H}_{m_-}$ we have $\eta(v_{m_+}, v_{m_-}) = 0$ unless $m_+m_- = 1$.

G-graded G-modules form a category whose objects are G-graded G-modules and whose morphisms are homomorphisms of G-modules which respect the G-grading. Furthermore, the dual of a G-graded G-module inherits the structure of a G-graded G-module.

Example 3.2. Any finite-dimensional G-module V is a G-graded G-module where $\mathcal{H}_1 := V$ and $\mathcal{H}_m := 0$ for all m not equal to 1 in G.

Example 3.3. The simplest example of a nontrivial G-graded G-module is $\mathbb{C}[G]$, the free vector space generated by G, with its natural G-grading, endowed with the G-action $\rho(\gamma)m := \gamma^{-1}m\gamma$ for all γ, m in G.

Definition 3.4. Recall that $\overline{G}$ is the set of conjugacy classes of G, the conjugacy class of m in G is denoted by $\overline{m}$, and the conjugacy class of m^{-1} is denoted by $\overline{m}^{-1}$.

A section s of the natural map $G \longrightarrow \overline{G}$ is said to be *involutive* if $s(\overline{m}^{-1}) = s(\overline{m})^{-1}$ for all $\overline{m}$.

Definition 3.5. Let $(\mathcal{H}, \rho)$ be a G-graded G-module. Let $\pi_G : \mathcal{H} \to \mathcal{H}$ be the averaging map

$$\pi_G(v) := \frac{1}{|G|}\sum_{\gamma\in G}\rho(\gamma)v$$

for all v in $\mathcal{H}$. Let $\overline{\mathcal{H}}$ be the image of π_G. The vector space $\overline{\mathcal{H}}$ is called the *space of* G*-coinvariants of* $\mathcal{H}$, and it inherits a grading by $\overline{G}$, denoted by

$$\overline{\mathcal{H}} = \bigoplus_{\overline{\gamma}\in\overline{G}}\overline{\mathcal{H}}_{\overline{\gamma}}.$$

If η is a metric on $\mathcal{H}$, then let $\overline{\eta}$ be the restriction of the metric $\frac{1}{|G|}\eta$ to $\overline{\mathcal{H}}$.

Remark 3.6. The reason for the factor of $\frac{1}{|G|}$ in the definition of $\overline{\eta}$ will become evident when we discuss the geometry of G-CohFTs.

Let us describe $\overline{\mathcal{H}}$ in terms of $\mathcal{H}$.

Proposition 3.7. *Let $(\mathcal{H}, \rho)$ be a G-graded G-module with a G-invariant metric* η.

(1) *Consider $v_{\overline{m}}$ in $\overline{\mathcal{H}}_{\overline{m}}$, where $v_{\overline{m}} = \sum_{m'\in\overline{m}} v_{m'}$. For all m' in $\overline{m}$, $v_{m'}$ belongs to $\mathcal{H}_{m'}^{C(m')}$, the $C(m')$-invariant subspace of $\mathcal{H}_{m'}$. In particular, for all v_m in $\mathcal{H}_m$,*

$$\pi_G(v_m) = \pi_G(\pi_{C(m)}(v_m)),$$

where $\pi_{C(m)} : \mathcal{H}_m \longrightarrow \mathcal{H}_m^{C(m)}$ is the averaging map

$$\pi_{C(m)}(v_m) := \frac{1}{|C(m)|}\sum_{\gamma\in C(m)}\rho(\gamma)v_m.$$

(2) *For all m in G, the map $\pi_m : \mathcal{H}_m^{C(m)} \longrightarrow \overline{\mathcal{H}}_{\overline{m}}$, defined as*

$$\pi_m(v_m) := \pi_G(v_m),$$

is an isomorphism of vector spaces.

(3) *For all $m_\pm$ in G and $v_{m_\pm}$ in $\mathcal{H}_{m_\pm}^{C(m_\pm)}$, where $m_+m_- = 1$, we have*

$$\eta(\pi_{m_+}(v_{m_+}), \pi_{m_-}(v_{m_-})) = \eta(\pi_G(v_{m_+}), \pi_G(v_{m_-})) = \frac{|C(m_+)|}{|G|}\eta(v_{m_+}, v_{m_-}).$$

(4) *If s is an involutive section of the natural map $G \longrightarrow \overline{G}$, then*

$$\bigoplus_{\overline{m}\in\overline{G}}\mathcal{H}_{s(\overline{m})}^{C(s(\overline{m}))} \longrightarrow \overline{\mathcal{H}},$$

taking $v_{s(\overline{m})} \mapsto \pi_G(v_{s(\overline{m})})$*, is an isomorphism of vector spaces which is not an isometry.*

(5) $\overline{\eta}$ *is nondegenerate, i.e.,* $\overline{\mathcal{H}}$ *is a* $\overline{G}$*-graded vector space with metric* $\overline{\eta}$*.*

Proof. To prove part (1), consider w_m in $\mathcal{H}_m$. We have

$$\begin{aligned}
\pi_G(w_m) &= \frac{1}{|G|}\sum_{\gamma'\in G}\rho(\gamma')w_m \\
&= \frac{1}{|G|}\sum_{[\gamma]\in(C(m)\backslash G)}\sum_{c\in C(m)}\rho(c\gamma)w_m \\
&= \frac{|C(m)|}{|G|}\sum_{[\gamma]\in C(m)\backslash G}\rho(\gamma)\frac{1}{|C(m)|}\sum_{c\in C(m)}\rho(c)w_m \\
&= \frac{|C(m)|}{|G|}\sum_{[\gamma]\in C(m)\backslash G}\rho(\gamma)\pi_{C(m)}(w_m).
\end{aligned}$$

We conclude that

$$\pi_G(w_m) = \frac{|C(m)|}{|G|}\sum_{[\gamma]\in C(m)\backslash G}\pi_{C(\gamma^{-1}m\gamma)}(\rho(\gamma)w_m), \tag{21}$$

which finishes the proof.

We prove part (2) by showing that the map $f_m : \overline{\mathcal{H}}_{\overline{m}} \longrightarrow \mathcal{H}_m^{C(m)}$, defined by

$$f_m(\sum_{m'\in\overline{m}} v_{m'}) := \frac{|G|}{|C(m)|}v_m, \tag{22}$$

is the inverse of π_m. Notice that the right hand side is $C(m)$-invariant by part (1). Consider w_m in $\mathcal{H}_m^{C(m)}$. We have

$$f_m(\pi_G(w_m)) = f_m\left(\frac{|C(m)|}{|G|}\sum_{[\gamma]\in C(m)\backslash G}\rho(\gamma)w_m\right)$$

$$= \frac{|C(m)|}{|G|}f_m(w_m) = \frac{|C(m)|}{|G|}\frac{|G|}{|C(m)|}w_m = w_m.$$

Therefore, π_m is an isomorphism.

To prove part (3), observe that

$$\begin{aligned}
\eta(\pi_G(v_{m_+}),\pi_G(v_{m_-})) &= \frac{1}{|G|^2}\sum_{\gamma_\pm\in G}\eta(\rho(\gamma_+)v_{m_+},\rho(\gamma_-)v_{m_-})\\
&= \frac{1}{|G|^2}\sum_{\gamma_\pm\in G}\eta(\rho(\gamma_-^{-1})\rho(\gamma_+)v_{m_+},v_{m_-})\\
&= \frac{1}{|G|^2}\sum_{\gamma_\pm\in G}\eta(\rho(\gamma_+\gamma_-^{-1})v_{m_+},v_{m_-})\\
&= \frac{1}{|G|^2}\sum_{\gamma\in G}\sum_{\gamma_+\in G}\eta(\rho(\gamma)v_{m_+},v_{m_-})\\
&= \frac{1}{|G|}\sum_{\gamma\in G}\eta(\rho(\gamma)v_{m_+},v_{m_-})\\
&= \frac{1}{|G|}\sum_{\gamma\in C(m_+)}\eta(\rho(\gamma)v_{m_+},v_{m_-})\\
&= \frac{|C(m_+)|}{|G|}\eta(v_{m_+},v_{m_-}),
\end{aligned}$$

where we used the $C(m_+)$-invariance of v_{m_+} in the last equality.

Part (4) follows immediately from (2) and (3). Involutivity of s is needed to insure that $\bigoplus_{\overline{m}\in\overline{G}}\mathcal{H}_{s(\overline{m})}^{C(s(\overline{m}))}$ inherits a metric from $\mathcal{H}$ compatible with its grading.

To prove part (5), let $m_+m_- = 1$, so that η restricted to $\mathcal{H}_{m_+}\oplus\mathcal{H}_{m_-}$ is nondegenerate. $\mathcal{H}_{m_\pm}$ is a $C(m_\pm)$-module, so one can write

$$\mathcal{H}_{m_\pm} = \mathcal{H}_{m_\pm}^{C(m_\pm)}\oplus\mathcal{H}'_{m_\pm} \tag{23}$$

as $C(m_\pm)$-modules, where $\mathcal{H}'_{m_\pm}$ is the direct sum of all nontrivial irreducible representations of $C(m_\pm)$ appearing in $\mathcal{H}_{m_\pm}$. Notice that since $m_+m_- = 1$, we have $C(m_+) = C(m_-)$.

Since η is $C(m_+)$-invariant, η restricted to $\mathcal{H}_{m_+}\oplus\mathcal{H}_{m_-}$ is the direct sum of η restricted to $\mathcal{H}_{m_+}^{C(m_+)}\oplus\mathcal{H}_{m_-}^{C(m_+)}$ and η restricted to $\mathcal{H}'_{m_+}\oplus\mathcal{H}'_{m_-}$. Therefore, η restricted to $\mathcal{H}_{m_+}^{C(m_+)}\oplus\mathcal{H}_{m_-}^{C(m_-)}$ is nondegenerate.

Let v_{m_+} be in $\mathcal{H}_{m_+}^{C(m_+)}$. Suppose that $\eta(\pi_G(v_{m_+}),\pi_G(v_{m_-})) = 0$ for all v_{m_-} in $\mathcal{H}_{m_-}^{C(m_-)}$. By part (3), this is equivalent to the condition $\eta(v_{m_+},v_{m_-}) = 0$ for all v_{m_-} in $\mathcal{H}_{m_-}^{C(m_-)}$. However, η restricted to $\mathcal{H}_{m_+}^{C(m_+)}\oplus\mathcal{H}_{m_-}^{C(m_-)}$ is nondegenerate, therefore, $v_{m_+} = 0$. Thus η restricted to $\overline{\mathcal{H}}$ is also non-degenerate. □

3.2. **Tensor products and the braid group.**

As is usual in the representation theory of groups, there are two kinds of tensor products associated to G-graded G-modules,

Definition 3.8. Let $\mathcal{H}'$ be a G'-graded G'-module and $\mathcal{H}''$ be a G''-graded G''-module. Their vector space tensor product $\mathcal{H}'\otimes\mathcal{H}''$ is naturally a $G'\times G''$-graded $G'\times G''$-module called the *external tensor product of $\mathcal{H}'$ and $\mathcal{H}''$.*

On the other hand, the category of G-graded G-modules has a natural tensor product which differs from the tensor product of their underlying vector spaces.

Definition 3.9. Let $\mathcal{H}' := \bigoplus_{m\in G} \mathcal{H}'_m$ and $\mathcal{H}'' := \bigoplus_{m\in G} \mathcal{H}''_m$ be two G-graded G-modules. Let
$$\mathcal{H}' \odot \mathcal{H}'' := \bigoplus_{m\in G} \mathcal{H}'_m \otimes \mathcal{H}''_m,$$
with the induced G-module structure, where G acts diagonally. We call $\mathcal{H}' \odot \mathcal{H}''$ the *tensor product of* $\mathcal{H}'$ *and* $\mathcal{H}''$.

Remark 3.10. The G-graded G-module $\mathbb{C}[G]$ has the important property that
$$\mathcal{H} \odot \mathbb{C}[G] \cong \mathbb{C}[G] \odot \mathcal{H} \cong \mathcal{H}$$
for any G-graded G-module $\mathcal{H}$.

Finally, we note that this category has a natural action of the braid group, which we now describe.

Definition 3.11. Let $\mathcal{H}$ be a G-graded G-module. Its n-fold tensor product $\mathcal{H}^{\otimes n}$ inherits the structure of a right G^n-module. Furthermore, the permutation group S_n acts on $\mathcal{H}^{\otimes n}$ by permuting its factors.

For all $i = 1, \dots, n-1$, let $b_i : \mathcal{H}^{\otimes n} \longrightarrow \mathcal{H}^{\otimes n}$ be defined by
$$b_i(v_{m_1} \otimes \cdots \otimes v_{m_i} \otimes v_{m_{i+1}} \cdots \otimes v_{m_n}) := v_{m_1} \otimes \cdots \otimes (\rho(m_i^{-1}) v_{m_{i+1}}) \otimes v_{m_i} \otimes \cdots \otimes v_{m_n}$$
for all v_{m_j} in $\mathcal{H}_{m_j}$, m_j in G, and $j = 1, \dots, n-1$.

The following proposition is immediate.

Proposition 3.12. *The elements* $\{b_1, \dots, b_n\}$ *on* $\mathcal{H}^{\otimes n}$ *give an action of* B_n *on* $\mathcal{H}^{\otimes n}$.

4. G-equivariant cohomological field theories

In this section, we introduce the notion of a G-CohFT , defined in terms of $\overline{\mathcal{M}}^G_{g,n}$, and prove some of its basic properties.

4.1. G-CohFTs and G-Frobenius algebras.

Definition 4.1. A tuple $((\mathcal{H}, \rho), \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ is said to be a G*-equivariant Cohomological Field Theory (*G*-CohFT)* if the following axioms hold:

(1) (G-Graded G-module) $(\mathcal{H}, \rho)$ is a G-graded G-module. The subspace $\mathcal{H}_1$ is called the *untwisted sector of the* G*-CohFT*, and $\mathcal{H}_m$, where $m \neq 1$, is called a *twisted sector of the* G*-CohFT.*

(2) (G^n and S_n Invariance) For all $\mathbf{m} := (m_1, \dots, m_n)$ in G^n and all stable pairs (g, n), if we denote $\mathcal{H}_{\mathbf{m}} := \bigotimes_{i=1}^n \mathcal{H}_{m_i}$, then $\Lambda_{g,n}$ is an element of $\bigoplus_{\mathbf{m}} H^\bullet(\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})) \otimes \mathcal{H}^*_{\mathbf{m}}$ which is both S_n-invariant and G^n-invariant.

(3) (Identity) The element $\mathbf{1}$ in $\mathcal{H}_1$ is non-zero, and is called the *flat identity* or *vacuum vector.*

   (a) (G-Invariance of the Identity) The vacuum vector $\mathbf{1}$ is G invariant, i.e., $\rho(\gamma)\mathbf{1} = \mathbf{1}$ for all γ in G.

   (b) (Flat Identity) Under the forgetting tails morphism $\widetilde{\tau} : \overline{\mathcal{M}}^G_{g,n+1}(\mathbf{m}, 1) \longrightarrow \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$, we have
$$\Lambda_{g,n+1}(v_{m_1}, \dots, v_{m_n}, \mathbf{1}) = \widetilde{\tau}^* \Lambda_{g,n}(v_{m_1}, \dots, v_{m_n})$$
   for all $\mathbf{m}$ in G^n, and v_{m_i} in $\mathcal{H}_{m_i}$ for all $i = 1, \dots, n$.

(4) (Metric) η is a symmetric, nondegenerate, bilinear form on $\mathcal{H}$ such that

$$\eta(v_{m_1}, v_{m_2}) := \int_{[\![\xi(m_1,m_2,1)]\!]} \Lambda_{0,3}(v_{m_1}, v_{m_2}, \mathbf{1}).$$

It follows that $\eta(v_{m_1}, v_{m_2}) = 0$ unless $m_1 m_2 = 1$. Recall that ξ is defined in Subsection 2.5.1 and the scaled class $[\![Q]\!]$ in Definition 2.28.

(5) (Factorization) Fix any $m_+ \in G$ and $m_- := (m_+)^{-1}$. Let the set $\{e_\alpha\}$ be a basis for $\mathcal{H}_{m_+}$, the set $\{\hat{e}_\beta\}$ be a basis for $\mathcal{H}_{m_-}$, and $\eta^{\alpha\beta}$ be the inverse of the metric

$$\eta : \mathcal{H}_{m_+} \otimes \mathcal{H}_{m_-} \longrightarrow \mathbb{C}$$

relative to these bases.

(a) For all stable pairs $(g_1, n_1 + 1)$ and $(g_2, n_2 + 1)$ let $g = g_1 + g_2$ and $n = n_1 + n_2$. For all $\mathbf{m}$ in G^n and all $(v_{m_1} \dots, v_{m_n}) \in \mathcal{H}_{\mathbf{m}}$ we require

$$(\rho^*_{tree}\Lambda_{g,n})(v_{m_1}, \dots, v_{m_n}) = \sum_{\alpha,\beta} \Lambda_{g_1,n_1+1}(v_{m_{i_1}}, \dots, v_{m_{i_{n_1}}}, e_\alpha)\eta^{\alpha\beta}\Lambda_{g_2,n_2+1}(\hat{e}_\beta, v_{m_{j_1}}, \dots, v_{m_{j_{n_2}}})$$

for all partitions $\{i_1, \dots, i_{n_1}\} \sqcup \{j_1, \dots, j_{n_2}\}$ of the set $\{1, \dots, n\}$.

(b) For all stable pairs $(g-1, n+2)$, all $\mathbf{m} \in G^n$, and all $(v_{m_1}, \dots, v_{m_n}) \in \mathcal{H}_{\mathbf{m}}$, the classes Λ must satisfy

$$(\rho^*_{loop}\Lambda_{g,n})(v_{m_1}, \dots, v_{m_n}) = \sum_{\alpha,\beta} \Lambda_{g-1,n+2}(v_{m_1}, \dots, v_{m_n}, e_\alpha, \hat{e}_\beta)\eta^{\alpha\beta}.$$

Remark 4.2. If G is the trivial group, then a G-CohFT coincides with a CohFT in the sense of Kontsevich-Manin [KoMa].

Example 4.3. The simplest example of a G-CohFT has as its state space $\mathcal{H} = \bigoplus_{m\in G} \mathcal{H}_m := H^\bullet(G) = H^0(G) \cong \mathbb{C}[G]$ as G-graded G-modules, i.e., if $\{e_m\}_{m\in G}$ denotes the obvious basis in $\mathcal{H}$, then the G-action $\rho(\gamma) : \mathcal{H}_m \to \mathcal{H}_{\gamma^{-1}m\gamma}$ is $\rho(\gamma)(e_m) := e_{\gamma^{-1}m\gamma}$ for all γ, m in G.

For all $\mathbf{m} = (m_1, \dots, m_n)$ in G^n, let

$$\Lambda_{g,n}(e_{m_1}, \dots, e_{m_n}) := \mathbf{e}^*\mathbf{1}_{\mathbf{m}},$$

where $\mathbf{e} : \overline{\mathcal{M}}^G_{g,n} \longrightarrow G^n$, and $\mathbf{1}_{\mathbf{m}}$ in $H^0(G^n)$ denotes the fundamental class of the point $\mathbf{m}$ in G^n.

It follows that

$$\eta(e_{m_1}, e_{m_2}) := \int_{[\![\xi(m_1,m_2,1)]\!]} \Lambda_{0,3}(e_{m_1}, e_{m_2}, \mathbf{1}) = \delta_{m_1, m_2^{-1}}.$$

Definition 4.4. We will call the G-CohFT of the previous example the *group algebra* G-*CohFT*, and we will denote it simply by $\mathbb{C}[G]$ whenever it is clear from context that we mean the group algebra G-CohFT and not just the algebra itself.

Remark 4.5. We will see in the next section that this G-CohFT induces the G-Frobenius algebra $\mathbb{C}[G]$, and a standard argument (along the lines of [Tu]) shows that the two constructions are actually equivalent, thus we are justified in the terminology and notation of the previous definition.

4.2. **Tensor products of equivariant CohFTs.**

Given two equivariant CohFTs, one can construct a new one by taking their tensor product. As in the case of G-graded G-modules, there are two tensor products associated to G-CohFTs. The first, the external tensor product, associates to a G-CohFT and a G'-CohFT a $\mathsf{G}\times\mathsf{G}'$-CohFT. The second is a tensor product in the category of G-CohFTs.

