Finite free resolutions and homology of finitely generated torsion-free nilpotent groups

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Abstract

For any f.g. nilpotent group G such that Lie ring $\mathcal{G}$ associated to the lower central series of G is $\mathbf{Z}$-torsion-free we construct a finite free resolution of minimal length of the kind $\mathbf{Z}G \otimes \Lambda^*\mathcal{G} \to \mathbf{Z} \to 0$. Moreover, for any f.g. torsion-free nilpotent group G there exists a resolution of the kind $\mathbf{Z}G \otimes \Lambda^*\mathcal{G} \to \mathbf{Z} \to 0$ where $\mathcal{G}$ is the Lie ring associated to the central series of G consisting of isolators of $\gamma_i G$. We prove also that $H_n G \cong H_n \mathcal{G}$ for any free 2-step nilpotent group G.

Introduction

Let G be a free abelian group on r generators. It is well known how to construct a quite simple resolution of the trivial G-module $\mathbf{Z}$ over $\mathbf{Z}G$. One can take the exterior algebra of G over $\mathbf{Z}G$, i.e. $\mathbf{Z}G \otimes \Lambda^* G$, supplied with a suitable differential d (see (3) below).

The general case of a finitely generated torsion-free (f.g.t.-f.) nilpotent group G is obviously more complicated. In 1989 J. Huebschmann **[Hu]** constructed small resolutions for 2-step nilpotent groups G (his perturbation theory construction admits a generalisation to groups of higher class of nilpotency). But the question of computing the *integral* homology groups for any f.g.t.-f. nilpotent group still remains unsolved.

In this paper we develop the approach to the construction of finite free resolutions for f.g.t.-f. nilpotent groups that appeared in **[K-S]** for the case of 2-step free nilpotent groups. Our main result is the following

Theorem 4. Let G be a f.g.t.-f. nilpotent group and $\mathcal{G}$ be the Lie *ring associated to the central series of G consisting of the isolators of the lower central series terms $\gamma_i G$. Then there exists a finite free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$ of the kind $\mathbf{Z}G \otimes \Lambda^* \mathcal{G}$ where $\Lambda^* \mathcal{G}$ is the exterior algebra of $\mathcal{G}$. In particular, it is of minimal length $r = cd(G)$ (= Hirsch number of G).*

The proof of this theorem provides an inductive procedure for constructing such resolutions. Moreover, it is valid not only for nilpotent groups but also for nilpotent Lie rings (in this case it yields well-known resolutions of the kind $U(\mathcal{G}) \otimes \Lambda^* \mathcal{G}$). The parallels are depicted in the notion of an *E-triple* and in the theorem 3.

In **[K-S]** we used similar construction to prove that the integral homology groups $H_n G$ and $H_n \mathcal{G}$ are isomorphic for free 2-step nilpotent groups G (see also sections 3,4). In addition to this we should mention a paper by St.Sigg **[Si]** where he computes the groups $H_n(\mathcal{G}, \mathbf{C})$. In general, there are torsion elements in these integral homology groups ([**L**]).

Question. *Is it true that $H_n G \cong H_n \mathcal{G}$ for f.g.t.-f. nilpotent groups G?*

One knows that $H_n(G, \mathbf{Q}) \cong H_n(\mathcal{G}, \mathbf{Q})$ (see [**Ku**]). In view of this result the eventual difference between $H_n G$ and $H_n \mathcal{G}$ must be of torsion.

The paper is organised as follows. In the section 1 we present the main construction of a resolution for the case of a f.g.t.-f. nilpotent group G such that the Lie ring $\mathcal{G}$ associated to the lower central series of G is $\mathbf{Z}$-torsion-free. In the section 2 we generalise this construction for any f.g.t.-f. nilpotent group G. We consider the case of a free 2-step nilpotent group G as an example of how our construction works and we prove in sections 3,4 that there exists an isomorphism between $H_n G$ and $H_n \mathcal{G}$ (the proof differs from that of **[K-S]** in some details).