Proposition 4.6. *For all* $\mathbf{m}'$ *in* ${\mathsf{G}'}^n$ *and* $\mathbf{m}''$ *in* ${\mathsf{G}''}^n$, *let* $\mathbf{m}'\times\mathbf{m}''$ *denote the element* $((m_1', m_2''), \dots, (m_n', m_n''))$ *in* $(\mathsf{G}'\times\mathsf{G}'')^n$. *Consider the commuting diagram*

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathsf{G}'\times\mathsf{G}''}_{g,n}(\mathbf{m}'\times\mathbf{m}'') \arrow[r, "\Upsilon"] & \overline{\mathcal{M}}^{\mathsf{G}'}_{g,n}(\mathbf{m}') \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{\mathsf{G}''}_{g,n}(\mathbf{m}'') \arrow[r, "\mathrm{pr}''"] \arrow[d, "\mathrm{pr}'"'] & \overline{\mathcal{M}}^{\mathsf{G}''}_{g,n}(\mathbf{m}'') \arrow[d, "\mathrm{st}''"'] \\
& \overline{\mathcal{M}}^{\mathsf{G}'}_{g,n}(\mathbf{m}') \arrow[r, "\mathrm{st}'"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

where st' *and* st'' *forget the pointed admissible covers and* $\overline{\mathcal{M}}^{\mathsf{G}'}_{g,n}(\mathbf{m}')\times_{\overline{\mathcal{M}}_{g,n}}\overline{\mathcal{M}}^{\mathsf{G}''}_{g,n}(\mathbf{m}'')$ *is the fibered product with projections* pr' *and* pr''. *The map* Υ *takes an object* $(E \longrightarrow C; \widetilde{p}_1, \dots, \widetilde{p}_n)$ *to* $((E' \longrightarrow C; \widetilde{p}_1', \dots, \widetilde{p}_n'), (E'' \longrightarrow C; \widetilde{p}_1'', \dots, \widetilde{p}_n''))$, *where* E' *is the variety* E/G'' *and* $\widetilde{p}_i'$ *is the marked point on* E' *induced by* $\widetilde{p}_i$, E'' *is the variety* E/G' *and* $\widetilde{p}_i''$ *is the marked point on* E'' *induced by* $\widetilde{p}_i$.

1. Υ *is an isomorphism, preserving the* $(\mathsf{G}'\times\mathsf{G}'')^n$ *and* S_n *actions.*
2. *The morphism* pr' *is* ${\mathsf{G}'}^n$*-equivariant and* pr'' *is* ${\mathsf{G}''}^n$*-equivariant.*
3. *The morphisms* $\mathrm{pr}', \mathrm{pr}'', \mathrm{st}', \mathrm{st}''$ *are* S_n*-equivariant.*
4. *The morphisms* $\Upsilon, \mathrm{pr}', \mathrm{pr}'', \mathrm{st}', \mathrm{st}''$ *commute with the gluing morphisms.*

Proof. Part (1) follows from the fact that both $E' \longrightarrow C$ (resp. $E'' \longrightarrow C$) are admissible G'-covers (resp. G''-covers) with the proper monodromies. Conversely, given a pointed admissible G'-cover $(E' \longrightarrow C; \widetilde{p}_1', \dots, \widetilde{p}_n')$ and a pointed admissible G''-cover $(E'' \longrightarrow C; \widetilde{p}_1'', \dots, \widetilde{p}_n'')$, it is clear that $(E'\times_C E'' \longrightarrow C; (\widetilde{p}_1', \widetilde{p}_1''), \dots, (\widetilde{p}_n', \widetilde{p}_n''))$ is a pointed admissible $\mathsf{G}'\times\mathsf{G}''$-cover, and thus Υ is an isomorphism. The equivariance under the actions of $(\mathsf{G}'\times\mathsf{G}'')^n$ and S_n is manifest.

Similarly, Parts (2) and (3) are manifest.

We now treat part (4) in the case of the loop for the morphism pr'. For all $m_\pm'$ in G' and $m_\pm''$ in G'' such that $m_+'m_-' = m_+''m_-'' = 1$, consider the diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\mathsf{G}'\times\mathsf{G}''}_{g-1,n+2}(\mathbf{m}'\times\mathbf{m}'', (m_+', m_+''), (m_-', m_-'')) \arrow[r, "\widetilde{\rho}"] \arrow[d, "\mathrm{pr}'_{\mathrm{cut}}"'] & \overline{\mathcal{M}}^{\mathsf{G}'\times\mathsf{G}''}_{g,n}(\mathbf{m}'\times\mathbf{m}'') \arrow[d, "\mathrm{pr}'"'] \\
\overline{\mathcal{M}}^{\mathsf{G}'}_{g-1,n+2}(\mathbf{m}', m_+', m_-') \arrow[r, "\widetilde{\rho}'"] & \overline{\mathcal{M}}^{\mathsf{G}'}_{g,n}(\mathbf{m}')
\end{tikzcd} \tag{24}$$

where $\widetilde{\rho}$ and $\widetilde{\rho}'$ are the gluing morphisms and $\mathrm{pr}'_{\mathrm{cut}}$ and pr' are the canonical projections. Part (4) states that this diagram commutes, which follows immediately from the definition of the morphisms involved. Similarly, the analogous diagrams for Υ, pr'', st', and st'' also commute. The proof in the case of the tree is identical and will be omitted. ∎

Corollary 4.7. *Let* $(\mathcal{H}', \eta', \{\Lambda'_{g,n}\}, \mathbf{1}')$ *be a* G'*-CohFT and* $(\mathcal{H}'', \eta'', \{\Lambda''_{g,n}\}, \mathbf{1}'')$ *be a* G''*-CohFT. If we define*

(25)

$$\Lambda_{g,n}(v'_{m'_1} \otimes v''_{m''_1}, \ldots, v'_{m'_n} \otimes v''_{m''_n}) := \Upsilon^*((\mathrm{pr}'^* \Lambda'_{g,n}(v'_{m'_1}, \ldots, v'_{m'_n})) \cup (\mathrm{pr}''^* \Lambda''_{g,n}(v''_{m''_1}, \ldots, v''_{m''_n})))$$

for all $v'_{m'_i}$ *in* $\mathcal{H}'_{m'_i}$ *and* $v''_{m''_i}$ *in* $\mathcal{H}''_{m''_i}$*, where the morphisms* pr' *and* pr'' *are defined as in Proposition 4.6, then* $(\mathcal{H}' \otimes \mathcal{H}'', \eta' \otimes \eta'', \{\Lambda_{g,n}\}, \mathbf{1}' \otimes \mathbf{1}'')$ *is a* $G' \times G''$*-CohFT.*

Proof. Let $G := G' \times G''$. Using the tensor product of a G'-graded G'-module and G''-graded G''-module, $\mathcal{H}' \otimes \mathcal{H}''$ inherits the structure of a G-graded G-module. The G-invariance of $\mathbf{1}' \otimes \mathbf{1}''$ follows.

The G^n- and S_n-invariance follow from Proposition 4.6(2) and (3), respectively.

The flatness of the identity follows immediately from the definition of $\Lambda_{g,n}$.

The metric axiom follows from observation that since $\overline{\mathcal{M}}_{0,3}$ is a point, the fibered product $\overline{\mathcal{M}}^{G'}_{0,3}(\mathbf{m}') \times_{\overline{\mathcal{M}}_{0,3}} \overline{\mathcal{M}}^{G''}_{0,3}(\mathbf{m}'')$ is equal to $\overline{\mathcal{M}}^{G'}_{0,3}(\mathbf{m}') \times \overline{\mathcal{M}}^{G''}_{0,3}(\mathbf{m}'')$.

We prove the factorization axiom in the case of the loop—the case of the tree is similar. Let us adopt the notation from Proposition 4.6 and define $v'_{\mathbf{m}'} \times v''_{\mathbf{m}''}$ to be $(v'_{m'_1} \otimes v''_{m''_1}, \ldots, v'_{m'_n} \otimes v''_{m''_n})$ for all $v'_{\mathbf{m}'}$ in $\mathcal{H}'_{\mathbf{m}'}$ and $v''_{\mathbf{m}''}$ in $\mathcal{H}''_{\mathbf{m}''}$.

¿From the definition of Λ we have

$$\widetilde{\rho}^* \Lambda_{g,n}(v'_{\mathbf{m}'} \times v''_{\mathbf{m}''}) = ((\mathrm{pr}' \times \mathrm{pr}'') \circ \widetilde{\Delta} \circ \Upsilon \circ \widetilde{\rho})^* (\Lambda'_{g,n}(v'_{\mathbf{m}'}) \otimes \Lambda''_{g,n}(v''_{\mathbf{m}''})),$$

where $\widetilde{\Delta}$ is the diagonal morphism

$$\widetilde{\Delta} : \overline{\mathcal{M}}^{G'}_{g,n}(\mathbf{m}') \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{G''}_{g,n}(\mathbf{m}'') \longrightarrow \overline{\mathcal{M}}^{G'}_{g,n}(\mathbf{m}') \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{G''}_{g,n}(\mathbf{m}'') \times \overline{\mathcal{M}}^{G'}_{g,n}(\mathbf{m}') \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{G''}_{g,n}(\mathbf{m}'').$$

Let Δ denote the diagonal morphism

$$\Delta : \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m} \times \mathbf{m}') \longrightarrow \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m} \times \mathbf{m}') \times \overline{\mathcal{M}}^{G}_{g,n}(\mathbf{m} \times \mathbf{m}')$$

and Δ_{cut} denote the diagonal morphism associated to $\overline{\mathcal{M}}^{G}_{g-1,n+2}(\mathbf{m}' \times \mathbf{m}'', (m'_+, m''_+), (m'_-, m''_-))$ for any $m'_\pm$ in G' and $m''_\pm$ in G'', such that $m'_+ m'_- = m''_+ m''_- = 1$. We have

$$\begin{aligned}
(\mathrm{pr}' \times \mathrm{pr}'') \circ \widetilde{\Delta} \circ \Upsilon \circ \widetilde{\rho} &= (\mathrm{pr}' \times \mathrm{pr}'') \circ (\Upsilon \times \Upsilon) \circ \widetilde{\Delta} \circ \widetilde{\rho} \\
&= (\mathrm{pr}' \times \mathrm{pr}'') \circ (\Upsilon \times \Upsilon) \circ (\widetilde{\rho} \times \widetilde{\rho}) \circ \Delta_{\mathrm{cut}} \\
&= (\widetilde{\rho}' \times \widetilde{\rho}'') \circ (\mathrm{pr}' \times \mathrm{pr}'') \circ (\Upsilon \times \Upsilon) \circ \Delta_{\mathrm{cut}},
\end{aligned}$$

where the first equality follows from the identity $(\Upsilon \times \Upsilon) \circ \Delta = \widetilde{\Delta} \circ \Upsilon$, the second from the identity $(\widetilde{\rho} \times \widetilde{\rho}) \circ \Delta_{\mathrm{cut}} = \Delta \circ \widetilde{\rho}$, and the third from Proposition 4.6(4). Putting these together, we obtain

$$\begin{aligned}
\widetilde{\rho}^* \Lambda_{g,n}(v'_{\mathbf{m}'} \times v''_{\mathbf{m}''}) &= ((\mathrm{pr}' \circ \Upsilon)^* \rho'^* \Lambda'_{g,n}(v'_{\mathbf{m}'})) \cup ((\mathrm{pr}'' \circ \Upsilon)^* \rho''^* \Lambda''_{g,n}(v''_{\mathbf{m}''})) \\
&= ((\mathrm{pr}' \times \mathrm{pr}'') \circ (\Upsilon \times \Upsilon) \circ \Delta)^* (\rho'^* \Lambda'_{g,n}(v'_{\mathbf{m}'})) \otimes \rho''^* \Lambda''_{g,n}(v''_{\mathbf{m}''}))) \\
&= ((\mathrm{pr}' \times \mathrm{pr}'') \circ \widetilde{\Delta} \circ \Upsilon)^* (\rho'^* \Lambda'_{g,n}(v'_{\mathbf{m}'})) \otimes \rho''^* \Lambda''_{g,n}(v''_{\mathbf{m}''}))) \\
&= \Upsilon^* ((\mathrm{pr}'^* \rho'^* (\Lambda'_{g,n}(v'_{\mathbf{m}'}))) \cup (\mathrm{pr}''^* \rho''^* (\Lambda''_{g,n}(v''_{\mathbf{m}''})))) \\
&= \Upsilon^* ((\mathrm{pr}'^* \Lambda'_{g-1,n+2}(v'_{\mathbf{m}'}, e'_{\alpha[m'_+]}, e'_{\alpha[m'_-]})) \cup \\
&\qquad (\mathrm{pr}''^* \Lambda''_{g-1,n+2}(v''_{\mathbf{m}''}, e''_{\beta[m''_+]}, e''_{\beta[m''_-]}))) \eta'^{\alpha[m'_+]\alpha[m'_-]} \eta''^{\beta[m''_+]\beta[m''_-]}
\end{aligned}$$

as desired, where $\{e'_{\alpha[m'_\pm]}\}$ is a basis for $\mathcal{H}'_{m'_\pm}$ and $\{e''_{\beta[m''_\pm]}\}$ is a basis for $\mathcal{H}''_{m''_\pm}$.

This completes the case of the loop. The case of the tree is identical and will be omitted. □

Definition 4.8. Let $\mathfrak{G}' = (\mathcal{H}', \eta', \{\Lambda'_{g,n}\}, \mathbf{1}')$ be a G'-CohFT and $\mathfrak{G}'' := (\mathcal{H}'', \eta'', \{\Lambda''_{g,n}\}, \mathbf{1}'')$ be a G''-CohFT. Their *external tensor product* $\mathfrak{G}' \otimes \mathfrak{G}''$ is the $G' \times G''$-CohFT $(\mathcal{H}' \otimes \mathcal{H}'', \eta' \otimes \eta'', \{\Lambda_{g,n}\}, \mathbf{1}' \otimes \mathbf{1}'')$, where $\Lambda_{g,n}$ is defined by Equation (25).

The category of G-CohFTs also has a tensor product induced from the diagonal morphism on $\overline{\mathcal{M}}^G_{g,n}$.

Definition 4.9. Let $\mathfrak{G}' = (\mathcal{H}', \eta', \{\Lambda'_{g,n}\}, \mathbf{1}')$ and $\mathfrak{G}'' = (\mathcal{H}'', \eta'', \{\Lambda''_{g,n}\}, \mathbf{1}'')$ be G-CohFTs, then consider the tuple $(\mathcal{H}, \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ given by

(1) $\mathcal{H} = \mathcal{H}' \odot \mathcal{H}''$ as G-graded G-modules,
(2) For all $v'_{m_1} \otimes v''_{m_1}$ in $\mathcal{H}_{m_1}$ and $v'_{m_2} \otimes v''_{m_2}$ in $\mathcal{H}_{m_2}$,
$$\eta(v'_{m_1} \otimes v''_{m_1}, v'_{m_2} \otimes v''_{m_2}) := \eta'(v'_{m_1}, v'_{m_2})\eta''(v''_{m_1}, v''_{m_2}),$$
(3) $\mathbf{1} := \mathbf{1}' \otimes \mathbf{1}''$, and
(4)
$$\Lambda_{g,n}(v'_{m_1} \otimes v''_{m_1}, \dots, v'_{m_n} \otimes v''_{m_n}) := \Lambda'_{g,n}(v'_{m_1}, \dots, v'_{m_n}) \cup \Lambda''_{g,n}(v''_{m_1}, \dots, v''_{m_n}).$$

$(\mathcal{H}, \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ is said to be the *tensor product of the* G*-CohFTs* $(\mathcal{H}', \eta', \{\Lambda'_{g,n}\}, \mathbf{1}')$ and $(\mathcal{H}'', \eta'', \{\Lambda''_{g,n}\}, \mathbf{1}'')$ and is denoted $\mathfrak{G}' \odot \mathfrak{G}''$.

Proposition 4.10. *The tensor product of two* G*-CohFTs is a* G*-CohFT.*

Proof. The proof follows, first, from the fact that the diagonal morphism
$$\Delta : \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \to \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \times \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$$
induces a morphism
$$H_\bullet(\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})) \to H_\bullet(\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})) \otimes H_\bullet(\overline{\mathcal{M}}^G_{g,n}(\mathbf{m})),$$
which respects the gluing, the S_n actions, and the G^n action, and second, from the fact that the cup product is induced via pullback of the diagonal morphism. The definitions of the flat identity and the metric are easily verified. $\square$

Remark 4.11. Let $\mathcal{H}'$ and $\mathcal{H}''$ be two G-graded G-modules. The G-module structure on $\mathcal{H}' \odot \mathcal{H}''$ is induced from the $G \times G$-module structure on the external tensor product $\mathcal{H}' \otimes \mathcal{H}''$ via the diagonal homomorphism $G \hookrightarrow G \times G$. An analogous phenomenon occurs in the category of $G-\mathsf{CohFTs}$, where the role of the homomorphism $G \hookrightarrow G \times G$ is replaced by a natural inclusion $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \hookrightarrow \overline{\mathcal{M}}^{G\times G}_{g,n}(\mathbf{m} \times \mathbf{m})$ for all stable pairs (g, n) and $\mathbf{m}$ in G^n. This inclusion respects the actions of G^n and S_n as well as the gluing morphisms. Consequently, the tensor product in the category of G-CohFTs "factors through" the external tensor product.

This natural inclusion is obtained as follows. The diagonal morphism $\Delta : \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \hookrightarrow \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \times \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ can be written as the composition
$$\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \xhookrightarrow{\widehat{\Delta}} \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \xhookrightarrow{\widehat{j}} \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \times \overline{\mathcal{M}}^G_{g,n}(\mathbf{m}),$$
where $\widehat{\Delta}$ is the diagonal morphism into the fibered product, and $\widehat{j}$ is the obvious inclusion. However, $\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}) \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ is isomorphic to $\overline{\mathcal{M}}^{G\times G}_{g,n}(\mathbf{m} \times \mathbf{m})$ via Υ. Observe that $\widehat{\Delta}$ and $\widehat{j}$ both preserve the actions of S_n and G^n and the gluing operations.

The G-CohFT $\mathbb{C}[G]$ is initial among all G-CohFTs, in the following sense.

Proposition 4.12. *Let* $\mathfrak{G} := (\mathcal{H}, \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ *be any* G*-CohFT. The tensor product of* $\mathbb{C}[G]$ *with* $\mathfrak{G}$ *satisfies*

$$\mathbb{C}[G] \odot \mathfrak{G} \cong \mathfrak{G} \odot \mathbb{C}[G] \cong \mathfrak{G}.$$

The proof is immediate from the definition of tensor product.

4.3. G-Frobenius algebras.

Recall that a Frobenius algebra is a special CohFT. This statement admits a generalization to G-CohFTs and G-Frobenius algebras, as we will see in Theorem 4.15.

Definition 4.13. Let us adopt the notation that v_m is a vector in $\mathcal{H}_m$ for any $m \in G$. A tuple $((\mathcal{H}, \rho), \cdot, \mathbf{1}, \eta)$ is said to be a *(non-projective)* G*-Frobenius algebra* [Kau, Tu] provided that the following hold:

(1) (G-graded G-module) $(\mathcal{H}, \rho)$ is a G-graded G-module.
(2) (Self-invariance) For all γ in G, $\rho(\gamma) : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is the identity map.
(3) (Metric) η is a symmetric, nondegenerate, bilinear form on $\mathcal{H}$ such that $\eta(v_{m_1}, v_{m_2})$ is nonzero only if $m_1 m_2 = 1$.
(4) (G-graded Multiplication) The binary product $(v_1, v_2) \mapsto v_1 \cdot v_2$, called the *multiplication* on $\mathcal{H}$, preserves the G-grading (i.e., the multiplication takes $\mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$ to $\mathcal{H}_{m_1 m_2}$) and is distributive over addition.
(5) (Associativity) The multiplication is associative; i.e.,
$$(v_1 \cdot v_2) \cdot v_3 = v_1 \cdot (v_2 \cdot v_3)$$
for all v_1, v_2, and v_3 in $\mathcal{H}$.
(6) (Braided Commutativity) The multiplication is invariant with respect to the braiding:
$$v_{m_1} \cdot v_{m_2} = (\rho(m_1^{-1}) v_{m_2}) \cdot v_{m_1},$$
for all $m_i \in G$ and all $v_{m_i} \in \mathcal{H}_{m_i}$ with $i = 1, 2$.
(7) (G-equivariance of the Multiplication)
$$(\rho(\gamma)v_1) \cdot (\rho(\gamma)v_2) = \rho(\gamma)(v_1 \cdot v_2)$$
for all γ in G, and all $v_1, v_2 \in \mathcal{H}$.
(8) (G-invariance of the Metric)
$$\eta(\rho(\gamma)v_1, \rho(\gamma)v_2) = \eta(v_1, v_2)$$
for all γ in G, and all $v_1, v_2 \in \mathcal{H}$.
(9) (Invariance of the Metric)
$$\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$$
for all $v_1, v_2, v_3 \in \mathcal{H}$.
(10) (G-invariant Identity) The element $\mathbf{1}$ in $\mathcal{H}_1$ is the identity element under the multiplication, and which satisfies
$$\rho(\gamma)\mathbf{1} = \mathbf{1}$$
for all γ in G.
(11) (Trace Axiom) For all a, b in G and v in $\mathcal{H}_{[a,b]}$, let L_v denote left multiplication by v:
$$\mathrm{Tr}_{\mathcal{H}_a}(L_v \rho(b^{-1})) = \mathrm{Tr}_{\mathcal{H}_b}(\rho(a) L_v).$$

Remark 4.14. When G is the trivial group, a G-Frobenius algebra is a *Frobenius algebra*, a unital, commutative, associative algebra with an invariant metric. Given a general G-Frobenius algebra $\mathcal{H}$, there are two ways that one can construct a Frobenius algebra from it. The first Frobenius algebra is obtained by considering the subalgebra $\mathcal{H}_1$. The second approach is to consider $\overline{\mathcal{H}}$, the algebra of G-coinvariants of $\mathcal{H}$, with its induced multiplication and identity. The metric on $\mathcal{H}$ induces a metric on $\overline{\mathcal{H}}$ which makes $\overline{\mathcal{H}}$ into a Frobenius algebra.

Theorem 4.15. *Let* $((\mathcal{H}, \rho), \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ *be a* G*-CohFT. Define a multiplication* $\cdot$ *on* $\mathcal{H}$ *as follows: For any* $m_1, m_2 \in G$*, let* $m_3 = (m_1 m_2)^{-1}$*. For all* v_{m_1} *in* $\mathcal{H}_{m_1}$ *and* v_{m_2} *in* $\mathcal{H}_{m_2}$*, define*

$$v_{m_1} \cdot v_{m_2} := \int_{[\![\xi(m_1,m_2,m_3)]\!]} \Lambda_{0,3}(v_{m_1}, v_{m_2}, e_\alpha)\eta^{\alpha\beta} f_\beta,$$

where $\{e_\alpha\}$ *is a basis for* $\mathcal{H}_{m_3}$*,* $\{f_\beta\}$ *is a basis for* $\mathcal{H}_{m_3^{-1}}$*, and* $\eta^{\alpha\beta}$ *is the inverse of the metric in those bases.*

The tuple $((\mathcal{H}, \rho), \cdot, \mathbf{1}, \eta)$ *is a* G-Frobenius algebra.

Proof. The G-module $(\mathcal{H}, \rho)$, the metric η, and the identity element $\mathbf{1}$ in the G-CohFT are the same for the G-Frobenius algebra.

The invariance of the metric follows from the fact that

$$s\xi(m_1, m_2, m_3) = \xi(m_2, m_3, m_1),$$

where s is the isomorphism induced from the cyclic permutation in S_3 (this is proved in Lemma 2.24).

Notice also that since $\xi(m_1, m_2, m_3)$ is empty unless $m_1 m_2 m_3 = 1$, the product is naturally graded.

The product is not commutative, in general, because $\xi(m_1, m_2, m_3) \neq \xi(m_2, m_1, m_3)$. However, it is braided commutative, because

$$\xi(m_1, m_2, m_3) = b_1^{-1}\xi(m_1 m_2 m_1^{-1}, m_1, m_3) = \sigma\rho_1(m_1)\xi(m_1 m_2 m_1^{-1}, m_1, m_3), \tag{26}$$

with σ the transposition $(1, 2) \in S_3$, as shown in Lemma 2.24. The relation (26) on ξ implies the braided commutativity via the equation

$$\begin{aligned}
v_{m_1} \cdot v_{m_2} &= \int_{[\![\xi(m_1,m_2,m_3)]\!]} \Lambda_{0,3}(v_{m_1}, v_{m_2}, e_\alpha)\eta^{\alpha\beta} f_\beta \\
&= \int_{[\![(\sigma\rho_1(m_1))\xi(m_1 m_2 m_1^{-1},m_1,m_3)]\!]} \Lambda_{0,3}(v_{m_1}, v_{m_2}, e_\alpha)\eta^{\alpha\beta} f_\beta \\
&= \int_{[\![\xi(m_1 m_2 m_1^{-1},m_1,m_3)]\!]} (\sigma\rho_1(m_1))^*(\Lambda_{0,3}(v_{m_1}, v_{m_2}, e_\alpha))\eta^{\alpha\beta} f_\beta \\
&= \int_{[\![\xi(m_1 m_2 m_1^{-1},m_1,m_3)]\!]} \rho_1(m_1)^*\sigma^*(\Lambda_{0,3}(v_{m_1}, v_{m_2}, e_\alpha))\eta^{\alpha\beta} f_\beta \\
&= \int_{[\![\xi(m_1 m_2 m_1^{-1},m_1,m_3)]\!]} \rho_1(m_1)^*(\Lambda_{0,3}(v_{m_2}, v_{m_1}, e_\alpha))\eta^{\alpha\beta} f_\beta \\
&= \int_{[\![\xi(m_1 m_2 m_1^{-1},m_1,m_3)]\!]} \Lambda_{0,3}(\rho_1(m_1^{-1})v_{m_2}, v_{m_1}, e_\alpha)\eta^{\alpha\beta} f_\beta \\
&= (\rho(m_1^{-1})v_{m_2}) \cdot v_{m_1}.
\end{aligned}$$

Again, using the braided commutativity for the classes ξ and the invariance of $\mathbf{1}$, we can show that $\mathbf{1}$ is indeed a unit for the multiplication, since

$$\begin{aligned}
\nu_{m_1}\cdot\mathbf{1} &= \int_{[\![\xi(m_1,1,m_1^{-1})]\!]}\Lambda_{0,3}(\nu_{m_1},\mathbf{1},e_\alpha)\eta^{\alpha\beta}f_\beta\\
&= \int_{[\![\xi(m_1,m_1^{-1},1)]\!]}\Lambda_{0,3}(\nu_{m_1},e_\alpha,\mathbf{1})\eta^{\alpha\beta}f_\beta\\
&= \eta(\nu_{m_1},\mathbf{1})\eta^{\alpha\beta}f_\beta=\nu_{m_1},
\end{aligned}$$

where we introduced a basis (e_α) of $\mathcal{H}_{m_1^{-1}}$ and a basis (f_β) of $\mathcal{H}_{m_1}$.

The property that $\mathbf{1}$ is a unit implies that the invariance of the metric follows from

$$\eta(\nu_{m_1},\nu_{m_2})=\eta(\nu_{m_1}\cdot\nu_{m_2},\mathbf{1}). \tag{27}$$

Equation (27) in turn follows from

$$\begin{aligned}
\eta(\nu_{m_1}\cdot\nu_{m_2},\mathbf{1}) &= \int_{[\![\xi(m_1,m_2,m_3)]\!]}\Lambda_{0,3}(\nu_{m_1},\nu_{m_2},e_\alpha)\eta^{\alpha\beta}\int_{[\![\xi(m_3,1,1)]\!]}\Lambda_{0,3}(f_\beta,\mathbf{1},\mathbf{1})\\
&= \int_{[\![\xi(m_1,m_2,m_3)]\!]}\Lambda_{0,3}(\nu_{m_1},\nu_{m_2},e_\alpha)\eta^{\alpha\beta}\eta(f_\beta,\mathbf{1})\\
&= \int_{[\![\xi(m_1,m_2,m_3)]\!]}\Lambda_{0,3}(\nu_{m_1},\nu_{m_2},\mathbf{1})\\
&= \eta(\nu_{m_1},\nu_{m_2}),
\end{aligned}$$

where we use the notation $m_3:=(m_1m_2)^{-1}$, and we let $\{e_\alpha\}$ be a basis of $\mathcal{H}_{m_3}$ and $\{f_\beta\}$ be a basis of $\mathcal{H}_{m_3^{-1}}$.

The $\rho(\gamma)$-invariance of $\mathcal{H}_\gamma$ follows from the second part of Lemma 2.24:

$$\begin{aligned}
\rho(\gamma)\nu_\gamma &= \rho(\gamma)\nu_\gamma\cdot\mathbf{1}=\int_{[\![\xi(\gamma,1,\gamma^{-1})]\!]}\Lambda_{0,3}(\rho(\gamma)\nu_\gamma,\mathbf{1},e_\alpha)\eta^{\alpha\beta}f_\beta\\
&= \int_{[\![\rho(\gamma,1,1)\xi(\gamma,1,\gamma^{-1})]\!]}\Lambda_{0,3}(\rho(\gamma)\nu_\gamma,\mathbf{1},e_\alpha)\eta^{\alpha\beta}f_\beta\\
&= \int_{[\![\xi(\gamma,1,\gamma^{-1})]\!]}\Lambda_{0,3}(\nu_\gamma,\mathbf{1},e_\alpha)\eta^{\alpha\beta}f_\beta\\
&= \nu_\gamma.
\end{aligned}$$

Again we use bases $\{e_\alpha\}$ of $\mathcal{H}_{\gamma^{-1}}$ and $\{f_\beta\}$ of $\mathcal{H}_\gamma$.

The self invariance, together with the invariance of the metric, imply the symmetry of the metric:

$$\eta(\nu_m,\nu_{m^{-1}})=\eta(\nu_m\nu_{m^{-1}},\mathbf{1})=\eta(\rho(m^{-1})(\nu_{m^{-1}})\nu_m,\mathbf{1})=\eta(\nu_{m^{-1}},\nu_m).$$

The G-invariance of the metric follows from the G^n-invariance of Λ and the ρ-invariance of the unit $\mathbf{1}$ via

$$\begin{aligned}\eta(\rho(\gamma)v_{m_1},\rho(\gamma)v_{m_2}) &= \int_{[\![\xi(\gamma^{-1}m_1\gamma,\gamma^{-1}m_2\gamma,1)]\!]}\Lambda_{0,3}(\rho(\gamma)v_{m_1},\rho(\gamma)v_{m_2},\mathbf{1})\\ &= \int_{[\![\rho(\gamma,\gamma,\gamma)\xi(m_1,m_2,1)]\!]}\Lambda_{0,3}(\rho(\gamma)v_{m_1},\rho(\gamma)v_{m_2},\rho(\gamma)\mathbf{1})\\ &= \int_{[\![\xi(m_1,m_2,1)]\!]}\Lambda_{0,3}(v_{m_1},v_{m_2},\mathbf{1})\\ &= \eta(v_{m_1},v_{m_2}),\end{aligned}$$

where we used the first property of ξ of Lemma 2.24.

The above in turn gives the G-equivariance of the multiplication

$$\begin{aligned}\rho(\gamma)v_{m_1}\cdot\rho(\gamma)v_{m_2} &= \int_{[\![\xi(\gamma^{-1}m_1\gamma,\gamma^{-1}m_2\gamma,(\gamma^{-1}m_3\gamma)^{-1})]\!]}\Lambda_{0,3}(\rho(\gamma)v_{m_1},\rho(\gamma)v_{m_2},e_\alpha)\eta^{\alpha\beta}f_\beta\\ &= \int_{[\![\rho(\gamma,\gamma,\gamma)(\xi(m_1,m_2,m_3))]\!]}\Lambda_{0,3}(\rho(\gamma)v_{m_1},\rho(\gamma)v_{m_2},\rho(\gamma)e'_\alpha)\eta^{\alpha\beta}\rho(\gamma)f'_\beta\\ &= \int_{[\![\xi(m_1,m_2,m_3]\!]}\Lambda_{0,3}(v_{m_1},v_{m_2},e'_\alpha)\eta'^{\alpha\beta}\rho(\gamma)f'_\beta\\ &= \rho(\gamma)(v_{m_1}\cdot v_{m_2}),\end{aligned}$$

where we used $m_3 := (m_1m_2)^{-1}$, a basis $\{e_\alpha\}$ of $\mathcal{H}_{\gamma^{-1}m_3\gamma}$, $\{f_\beta\}$ of $\mathcal{H}_{\gamma^{-1}m_3^{-1}\gamma}$, and the transformed bases $\{e'_\alpha := \rho(\gamma^{-1})e_\alpha\}$ of $\mathcal{H}_{m_3}$ and $\{f'_\beta := \rho(\gamma^{-1})f_\beta\}$ of $\mathcal{H}_{m_3^{-1}}$. Also, we used the notation $\eta'^{\alpha\beta}$ for the inverse metric of $\eta_{\alpha\beta} = \eta(e'_\alpha, f'_\beta)$, the G-invariance of the metric $\eta'_{\alpha\beta} = \eta_{\alpha\beta}$, and the first property of Lemma 2.24.

Associativity follows from Lemma 2.29 in the following way:

$$\begin{aligned}(v_{m_1}\cdot v_{m_2})\cdot v_{m_3} &= \int_{[\![\xi(m_1,m_2,m_+]\!]}\Lambda_{0,3}(v_{m_1},v_{m_2},e_\alpha)\eta^{\alpha\beta}\int_{[\![\xi(m_-,m_3,m_4)]\!]}\Lambda_{0,3}(f_\beta,v_{m_3},k_\gamma)\eta^{\gamma\delta}l_\delta\\ &= \int_{[\![\xi(m_1,m_2,m_+)\times\xi(m_-,m_3,m_4)]\!]}\rho^*\Lambda_{0,4}(v_{m_1},v_{m_2},v_{m_3},k_\gamma)\eta^{\gamma\delta}l_\delta\\ &= \int_{[\![\rho(\xi(m_1,m_2,m_+)\times\xi(m_-,m_3,m_4)]\!]}\Lambda_{0,4}(v_{m_1},v_{m_2},v_{m_3},k_\gamma)\eta^{\gamma\delta}l_\delta\\ &= \int_{[\![\rho'(\xi(m_4,m_1,m'_+)\times\xi(m'_-,m_2,m_3))]\!]}\Lambda_{0,4}(v_{m_1},v_{m_2},v_{m_3},k_\gamma)\eta^{\gamma\delta}l_\gamma\\ &= \int_{[\![\xi(m_4,m_1,m'_+)\times\xi(m'_-,m_2,m_3)]\!]}\rho'^*\Lambda_{0,4}(v_{m_1},v_{m_2},v_{m_3},k_\gamma)\eta^{\gamma\delta}l_\delta\\ &= \int_{[\![\xi(m_4,m_1,m'_+)]\!]}\Lambda_{0,3}(k_\gamma,v_{m_1},e_\alpha)\eta^{\alpha\beta}\int_{[\![\xi(m'_-,m_2,m_3)]\!]}\Lambda_{0,3}(f_\beta,v_{m_2},v_{m_3})\eta^{\gamma\delta}l_\delta\\ &= \int_{[\![\xi(m_2,m_3,m'_-)]\!]}\Lambda_{0,3}(v_{m_2},v_{m_3},f_\beta)\eta^{\beta\alpha}\int_{[\![\xi(m_1,m'_+,m_4]\!]}\Lambda_{0,3}(v_{m_1},e_\alpha,k_\gamma)\eta^{\gamma\delta}l_\delta\\ &= v_{m_1}\cdot(v_{m_2}\cdot v_{m_3}),\end{aligned}$$

where we used the S_3-invariance of Λ, the fourth property of 2.24, and the symmetry of the metric. Also, we introduced the notation $m_4 = (m_1m_2m_3)^{-1}$ and used the notations of Lemma 2.29 for $m_\pm, m'_\pm, \rho, \rho'$; i.e., $m_+ := (m_1m_2)^{-1}, m_- :=$

$m_1m_2, m'_+ = m_2m_3$, and $m'_- := (m_2m_3)^{-1}$. Furthermore, $\{e_\alpha\}$ is a basis of $\mathcal{H}_{m_+}$, $\{f_\beta\}$ is a basis of $\mathcal{H}_{m_-}$, $\{k_\gamma\}$ is a basis of $\mathcal{H}_{m_4}$, and $\{l_\delta\}$ is a basis of $\mathcal{H}_{m_4^{-1}}$.

Lastly, the proof of the trace axiom follows using Lemma 2.31:

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}_a}(L_\nu\rho(b^{-1})) &= \eta(\eta^{\alpha\beta}f_\beta, \nu_{aba^{-1}b^{-1}}\cdot\rho(b^{-1})e_\alpha)\\
&= \eta(\eta^{\alpha\beta}f_\beta, \int_{[\![\xi(m_1,bab^{-1},a^{-1})]\!]}\Lambda_{0,3}(\nu_{m_1},\rho(b^{-1})e_\alpha,f_\gamma)\eta^{\gamma\delta}e_\delta)\\
&= \int_{[\![\xi(a^{-1},a,1)]\!]}\Lambda_{0,3}(f_\beta,e_\delta,\mathbf{1})\eta^{\alpha\beta}\int_{[\![\xi(m_1,bab^{-1},a^{-1})]\!]}\Lambda_{0,3}(\nu_{m_1},\rho(b^{-1})e_\alpha,f_\gamma)\eta^{\gamma\delta}\\
&= \int_{[\![\xi(m_1,bab^{-1},a^{-1})]\!]}\Lambda_{0,3}(k_\lambda,\rho(b^{-1})e_\alpha,f_\beta)\eta^{\alpha\beta}\int_{[\![\xi(1,m_1,m_1^{-1})]\!]}\Lambda_{0,3}(\mathbf{1},\nu_{m_1},l_\mu)\eta^{\lambda\mu}\\
&= \int_{[\![\rho_{b*}\xi(m_1,bab^{-1},a^{-1})]\!]}\Lambda_{1,1}(k_\lambda)\int_{[\![\xi(1,m_1,m_1^{-1})]\!]}\Lambda_{0,3}(\mathbf{1},\nu_{m_1},l_\mu)\eta^{\lambda\mu}\\
&= \int_{[\![\rho_{a*}\xi(m_1,b,ab^{-1}a^{-1})]\!]}\Lambda_{1,1}(k_\lambda)\int_{[\![\xi(1,m_1,m_1^{-1})]\!]}\Lambda_{0,3}(\mathbf{1},\nu_{m_1},l_\mu)\eta^{\lambda\mu}\\
&= \int_{[\![\xi(m_1,b,ab^{-1}a^{-1})]\!]}\Lambda_{0,3}(k_\lambda,g_\gamma,\rho(a)^{-1}h_\delta)\eta^{\gamma\delta}\int_{[\![\xi(1,aba^{-1}b^{-1},m_1^{-1})]\!]}\Lambda_{0,3}(\mathbf{1},\nu_{m_1},l_\mu)\eta^{\lambda\mu}\\
&= \int_{[\![\xi(b^{-1},b,1)]\!]}\Lambda_{0,3}(h_\tau,g_\gamma,\mathbf{1})\eta^{\sigma\tau}\int_{[\![\xi(m_1,b,ab^{-1}a^{-1})]\!]}\Lambda_{0,3}(\nu_{m_1},g_\sigma,\rho(a^{-1})h_\delta)\eta^{\gamma\delta}\\
&= \mathrm{Tr}_{\mathcal{H}_b}(\rho(a)L_\nu),
\end{aligned}$$

where we used Lemma 2.29, as well as Lemma 2.31 with its notation for the maps ρ_a, ρ_b and $m_1 = [a,b]$, and introduced the bases $\{e_\alpha\}$ of $\mathcal{H}_a$, $\{f_\beta\}$ of $\mathcal{H}_{a^{-1}}$, $\{g_\gamma\}$ of $\mathcal{H}_b$, $\{h_\delta\}$ of $\mathcal{H}_{b^{-1}}$, $\{k_\lambda\}$ of $\mathcal{H}_{m_1}$, and $\{l_\mu\}$ of $\mathcal{H}_{m_1^{-1}}$. □

We can now justify naming the G-CohFT $\mathbb{C}[G]$ of Example 4.3 the *group ring* G-*CohFT*.

Proposition 4.16. *In the group ring* G*-CohFT, the metric* η *on* $\mathcal{H} = \bigoplus_{g\in G}\mathbb{C}$ *satisfies*

$$\eta(e_{m_1}, e_{m_2}) = \delta_{m_1,m_2^{-1}}$$

for all m_1, m_2 *in* G*.*

The multiplication is given by

$$e_{m_1}\cdot e_{m_2} = e_{m_1m_2}$$

for all m_1, m_2 *in* G*. The identity element is* $\mathbf{1} := e_1$*. The resulting* G*-Frobenius algebra is isomorphic to the group ring* $\mathbb{C}[G]$*.*

Proof. The multiplication operation is

$$e_{m_1}\cdot e_{m_2} = \int_{[\![\xi(m_1,m_2,(m_1m_2)^{-1})]\!]}\Lambda_{0,3}(e_{m_1},e_{m_2},e_{(m_1m_2)^{-1}})e_{m_1m_2} = e_{m_1m_2}.$$

The metric and identity element follow by a similar calculation. □

5. CohFTs and Quotients of G-CohFTs

In this subsection, we explain how to obtain a CohFT from a G-CohFT by taking the appropriate quotient with respect to G. Geometrically, going from a G-CohFT to a CohFT corresponds to going from $\overline{\mathcal{M}}^G_{g,n}$ to $\overline{\mathcal{M}}_{g,n}$, where the $\Lambda_{g,n}$ are allowed to only act upon elements of $\overline{\mathcal{H}}$. We perform this procedure in two steps. The first step is to go from $\overline{\mathcal{M}}^G_{g,n}$ to $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$. The second step is to go from $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ to $\overline{\mathcal{M}}_{g,n}$.

5.1. From $\overline{\mathcal{M}}^G_{g,n}$ to $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$.

We begin with a useful lemma.

Lemma 5.1. *For all $\overline{\mathbf{m}}$ in $\overline{G}^n$, the forgetful morphism $\widetilde{\mathrm{st}}_{\overline{\mathbf{m}}} : \overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}) \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}})$ induces a ring isomorphism $\widetilde{\mathrm{st}}^*_{\overline{\mathbf{m}}} : H^\bullet(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}})) \longrightarrow H^\bullet(\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}))^{G^n}$.*

Proof. Let $\mathcal{C}$ be the constant sheaf of complex numbers on $\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}})$, and let $\mathcal{C}'$ be the constant sheaf on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}})$.