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1 Main construction

Let G be a finitely generated nilpotent group of class c. We'll suppose that Lie ring $\mathcal{G}$ associated to the lower central series of G is torsion-free as a $\mathbf{Z}$-module (then G is called *Magnus* nilpotent group). Consider the central extension

$$1 \to \gamma_c G \to G \to \overline{G} \to 1, \tag{0}$$

where $\gamma_c G = A$ is the last term of the lower central series of the group G. We also consider A as an abelian Lie ring; then there is the extension

$$0 \to A \to \mathcal{G} \to \overline{\mathcal{G}} \to 0,$$

where Lie ring $\overline{\mathcal{G}}$ is associated to quotient group $\overline{G}$. As a $\mathbf{Z}$-module $\mathcal{G} \cong A \oplus \overline{\mathcal{G}}$. We fix $\mathbf{Z}$-bases $a_1, \ldots, a_k$ in A and $x_1, \ldots, x_m$ in $\overline{\mathcal{G}}$. In the sequel the usual notation $\Lambda^p M$ for the p-th exterior power of a $\mathbf{Z}$-module M is used ($\Lambda^0 M \cong \mathbf{Z}$).

Theorem 1. Let G be a f.g. Magnus nilpotent group of class c. Given the extension (0) let's suppose that there is a free resolution of the trivial $\overline{G}$-module $\mathbf{Z}$ over $\mathbf{Z}\overline{G}$ of the kind

$$\ldots \xrightarrow{\overline{d}} \mathbf{Z}\overline{G} \otimes \Lambda^p \overline{\mathcal{G}} \xrightarrow{\overline{d}} \ldots \xrightarrow{\overline{d}} \mathbf{Z}\overline{G} \xrightarrow{\overline{\pi}} \mathbf{Z} \to 0 \tag{1}$$

Here $\overline{d}$ is a differential and $\overline{\pi}$ is the canonical augmentation epimorphism.

Then there exists a free resolution of the trivial G-module $\mathbf{Z}$ over $\mathbf{Z}G$ of the similar kind

$$\ldots \xrightarrow{d} \mathbf{Z}G \otimes \Lambda^n \mathcal{G} \xrightarrow{d} \ldots \xrightarrow{d} \mathbf{Z}G \xrightarrow{\pi} \mathbf{Z} \to 0 \tag{2}$$

for some differential d depending on $\overline{d}$.

Proof. It is well known that for a free abelian group A there is a resolution

$$\ldots \xrightarrow{d_0} \mathbf{Z}A \otimes \Lambda^q A \xrightarrow{d_0} \ldots \xrightarrow{d_0} \mathbf{Z}A \xrightarrow{\pi} \mathbf{Z} \to 0, \tag{3}$$

where A-homomorphism d_0 carries the basic element $1 \otimes a_{j_1} \wedge \ldots \wedge a_{j_q}$ to

$$\sum_{t=1}^{q} (-1)^{t-1} (a_{j_t} - 1) \otimes a_{j_1} \wedge \ldots \widehat{a}_{j_t} \ldots \wedge a_{j_q}.$$

Applying to (3) the functor $\mathbf{Z}G \otimes_{\mathbf{Z}A}$ we get the exact sequence

$$\ldots \xrightarrow{d_0} \mathbf{Z}G \otimes \Lambda^q A \xrightarrow{d_0} \ldots \xrightarrow{d_0} \mathbf{Z}G \xrightarrow{\mu} \mathbf{Z}\overline{G} \to 0, \tag{4}$$

where μ is induced by the canonical epimorphism $G \to \overline{G}$ and d_0 is a G-homorphism. Note that $Ker\,\mu$ is generated by $a_j - 1$, $j = 1, \dots, k$ as a left G-module.

Tensoring (4) by $\Lambda^p \overline{\mathcal{G}}$ we obtain the following family of exact sequences ($p = 0, \dots, m = rk_{\mathbf{Z}} \overline{\mathcal{G}}$):

$$\dots \xrightarrow{d_0 \otimes 1} \mathbf{Z}G \otimes \Lambda^q A \otimes \Lambda^p \overline{\mathcal{G}} \xrightarrow{d_0 \otimes 1} \dots \xrightarrow{d_0 \otimes 1} \mathbf{Z}G \otimes \Lambda^p \overline{\mathcal{G}} \xrightarrow{\mu \otimes 1} \mathbf{Z}\overline{G} \otimes \Lambda^p \overline{\mathcal{G}} \to 0. \tag{4p}$$

Now let $E_{pq} = \mathbf{Z}G \otimes \Lambda^q A \otimes \Lambda^p \overline{\mathcal{G}}$. This notation agrees with that used for spectral sequences: p is the horizontal and q is the vertical coordinate (thus, every sequence (4p) becomes a column with the "underground floor" term $\mathbf{Z}\overline{G} \otimes \Lambda^p \overline{\mathcal{G}}$ which is not included in E but should be always kept in mind in what follows). Having chosen an additive basis in $\overline{\mathcal{G}}$ $x_1, \dots, x_m$ we order it so that $x_1 < \dots < x_m$. The typical basic element of E_{po} is $1 \otimes x_{i_1} \wedge \dots \wedge x_{i_p}$, $i_1 < \dots < i_p$. To simplify the notation it will be usually denoted by $1 \otimes X$ (if there is no need to specify it).

We consider $E = \oplus_{p,q \geq 0} E_{pq}$ as a bi-graded G-module. Now we have a differential (G-homorphism) $d_0 : E_{pq} \to E_{p,q-1}$ which is the same as $d_0 \otimes 1$ in (4p) and equal to 0 when $q = 0$ (the vertical differential). It also may be viewed as an element of $Homgr_G^{0,-1}(E, E) \subset Homgr_G(E, E)$. For a fixed basic element $a_j \in A$ consider a G-homorphism $R_j \in Homgr_G^{01}(E, E)$ such that

$$R_j(1 \otimes a_{j_1} \wedge \dots a_{j_q} \wedge x_{i_1} \wedge \dots \wedge x_{i_p}) = 1 \otimes a_j \wedge a_{j_1} \wedge \dots a_{j_q} \wedge x_{i_1} \wedge \dots \wedge x_{i_p}.$$

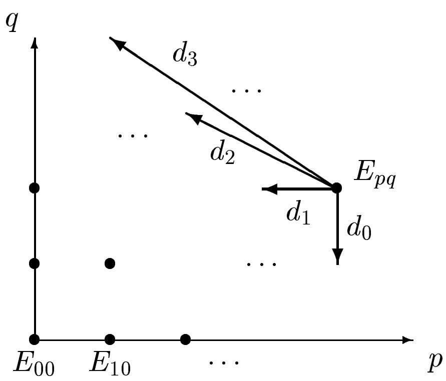
It is not hard to see that in $Homgr_G(E, E)$ one has

$$d_0 R_j + R_j d_0 = (a_j - 1) Id \tag{5}$$

since all the a_j belong to the centre of G.

Given d_0 we are interested in the following problem: how to find the elements $d_s \in Homgr_G^{-s,s-1}(E, E)$, $s = 1, \dots, m$ so that one has the equality

$$d^2 = (d_0 + d_1 + \dots + d_m)^2 = 0. \tag{6}$$



Pic.1

Let's consider the equation (6) as a system of equations

$$d_0 d_s + d_s d_0 = -\sum_{i=1}^{s-1} d_i d_{s-i}\,, \quad s = 1, \ldots, m. \tag{7}$$

First of all, we'll assume that all the desired d_s commute with all R_j (in the graded sense): $d_s R_j + R_j d_s = 0$. This assumption enables to define d_s only on the elements of the type $1 \otimes x_{i_1} \wedge \ldots \wedge x_{i_p} = 1 \otimes X$. The correctness is based on the following lemma.

Lemma 1. Suppose that the endomorphism $F_s = d_0 d_s + d_s d_0$ coincides with $K_s = -\sum\limits_{i=1}^{s-1} d_i d_{s-i}$ on E_{p0}. Then the system (7) holds in $Homgr_G(E, E)$.

Proof. It is enough to prove that both F_s and K_s commute with any R_j. The equality $K_s R_j = R_j K_s$ follows immediately from the above assumption and $F_s R_j = R_j F_s$ from (5). ■

In order to solve (7) we'll proceed inductively. Since d_1 should be related with the differential $\overline{d}$ in (1) we define d_1 in the following manner. Let's construct a commutative diagram

$$\begin{array}{ccccccc}
\ldots & \to & \mathbf{Z}G \otimes \Lambda^p \overline{\mathcal{G}} & \xrightarrow{d_1} & \mathbf{Z}G \otimes \Lambda^{p-1} \overline{\mathcal{G}} & \to & \ldots \\
 & & \downarrow & & \downarrow & & \\
\ldots & \to & \mathbf{Z}\overline{G} \otimes \Lambda^p \overline{\mathcal{G}} & \xrightarrow{\overline{d}} & \mathbf{Z}\overline{G} \otimes \Lambda^{p-1} \overline{\mathcal{G}} & \to & \ldots,
\end{array}$$

where the vertical arrows are homomorphisms $\mu \otimes 1$. This diagram is a diagram of G-modules (via μ). Thus we can define d_1 since all the G-modules $\mathbf{Z}G \otimes \Lambda^p \overline{\mathcal{G}}$ are free and the vertical arrows are surjective.