Since $\widetilde{\mathrm{st}}$ is finite, the Leray spectral sequence degenerates, giving

$$H^p(\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}),\mathcal{C}) = H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),\widetilde{\mathrm{st}}_*(\mathcal{C})).$$

Since these sheaves are all sheaves of vector spaces over $\mathbb{C}$, they are all divisible, hence the coinvariant map π_{G^n} is well defined and preserves invariants; i.e., if $i : (\widetilde{\mathrm{st}}_*\mathcal{C})^{G^n} \longrightarrow \widetilde{\mathrm{st}}_*\mathcal{C}$ is the natural inclusion, then $\pi_{G^n} \circ i = 1$. Thus taking G^n-invariants is the same as applying the map π_{G^n}, and is exact. So a general homological argument gives that

$$(H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),\widetilde{\mathrm{st}}_*\mathcal{C}))^{G^n} = H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),(\mathrm{st}_*\mathcal{C})^{G^n}),$$

and we have

$$(H^p(\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}),\mathcal{C}))^{G^n} = H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),(\widetilde{\mathrm{st}}_*\mathcal{C})^{G^n}).$$

On the other hand, we have $\widetilde{\mathrm{st}}^*(\mathcal{C}') = \mathcal{C}$, so by adjointness we have a map $j : \mathcal{C}' \longrightarrow \widetilde{\mathrm{st}}_*\mathcal{C}$. Composing with π_{G^n}, we get a map of sheaves $\pi_{G^n} \circ j : \mathcal{C}' \longrightarrow (\widetilde{\mathrm{st}}_*\mathcal{C})^{G^n}$. On each fiber this map is an isomorphism, since for a fixed admissible cover $E \longrightarrow C$ the fiber $F := \widetilde{\mathrm{st}}^{-1}([E \longrightarrow C])$ is a disjoint union of points with transitive G^n-action inducing the G^n-action on $\widetilde{\mathrm{st}}_*\mathcal{C} = \bigoplus_{f\in F}\mathcal{C}'$, and j is just given by $q \mapsto (q,q,...,q)$. Some straightforward work shows that for any vector space V and any set F with transitive G^n-action, the vector space $V \times F$ has as its G^n-invariants exactly the image of the map $j : V \longrightarrow V \times F$, taking v to $(v,v,...,v)$. In particular, this holds for $V = \mathcal{C}'$. Since the fiber F and the G^n-action on F are unchanged under small deformation, this shows that the morphism of sheaves $\pi_{G^n} \circ j$ induces an isomorphism on stalks, and thus is an isomorphism of sheaves. So we have

$$\begin{aligned} H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),\mathcal{C}') &= H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),(\widetilde{\mathrm{st}}_*(\mathcal{C}))^{G^n}) \\ &= H^p(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}),\widetilde{\mathrm{st}}_*\mathcal{C})^{G^n} = H^p(\overline{\mathcal{M}}^G_{g,n},\mathcal{C})^{G^n} \end{aligned}$$

as desired.

∎

Proposition 5.2. *Let $(\mathcal{H}, \eta, \Lambda_{g,n}, \mathbf{1})$ be a* G*-CohFT. There exist uniquely-determined classes $\widehat{\Lambda}_{g,n}$ in $\bigoplus_{\overline{\mathbf{m}} \in \overline{G}^n} H^\bullet(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}})) \otimes \overline{\mathcal{H}}^*_{\overline{\mathbf{m}}}$ such that*

$$\widetilde{\mathrm{st}}^*_{\overline{\mathbf{m}}} \widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \Lambda_{g,n}(v_{\overline{\mathbf{m}}})$$

for all $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$.

Proof. Consider $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$ for $\overline{\mathbf{m}}$ in $\overline{G}^n$. For all $\boldsymbol{\gamma}$ in G^n we have

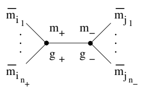
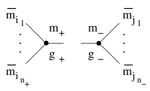
$$\rho(\boldsymbol{\gamma})^*(\Lambda_{g,n}(v_{\overline{\mathbf{m}}})) = \Lambda_{g,n}(\rho(\boldsymbol{\gamma}^{-1})_* v_{\overline{\mathbf{m}}})) = \Lambda_{g,n}(v_{\overline{\mathbf{m}}})),$$

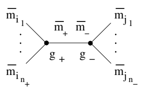
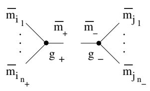
where the first equality is by the (diagonal) G^n-invariance of $\Lambda_{g,n}$ and the second is by the definition of $\overline{\mathcal{H}}$. Therefore, $\Lambda_{g,n}(v_{\overline{\mathbf{m}}})$ belongs to $H^\bullet(\overline{\mathcal{M}}_{g,n}(\overline{\mathbf{m}}))^{G^n}$, and we are done by the previous lemma. □

Fix an element m_+ in G and let $m_- := m_+^{-1}$. To each such choice, we have the following associated commutative diagram which we will use extensively hereafter, and whose morphisms and other terms we explain below:

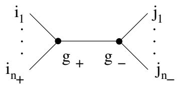
$$\begin{tikzcd}
 & F^G_{\widehat{\Gamma}} \arrow[r, "\widetilde{\mu}"] \arrow[dd, "\widetilde{\mathrm{pr}}"'] & \overline{\mathcal{M}}^G_{\widehat{\Gamma}} \arrow[r, hook, "\widetilde{i}"] \arrow[dd, "\widetilde{\mathrm{st}}'"'] & \overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}) \arrow[dd, "\widetilde{\mathrm{st}}"'] \\
\overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}} \arrow[ur, "\widetilde{r}"'] \arrow[dd, "\widetilde{\mathrm{st}}''"] & & & \\
 & F_{\widehat{\Gamma}}(\mathcal{B}G) \arrow[r, "\widehat{\mu}"] \arrow[dl, "\widehat{r}"] \arrow[dd, "\widehat{\mathrm{pr}}"'] & \overline{\mathcal{M}}_{\widehat{\Gamma}}(\mathcal{B}G) \arrow[r, hook, "\widehat{i}"] \arrow[dd, "\widehat{\mathrm{st}}'"'] & \overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}}) \arrow[dd, "\widehat{\mathrm{st}}"'] \\
\overline{\mathcal{M}}_{\widehat{\Gamma}_{\mathrm{cut}}} \arrow[dr, "\widehat{\mathrm{st}}''"'] & & & \\
 & \overline{\mathcal{M}}_{\Gamma_{\mathrm{cut}}} \arrow[r, "\mu"] & \overline{\mathcal{M}}_{\Gamma} \arrow[r, hook, "i"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd} \tag{28}$$

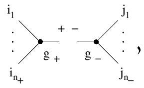
The above diagram has two cases. The first case corresponds to the situation where all graphs are decorated stable graphs of genus g with n tails which are trees of the form

(29) $\widetilde{\Gamma} =$  and $\widetilde{\Gamma}_{\mathrm{cut}} =$ 

(30) $\widehat{\Gamma} =$  and $\widehat{\Gamma}_{\mathrm{cut}} =$ 

and

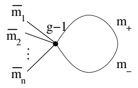
(31) $\Gamma =$ 

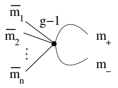
and $\Gamma_{\mathrm{cut}} =$ 

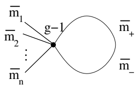
,

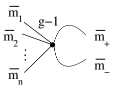
where $N_+ := \{i_1, \ldots, i_{n_+}\}$, is the index set of the labels for the tails on the left half of each graph above, $N_- := \{j_1, \ldots, j_{n_-}\}$, is the index set of the labels for the tails on the right half of each graph above, $N_+ \sqcup N_- = \{1, \ldots, n\}$, and $g_+ + g_- = g$.

The second case corresponds to the situation where all graphs are decorated stable graphs of genus g with n tails which are loops of the following form:

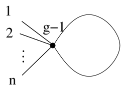
(32) $\widetilde{\Gamma} =$ 

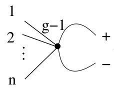
and $\widetilde{\Gamma}_{\mathrm{cut}} =$ 

(33) $\widehat{\Gamma} =$ 

and $\widehat{\Gamma}_{\mathrm{cut}} =$ 

and

(34) $\Gamma =$ 

and $\Gamma_{\mathrm{cut}} =$ 

.

Note that in both cases, the graph $\widetilde{\Gamma}$ has tails labeled by conjugacy classes $\overline{m}_i$, but its one edge is labeled by a specific choice of m_+ and m_-.

Now we explain the various terms and morphisms. $\overline{\mathcal{M}}_\Gamma$ is the closure of the locus in $\overline{\mathcal{M}}_{g,n}$ whose dual graph is Γ, i is the inclusion morphism, and μ is the normalization morphism associated to cutting the internal edge of Γ. Similarly, $\overline{\mathcal{M}}_{\widehat{\Gamma}}(\mathcal{B}G)$ denotes the closure of the locus in $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}})$ whose associated dual graph has tails decorated by $\overline{\mathbf{m}}$ and whose monodromies around one side of the node lie in $\overline{m}_+$ and whose monodromies around the other side of the node lie in $\overline{m}_-$. The morphism $\widehat{i}$ is the inclusion, and $F_{\widehat{\Gamma}}(\mathcal{B}G)$ is the fibered product $\overline{\mathcal{M}}_{\Gamma_{\mathrm{cut}}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}_{\widehat{\Gamma}}(\mathcal{B}G)$. The morphisms $\widehat{\mu}$ and $\widehat{\mathrm{pr}}$ are the canonical projections of the fibered product. To explain $\widehat{r}$, we first note that $F_{\widehat{\Gamma}}(\mathcal{B}G)$ is the the stack of triples consisting of a cut curve C' in $\overline{\mathcal{M}}_{\Gamma_{\mathrm{cut}}}$, an admissible G-cover $E \to C$ in $\overline{\mathcal{M}}_{\widehat{\Gamma}}(\mathcal{B}G)$, and an isomorphism α in $\overline{\mathcal{M}}_\Gamma$ from the glued curve $\mu(C')$ to C. The morphism $\widehat{r}$ takes such a triple to the pullback of E along the composition $\alpha \circ \mu$. The morphisms $\widehat{\mathrm{st}}$, $\widehat{\mathrm{st}}'$, and $\widehat{\mathrm{st}}''$ simply forget their respective twisted curve structures.

Similarly, $\overline{\mathcal{M}}^G_{\widetilde{\Gamma}}$ is the closure of the locus of pointed G-covers with dual graph $\widehat{\Gamma}$, so all tails are labeled by conjugacy classes $\overline{m}_i$; and $\overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}}$ is the closure of the locus of pointed G-covers with dual graph $\widetilde{\Gamma}_{\mathrm{cut}}$, so their tails are labeled with conjugacy classes $\overline{m}_i$, but on the two sides of the node their monodromies are the specific group elements m_+ and m_-. The morphisms $\widetilde{\mathrm{st}}'$ and $\widetilde{\mathrm{st}}''$ simply forget the marked points in the G-cover.

The stack $F^G_{\widetilde{\Gamma}}$ is the fibered product $F_{\widehat{\Gamma}}(\mathcal{B}G) \times_{\overline{\mathcal{M}}_{\widehat{\Gamma}}(\mathcal{B}G)} \overline{\mathcal{M}}^G_{\widetilde{\Gamma}} = \overline{\mathcal{M}}_{\Gamma_{\mathrm{cut}}} \times_{\overline{\mathcal{M}}_\Gamma} \overline{\mathcal{M}}^G_{\widetilde{\Gamma}}$, and the morphisms $\widetilde{\mu}$ and $\widetilde{\mathrm{pr}}$ are the canonical projections. The morphism $\widetilde{r}$ is induced by the pair of the gluing map $\rho : \overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}} \longrightarrow \overline{\mathcal{M}}^G_{\widetilde{\Gamma}}$ and the map $\widehat{\mathrm{st}}'' \circ \widetilde{\mathrm{st}}'' : \overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}} \longrightarrow \overline{\mathcal{M}}_{\Gamma_{\mathrm{cut}}}$ (actually the gluing map has as its target $\overline{\mathcal{M}}^G_{g,n}$, but it factors through the substack $\overline{\mathcal{M}}^G_{\widetilde{\Gamma}}$). In particular, we can write the gluing morphism on $\overline{\mathcal{M}}^G$ as

$$\widetilde{\rho}_{\widetilde{\Gamma}} = \widetilde{i} \circ \widetilde{\mu} \circ \widetilde{r}, \tag{35}$$

while the corresponding gluing morphism on $\overline{\mathcal{M}}$ can be written as

$$\rho_\Gamma = i \circ \mu. \tag{36}$$

Remark 5.3. The morphisms i, $\widehat{i}$, $\widetilde{i}$ are regular embeddings. The remaining morphisms in the diagram are both flat and proper.

Notation 5.4. For all $\overline{m}$ in $\overline{G}$, let $|C(\overline{m})|$ denote the order of the subgroup $C(m')$ of G for any m' in $\overline{m}$, as it is independent of the choice of m'.

Theorem 5.5. *Let $\{\widehat{\Lambda}_{g,n}\}$ be a collection of classes associated to a G-CohFT $\{\Lambda_{g,n}\}$, as in Proposition 5.2. Fix any conjugacy class $\overline{m}_+$, and let $\overline{m}_- := \overline{m}_+^{-1}$. Let $\widehat{\Gamma}$ be a decorated stable graph of genus g with n-tails which is either a tree, as in Equation (30), or a loop, as in Equation (33). Let $v_{\overline{\mathbf{m}}}$ belong to $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$.*

When $\widehat{\Gamma}$ is a tree then

$$\widehat{r}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \frac{\deg(\widehat{st}')}{\deg(\widehat{st}'')} \sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]} \widehat{\Lambda}_{g_+,n_++1}(v_{\overline{\mathbf{m}}_{N_+}}, e_{\beta[\overline{m}_+]})\widehat{\eta}^{\beta[\overline{m}_+],\beta[\overline{m}_-]}\widehat{\Lambda}_{g_-,n_-+1}(e_{\beta[\overline{m}_-]}, v_{\overline{\mathbf{m}}_{N_-}}), \tag{37}$$

where $N_+ \sqcup N_- = \{1,\dots,n\}$ is the partition corresponding to the tree, $n_\pm = |N_\pm|$ and $v_{\overline{\mathbf{m}}_{N_\pm}}$ denotes the $n_\pm$-tuple $\prod_{i\in N_\pm} v_{m_i}$, the collection $\{e_{\beta[\overline{m}_\pm]}\}$ is a basis of $\overline{\mathcal{H}}_{\mathbf{m}_\pm}$, and $g_+ + g_- = g$. And $\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}$ is the inverse of the metric $\widehat{\eta}$ on $\overline{\mathcal{H}}$ in the basis $\{e_{\beta[\overline{m}_\pm]}\}$, where

$$\widehat{\eta}(v_{\overline{m}_+}, v_{\overline{m}_-}) := |C(\overline{m}_+)|\overline{\eta}(v_{\overline{m}_+}, v_{\overline{m}_-}) \tag{38}$$

for all $v_{\overline{m}_\pm}$ in $\overline{\mathcal{H}}_{\overline{m}_\pm}$.

When $\widehat{\Gamma}$ is a loop then

$$\widehat{r}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \frac{\deg(\widehat{st}')}{\deg(\widehat{st}'')} \sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]} \widehat{\Lambda}_{g-1,n+2}(v_{\overline{\mathbf{m}}}, e_{\beta[\overline{m}_+]}, e_{\beta[\overline{m}_-]})\widehat{\eta}^{\beta[\overline{m}_+],\beta[\overline{m}_-]}. \tag{39}$$

In either case, denote the right hand side of equations (37) and (39) by $\frac{\deg(\widehat{st}')}{\deg(\widehat{st}'')}\widehat{\Lambda}_{\widehat{\Gamma}_{cut}}$.

Remark 5.6. This theorem suggests that $\widehat{\Lambda}_{g,n}$ should be regarded as an analog of the virtual class $c^{1/r}_{g,n}$ on $\overline{\mathcal{M}}^{1/r}_{g,n}$, the moduli stack of r-spin curves [JKV]. Equations (37) and (39) should be regarded as an analog of the Cutting-Edges axiom.

Proof: (of Theorem 5.5). Let m_+ be any representative of the conjugacy class $\overline{m}_+$ and let $m_- := m_+^{-1}$. Consider the associated commuting diagram (28) and graphs (29) to (34).

For $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$, let

$$I := \widetilde{st}''_* \widetilde{\rho}^*_{\widetilde{\Gamma}} \Lambda_{g,n}(v_{\overline{\mathbf{m}}}).$$

We have

$$\begin{aligned}\widetilde{\rho}^*_{\widetilde{\Gamma}}\Lambda_{g,n}(\nu_{\overline{\mathbf{m}}}) &= \widetilde{\rho}^*_{\widetilde{\Gamma}}\widetilde{\mathrm{st}}^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}})\\ &= (\widetilde{\mathrm{st}}\circ\widetilde{\mathfrak{i}}\circ\widetilde{\mu}\circ\widetilde{\mathrm{r}})^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}})\\ &= (\widehat{\mathfrak{i}}\circ\widehat{\mu}\circ\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}})^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}}).\end{aligned}$$

Therefore,

$$\begin{aligned}I &= \widetilde{\mathrm{st}}''_*(\widehat{\mathfrak{i}}\circ\widehat{\mu}\circ\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}})^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}})\\ &= (\widehat{\mathrm{r}}\circ\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}})_*(\widehat{\mathfrak{i}}\circ\widehat{\mu}\circ\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}})^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}})\\ &= \deg(\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}})\widehat{\mathrm{r}}_*\widehat{\mu}^*\widehat{\mathfrak{i}}^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}}) \qquad (40)\end{aligned}$$

because $\widetilde{\mathrm{pr}}\circ\widetilde{\mathrm{r}}$ is finite and surjective.

For all m in G, let $\{e_{\alpha[m]}\}$ be a basis for $\mathcal{H}_m$ such that $\{e_{\alpha[m]}\}$ is the disjoint union of a basis $\{e_{\mu[m]}\}$ for $\mathcal{H}_m^{C(m)}$ and a basis $\{e_{\nu[m]}\}$ for $\mathcal{H}'_m$ as in Equation (23), such that for all γ in G,

$$\rho(\gamma)e_{\mu[m]} = e_{\mu[\gamma^{-1}m\gamma]}. \qquad (41)$$

Assume that $\widehat{\Gamma}$ is a tree, then let $\overline{\mathcal{M}}^G_+(m'_+) := \overline{\mathcal{M}}^G_{g_+,n_++1}(\overline{\mathbf{m}}_{N_+}, m'_+)$ and $\overline{\mathcal{M}}^G_-(m'_-) := \overline{\mathcal{M}}^G_{g_-,n_-+1}(m'_-, \overline{\mathbf{m}}_{N_-})$ for all $m'_\pm$ in $\overline{m}_\pm$. Let $\overline{\mathcal{M}}^G_\pm(\overline{m}_\pm) := \coprod_{m'_\pm\in\overline{m}_\pm}\overline{\mathcal{M}}^G_\pm(m'_\pm)$. We can write $\overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}} = \overline{\mathcal{M}}^G_+(m_+)\times\overline{\mathcal{M}}^G_-(m_-)$. Similarly, let

$$\Lambda_+(\nu_{m_+}) := \Lambda_{g_+,n_++1}(\nu_{\overline{\mathbf{m}}_{N_+}}, \nu_{m_+})$$

and

$$\Lambda_-(\nu_{m_-}) := \Lambda_{g_-,n_-+1}(\nu_{m_-}, \nu_{\overline{\mathbf{m}}_{N_-}})$$

for all $\nu_{m_\pm}$ in $\mathcal{H}_{m_\pm}$. Furthermore, let $\widehat{\Lambda}_\pm(\nu_{\overline{m}_\pm})$ be defined by

$$\Lambda_\pm(\nu_{\overline{m}_\pm}) = \widetilde{\mathrm{st}}^*\widehat{\Lambda}_\pm(\nu_{\overline{m}_\pm})$$

for all $\nu_{\overline{m}_\pm}$ in $\mathcal{H}_{\overline{m}_\pm}$.

The G-CohFT axioms imply that

$$\begin{aligned}I &= \sum_{\alpha[m_\pm]}\widetilde{\mathrm{st}}''_*(\Lambda_+(e_{\alpha[m_+]})\eta^{\alpha[m_+]\alpha[m_-]}\Lambda_-(e_{\alpha[m_-]}))\\ &= \sum_{\alpha[m_\pm]}\widetilde{\mathrm{st}}_{\overline{m}_+*}\Lambda_+(e_{\alpha[m_+]})\eta^{\alpha[m_+]\alpha[m_-]}\widetilde{\mathrm{st}}_{\overline{m}_-*}\Lambda_-(e_{\alpha[m_-]}),\end{aligned}$$

where we have the natural forgetful morphisms $\widetilde{\mathrm{st}}_{\overline{m}_+} : \overline{\mathcal{M}}^G_+(\overline{m}_+) \longrightarrow \overline{\mathcal{M}}_{g_+,n_++1}(\mathcal{B}G;\overline{\mathbf{m}}_{N_+},\overline{m}_+)$ and $\widetilde{\mathrm{st}}_{\overline{m}_-} : \overline{\mathcal{M}}^G_-(\overline{m}_-) \longrightarrow \overline{\mathcal{M}}_{g_-,n_-+1}(\mathcal{B}G;\overline{m}_-,\overline{\mathbf{m}}_{N_-})$.