Obviously, $d_0d_1+d_1d_0 = 0$ on E_{p0}. By the lemma 1 $d_0d_1+d_1d_0 \equiv 0$.

A G-homorphism d_2 should be defined from the equality (X is fixed)

$$(d_0d_2 + d_2d_0)(1 \otimes X) = -d_1^2(1 \otimes X).$$

Since $\overline{d}^2 = 0$ then $z = -d_1^2(1 \otimes X)$ belongs to $Ker\,\mu \otimes \Lambda^{p-2}\overline{\mathcal{G}}$. It follows that z can be written in the form

$$z = \sum_{i=1}^{k} \sum_{t} w_{it}(a_i - 1) \otimes X_{it}$$

for some $w_{it} \in \mathbf{Z}G$ and basic elements X_{it} of $\Lambda^{p-2}\overline{\mathcal{G}}$.

The differential $d_0 = 0$ on E_{p0}. Consequently, it remains only to prove that z is a boundary. In fact,

$$z = d_0(\sum_{i=1}^{k} \sum_{t} w_{it} \otimes a_i \wedge X_{it}),$$

and we can define

$$d_2(1 \otimes X) = \sum_{i=1}^{k} \sum_{t} w_{it} \otimes a_i \wedge X_{it}.$$

Suppose that we have already defined $d_1, \dots, d_{s-1}$, $s \geq 3$, so that the first $s-1$ equations of the system (7) hold. Consider $K_s = -\sum\limits_{i=1}^{s-1} d_i d_{s-i}$.

Lemma 2. The equality $K_s d_0 = d_0 K_s$ holds.

Proof. Using (7) we have on the one hand

$$K_s d_0 = -\sum_{i=1}^{s-1} d_i d_{s-i} d_0 = -\sum_{i=1}^{s-1} d_i(-d_0 d_{s-i} - \sum_{j=1}^{s-i-1} d_j d_{s-i-j}) =$$
$$= \sum_{i=1}^{s-1} d_i d_0 d_{s-i} + \sum_{i=1}^{s-1} \sum_{j=1}^{s-i-1} d_i d_j d_{s-i-j} = \sum_{i=1}^{s-1} d_i d_0 d_{s-i} + \sum_{\substack{p+q+r=s \\ p,q,r>0}} d_p d_q d_r.$$

On the other hand,

$$d_0 K_s = -\sum_{i=1}^{s-1} d_0 d_i d_{s-i} = -\sum_{i=1}^{s-1}(-d_i d_0 - \sum_{j=1}^{i-1} d_j d_{i-j} d_{s-i-j}) =$$
$$= \sum_{i=1}^{s-1} d_i d_0 d_{s-i} + \sum_{i=1}^{s-1} \sum_{j=1}^{i-1} d_j d_{i-j} d_{s-i} = \sum_{i=1}^{s-1} d_i d_0 d_{s-i} + \sum_{\substack{p+q+r=s \\ p,q,r>0}} d_p d_q d_r. \blacksquare$$

In other words, this lemma asserts that K_s is a cycle relative to a differential D in $Homgr_G(E,E)$ such that $Df = d_0 f - (-1)^{\tilde{f}} f d_0$ where $\tilde{f}$ is the total degree of f. To find a solution of the system (7) is equivalent to prove that all K_s are boundaries.

By the lemma 2

$$0 = (d_0 K_s - K_s d_0)(1 \otimes X) = d_0 K_s (1 \otimes X).$$

It follows that $K_s(1 \otimes X)$ is a cycle of $E_{p-s,s-2}$. Since all the sequences (4p) are exact we deduce that there is an element $Y \in E_{p-s,s-1}$ such that $d_0(Y) = K_s(1 \otimes X)$ (recall that $s \geq 3$). Set $d_s(1 \otimes X) = Y$. Thus we have

$$(d_0 d_s + d_s d_0)(1 \otimes X) = d_0(Y) = K_s(1 \otimes X).$$

By the lemma 1 we obtain the equality $d_0 d_s + d_s d_0 = K_s$.