Therefore, since Λ is G-equivariant and the fibers of $\widetilde{\mathrm{st}}_{\overline{m}_\pm}$ are G-orbits, we have

$$\begin{aligned}
I &= \sum_{\alpha[m_\pm]} \widetilde{\mathrm{st}}_{\overline{m}_+ *}\Lambda_+(\pi_G(e_{\alpha[m_+]}))\eta^{\alpha[m_+]\alpha[m_-]}\widetilde{\mathrm{st}}_{\overline{m}_- *}\Lambda_-(\pi_G(e_{\alpha[m_-]}))\\
&= \sum_{\mu[m_\pm]} \widetilde{\mathrm{st}}_{\overline{m}_+ *}\Lambda_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widetilde{\mathrm{st}}_{\overline{m}_- *}\Lambda_-(\pi_G(e_{\mu[m_-]}))\\
&= \sum_{\mu[m_\pm]} \widetilde{\mathrm{st}}_{\overline{m}_+ *}\widetilde{\mathrm{st}}^*_{\overline{m}_+}\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widetilde{\mathrm{st}}_{\overline{m}_- *}\widetilde{\mathrm{st}}^*_{\overline{m}_-}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]}))\\
&= \sum_{\mu[m_\pm]} \deg(\widetilde{\mathrm{st}}_{\overline{m}_+})\deg(\widetilde{\mathrm{st}}_{\overline{m}_-})\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]}))\\
&= \sum_{\mu[m_\pm]} \deg(\widetilde{\mathrm{st}}_{\overline{m}_+}\times\widetilde{\mathrm{st}}_{\overline{m}_-})\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]})),
\end{aligned}$$

where the first equality holds because $\widetilde{\mathrm{st}}_*\Lambda_{g,n}$ belongs to $H^\bullet(\overline{\mathcal{M}}_{g,n}(\mathcal{B}G))\otimes\overline{\mathcal{H}}^{*\otimes n}$, and the second follows from the choice of basis and Proposition 3.7(1).

Furthermore, let $\widetilde{\mathrm{st}}''_{(m'_+,m'_-)}$ denote the forgetful morphism

$$\overline{\mathcal{M}}^G_+(m'_+)\times\overline{\mathcal{M}}^G_-(m'_-) \longrightarrow \overline{\mathcal{M}}_{g_+,n_++1}(\mathcal{B}G;\overline{\mathbf{m}}_{N_+},\overline{m}_+)\times\overline{\mathcal{M}}_{g_-,n_-+1}(\mathcal{B}G;\overline{m}_-,\overline{\mathbf{m}}_{N_-})$$

for all $m'_\pm$ in $\overline{m}_\pm$, then

$$\deg(\widetilde{\mathrm{st}}_{\overline{m}_+}\times\widetilde{\mathrm{st}}_{\overline{m}_-}) = \sum_{m'_\pm\in\overline{m}_\pm}\deg(\widetilde{\mathrm{st}}''_{(m'_+,m'_-)}) = \frac{|G|^2}{|C(\overline{m}_+)|^2}\deg(\widetilde{\mathrm{st}}''_{(m_+,m_-)}),$$

where in the second equality, we have used that $\deg(\widetilde{\mathrm{st}}''_{(m'_+,m'_-)})$ is independent of the choice $m'_\pm$ in $\overline{m}_\pm$, the fact that $\overline{m}'_\pm$ contains $\frac{|G|}{|C(\overline{m}_\pm)|}$ elements, and that $|C(\overline{m})| = |C(\overline{m}^{-1})|$ for all conjugacy classes $\overline{m}$ in G. Thus,

$$I = \sum_{\mu[m_\pm]}\deg(\widetilde{\mathrm{st}}'')\frac{|G|^2}{|C(\overline{m}_+)|^2}\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]})), \tag{42}$$

but $\widetilde{\mathrm{st}}'' = \widehat{r}\circ\widetilde{\mathrm{pr}}\circ\widetilde{r}$, hence,

$$I = \sum_{\mu[m_\pm]}\deg(\widehat{r})\deg(\widetilde{\mathrm{pr}}\circ\widetilde{r})\frac{|G|^2}{|C(\overline{m}_+)|^2}\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[m_+]\mu[m_-]}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]})). \tag{43}$$

Equating Equations (40) and (43) and canceling factors of $\deg(\widetilde{\mathrm{pr}}\circ\widetilde{r})$, we obtain

$$\widehat{r}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}}) = \frac{|G|^2}{|C(\overline{m}_+)|^2}\deg(\widehat{r})\sum_{\mu[\overline{m}_\pm]}\widehat{\Lambda}_+(\pi_G(e_{\mu[m_+]}))\eta^{\mu[\overline{m}_+]\mu[\overline{m}_-]}\widehat{\Lambda}_-(\pi_G(e_{\mu[m_-]})). \tag{44}$$

Let $\epsilon_{\mu[\overline{m}_\pm]} := \pi_G(e_{\mu[m_\pm]})$. Notice that the left hand side only depends upon $\overline{m}_\pm$ because of Equation (41). Since $\{e_{\mu[m_\pm]}\}$ is a basis for $\mathcal{H}^{C(m_\pm)}_{m_\pm}$, then by Proposition 3.7(2), $\{\epsilon_{\mu[\overline{m}_\pm]}\}$ is a basis for $\overline{\mathcal{H}}_{\overline{m}_\pm}$. Let

$$\overline{\eta}_{\mu[\overline{m}_+]\mu[\overline{m}_-]} := \overline{\eta}(\epsilon_{\mu[\overline{m}_+]},\epsilon_{\mu[\overline{m}_-]}) = \frac{1}{|G|}\eta(\epsilon_{\mu[\overline{m}_+]},\epsilon_{\mu[\overline{m}_-]}) = \frac{|C(m_+)|}{|G|^2}\eta_{\mu[m_+]\mu[m_-]}$$

where $\eta_{\mu[m_+]\mu[m_-]} = \eta(e_{\mu[m_+]}, e_{\mu[m_-]})$. Therefore, taking inverses,

$$\overline{\eta}^{\mu[\overline{m}_+]\mu[\overline{m}_-]} = \frac{|G|^2}{|C(\overline{m}_+)|}\eta^{\mu[m_+]\mu[m_-]} \tag{45}$$

and

$$\widehat{\eta}^{\mu[\overline{m}_+]\mu[\overline{m}_-]} = \frac{|G|^2}{|C(\overline{m}_+)|^2}\eta^{\mu[m_+]\mu[m_-]}$$

by Equation (38), so

$$\widehat{r}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}}) = \deg(\widehat{r}) \sum_{\mu[\overline{m}_\pm]} \widehat{\Lambda}_{g_+,n_++1}(\nu_{\overline{\mathbf{m}}_{N_+}}, \epsilon_{\mu[\overline{m}_+]}))\widehat{\eta}^{\mu[\overline{m}_+]\mu[\overline{m}_-]}\widehat{\Lambda}_{g_-,n_-+1}(\epsilon_{\mu[\overline{m}_-]}, \nu_{\overline{\mathbf{m}}_{N_-}}). \tag{46}$$

To conclude, note that $\widehat{\mathrm{pr}} = \widehat{\mathrm{st}}'' \circ \widehat{r}$ and $\deg(\widehat{\mathrm{pr}}) = \deg(\widehat{\mathrm{st}}')$, so

$$\deg(\widehat{r}) = \frac{\deg(\widehat{\mathrm{pr}})}{\deg(\widehat{\mathrm{st}}'')} = \frac{\deg(\widehat{\mathrm{st}}')}{\deg(\widehat{\mathrm{st}}'')}.$$

This finishes the tree case.

Suppose now that $\widehat{\Gamma}$ is a loop and that $\overline{\mathcal{M}}^G_{\widetilde{\Gamma}_{\mathrm{cut}}} = \overline{\mathcal{M}}^G_{g-1,n+2}(\overline{\mathbf{m}}, m_+, m_-)$. Following the analogous steps to the case of the tree, we obtain the counterpart of Equation (43):

$$I = \sum_{\mu[m_\pm]} \deg(\widehat{r})\deg(\widetilde{\mathrm{pr}}\circ\widetilde{r})\frac{|G|^2}{|C(\overline{m}_+)|^2}\widehat{\Lambda}_{g-1,n+2}(\nu_{\overline{\mathbf{m}}}, \pi_G(e_{\mu[m_+]}), \pi_G(e_{\mu[m_-]}))\eta^{\mu[m_+]\mu[m_-]}. \tag{47}$$

Proceeding further, the counterpart of Equation (46) is

$$\widehat{r}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(\nu_{\overline{\mathbf{m}}}) = \deg(\widehat{r}) \sum_{\mu[\overline{m}_\pm]} \widehat{\Lambda}_{g-1,n+2}(\nu_{\overline{\mathbf{m}}}, \epsilon_{\mu[\overline{m}_+]}), \epsilon_{\mu[\overline{m}_-]})\widehat{\eta}^{\mu[\overline{m}_+]\mu[\overline{m}_-]}. \tag{48}$$

The rest of the proof is essentially the same as in the case of the tree. □

5.2. From $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ to $\overline{\mathcal{M}}_{g,n}$.

Definition 5.7. Let $(\mathcal{H}, \eta, \Lambda_{g,n}, \mathbf{1})$ be a G-CohFT. Define $\overline{\Lambda}_{g,n} := \widehat{\mathrm{st}}_*\widehat{\Lambda}_{g,n}$ in $H^\bullet(\overline{\mathcal{M}}_{g,n}) \otimes \mathcal{H}^{*\otimes n}$.

Theorem 5.8. *If $(\mathcal{H}, \eta, \Lambda_{g,n}, \mathbf{1})$ is a G-CohFT, then $(\overline{\mathcal{H}}, \overline{\eta}, \overline{\Lambda}_{g,n}, \mathbf{1})$ forms a CohFT.*

Proof. We begin by observing that

$$\frac{1}{\deg\widehat{\mathrm{st}}}i^*\widehat{\mathrm{st}}_* = \frac{1}{\deg\widetilde{\mathrm{st}}'}\widehat{\mathrm{st}}'_*\widehat{i}^*, \tag{49}$$

since the lower right square is not Cartesian, due to ramification over $\overline{\mathcal{M}}_\Gamma$.

Next, we observe that

$$\mu^*\widehat{\mathrm{st}}'_* = \widehat{\mathrm{pr}}_*\widehat{\mu}^*, \tag{50}$$

since the lower left square is Cartesian by definition.

Therefore, for all $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$,

$$\begin{aligned}
\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) &= \rho_\Gamma^*\widehat{\mathrm{st}}_*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \mu^* i^*\widehat{\mathrm{st}}_*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \mu^*\left(\frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}')}\widehat{\mathrm{st}}'_*\widehat{i}^*\right)\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}')}\widehat{\mathrm{pr}}_*\widehat{\mu}^*\widehat{i}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}')}\widehat{\mathrm{st}}''_*(\widehat{r}_*\widehat{\mu}^*\widehat{i}^*)\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}')}\widehat{\mathrm{st}}''_*\left(\frac{\deg(\widehat{\mathrm{st}}')}{\deg(\widehat{\mathrm{st}}'')}\widehat{\Lambda}_{\widehat{\Gamma}_{\mathrm{cut}}}\right) \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}'')}\widehat{\mathrm{st}}''_*\widehat{\Lambda}_{\widehat{\Gamma}_{\mathrm{cut}}},
\end{aligned}$$

where Equations (37) and (39) have been used in the sixth equality.

Assume that $\widehat{\Gamma}$ is a tree. Adopting the notation from the proof of Theorem 5.5, we obtain

$$\begin{aligned}
\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) &= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}'')}\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]}\widehat{\mathrm{st}}''_*(\widehat{\Lambda}_+(e_{\beta[\overline{m}_+]})\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}\widehat{\Lambda}_-(e_{\beta[\overline{m}_-]})) \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}'')}\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]}\overline{\Lambda}_+(e_{\beta[\overline{m}_+]})\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}\overline{\Lambda}_-(e_{\beta[\overline{m}_-]}),
\end{aligned}$$

where $\overline{\Lambda}_\pm := \mathrm{st}_{\overline{m}_\pm *}\widehat{\Lambda}_\pm$. This can be rewritten as

(51)
$$\deg(\widehat{\mathrm{st}}'')\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \deg(\widehat{\mathrm{st}})\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]}\overline{\Lambda}_+(e_{\beta[\overline{m}_+]})\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}\overline{\Lambda}_-(e_{\beta[\overline{m}_-]}).$$

Following [JK], let

$$\Omega_{g,n}(\overline{\mathbf{m}}) := \deg(\widehat{\mathrm{st}}) \tag{52}$$

for all $\overline{\mathbf{m}}$ in $\overline{G}^n$. We have

$$\deg(\widehat{\mathrm{st}}'') = \Omega_{g_+,n_++1}(\overline{m}_{N_+},\overline{m}_+)\Omega_{g_-,n_-+1}(\overline{m}_-,\overline{m}_{N_-}),$$

and Equation (51) becomes, after multiplying both sides by $|C(\overline{m}_+)|$, using the definition of $\widehat{\eta}$, and summing over all conjugacy classes $\overline{m}_\pm$ such that $\overline{m}_- = \overline{m}_+^{-1}$,

$$\sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}}|C(\overline{m}_+)|\quad\Omega_{g_+,n_++1}(\overline{m}_{N_+},\overline{m}_+)\Omega_{g_-,n_-+1}(\overline{m}_-,\overline{m}_{N_-})\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) =$$
$$\Omega_{g,n}(\overline{\mathbf{m}})\sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}}\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]}\overline{\Lambda}_+(e_{\beta[\overline{m}_+]})\overline{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}\overline{\Lambda}_-(e_{\beta[\overline{m}_-]}).$$

But Lemma 3.5(1) from [JK] states that

$$\sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}}|C(\overline{m}_+)|\Omega_{g_+,n_++1}(\overline{m}_{N_+},\overline{m}_+)\Omega_{g_-,n_-+1}(\overline{m}_-,\overline{m}_{N_-}) = \Omega_{g,n}(\overline{\mathbf{m}}).$$

Therefore, by canceling $\Omega_{g,n}(\overline{\mathbf{m}})$ from both sides, we obtain the desired result.

In the case of the loop, we have

$$\begin{aligned}
\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) &= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}'')}\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]} \widehat{\mathrm{st}}''_*(\widehat{\Lambda}_{g-1,n+2}(v_{\overline{\mathbf{m}}}, e_{\beta[\overline{m}_+]}, e_{\beta[\overline{m}_-]}))\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]} \\
&= \frac{\deg(\widehat{\mathrm{st}})}{\deg(\widehat{\mathrm{st}}'')}\sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]} \overline{\Lambda}_{g-1,n+2}(v_{\overline{\mathbf{m}}}, e_{\beta[\overline{m}_+]}, e_{\beta[\overline{m}_-]})\widehat{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}.
\end{aligned}$$

Multiplying both sides by $\deg(\widehat{\mathrm{st}}'')|C(\overline{m}_+)|$, plugging in $\deg(\widehat{\mathrm{st}}'') = \Omega_{g-1,n+2}(\overline{\mathbf{m}}, \overline{m}_+, \overline{m}_-)$, $\deg(\widehat{\mathrm{st}}) = \Omega_{g,n}(\overline{\mathbf{m}})$, and then summing over all conjugacy classes $\overline{m}_\pm$ such that $\overline{m}_- = \overline{m}_+^{-1}$, we obtain

$$\begin{aligned}
&\sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}} |C(\overline{m}_+)| \quad \Omega_{g-1,n+2}(\overline{\mathbf{m}}, \overline{m}_+, \overline{m}_-)\rho_\Gamma^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \\
&\qquad \Omega_{g,n}(\overline{\mathbf{m}}) \sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}} \sum_{\beta[\overline{m}_+],\beta[\overline{m}_-]} \overline{\Lambda}_{g-1,n+2}(v_{\overline{\mathbf{m}}}, e_{\beta[\overline{m}_+]}, e_{\beta[\overline{m}_-]})\overline{\eta}^{\beta[\overline{m}_+]\beta[\overline{m}_-]}.
\end{aligned}$$

Since Lemma 3.5(2) from [JK] states that

$$\sum_{\overline{m}_\pm:\overline{m}_-=\overline{m}_+^{-1}} |C(\overline{m}_+)|\Omega_{g-1,n+2}(\overline{\mathbf{m}}, \overline{m}_+, \overline{m}_-) = \Omega_{g,n}(\overline{\mathbf{m}}),$$

we may cancel $\Omega_{g,n}(\overline{\mathbf{m}})$ from both sides to obtain the desired result.

This completes the proof of the factorization axiom of the CohFT.

The invariance under the symmetric group is manifest.

The flat identity axiom follows from considering the following commuting diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}^G_{g,n+1}(\overline{\mathbf{m}}, 1) \arrow[r, "\widetilde{\tau}"] \arrow[d, "\widetilde{\mathrm{st}}_1"] & \overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}) \arrow[d, "\widetilde{\mathrm{st}}"] \\
\overline{\mathcal{M}}_{g,n+1}(\mathcal{B}G; \overline{\mathbf{m}}, 1) \arrow[r, "\widehat{\tau}"] \arrow[d, "\widehat{\mathrm{st}}_1"] & \overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}}) \arrow[d, "\widehat{\mathrm{st}}"] \\
\overline{\mathcal{M}}_{g,n+1} \arrow[r, "\tau"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

The horizontal morphisms are forgetting-tails morphisms and are both flat and proper. The vertical morphisms are forgetful morphisms and are all quasi-finite, flat, and proper.

By Lemma 5.1 We have

$$\begin{aligned}
\widetilde{\mathrm{st}}_1^*\widehat{\Lambda}_{g,n+1}(v_{\overline{\mathbf{m}}}, \mathbf{1}) &= \Lambda_{g,n+1}(v_{\overline{\mathbf{m}}}, \mathbf{1}) \\
&= \widetilde{\tau}^*\Lambda_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \widetilde{\tau}^*\widetilde{\mathrm{st}}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \\
&= \widetilde{\mathrm{st}}_1^*\widehat{\tau}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}})
\end{aligned}$$

By the uniqueness of the classes $\widehat{\Lambda}$ (again, see Lemma 5.1) we conclude that

$$\widehat{\tau}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \widehat{\Lambda}_{g,n+1}(v_{\overline{\mathbf{m}}}, \mathbf{1}). \tag{53}$$

On the other hand, while the bottom square of this diagram is not Cartesian, it is almost so—the stack $\overline{\mathcal{M}}^G_{g,n+1}(\mathcal{B}G; \overline{\mathbf{m}}, 1)$ is the universal *orbicurve* over $\overline{\mathcal{M}}^G_{g,n}(\mathcal{B}G; \overline{\mathbf{m}})$, and it is birational to its coarse moduli space, the universal curve over $\overline{\mathcal{M}}^G_{g,n}(\mathcal{B}G; \overline{\mathbf{m}})$. Thus, we have

$$\tau^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \tau^*\widehat{\mathrm{st}}_*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \mathrm{st}_{1*}\widehat{\tau}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \widehat{\mathrm{st}}_{1*}\widehat{\Lambda}_{g,n+1}(v_{\overline{\mathbf{m}}}, \mathbf{1}) = \overline{\Lambda}_{g,n+1}(v_{\overline{\mathbf{m}}}, \mathbf{1}).$$

The last property that must be verified is

$$\overline{\eta}(v_{\overline{m}_+}, v_{\overline{m}_-}) = \overline{\Lambda}_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) \tag{54}$$

for all $v_{\overline{m}_\pm}$ in $\overline{\mathcal{H}}_\pm$, where we have identified $H^\bullet(\overline{\mathcal{M}}_{0,3})$ with the ground ring $\mathbb{C}$. Since this identity holds trivially if $\overline{m}_- \neq \overline{m}_+^{-1}$, let us assume that $\overline{m}_- = \overline{m}_+^{-1}$.

We have the morphisms

$$\coprod_{m'_+ \in \overline{\mathbf{m}}_+} \xi_{0,3}(m'_+, m'^{-1}_+, 1) \xrightarrow{\widetilde{\mathrm{st}}_\xi} \overline{\mathcal{M}}_{0,3}(\mathcal{B}G; \overline{m}_+, \overline{m}_+^{-1}, 1) \xrightarrow{\widehat{\mathrm{st}}} \overline{\mathcal{M}}_{0,3},$$

and we let $\mathrm{st}_\xi := \widehat{\mathrm{st}} \circ \widetilde{\mathrm{st}}_\xi$. Since η is defined by

$$\eta(v_{m_+}, v_{m_-}) = \int_{[\![\xi(m_+,m_-,1)]\!]} \Lambda_{0,3}(v_{m_+}, v_{m_-}, \mathbf{1})\mathrm{st}_{\xi*}\Lambda_{0,3}(v_{m_+}, v_{m_-}, \mathbf{1}),$$

we have

$$\begin{aligned}
\overline{\Lambda}_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) &= \widehat{\mathrm{st}}_*\widehat{\Lambda}_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) \\
&= \widehat{\mathrm{st}}_*(\widetilde{\mathrm{st}}_{\xi*}\widetilde{\mathrm{st}}^*_\xi \frac{1}{\deg \widetilde{\mathrm{st}}_\xi})\widehat{\Lambda}_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) \\
&= \frac{1}{\deg \widetilde{\mathrm{st}}_\xi}\widehat{\mathrm{st}}_*\widetilde{\mathrm{st}}_{\xi*}\Lambda_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) \\
&= \frac{1}{\deg \widetilde{\mathrm{st}}_\xi}\mathrm{st}_{\xi*}\Lambda_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1}) \\
&= \frac{1}{\deg \widetilde{\mathrm{st}}_\xi}\eta(v_{\overline{m}_+}, v_{\overline{m}_-}),
\end{aligned}$$

but

$$\deg(\widetilde{\mathrm{st}}_\xi) = |\overline{m}_+||C(\overline{m}_+)| = |G|,$$

since a generic point of $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{m}_+, \overline{m}_-, 1)$ has automorphism group isomorphic to $C(m_+)$. Therefore, Equation (54) is satisfied. $\square$

Remark 5.9. The CohFT $(\overline{\mathcal{H}}, \overline{\eta}, \{\overline{\Lambda}_{g,n}\}, \mathbf{1})$ constructed above has more structure than a generic CohFT, as it is $\overline{G}$-graded; that is, $(\overline{\mathcal{H}}, \overline{\eta})$ is a $\overline{G}$-graded vector space with metric, and for all $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$, the class $\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}})$ vanishes unless there exist representatives m'_i in $\overline{m}_i$ for all $i = 1, \ldots, n$ such that $\prod_{i=1}^n m'_i$ belongs to the subgroup $[G, G]^g$. This is follows from the fact that $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G; \overline{\mathbf{m}})$ is empty unless this holonomy condition holds.

Proposition 5.10. *Let $(\mathcal{H}, \eta, \{\Lambda_{g,n}\}, \mathbf{1})$ be a G-CohFT. For all nonzero λ in $\mathbb{C}$, $(\mathcal{H}, \lambda^{-2}\eta, \{\lambda^{2g-2}\Lambda_{g,n}\}, \mathbf{1})$ is a G-CohFT.*

The proof is immediate from the definition.

Remark 5.11. One can eliminate the annoying factor of $\frac{1}{|G|}$ in the definition of $\overline{\eta}$ by choosing λ such that $\lambda^2 = \frac{1}{|G|}$. In this case, the associated "quotient" by G of the G-CohFT $(\mathcal{H}, |G|\eta, \{|G|^{1-g}\Lambda_{g,n}\}, \mathbf{1})$ is the CohFT $(\overline{\mathcal{H}}, |G|\overline{\eta}, \{|G|^{1-g}\Lambda_{g,n}\}, \mathbf{1})$, but $|G|\overline{\eta}$ is equal to the restriction of η to $\overline{\mathcal{H}}$.

5.3. **Coinvariants of G-Frobenius algebras.**

Let $((\mathcal{H}, \rho), \cdot, \mathbf{1}, \eta)$ be a G-Frobenius algebra. We now have two ways to endow its space of coinvariants $\overline{\mathcal{H}}$ with the structure of a Frobenius algebra. The first is purely algebraic. The tuple $(\overline{\mathcal{H}}, \cdot, \mathbf{1}, \eta)$ is a Frobenius algebra where the multiplication on $\overline{\mathcal{H}}$ is inherited by restriction from the multiplication on $\mathcal{H}$ and the metric η is the restriction of the metric on $\mathcal{H}$.

The second is to apply the geometric procedure described in the previous subsection to $\mathcal{H}$, regarded as a G-CohFT, to induce the structure of a Frobenius algebra on $\overline{\mathcal{H}}$. It turns out that these two Frobenius structures are identical after a rescaling.

In order to simplify the proof, we note that the structure of the G-Frobenius algebra $((\mathcal{H}, \rho), \cdot, \mathbf{1}, \eta)$ can also be described as the tuple $((\mathcal{H}, \rho), \mu, \mathbf{1})$, where μ belongs to $\mathcal{H}^{*\otimes 3}$ and is defined by

$$\mu(v_{m_1}, v_{m_2}, v_{m_3}) := \eta(v_{m_1}, v_{m_2} \cdot v_{m_3}), \tag{55}$$

since it follows that $\eta(v_{m_1}, v_{m_2}) = \eta(v_{m_1}, v_{m_2} \cdot \mathbf{1})$. If $\widetilde{\mu}$ denotes the restriction of μ to $\overline{\mathcal{H}}$, then the data $(\overline{\mathcal{H}}, \widetilde{\mu}, \mathbf{1})$ is an equivalent description of the Frobenius algebra structure on $\overline{\mathcal{H}}$ induced by restriction.

Proposition 5.12. *Let $((\mathcal{H}, \rho), \mu, \mathbf{1})$ be a G-Frobenius algebra arising from a G-CohFT $(\mathcal{H}, \eta, \{\Lambda_{g,n}\}, \mathbf{1})$. The Frobenius algebra structure on $\overline{\mathcal{H}}$ arising from the CohFT $(\mathcal{H}, \overline{\eta}, \{\overline{\Lambda}_{g,n}\}, \mathbf{1})$ is $(\overline{\mathcal{H}}, \overline{\mu}, \mathbf{1})$, where*

$$\overline{\mu} = \frac{1}{|G|}\widetilde{\mu},$$

and $\widetilde{\mu}$ is the restriction of μ to $\overline{\mathcal{H}}$.

Remark 5.13. The Frobenius algebra $(\overline{\mathcal{H}}, \overline{\mu}, \mathbf{1})$ can also be described as the tuple $(\overline{\mathcal{H}}, \cdot, \overline{\eta}, \mathbf{1})$, where the multiplication $\cdot$ on $\overline{\mathcal{H}}$ is inherited from the multiplication on $\mathcal{H}$, but where $\overline{\eta}$ is the restriction of $\frac{1}{|G|}\eta$ to $\overline{\mathcal{H}}$.

Proof. (of Proposition 5.12)

Since $\overline{\mu}(v_{\overline{\mathbf{m}}}) = \overline{\Lambda}_{0,3}(v_{\overline{\mathbf{m}}})$, after identifying $H^\bullet(\overline{\mathcal{M}}_{0,3})$ with $\mathbb{C}$, we need only prove that

$$\overline{\Lambda}_{0,3}(v_{\overline{\mathbf{m}}}) = \frac{1}{|G|}\mu(v_{\overline{\mathbf{m}}})$$

for all $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$. In order to proceed, let us introduce some notation.

For all $\mathbf{m} := (m_1, m_2, m_3)$ belonging to G^3 such that $m_1 m_2 m_3 = 1$, we have the following forgetful morphisms

$$\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}}) \xrightarrow{\widetilde{\mathrm{st}}} \overline{\mathcal{M}}_{0,3}(\mathcal{B}G; \overline{\mathbf{m}}) \xrightarrow{\widehat{\mathrm{st}}} \overline{\mathcal{M}}_{0,3},$$

and we let $\mathrm{st} := \widehat{\mathrm{st}} \circ \widetilde{\mathrm{st}}$.

Furthermore, if Q is a substack of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$ then we let $\widetilde{\mathrm{st}}_Q$ denote the restriction of $\widetilde{\mathrm{st}}$ to Q. Let

$$\xi := \coprod_{\mathbf{m}' \in \chi(\overline{\mathbf{m}})} \xi(\mathbf{m}'),$$

where

$$\chi(\overline{\mathbf{m}}) := \{(m_1', m_2', m_3') \in \overline{\mathbf{m}} | m_1' m_2' m_3' = 1\}.$$

Henceforth, fix an element $\mathbf{m}$ in $\chi(\overline{\mathbf{m}})$ once and for all.

Let us adopt the notation that for any $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$, and for any $\mathbf{m}' \in \overline{\mathbf{m}}$, the vector $v_{\mathbf{m}'}$ denotes the $\mathbf{m}'$-graded component of $v_{\overline{\mathbf{m}}}$; that is,

$$v_{\overline{\mathbf{m}}} =: \sum_{\mathbf{m}' \in \overline{\mathbf{m}}} v_{\mathbf{m}'}.$$

Note that $v_{\mathbf{m}'}$ belongs to the subspace of $C(\mathbf{m}')$-invariant vectors in $\mathcal{H}_{\mathbf{m}'}$, where $C(\mathbf{m}') := C(m_1') \times C(m_2') \times C(m_3')$.

For all $\mathbf{m}'$ in $\chi(\overline{\mathbf{m}})$, we have

$$\mu(v'_{\mathbf{m}}) = \frac{1}{\deg(\mathrm{st}_{\xi(\mathbf{m}')})} \mathrm{st}_{\xi(\mathbf{m}')*} \Lambda_{0,3}(v'_{\mathbf{m}}).$$

Otherwise, $\mu(v'_{\mathbf{m}}) = 0$. Since $\widetilde{\mu}$ is the restriction of μ to $\overline{\mathcal{H}}$,

$$\begin{aligned} \widetilde{\mu}(v_{\overline{\mathbf{m}}}) &= \sum_{\mathbf{m}' \in \chi(\overline{\mathbf{m}})} \frac{1}{\deg(\mathrm{st}_{\xi(\mathbf{m}')})} \mathrm{st}_{\xi(\mathbf{m}')*} \Lambda_{0,3}(v_{\mathbf{m}'}) \\ &= \frac{1}{\deg(\mathrm{st}_{\xi(\mathbf{m})})} \sum_{\mathbf{m}' \in \chi(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m}')*} \Lambda_{0,3}(v_{\mathbf{m}'}) \\ &= \frac{1}{\deg(\mathrm{st}_{\xi(\mathbf{m})})} \mathrm{st}_{\xi*} \Lambda_{0,3}(v_{\overline{\mathbf{m}}}), \end{aligned}$$

where the second equality comes from the fact that the degree of st restricted to any connected component of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$ is independent of the choice of connected component. This statement follows from the fact that every connected component of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$ is $\rho(\boldsymbol{\gamma})\xi(\mathbf{m})$ for some $\boldsymbol{\gamma}$ in G^3, but $\rho(\boldsymbol{\gamma})$ is an isomorphism.

However, we have

$$\begin{aligned} \mathrm{st}_{\xi*} \Lambda_{0,3}(v_{\overline{\mathbf{m}}}) &= \sum_{\mathbf{m}' \in \chi(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m}')*} \Lambda_{0,3}(v_{\mathbf{m}'}) \\ &= |\chi(\overline{\mathbf{m}})| \mathrm{st}_{\xi(\mathbf{m})*} \Lambda_{0,3}(v_{\mathbf{m}}) \\ &= |G| \Omega_{0,3}(\overline{\mathbf{m}}) \mathrm{st}_{\xi(\mathbf{m})*} \Lambda_{0,3}(v_{\mathbf{m}}), \end{aligned}$$

where the second equality follows from the observation that every connected component of ξ can be obtained by the action of some element of G^3, and the fact that $v_{\overline{\mathbf{m}}}$ and $\Lambda_{0,3}$ are G^3-invariant. The third equality is from Proposition 3.4 of [JK], where $\Omega_{0,3}$ is defined in Equation (52). Therefore, we obtain

$$\widetilde{\mu}(v_{\overline{\mathbf{m}}}) = \frac{|G| \Omega_{0,3}(\overline{\mathbf{m}})}{\deg(\mathrm{st}_{\xi(\mathbf{m})})} \mathrm{st}_{\xi(\mathbf{m})*} \Lambda_{0,3}(v_{\mathbf{m}}). \tag{56}$$

On the other hand, the definition of $\widehat{\Lambda}_{0,3}$ implies that

$$\widehat{\Lambda}_{0,3}(v_{\overline{\mathbf{m}}}) = \frac{1}{\deg(\widetilde{\mathrm{st}})} \widetilde{\mathrm{st}}_* \Lambda_{0,3}(v_{\overline{\mathbf{m}}}),$$

hence

$$\begin{aligned}\overline{\Lambda}_{0,3}(\nu_{\overline{\mathbf{m}}}) &= \widehat{\mathrm{st}}_*\widehat{\Lambda}_{0,3}(\nu_{\overline{\mathbf{m}}}) \\ &= \frac{1}{\deg(\widetilde{\mathrm{st}})}\widehat{\mathrm{st}}_*\widetilde{\mathrm{st}}_*\Lambda_{0,3}(\nu_{\overline{\mathbf{m}}}),\end{aligned}$$

and we obtain

$$\overline{\Lambda}_{0,3}(\nu_{\overline{\mathbf{m}}}) = \frac{1}{\deg(\widetilde{\mathrm{st}})}\mathrm{st}_*\Lambda_{0,3}(\nu_{\overline{\mathbf{m}}}). \tag{57}$$

Using the fact that $\deg(\mathrm{st}_Q)$ is independent of the choice of connected component Q of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$, we can write

$$\deg(\widetilde{\mathrm{st}}) = A(\overline{\mathbf{m}})\deg(\widetilde{\mathrm{st}}_{\xi(\mathbf{m})}), \tag{58}$$

where $A(\overline{\mathbf{m}})$ is the number of connected components of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$. Similarly, let $I(\overline{\mathbf{m}})$ consist of all elements $\boldsymbol{\gamma}$ in G^3 such that the collection $\{\rho(\boldsymbol{\gamma})\xi(\mathbf{m})\}$ is in one-to-one correspondence with the connected components of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$, then

$$\begin{aligned}\mathrm{st}_*\Lambda_{0,3}(\nu_{\overline{\mathbf{m}}}) &= \sum_{\boldsymbol{\gamma}\in I(\overline{\mathbf{m}})} \mathrm{st}_{\rho(\boldsymbol{\gamma})\xi(\mathbf{m})*}\Lambda_{0,3}(\nu_{\rho(\boldsymbol{\gamma})\mathbf{m}}) \\ &= \sum_{\boldsymbol{\gamma}\in I(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m})*}\rho(\boldsymbol{\gamma}^{-1})_*\Lambda_{0,3}(\nu_{\rho(\boldsymbol{\gamma})\mathbf{m}}) \\ &= \sum_{\boldsymbol{\gamma}\in I(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m})*}\rho(\boldsymbol{\gamma})^*\Lambda_{0,3}(\nu_{\rho(\boldsymbol{\gamma})\mathbf{m}}) \\ &= \sum_{\boldsymbol{\gamma}\in I(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(\rho(\boldsymbol{\gamma}^{-1})\nu_{\rho(\boldsymbol{\gamma})\mathbf{m}}) \\ &= \sum_{\boldsymbol{\gamma}\in I(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(\nu_{\mathbf{m}}) \\ &= A(\overline{\mathbf{m}})\mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(\nu_{\mathbf{m}}),\end{aligned} \tag{59}$$

where the first equality is the sum over contributions from each connected components of $\overline{\mathcal{M}}^G_{0,3}(\overline{\mathbf{m}})$, and the second is from the fact that, for all $\boldsymbol{\gamma}$ in G^3, we have

$$\mathrm{st}_{\xi(\mathbf{m})} = \mathrm{st}_{\rho(\boldsymbol{\gamma})\xi(\mathbf{m})} \circ \rho(\boldsymbol{\gamma}). \tag{60}$$

The fourth equality is from the G^3-invariance of $\Lambda_{0,3}$ and the fifth is from the G^3-invariance of $\nu_{\overline{\mathbf{m}}}$. Putting together Equations (57), (58), and (59), we obtain

$$\begin{aligned}\overline{\Lambda}_{0,3}(\nu_{\overline{\mathbf{m}}}) &= \frac{1}{\deg(\widetilde{\mathrm{st}}_{\xi(\mathbf{m})})}\mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(\nu_{\mathbf{m}}) \\ &= \frac{\Omega_{0,3}(\overline{\mathbf{m}})}{\deg(\mathrm{st}_{\xi(\mathbf{m})})}\mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(\nu_{\mathbf{m}}),\end{aligned}$$

since

$$\deg(\mathrm{st}_{\xi(\mathbf{m})}) = \deg(\widetilde{\mathrm{st}}_{\xi(\mathbf{m})})\deg(\widehat{\mathrm{st}}) = \deg(\widetilde{\mathrm{st}}_{\xi(\mathbf{m})})\Omega_{0,3}(\overline{\mathbf{m}}).$$

However,

$$\begin{aligned}
\mathrm{st}_*\Lambda_{0,3}(v_{\overline{\mathbf{m}}}) &= \sum_{\mathbf{m}'\in\chi(\overline{\mathbf{m}})} \mathrm{st}_{\xi(\mathbf{m}')*}\Lambda_{0,3}(v_{\mathbf{m}'}) \\
&= |\chi(\overline{\mathbf{m}})|\mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(v_{\mathbf{m}}) \\
&= |G|\Omega_{0,3}(\overline{\mathbf{m}})\mathrm{st}_{\xi(\mathbf{m})*}\Lambda_{0,3}(v_{\mathbf{m}}).
\end{aligned}$$

Putting this all together, we obtain the desired result

$$\begin{aligned}
\overline{\Lambda}_{0,3}(v_{\overline{\mathbf{m}}}) &= \frac{\Omega_{0,3}(\overline{\mathbf{m}})}{\deg(\mathrm{st}_{\xi(\mathbf{m})})}\frac{1}{|G|\Omega_{0,3}(\overline{\mathbf{m}})}\mathrm{st}_{\xi*}\Lambda_{0,3}(v_{\overline{\mathbf{m}}}) \\
&= \frac{1}{|G|}\widetilde{\mu}(v_{\overline{\mathbf{m}}}).
\end{aligned}$$

□

The results of this section can be applied to the example of the group algebra G-CohFT and its associated G-Frobenius algebra to yield the (stringy) orbifold cohomology of a point with trivial G-action.

Proposition 5.14. *The Frobenius algebra $\overline{\mathcal{H}}$ induced from the G-Frobenius algebra $\mathcal{H} = \mathbb{C}[G]$ is the Frobenius algebra $Z(\mathbb{C}[G])$, the center of the group ring, with its induced multiplication, identity, and the metric $\overline{\eta}$.*

The resulting Frobenius algebra is isomorphic to the orbifold (stringy) quantum cohomology of $\mathcal{B}G$, the classifying stack of G.

We refer the reader to [JK] where the calculation is worked through in detail.

5.4. The quotient stack $[\overline{\mathcal{M}}^G_{g,n}/G^n]$.

The process of obtaining a CohFT from a G-CohFT involved the stack $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$. However, there is another stack that one could have used instead, namely, the quotient stack $\mathcal{Q}_{g,n} := [\overline{\mathcal{M}}^G_{g,n}/G^n]$ and its substacks $\mathcal{Q}_{g,n}(\overline{\mathbf{m}}) := [\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}})/G^n]$. We will show that one can construct a CohFT by replacing $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ by $\mathcal{Q}_{g,n}(\overline{\mathbf{m}}) := [\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}})/G^n]$, but that the resulting CohFT is isomorphic to the original one.

We have the following sequence of forgetful morphisms

$$\overline{\mathcal{M}}^G_{g,n}(\overline{\mathbf{m}}) \xrightarrow{\check{\mathrm{st}}'} \mathcal{Q}_{g,n}(\overline{\mathbf{m}}) \xrightarrow{\check{\mathrm{st}}} \overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}}) \xrightarrow{\widehat{\mathrm{st}}} \overline{\mathcal{M}}_{g,n}, \tag{61}$$

where $\widetilde{\mathrm{st}} := \check{\mathrm{st}} \circ \check{\mathrm{st}}'$. The stack $\mathcal{Q}$ is a smooth, Deligne-Mumford stack, and all of these morphisms are proper and flat. Observe that while the morphism $\check{\mathrm{st}}$ induces an isomorphism at the level of the corresponding coarse moduli spaces, they are not isomorphic as stacks, since an object in $\mathcal{Q}_{g,n}(\overline{\mathbf{m}})$ has a larger automorphism group than the corresponding object in $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G;\overline{\mathbf{m}})$.

Definition 5.15. Let $((\mathcal{H},\rho),\eta,\{\Lambda_{g,n}\},\mathbf{1})$ be a G-CohFT. Define the elements $\check{\Lambda}_{g,n}$ in $\bigoplus_{\overline{\mathbf{m}}\in\overline{G}^n} H^\bullet(\mathcal{Q}_{g,n}(\overline{\mathbf{m}}))\otimes\overline{\mathcal{H}}^*_{\overline{\mathbf{m}}}$ via

$$\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) := \check{\mathrm{st}}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) \tag{62}$$

for all $v_{\overline{\mathbf{m}}}$ in $\overline{\mathcal{H}}_{\overline{\mathbf{m}}}$ and $\overline{\mathbf{m}}$ in $\overline{G}^n$. Define $\overline{\Lambda}'_{g,n}$ in $H^\bullet(\overline{\mathcal{M}}_{g,n})\otimes\overline{\mathcal{H}}^*_{\overline{\mathbf{m}}}$ via

$$\overline{\Lambda}'_{g,n}(v_{\overline{\mathbf{m}}}) := (\widehat{\mathrm{st}}\circ\check{\mathrm{st}})_*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}).$$

Let

$$\overline{\eta}'(v_{\overline{m}_+}, v_{\overline{m}_-}) := \overline{\Lambda}'_{0,3}(v_{\overline{m}_+}, v_{\overline{m}_-}, \mathbf{1})$$

for all $v_{\overline{m}_\pm}$ in $\overline{\mathcal{H}}_{\overline{m}_\pm}$.