Let

$$E_n = \bigoplus_{p+q=n} E_{pq} = \bigoplus_{p+q=n} \mathbf{Z}G \otimes \Lambda^q A \otimes \Lambda^p \overline{\mathcal{G}} \cong \mathbf{Z}G \otimes \Lambda^n (A \oplus \overline{\mathcal{G}}) \cong \mathbf{Z}G \otimes \Lambda^n \mathcal{G}.$$

The differential $d = d_0 + \ldots + d_m$ maps E_n to E_{n-1}, i.e. we have the augmented complex

$$\ldots \xrightarrow{d} E_n \xrightarrow{d} \ldots \xrightarrow{d} E_0 \xrightarrow{\pi} \mathbf{Z} \longrightarrow 0. \tag{8}$$

In order to prove the theorem we have to show that (8) is exact.

Step 1. Exactness in dimension 0. Since $E_0 = \mathbf{Z}G$ and π is the augmentation epimorphism let's show that $Im\, d : E_1 \to E_0$ coincides with the augmentation ideal $Ker\, \pi \subset \mathbf{Z}G$.

Consider a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Z}G \otimes \overline{\mathcal{G}} & \xrightarrow{d_1} & \mathbf{Z}G & \xrightarrow{\pi} & \mathbf{Z} & \to & 0 \\
\downarrow & & \mu \downarrow & & || & & \\
\mathbf{Z}\overline{G} \otimes \overline{\mathcal{G}} & \xrightarrow{\overline{d}} & \mathbf{Z}\overline{G} & \xrightarrow{\overline{\pi}} & \mathbf{Z} & \to & 0.
\end{array}$$

It is not hard to see that $Ker\, \pi = Ker\, \mu + Im\, d_1$. But $Ker\, \mu \cong Im\, d_0 : \mathbf{Z}G \otimes A \to \mathbf{Z}G$, i.e.

$$Ker\, \pi = Im\, d_0 + Im\, d_1 = Im\, d.$$

Step 2. Exactness in dimensions ≥ 1. Consider the following finite filtration on E:

$$F_p E_n = \bigoplus_{\substack{s+q=n \\ s\leq p}} E_{sq}.$$

The differential d respects this filtration. Let (E^r_{pq}, d^r_{pq}) be the associated spectral sequence converging to homology of E.

Since there is the decomposition $d = d_0 + \ldots + d_m$ then $E^0_{pq} \cong E_{pq}$ and $d^0_{pq} = d_0$. From the exact sequences (4p) we deduce that $E^1_{pq} = 0$ if $q > 0$, and $E^1_{p0} \cong \mathbf{Z}\overline{G} \otimes \Lambda^p \overline{\mathcal{G}}$. Due to these isomorphisms d^1_{p0} becomes $\overline{d}$. It follows from (1) that $E^2_{pq} = 0$ if $p + q > 0$, i.e. (8) is in fact a resolution of $\mathbf{Z}$ over $\mathbf{Z}G$. **Q.E.D.**

As a direct consequence of the theorem 1 we obtain
Theorem 2. Let G be finitely generated nilpotent Magnus group of class c (e.g. a free nilpotent group). Then there is a resolution of the kind $\mathbf{Z}G \otimes \Lambda^* \mathcal{G} \to \mathbf{Z} \to 0$.
Proof. It is well known for $c = 1$ (abelian case). Step of induction is provided by the theorem 1. ■
Question. One can prove for a f.g. *free* nilpotent group G of class 2 that there is an isomorphism between integral homology groups $H_n G$ and $H_n \mathcal{G}$ (sections 3,4). Is it true for arbitrary f.g. Magnus nilpotent groups?

2 Resolutions for torsion-free nilpotent groups.

Given a free $\mathbf{Z}$-module A of finite rank k with a basis $a_1, \ldots, a_k$, let $S(A)$, $\Lambda(A)$ be symmetric and exterior algebras of A over $\mathbf{Z}$ and $\varepsilon : S(A) \to \mathbf{Z}$ be the canonical augmentation epimorphism.