Proposition 5.16. *Let $((\mathcal{H},\rho),\eta,\{\Lambda_{g,n}\},\mathbf{1})$ be a G-CohFT.*

(1) *We have the identity*

$$\Lambda_{g,n}(v_{\overline{\mathbf{m}}}) = \check{\mathrm{st}}'^*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}).$$

(2) *We also have*

$$\overline{\Lambda}'_{g,n}(v_{\overline{\mathbf{m}}}) := \left(\prod_{i=1}^{n}\frac{1}{k_{\overline{m}_i}}\right)\Lambda_{g,n}(v_{\overline{\mathbf{m}}}),$$

where $k_{\overline{m}}$ is the order of the cyclic subgroup generated by any representative of $\overline{m}$ in $\overline{G}$.

(3) *$(\overline{\mathcal{H}},\overline{\eta}',\{\overline{\Lambda}'_{g,n}\},\mathbf{1})$ is a CohFT.*

(4) *The linear map $\phi:\overline{\mathcal{H}}\longrightarrow\overline{\mathcal{H}}$, where*

$$\phi(v_{\overline{m}}) := k_{\overline{m}}v_{\overline{m}}$$

for all $v_{\overline{m}}$ in $\overline{\mathcal{H}}_{\overline{m}}$ and $\overline{m}$ in $\overline{G}$, is an isomorphism between the CohFTs $(\overline{\mathcal{H}},\overline{\eta},\{\overline{\Lambda}_{g,n}\},\mathbf{1})$ and $(\overline{\mathcal{H}},\overline{\eta}',\{\overline{\Lambda}'_{g,n}\},\mathbf{1})$.

Proof. Since $\check{\mathrm{st}}'^*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \widetilde{\mathrm{st}}^*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \check{\mathrm{st}}'^*\check{\mathrm{st}}^*\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}})$, we obtain

$$\Lambda_{g,n}(v_{\overline{\mathbf{m}}}) = \check{\mathrm{st}}'^*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}).$$

For the second part, apply $\check{\mathrm{st}}_*$ to both sides of equation (62) and use the fact that

$$\deg(\check{\mathrm{st}}) = \prod_{i=1}^{n}\frac{1}{k_{\overline{m}_i}}$$

to get

$$\check{\mathrm{st}}_*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \left(\prod_{i=1}^{n}\frac{1}{k_{\overline{m}_i}}\right)\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}).$$

Thus,

$$\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) = \left(\prod_{i=1}^{n}k_{\overline{m}_i}\right)\check{\mathrm{st}}_*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}).$$

However,

$$\begin{aligned}\overline{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}}) &= \widehat{\mathrm{st}}_*\widehat{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}})\\ &= \left(\prod_{i=1}^{n}k_{\overline{m}_i}\right)\widehat{\mathrm{st}}_*\check{\mathrm{st}}_*\check{\Lambda}_{g,n}(v_{\overline{\mathbf{m}}})\\ &= \left(\prod_{i=1}^{n}k_{\overline{m}_i}\right)\overline{\Lambda}'_{g,n}(v_{\overline{\mathbf{m}}}).\end{aligned}$$

This establishes the second part of the proposition.

Clearly, $\phi^*\overline{\Lambda}'_{g,n} = \overline{\Lambda}_{g,n}$, $\phi^*\overline{\eta}' = \overline{\eta}$, and $\phi(\mathbf{1}) = \mathbf{1}$. Since $(\overline{\mathcal{H}},\overline{\eta},\{\overline{\Lambda}_{g,n}\},\mathbf{1})$ is a CohFT, so is $(\overline{\mathcal{H}},\overline{\eta}',\{\overline{\Lambda}'_{g,n}\},\mathbf{1})$, and ϕ is an isomorphism. □

Remark 5.17. A similar rescaling was observed in [AGV], and the previous proposition could be regarded as its origin in the framework of G-CohFTs.

6. G-stable maps

In this section we briefly describe the main source of examples of G-CohFTs; namely, Gromov-Witten style classes on the moduli space of G-stable maps.

Definition 6.1. A *genus* g*,* n*-pointed* G*-stable map* over a base T into a global quotient $[X/G]$ is a G-equivariant morphism $f : E \longrightarrow X$ from an admissible G-cover $\pi : E \longrightarrow C$ of a genus g prestable curve C/T with n sections $\widetilde{p}_i : T \longrightarrow E$ such that the induced morphism of stacks $\bar{f} : [E/G] \longrightarrow [X/G]$ with marked points $p_i := \pi \circ \widetilde{p}_i$ is an n-pointed orbifold (a.k.a. twisted) stable map of genus g (as defined in [CR1, AGV]).

We denote the stack of genus g, n-pointed G-stable maps by $\overline{\mathcal{M}}^G_{g,n}(X)$, and if $\beta \in H_2(X/G, \mathbb{Z})$, then we denote the substack of maps whose image lies in the homology class β by $\overline{\mathcal{M}}^G_{g,n}(X, \beta)$.

Theorem 6.2. *If the quotient* $[X/G]$ *admits a projective coarse moduli space* X/G*, then the stack* $\overline{\mathcal{M}}^G_{g,n}([X/G], \beta)$ *is a proper Deligne-Mumford stack, which itself admits a projective coarse moduli space.*

The proof follows from the results of [AGV] in essentially the same way that Theorem 2.4 follows from the results of [ACV].

There is a natural forgetful morphism $\mathrm{st}_{(X,\beta)} : \overline{\mathcal{M}}^G_{g,n}([X/G], \beta) \longrightarrow \overline{\mathcal{M}}^G_{g,n}$ obtained by forgetting the morphism f and contracting components in a manner similar to that described in the definition of the forgetting tails morphism of Section 2. There are also natural evaluation morphisms ev_i from $\overline{\mathcal{M}}^G_{g,n}([X/G], \beta)$ to the space

$$\widehat{X} := \{(x, g) | x \in X,\ g \in \mathrm{stab}(x)\} = \coprod_{g \in G} X^g \subseteq X \times G,$$

with $ev_i((f : E \longrightarrow X, \widetilde{p}_i)) = (f(\widetilde{p}_i), m_i)$, where m_i is the monodromy of E around $\widetilde{p}_i$ and X^g is the fixed point locus in X of the subgroup $\langle g \rangle \subseteq G$. These are compatible in the sense that the following diagram commutes

$$\begin{tikzcd}
\overline{\mathcal{M}}^G_{g,n}(X, \beta) \arrow[r, "ev_i"] \arrow[d, "\mathrm{st}_{(X,\beta)}"'] & \widehat{X} \arrow[d, "\mathrm{pr}_2"] \\
\overline{\mathcal{M}}^G_{g,n} \arrow[r, "ev_i"] & G
\end{tikzcd},$$

where the map pr_2 is the projection onto the second factor and the lower map ev_i is the ith component of the map $\mathbf{e}$ of Definition 2.3.

Definition 6.3. We denote by $\overline{\mathcal{M}}^G_{g,n}(X, \beta, \mathbf{m})$ the component $\mathrm{st}_{(X,\beta)}{}^{-1}(\overline{\mathcal{M}}^G_{g,n}(\mathbf{m}))$ that maps to $\mathbf{m} \in G^n$ via $\mathbf{e} \circ \mathrm{st}_{(X,\beta)}$.

Definition 6.4. Let $\mathcal{H}(X) := H^{2\bullet}(\widehat{X}; \Theta) = \bigoplus_{m \in G} \mathcal{H}(X)_m$, where $\mathcal{H}(X)_m := H^{2\bullet}(X^m; \Theta)$, and Θ is the usual ring (see [Ma]) associated to X with generators $\{q^\beta\}$ over $\mathbb{C}$, satisfying $q^{\beta+\beta'} = q^\beta q^{\beta'}$.

Remark 6.5. Of course, one could allow odd-dimensional cohomology classes as well, after inserting the necessary signs for skew-symmetry, but for simplicity we will work only with even-dimensional classes.

In a subsequent paper [IP], we will describe the details of how the classes

$$\Lambda^{G,X}_{g,n}(\nu_1,\ldots,\nu_n) := \sum_{\beta} st_{(X,\beta)_*}\Big(\prod_{i=1}^{n} ev_i^*(\nu_i)\cap[\overline{\mathcal{M}}^G_{g,n}(X)]^{vir}\Big)q^{\beta}$$

form a G-CohFT, and how the CohFT of coinvariants of $\{\Lambda^{G,X}_{g,n}\}$ agrees with the orbifold Gromov-Witten classes of Chen-Ruan [CR2].

In the remainder of this section we will briefly treat two special cases. In Subsection 6.1 we describe the case of $\beta = 0$, and show that it gives the ring $H^{\bullet}(X, G)$ of Fantechi and Göttsche—and therefore the stringy orbifold cohomology of Chen and Ruan—as special cases. In Subsection 6.2 we describe the G-CohFT $\{\Lambda^{G,X}_{g,n}\}$ for all β in the case that G acts trivially on X.

6.1. The degree zero case, the Fantechi-Göttsche ring, and Chen-Ruan orbifold cohomology.

We will now study the case of degree-zero G-stable maps in more detail. We will explicitly prove that the degree-zero G-stable maps endow $\mathcal{H}(X)$ with the structure of a G-Frobenius algebra, the genus-zero part of which agrees with the ring $H^{\bullet}(X, G)$ in [FG].

Throughout this section, we will use the ground ring $\mathbb{C}$ instead of Θ in the definition of $\mathcal{H}(X)$, since we are restricting to degree-zero maps. We will also assume that X is a smooth variety with projective coarse moduli space, unless otherwise stated.

Definition 6.6. Let $\widetilde{st}_X : \overline{\mathcal{M}}^G_{g,n}(X, 0, \mathbf{m}) \longrightarrow \overline{\mathcal{M}}^G_{g,n}(\mathbf{m})$ denote the morphism $st_{(X,\beta=0)}$. We define

$$\xi(X, 0, \mathbf{m}) := \widetilde{st}_X^{-1}(\xi(\mathbf{m})).$$

Similarly, if $m, a, b \in G$ are chosen such that $m \in [a, b]$, we let

$$\xi_{1,1}(X, 0, (m, a, b)) := \widetilde{st}_X^{-1}(\xi_{1,1}(m, a, b)).$$

We also define $X^{\langle \mathbf{m}\rangle}$ to be the locus in X of points fixed by the subgroup $\langle \mathbf{m}\rangle \le G$ generated by all of the elements $m_1, \ldots, m_n$ in $\mathbf{m}$.

Since the marked points $\widetilde{p}_i$ in the universal G-cover $\mathcal{E}$ over $\xi(\mathbf{m})$ all lie in the same connected component of $\mathcal{E}$, it is straightforward to see that any G-stable map f into X of degree 0 that maps by $\widetilde{st}_X$ to $\xi(\mathbf{m})$ is determined only by the underlying G-cover (the point $\widetilde{st}_X([E \to C] \in \xi(\mathbf{m}))$ and by the point $f(\widetilde{p}_1) = \cdots = f(\widetilde{p}_n)$. Moreover, the point $f(\widetilde{p}_i)$ must have a stabilizer that includes the monodromy element m_i, so the following proposition is now easy to see.

Lemma 6.7. *The substack $\xi(X, 0, \mathbf{m})$ of $\overline{\mathcal{M}}^G_{0,3}(X, 0, \mathbf{m})$ is canonically isomorphic to the product*

$$\xi(X, 0, \mathbf{m}) = \xi(\mathbf{m}) \times X^{\langle \mathbf{m}\rangle},$$

and the substack $\xi_{1,1}(X, 0, (m, a, b))$ is canonically isomorphic to the product

$$\xi_{1,1}(X, 0, (m, a, b)) = \xi_{1,1}(m, a, b) \times X^{m}.$$

Proof. For an object in $\overline{\mathcal{M}}^G_{0,3}(X, 0, \mathbf{m})$, the isomorphism is given by the morphism $(E \xrightarrow{f} X^{\langle \mathbf{m} \rangle}; \widetilde{p}_1, \ldots, \widetilde{p}_n) \mapsto (E; \widetilde{p}_1, \ldots, \widetilde{p}_n) \times f(\widetilde{p}_1)$, where E is the G-cover (we have suppressed the underlying curve C since it is determined by E), and its inverse is given by $(E; \widetilde{p}_1, \ldots, \widetilde{p}_n) \times q \mapsto (E \xrightarrow{f} X^{\langle \mathbf{m} \rangle}; \widetilde{p}_1, \ldots, \widetilde{p}_n)$, where if E' is the connected component of E containing the marked points $\widetilde{p}_1, \ldots, \widetilde{p}_n$, then $f(E') := q$ and $f(\rho(\gamma)\widetilde{p}') := \rho(\gamma) f(\widetilde{p}')$ for all γ in G and $\widetilde{p}'$ in E'. The maps for $\xi_{1,1}(X,), (m, a, b))$ are similar. $\square$

6.1.1. *The minimal cover $\xi'(\mathbf{m})$.*

Definition 6.8. Let G be a finite group and fix $\mathbf{m}$ in G^n such that $\prod_{i=1}^{3} m_i = 1$, and let $G' := \langle \mathbf{m} \rangle$ denote the subgroup of G generated by the components of $\mathbf{m}$. Let $\xi'(\mathbf{m})$ denote the connected component of $\overline{\mathcal{M}}^{G'}_{0,3}(\mathbf{m})$ which is defined in the same way as $\xi(\mathbf{m})$ but with the group G replaced by G'.

Lemma 6.9. *Let G be a finite group $\mathbf{m} \in G^3$ with $\prod_{i=1}^{3} m_i = 1$, and $G' = \langle \mathbf{m} \rangle$. Consider the morphism $\hat{I} : \xi(\mathbf{m}) \longrightarrow \xi'(\mathbf{m})$ taking the object $(E \longrightarrow C; \widetilde{p}_1, \ldots, \widetilde{p}_n)$ to the object $(E' \longrightarrow C; \widetilde{p}_1, \ldots, \widetilde{p}_n)$, where E' is the connected component of E which contains $\widetilde{p}_i$ for all $i = 1, \ldots, n$. The morphism $\hat{I}$ is an isomorphism.*

Proof. Since E' is a G'-cover (see Appendix of [FG]), $\hat{I}$ is a morphism.

The inverse morphism takes $(E' \longrightarrow C; \widetilde{p}'_1, \ldots, \widetilde{p}'_n)$ to $(E \longrightarrow C; \widetilde{p}_1, \ldots, \widetilde{p}_n)$, where $E = E' \times_{G'} G$ and G' acts on E from the right in the usual way, G' acts on G by left multiplication, and $\widetilde{p}_i := [\widetilde{p}'_i, 1]$ for all $i = 1, \ldots, n$. $\square$

Consider the following commutative diagram

(63)
$$
\begin{tikzcd}
 & X & & \\
\mathcal{E}' \times X^{\langle \mathbf{m} \rangle} \arrow[ur, "f'"] \arrow[rr, "\widetilde{I}"] \arrow[dd, "\widetilde{\pi}'"'] \arrow[dr] & & \mathcal{E} \times X^{\langle \mathbf{m} \rangle} \arrow[ul, "f"'] \arrow[dd, "\widetilde{\pi}"'] \arrow[dr] & \\
 & \mathcal{C}' \times X^{\mathbf{m}} \arrow[dl] & & \mathcal{C} \times X^{\mathbf{m}} \arrow[dl] \\
\xi'(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \arrow[rr, "I"] \arrow[d, "\mathrm{pr}_{\xi'}"'] & & \xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \arrow[d, "\mathrm{pr}_{\xi}"'] & \\
\xi'(\mathbf{m}) \arrow[rr, "\hat{I}"] & & \xi(\mathbf{m}) &
\end{tikzcd}
$$

where $\widetilde{I}$ and I are the isomorphisms induced by $\hat{I}$, $\mathcal{C}'$ and $\mathcal{C}$ are the universal curves, $\mathcal{E}'$ and $\mathcal{E}$ are the universal G' and G covers, respectively, and f, f' are the universal stable maps.

Proposition 6.10. $I^*R\widetilde{\pi}^G_*(f^*TX)$ *is canonically isomorphic to* $R\widetilde{\pi}'^{G'}_*(f'^*(TX))$ *in the* K*-theory of* $\xi'(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle}$*, where* $R\widetilde{\pi}^G_*$ *denotes the* G*-invariant derived push-forward, and* $R\widetilde{\pi}'^{G'}_*$ *denotes the* G'*-invariant derived push-forward.*

Proof. The fiber of $I^*R\widetilde{\pi}^G_*(f^*TX)$ over $\xi'(\mathbf{m}) \times q$ for all q in $X^{\langle \mathbf{m} \rangle}$ is $H^\bullet(\mathcal{E} \times q, \mathcal{T})$, where the sheaf $\mathcal{T}$ over $\mathcal{E} \times q$ is $f^*(TX)$. Since $\mathcal{E}'$ is the connected component of $\mathcal{E}$ containing $\widetilde{p}_i$ for all $i = 1, \ldots, n$ we have $\mathcal{T}|_{\widetilde{p}\times q} = T_qV$ for all $\widetilde{p}$ in $\mathcal{E}'$. Henceforth, let us regard $\mathcal{T}$ as a bundle over $\mathcal{E}$ to avoid notational clutter.

Observe that $\mathcal{T}$ is a G-equivariant trivial bundle on $\mathcal{E}$. Denote the restriction of $\mathcal{T}$ to $\mathcal{E}'$ by $\mathcal{T}'$ and observe that it is a G'-equivariant bundle. We will now construct a bundle from $\mathcal{T}'$ on $\mathcal{E}'$ which is isomorphic as a G-equivariant bundle to $\mathcal{T}$ on $\mathcal{E}$ as follows.

Consider the bundle $\mathcal{T}' \otimes \mathcal{O}_G$ on $\mathcal{E}' \times G$. We observe that $\mathcal{E}'$ is a right G'-space and G is a left G-space by left multiplication. Similarly, there is a right G' action on $\mathcal{T}'$ and a left G'-action on $\mathcal{O}_G$. Therefore, $\mathcal{T}' \otimes \mathcal{O}_G$ over $\mathcal{E}' \times G$ is a G'-equivariant bundle with respect to the diagonal G' action. Quotienting by G' and using the identification of $\mathcal{O}_G$ with $\mathbb{C}[G]$, we obtain $\mathcal{T}' \otimes_{\mathbb{C}[G']} \mathbb{C}[G]$ over $\mathcal{E}' \times_{G'} G$, which is a G-equivariant bundle, where an element $\widetilde{\gamma}$ in G acts upon an element of the base as $[e', \gamma] \mapsto [e', \gamma\widetilde{\gamma}]$, and similarly in the bundle. We now have the isomorphism of G-equivariant vector bundles

$$\begin{tikzcd}
\mathcal{T}' \otimes_{\mathbb{C}[G']} \mathbb{C}[G] \arrow[r, "\widetilde{\lambda}"] \arrow[d] & \mathcal{T} \arrow[d] \\
\mathcal{E}' \times_{G'} G \arrow[r, "\lambda"] & \mathcal{E},
\end{tikzcd}$$

where $\widetilde{\lambda}([\nu', \gamma]) := \rho(\gamma)\nu'$, and $\lambda([e', \gamma]) := \rho(\gamma)\nu'$, and where $\rho(\gamma)$ indicates the right G action.

Therefore,

$$\begin{aligned}
H^\bullet(\mathcal{E}, \mathcal{T}) &= H^\bullet(\mathcal{E}' \times_{G'} G, \mathcal{T}' \otimes_{\mathbb{C}[G']} \mathcal{O}_G) \\
&= H^\bullet(\mathcal{E}' \times G, \mathcal{T}' \otimes \mathbb{C}[G])^{G'} \\
&= (H^\bullet(\mathcal{E}', \mathcal{T}') \otimes H^\bullet(G, \mathcal{O}_G))^{G'} \\
&= (H^\bullet(\mathcal{E}', \mathcal{T}') \otimes \mathbb{C}[G])^{G'} \\
&= H^\bullet(\mathcal{E}', \mathcal{T}') \otimes_{\mathbb{C}[G']} \mathbb{C}[G].
\end{aligned}$$

Taking G-invariants, we have

$$H^\bullet(\mathcal{E}, \mathcal{T})^G = (H^\bullet(\mathcal{E}', \mathcal{T}') \otimes_{\mathbb{C}[G']} \mathbb{C}[G])^G \cong H^\bullet(\mathcal{E}', \mathcal{T}')^{G'}.$$

The latter is precisely the fiber of $R\widetilde{\pi}'^{G'}_*(f'^*(TX))$ over $\xi'(\mathbf{m}) \times q$. □

Proposition 6.11. *When* $\beta = 0$*, the sheaf* $R^1\widetilde{\pi}^G_*(f^*TX)$ *is locally free on* $\xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} = \xi(X, 0, \mathbf{m})$ *and the virtual fundamental class of* $\xi(X, 0, \mathbf{m})$ *is simply the top Chern class* $c_{top}(R^1\widetilde{\pi}^G_*(f^*TX))$*.*

Proof. This follows immediately from the construction of $\overline{\mathcal{M}}^G_{g,n}(X)$ as a fibered product of sections over $\overline{\mathcal{M}}_{g,n}(X)$, the stack of orbifold stable maps to X, and the fact that the proposition holds there (see e.g., [AGV]). □

Definition 6.12. Let $c(\mathbf{m}) := c_{\mathrm{top}}(R^1\widetilde{\pi}^G_*(f^*TX))$ and $c'(\mathbf{m}) := c_{\mathrm{top}}(R^1\widetilde{\pi}'^{G'}_*(f'^*TX))$, where c_{top} denotes the top Chern class.