Let's suppose that we're also given an associative ring R with identity and a monomorphism $\varphi : S(A) \to Z(R)$ where $Z(R)$ is the centre of R. By $\overline{R}$ we denote the quotient ring $R \otimes_{S(A)} \mathbf{Z}$ (R is a right $S(A)$-module via φ and $\mathbf{Z}$ is a left $S(A)$-module via ε). Let μ be the canonical epimorphism $R \to \overline{R}$. In the sequel we'll suppose that both R, $\overline{R}$ are free $\mathbf{Z}$-modules.

One knows ([**Bo**]) that there is an exact sequence of $S(A)$-modules

(Koszul resolution)

$$0 \longrightarrow S(A) \otimes \Lambda^k A \xrightarrow{\delta} \dots \xrightarrow{\delta} S(A) \xrightarrow{\varepsilon} \mathbf{Z} \to 0, \tag{9}$$

where δ is a $S(A)$-differential such that

$$\delta(1 \otimes a_{j_1} \wedge \dots \wedge a_{j_q}) = \sum_{s=1}^{q} (-1)^{s-1} a_{j_s} \otimes a_{j_1} \wedge \dots \widehat{a}_{j_s} \dots \wedge a_{j_q}.$$

Definition. The triple (A, R, φ) is said to have *E-property* (or to be an E-triple) if the sequence

$$0 \longrightarrow R \otimes \Lambda^k A \xrightarrow{d} \dots \xrightarrow{d} R \xrightarrow{\mu} \overline{R} \to 0 \tag{10}$$

obtained from (9) by applying the functor $R \otimes_{S(A)} -$ is exact.

Examples. Let R be a flat $S(A)$-module. Then (A, R, φ) has E-property. We are mostly interested in the following case: $\mathcal{G}$ is a $\mathbf{Z}$-torsion-free finite dimensional nilpotent Lie ring, A is its centre (supposed to be a direct summand of $\mathcal{G}$ as a $\mathbf{Z}$-module), $R = U(\mathcal{G})$ (universal enveloping ring), $\varphi : S(A) \to U(\mathcal{G})$ is induced by the inclusion $A \to \mathcal{G}$. It follows from Poincarè-Birkhoff-Witt theorem that $U(\mathcal{G})$ is a free $S(A)$-module, i.e. E-property holds.

Second example deals with torsion-free nilpotent groups. Let A be a central subgroup of a f.g. torsion-free nilpotent group G. If $a_1, \dots, a_k$ is a free basis of A (as a $\mathbf{Z}$-module) then we take $R = \mathbf{Z}G$ and $\varphi : S(A) \to \mathbf{Z}G$ is induced by the mapping which carries each $a_i \in A$ to $a_i - 1 \in \mathbf{Z}G$. The E-property easily follows (cf. (4)). (Note that the flatness of R over $S(A)$ also can be proved).

Theorem 3. Let (A, R, φ) be an E-triple. Suppose that there is a finite free resolution of an $\overline{R}$-module L of the kind

$$0 \longrightarrow \overline{R} \otimes B_m \xrightarrow{\overline{d}} \dots \xrightarrow{\overline{d}} \overline{R} \otimes B_0 \xrightarrow{\overline{\pi}} L \to 0 \tag{11}$$

Here $\overline{d}$ is a differential and $B_0, \dots, B_m$ are finitely generated free $\mathbf{Z}$-modules.

Then there exists a finite free resolution of L over R of the kind

$$0 \longrightarrow R \otimes C_n \xrightarrow{d} \dots \xrightarrow{d} R \otimes C_0 \xrightarrow{\pi} L \to 0 \tag{12}$$

where $C_n = \bigoplus_{p+q=n} \Lambda^q A \otimes B_p$ and π is the composition

$$R \otimes C_0 = R \otimes B_0 \xrightarrow{\mu \otimes 1} \overline{R} \otimes B_0 \xrightarrow{\overline{\pi}} L$$

(for a suitable differential d depending on $\overline{d}$).
Proof. The proof of this theorem is quite analogous to that of theorem 1 (we used only the properties of an E-triple). ■

Consequently, we obtain
Theorem 4. Let G be a f.g. torsion-free nilpotent group and $r = cdG$ be the Hirsch number of G. Then there exists a free $\mathbf{Z}$-module M of rank r and a finite free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$ of the kind

$$0 \longrightarrow \mathbf{Z}G \otimes \Lambda^r M \xrightarrow{d} \dots \xrightarrow{d} \mathbf{Z}G \xrightarrow{\pi} \mathbf{Z} \to 0. \tag{13}$$

Moreover, we can choose M to be $\mathcal{G}$ – the Lie ring associated to the central series of G consisting of the isolators of $\gamma_i G$ in G.

Proof. It is known that G admits a central series

$$1 = G_{l+1} \subset G_l \subset \dots \subset G_1 = G$$

with torsion-free abelian factors G_i/G_{i+1}, and

$$r = \sum_{i=1}^{l} rk_{\mathbf{Z}} G_i/G_{i+1}$$

is independent of the choice of such a central series.

Let $A = G_l$. Then $(A, \mathbf{Z}G, \varphi)$ is an E-triple as it was described in the second example before the theorem 3. We may assume by induction hypothesis that there is a resolution of desired type for the quotient group $\overline{G} = G/A$. Thus, theorem 3 provides such a resolution for a group G. Note that we can choose a module M to be $A \oplus \overline{M}$ where $\overline{M}$ had been chosen for $\overline{G}$.

Consider now a special kind of a the central series

$$1 = \sqrt{\gamma_{c+1} G} \subset \sqrt{\gamma_c G} \subset \dots \subset \sqrt{\gamma_1 G} = G. \tag{14}$$

Here $\sqrt{\gamma_i G} = \{g \in G | g^m \in \gamma_i G \text{ for some } m \geq 1\}$ is the isolator of $\gamma_i G$ in G. In fact, (14) is an N-series, i.e. $[\sqrt{\gamma_i G}, \sqrt{\gamma_j G}] \subset \sqrt{\gamma_{i+j} G}$ for all i, j. The associated Lie ring

$$\mathcal{G} = \bigoplus_{i \geq 1} \sqrt{\gamma_i G} / \sqrt{\gamma_{i+1} G}$$

is a free **Z**-module of rank r.

Let A be $\sqrt{\gamma_c G}$ and $\overline{G}$ be G/A. Since

$$\sqrt{\gamma_i \overline{G}} = \sqrt{\gamma_i G}/A$$

we obtain the isomorphism of **Z**-modules $\mathcal{G} \cong A \oplus \overline{\mathcal{G}}$. By induction hypothesis we may choose $\overline{M}$ to be $\overline{\mathcal{G}}$ and the theorem is proved. ■

Remark. Since **Z**-rank of M is equal to *Hirsch number of* G that is cdG ([**Bi**]), the resolution (13) has minimal possible length.

3 Example: a resolution for a free two-step nilpotent group

In this section we give an explicit construction of a free finite resolution for a free 2-step nilpotent group G.

3.1. Presentations of G **and** $\mathcal{G}$ **.** Let $G = G(r)$ be a free nilpotent group on r generators $x_1, \dots, x_r$. Consider the following presentation of G:

$$G =< x_i, c_{ij}; 1 \leq i < j \leq r \,|\, c_{ij} = [x_i, x_j],\, c_{ij}\, are\, central >,$$

where $[a, b] = a^{-1}b^{-1}ab$. Corresponding Lie ring $\mathcal{G} = \mathcal{G}(r)$ has the similar presentation

$$\mathcal{G} =< x_i, c_{ij}; 1 \leq i < j \leq r \,|\, c_{ij} = [x_i, x_j],\, c_{ij}\, are\, central > .$$

Remark. In fact, it is more convenient to use the same notation for generators and basic commutators of G and $\mathcal{G}$. In few doubtful cases we'll always mention what object is considered. ■

In what follows we will use a geometric interpretation of generators x_k and commutators c_{ij}. Let $\Delta = \Delta(r)$ be a standard simplex on r vertices embedded in the euclidian space $\mathbf{R}^r$, i.e.

$$\Delta = \{(a_1, \dots, a_r) \in \mathbf{R}^r | \sum_{i=1}^{r} a_i = 1,\, a_i \geq 0\}.$$

Let the generator x_i be represented by the vertex $v_i = (0.\dots, 1, \dots, 0)$ (1 is standing on the i-th position) and the commutator c_{ij} - by the directed edge e_{ij}, $i < j$, issuing from v_i and terminating in v_j (in coordinates $e_{ij} = (0, \ldots, -1, 0, \ldots, 1, 0, \ldots)$). If $v \in \Delta$ is a vertex (or e is a directed edge) then we denote by $x(v)$ ($c(e)$) the generator (commutator) corresponding to v (resp. e).