Corollary 6.13. *For all* $\mathbf{m}$ *in* G^3 *such that* $\prod_{i=1}^3 m_i = 1$*, we have*

$$I^*c(\mathbf{m}) = c'(\mathbf{m}). \tag{64}$$

We now prove that the 3-point correlator responsible for the multiplication in the G-Frobenius algebra can be identified by the isomorphism in Lemma 6.7.

Proposition 6.14. *For all* $\mathbf{m}$ *in* G^3 *such that* $\prod_{i=1}^3 m_i = 1$ *and* α_{m_i} *in* $H^\bullet(X^{m_i})$*, let* $\Lambda^{\xi'}_{0,3}(\alpha_{\mathbf{m}})$ *in* $H^\bullet(\xi'(\mathbf{m}))$ *and* $\Lambda^{\xi}_{0,3}(\alpha_{\mathbf{m}})$ *in* $H^\bullet(\xi'(\mathbf{m}))$ *be given by*

$$\Lambda^{\xi'}_{0,3}(\alpha_{\mathbf{m}}) := \mathrm{pr}_{\xi'*}(\prod_{i=1}^{3}(ev'^*_{m_i}\alpha_{m_i})) \cup c'(\mathbf{m}))$$

and

$$\Lambda^{\xi}_{0,3}(\alpha_{\mathbf{m}}) := \mathrm{pr}_{\xi*}((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i})) \cup c(\mathbf{m})).$$

We have

$$\hat{I}^*\Lambda^{\xi}_{0,3}(\alpha_{\mathbf{m}}) = \Lambda^{\xi'}_{0,3}(\alpha_{\mathbf{m}}),$$

where $ev_{m_i} : \xi(V, 0, \mathbf{m}) \longrightarrow X^{m_i}$ *and* $ev'_{m_i} : \xi'(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \longrightarrow X^{m_i}$ *are the evaluation morphisms, and* $\mathrm{pr}_{\xi'} : \xi'(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \longrightarrow \xi'(\mathbf{m})$ *and* $\mathrm{pr}_{\xi} : \xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \longrightarrow \xi(\mathbf{m})$ *are the projections, which can be identified with the morphism forgetting the* G*-stable maps.*

Proof.

$$
\begin{aligned}
\hat{I}^*\Lambda^{\xi}_{0,3}(\alpha_{\mathbf{m}}) &= \hat{I}^*\mathrm{pr}_{\xi *}((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i}))\cup c(\mathbf{m}))\\
&= \hat{I}^{-1}_*\mathrm{pr}_{\xi *}((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i}))\cup c(\mathbf{m}))\\
&= ((\hat{I}^{-1}\circ \mathrm{pr}_{\xi})_*((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i}))\cup c(\mathbf{m}))\\
&= (\mathrm{pr}_{\xi'}\circ I^{-1})_*((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i}))\cup c(\mathbf{m}))\\
&= \mathrm{pr}_{\xi' *}I^*((\prod_{i=1}^{3}(ev^*_{m_i}\alpha_{m_i}))\cup c(\mathbf{m}))\\
&= \mathrm{pr}_{\xi' *}((\prod_{i=1}^{3}((ev_{m_i}\circ I)^*\alpha_{m_i}))\cup I^*c(\mathbf{m}))\\
&= \mathrm{pr}_{\xi' *}((\prod_{i=1}^{3}((ev_{m_i}\circ I)^*\alpha_{m_i}))\cup I^*c(\mathbf{m}))\\
&= \mathrm{pr}_{\xi' *}((\prod_{i=1}^{3}(ev'^*_{m_i}\alpha_{m_i}))\cup c'(\mathbf{m}))\\
&= \Lambda^{\xi'}_{0,3}(\alpha_{\mathbf{m}}),
\end{aligned}
$$

where we have used Equation (64) in the penultimate equality. □

Corollary 6.15. *For all* $\mathbf{m}$ *in* G^3 *such that* $\prod_{i=1}^{3} m_i = 1$*, and for* α_{m_i} *in* $\mathcal{H}(X)_{m_i} = H^\bullet(X^{m_i})$*, we have*

$$\mu(\alpha_{\mathbf{m}}) = \int_{[\![\xi(\mathbf{m})]\!]} \Lambda^{\xi}_{0,3}(\alpha_{\mathbf{m}}) = \int_{[\![\xi'(\mathbf{m})]\!]} \Lambda^{\xi'}_{0,3}(\alpha_{\mathbf{m}}),$$

where μ *is defined as in Equation (55).*

μ completely determines the multiplication and metric. We will now prove that it yields a G-Frobenius algebra.

6.1.2. *The genus-zero part of the* G*-Frobenius algebra.*

For this subsection, we can assume, without loss of generality, that $G' = G$ in light of the results of the previous section.

Definition 6.16. Since the virtual fundamental class $c(\mathbf{m})$ belongs to $H^\bullet(\xi(\mathbf{m}) \times X^{\langle \mathbf{m}\rangle}) \cong H^\bullet(\xi(\mathbf{m})) \otimes H^\bullet(X^{\langle \mathbf{m}\rangle})$, define $\tilde{c}(\mathbf{m})$ in $H^\bullet(X^{\langle \mathbf{m}\rangle})$ to be the unique class such that

$$c(\mathbf{m}) = \mathbf{1}_{\xi(\mathbf{m})} \otimes \tilde{c}(\mathbf{m}),$$

where $\mathbf{1}_{\xi(\mathbf{m})}$ is the unit in $H^\bullet(\xi(\mathbf{m}))$.

We will now write $\mu(\nu_{\mathbf{m}})$ as an integral over $X^{\langle \mathbf{m}\rangle}$.

Proposition 6.17. *For all* $v_{\mathbf{m}}$ *in* $\mathcal{H}(X)_{\mathbf{m}}$ *where* $\prod_{i=1}^{3} m_i = 1$*, we have*

$$\mu(v_{\mathbf{m}}) = \int_{[X^{\langle \mathbf{m} \rangle}]} (\prod_{i=1}^{3} j_{m_i}^{*} v_{m_i}) \cup \widetilde{c}(\mathbf{m}), \tag{65}$$

where $j_{m_i} : X^{\langle \mathbf{m} \rangle} \hookrightarrow X^{m_i}$ *is the inclusion and* $[X^{\langle \mathbf{m} \rangle}]$ *is the fundamental class of the variety* $X^{\langle \mathbf{m} \rangle}$*. In particular, when* $m_i = 1$ *for all* $i = 1, 2, 3$*, then* $\widetilde{c}(1,1,1) = \mathbf{1}$*, the unit in* $H^{\bullet}(X)$*. The restriction of the multiplication and metric to the untwisted sector* $\mathcal{H}(X)_1$ *agree with the usual cup product and metric from* $H^{\bullet}(X)$*. Furthermore,* $(\overline{\mathcal{H}}(X), \overline{\mu}, \mathbf{1})$ *is isomorphic as a* $\overline{G}$*-graded Frobenius algebra to the Chen-Ruan orbifold cohomology of* $[X/G]$*.*

Proof. The first statement is a straightforward calculation. The second follows from the observation that the appropriate obstruction bundle vanishes when $m_1 = m_2 = m_3 = 1$. The third follows from the following remark and Section 2 of [FG]. □

Remark 6.18. The vector bundle $R^1\widetilde{\pi}_*^G(f^*TX) \longrightarrow \xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle}$ is *not* the pull-back of a vector bundle via the projection $\xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle} \longrightarrow X^{\langle \mathbf{m} \rangle}$ because the the automorphism group of a G-cover (which is isomorphic to $H(\mathbf{m})$) in $\xi(\mathbf{m})$ acts non-trivially on $R^1\widetilde{\pi}_*^G(f^*TX) \longrightarrow \xi(\mathbf{m})$ as the action of the automorphism group commutes with the action of G. Nevertheless, one can interpret the bundle $R^1\widetilde{\pi}_*^G(f^*TX) \longrightarrow \xi(\mathbf{m}) \times X^{\langle \mathbf{m} \rangle}$ as an $H(\mathbf{m})$-equivariant vector bundle $R^1\widetilde{\pi}_*^G(f^*TX) \longrightarrow X^{\langle \mathbf{m} \rangle}$. This bundle can be identified with the bundle $F(m_1, m_2) \longrightarrow X^{\langle \mathbf{m} \rangle}$ introduced in [FG]. Therefore, their cohomology class $c(m_1, m_2)$ can be identified with $\widetilde{c}(\mathbf{m})$, which is a class on $X^{\langle \mathbf{m} \rangle}$, so Equation (65) is consistent with their multiplication.

They also prove that the vector bundle $F(m_1, m_2)$ restricted to a connected component U of $X^{\langle \mathbf{m} \rangle}$ has rank $a(m_1, U) + a(m_2, U) - a(m_1 m_2, U) - \mathrm{codim}(U \subseteq X^{m_1 m_2})$. To explain this notation, let X have dimension D, q belong to X, and m belong to the isotropy subgroup of G at q. Denote the set of eigenvalues of the action of m on T_qX by $\{\exp(2\pi i r_1), \ldots, \exp(2\pi i r_D)\}$ for all $j = 1, \ldots, D$ where r_j belongs to the interval $[0, 1)$. The *age of* m *in* q, $a(m, q)$, is defined to be $\sum_{j=1}^{D} r_j$. Since $a(m, q)$ depends only upon the connected component containing q, $a(m, U)$ is defined to be $a(m, q)$ for any q in U.

Proposition 6.19. *The triple* $(\mathcal{H}(X), \mu, \mathbf{1})$ *satisfies all of the axioms of a* G*-Frobenius algebra except, perhaps, for the trace axiom. Our multiplication, metric, and identity agrees with that* [FG] *on the ring* $H^{\bullet}(X, G)$*. Furthermore,* η *on* $\mathcal{H}(X)$ *has nonzero homogeneous components* $\mathcal{H}(X)_{m_+} \otimes \mathcal{H}(X)_{m_-} \longrightarrow \mathbb{C}$ *only if* $m_+ m_- = 1$*, in which case* $\mathcal{H}(X)_{m_+} = \mathcal{H}(X)_{m_-} = H^{\bullet}(X^{m_+}, \mathbb{C})$*, and* η *agrees with the usual Poincaré pairing.*

Proof. This result follows from the previous remark and [FG]. □

Remark 6.20. Proposition 5.12 explains the origin of the factor of $\frac{1}{|G|}$ in the definition of $\overline{\eta}$ from the viewpoint of intersection theory. This factor may be removed, if desired, as per Remark 5.11.

6.1.3. *The trace axiom.*

We will now prove that the trace axiom, which is a genus-one condition, holds for $(\mathcal{H}(X), \mu, \mathbf{1}) = H^{\bullet}(X, G)$.

Proposition 6.21. *The Trace Axiom (Definition 4.13 (11)) holds for the triple* $(\mathcal{H}(X), \mu, \mathbf{1})$.

Proof. The proof of the trace axiom in Theorem 4.15 shows that it suffices for us to check that the cutting loops property 4.1(5a) holds in the special cases of $\rho_a : \xi(m_1, bab^{-1}, a^{-1}) \longrightarrow \xi_{1,1}(m_1, a, b)$ and $\rho_b : \xi(m_1, b, ab^{-1}a^{-1}) \longrightarrow \xi_{1,1}(m_1, a, b)$ for the virtual class. So it suffices to check that $\rho_a^* c_{top} R^1\pi_*^G f^*TX = c_{top} R^1\widetilde{\pi}_*^G \tilde{f}TX$ (and the same for ρ_b) where the morphisms are those of the following diagram

$$\begin{tikzcd}
 & X & \\
\mathcal{E}' \times X^{\langle m,a,b\rangle} \arrow[ur, "\tilde{f}"] \arrow[rr, "\widetilde{\rho}_a"] \arrow[d, "\widetilde{\pi}"] & & \mathcal{E} \times X^m \arrow[ul, "f"'] \arrow[d, "\pi"] \\
\xi(m, bab^{-1}a^{-1}) \times X^{\langle m,a,b\rangle} \arrow[rr, "\rho_a \times i"] \arrow[d, "\widetilde{\mathrm{pr}}_\xi"] & & \xi_{1,1}(m,a,b) \times X^m \arrow[d, "\mathrm{pr}_\xi"] \\
\xi(m, bab^{-1}, a^{-1}) \arrow[rr, "\rho_a"] & & \xi_{1,1}(m,a,b)
\end{tikzcd} \tag{66}$$

where f and f' are the universal stable maps from the universal admissible covers $\mathcal{E} \times X^m$ and $\mathcal{E}' \times X^{\langle m,a,b\rangle}$, respectively, i is the obvious inclusion $i : X^{\langle m,a,b\rangle} \longrightarrow X^m$, and $\widetilde{\rho}_a$ is the induced map making the top square Cartesian. Since ρ_a is the composition of a regular embedding and a flat morphism, we have

$$\begin{aligned}
(\rho_a \times i)^* R^1\pi_*^G f^*TX &= R^1\widetilde{\pi}_*^G \widetilde{\rho}_a^* f^*TX \\
&= R^1\pi_*^G \tilde{f}^*TX,
\end{aligned}$$

which gives the desired result. □

Remark 6.22. Finally, we note that the G-Frobenius algebra $(\mathcal{H}(X), \mu, \mathbf{1})$ enjoys some functoriality properties, as Fantechi-Göttsche have showed that it pulls back along étale maps [FG, pg. 11].

6.1.4. *Tensor products.*

We now work out the tensor products of the equivariant CohFTs described above and show that they reduce to the obvious notions of tensor products for G-Frobenius algebras.

Proposition 6.23. *Let X' be a smooth, projective variety with a G'-action and let $((\mathcal{H}(X'), \rho'), \mu', \mathbf{1}')$ be the G'-Frobenius algebra associated to contributions from maps of degree zero where μ' is defined by Equation (55). Let X'' be a smooth, projective variety with a G''-action and let $((\mathcal{H}(X''), \rho''), \mu'', \mathbf{1}'')$ be its similarly associated G''-Frobenius algebra.*

(1) *Consider* $X' \times X''$ *with its* $G' \times G''$ *action. The associated* $G' \times G''$*-Frobenius algebra* $((\mathcal{H}(X' \times X''), \rho), \mu, \mathbf{1})$ *is canonically isomorphic to the external tensor product of* $((\mathcal{H}(X'), \rho'), \mu', \mathbf{1}')$ *and* $((\mathcal{H}(X''), \rho''), \mu'', \mathbf{1}'')$.

(2) *Suppose that* $G' = G'' = G$*, and consider* $X' \times X''$ *with its diagonal* G *action. Its associated* G*-Frobenius algebra* $((\mathcal{H}(X' \times X''), \rho), \mu, \mathbf{1})$ *is canonically isomorphic to the tensor product of* $((\mathcal{H}(X'), \rho'), \mu', \mathbf{1}')$ *and* $((\mathcal{H}(X''), \rho''), \mu'', \mathbf{1}'')$.

Proof. To prove the first part, we need to understand the behavior of the obstruction bundle when $g = 0$, $n = 3$, and $\beta = 0$.

For all $\mathbf{m}'$ in G'^n and $\mathbf{m}''$ in G''^n, consider the substack $\xi^{G' \times G''}(\mathbf{m}' \times \mathbf{m}'')$ of $\overline{\mathcal{M}}_{0,3}^{G' \times G''}(\mathbf{m}' \times \mathbf{m}'')$, where we adopt the notation $\mathbf{m}' \times \mathbf{m}''$ from Proposition 4.6. Since $\overline{\mathcal{M}}_{0,3}$ is a point, $\xi^{G' \times G''}(\mathbf{m}' \times \mathbf{m}'')$ is canonically isomorphic to $\xi^{G'}(\mathbf{m}') \times \xi^{G''}(\mathbf{m}'')$. Similarly, $\xi^{G' \times G''}(X' \times X'', 0, \mathbf{m}' \times \mathbf{m}'')$ is canonically isomorphic to $\xi^{G'}(\mathbf{m}') \times X'^{\langle \mathbf{m}' \rangle} \times \xi^{G''}(\mathbf{m}'') \times X''^{\langle \mathbf{m}'' \rangle}$. Thus, we have

$$X' \times X'' \xleftarrow{f' \times f''} \mathcal{E}' \times X'^{\langle \mathbf{m}' \rangle} \times \mathcal{E}'' \times X''^{\langle \mathbf{m}'' \rangle} \xrightarrow{\widetilde{\pi}' \times \widetilde{\pi}''} \xi^{G'}(\mathbf{m}') \times X'^{\langle \mathbf{m}' \rangle} \times \xi^{G''}(\mathbf{m}'') \times X''^{\langle \mathbf{m}'' \rangle},$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are the universal curves, and f' and f'' are universal evaluation morphisms. It is easy to see that there is a canonical isomorphism

$$R^\bullet(\widetilde{\pi}' \times \widetilde{\pi}'')_*^{G' \times G''}(f' \times f'')^*(T(X' \times X'')) \cong R^\bullet \widetilde{\pi}_*'^{G'}(f'^*(TX')) \oplus R^\bullet \widetilde{\pi}_*''^{G''}(f''^*(TX'')).$$

By multiplicativity of the top Chern class, we obtain

$$c(\mathbf{m}' \times \mathbf{m}'') = c'(\mathbf{m}') \otimes c''(\mathbf{m}''), \tag{67}$$

where $c(\mathbf{m}' \times \mathbf{m}'')$ is the virtual fundamental class of $\xi^{G' \times G''}(X' \times X'', 0, \mathbf{m}' \times \mathbf{m}'') \cong \xi^{G'}(X', 0, \mathbf{m}') \times \xi^{G''}(X'', 0, \mathbf{m}'')$, $c'(\mathbf{m}')$ is the virtual fundamental class of $\xi^{G'}(X', 0, \mathbf{m}')$, and $c''(\mathbf{m}'')$ is the virtual fundamental class of $\xi^{G''}(X'', 0, \mathbf{m}'')$.

It is a straightforward exercise using Equation (67) and the identification $\mathcal{H}(X' \times X'') \cong \mathcal{H}(X') \otimes \mathcal{H}(X'')$ to show that

$$\mu = \mu' \otimes \mu''.$$

The trace axiom then follows immediately from the fact that trace of a tensor product is the product of the corresponding traces over each tensor factor. This finishes the proof of the first part of the proposition.

The second part of the proposition follows from Remark 4.11 and the first part of this proposition. □

6.2. **Trivial G-actions.**

In the special case that the action of G on X is trivial, the data of a G-stable map to X is the same as a stable map from the underlying curve C to X and the data of a positive admissible G-cover, that is

$$\overline{\mathcal{M}}_{g,n}^G(X) = \overline{\mathcal{M}}_{g,n}^G \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}(X).$$

Moreover, since $\overline{\mathcal{M}}_{g,n}^G$ is smooth, it is evident that the virtual fundamental class on $\overline{\mathcal{M}}_{g,n}^G(X)$ is simply the pullback of the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X)$

$$[\overline{\mathcal{M}}_{g,n}^G(X)]^{vir} = \mathrm{pr}_2^*[\overline{\mathcal{M}}_{g,n}(X)]^{vir},$$

and the evaluation map $\overline{\mathcal{M}}^{G}_{g,n}(X) \longrightarrow (\widehat{X})^n = (X \times G)^n$ is simply the product of the evaluation maps $\mathbf{e} : \overline{\mathcal{M}}^{G}_{g,n} \longrightarrow G^n$ and $\mathbf{ev} : \overline{\mathcal{M}}_{g,n}(X) \longrightarrow X^n$.

Thus in this special case, we have

$$\Lambda^{G,X}_{g,n} = \mathrm{st}^*\Lambda^{X}_{g,n},$$

where $\{\Lambda^X_{g,n}\}$ is the usual Gromov-Witten CohFT for X, and $\mathrm{st} : \overline{\mathcal{M}}^{G}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$ is the forgetful map ($\mathrm{st} := \widehat{\mathrm{st}} \circ \widetilde{\mathrm{st}}$).

Since G acts trivially, we have

$$\mathcal{H}(X)_{m_i} = H^\bullet(X^{m_i};\Theta) \cong H^\bullet(X;\Theta)$$

for every $m_i \in G$. So the state space $\mathcal{H}(X)$ is just

$$H^\bullet(X;\Theta) \otimes \mathbb{C}[G]$$

and

$$\Lambda^{G,X}_{g,n}((\nu_1 \otimes m_1), \ldots, (\nu_n \otimes m_n)) = \mathrm{st}^*\Lambda^{X}_{g,n}(\nu_1, \ldots, \nu_n) \cup \mathbf{e}^*(\mathbf{1}),$$

which is clearly just the external tensor product of $\Lambda^X_{g,n}$ with $\mathbb{C}[G]$. Thus we have proved the following.

Proposition 6.24. *If X is a smooth, projective variety with a trivial G action, then its associated G-CohFT is isomorphic to the external tensor product of the CohFT of stable maps associated to X (regarded as an equivariant CohFT for the trivial group) with $\mathbb{C}[G]$, the group ring G-CohFT (see Example 4.3).*

Remark 6.25. In the previous example, the induced CohFT on the space of G-coinvariants agrees with Proposition 3.7 in [JK].

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