Next we order the set of x_i according to their indices, i.e. $x_1 < \ldots < x_r$. We'll say that $c_{ij} < c_{mk}$ if and only if either $j < k$ or $j = k$ and $i < m$. Also $c_{ij} < x_k$ for all i, j, k. Due to the above correspondence the sets of vertices and edges of Δ are ordered, too.

3.2. Trees in Δ and their labels. First of all, let T be a maximal tree in the 1-skeleton of Δ. Evidently, T has r vertices and $r-1$ edges. Let $e_1(T) < \ldots < e_{r-1}(T)$ be vectors corresponding to the directed edges of the tree T. It's not hard to see that they form a basis of the subspace of $\mathbf{R}^r$ consisting of all vectors with the zero sum of coordinates. If we add to the above vectors some vector $e_0 = (a_1, \ldots, a_r)$ such that $a_1 + \ldots + a_r \neq 0$ we obtain a basis for $\mathbf{R}^r$. Let $\delta = \det(e_0, e_1(T), \ldots, e_{r-1}(T))$ be determinant composed from the coordinates of vectors $e_0, e_1(T), \ldots, e_{r-1}(T)$. It can be proved by induction on r that $\delta = (-1)^{\varepsilon(T)}(a_1 + \ldots + a_r)$ where $\varepsilon(T) = 0$ or 1 and depends only on the tree T. By definition, $\varepsilon(T)$ is said to be the parity of the tree T.

Let $V = \{v_{s_1}, \ldots, v_{s_p}\}$ be an ordered subset of vertices. Consider a face $\Gamma(V)$ of the simplex Δ spanned by V. $\Gamma(V)$ and $\Delta(p)$ are isomorphic as simplicial sets: vertex v_{s_j} is in a correspondence with v_j. To a maximal tree $T \in \Gamma(V)$ corresponds a tree $T' \in \Delta(p)$. By definition, $\varepsilon(T) = \varepsilon(T')$. Let's denote by $\max(V)$ the set of all maximal trees in $\Gamma(V)$. One knows (due to Cayley and others) that the number of maximal trees in $\max(V)$ is equal to p^{p-2}.

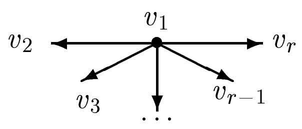
Let T be, as earlier, a maximal tree in $\Gamma(V)$. Since T is maximal then any complementary edge e in $\Gamma(V)$ defines a circuit $Z = Z(e)$ in the 1-skeleton of $\Gamma(V)$. We orient Z according to the direction pointed out by the senior edge of Z (recall that we have ordered all commutators, i.e. edges of Δ). If the orientation of Z given by e is the same (resp. opposite) then we call e positive relative to T (resp. negative).

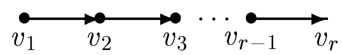
Let's define now the label $u(T) \in G$ of the tree T as

$$u(T) = \prod_{e>0} c(e),$$

where the product is taken over all positive edges relative to T. Note that $u(T)$ belongs to the centre of G.

Examples. Let $V = \{v_1, \ldots, v_r\}$ and T be the star tree with edges e_{1j}, $j = 2, \ldots, r$. Then all complementary edges are positive and $u(T) = \prod_{1<i<j\leq r} c_{ij}$. On the contrary, if T is the non-branched tree with edges $e_{i,i+1}$, $i = 1, \ldots, r-1$ then all complementary edges are negative and $u(T) = 1$.





Pic.2 Examples of trees.

3.3. Parity of partition. Let V be a finite ordered set, e.g. $V = \{1, \ldots, n\}$. Let's suppose that V is decomposed in a disjoint union of its components $V = V_1 \sqcup \ldots \sqcup V_h$ where the order of subsets is fixed. Consider a shuffle of elements of V corresponding to such a partition, i.e. a permutation of V such that first we write down the elements of V_1 (in the increasing order), next the elements of V_2 and so on. By definition, the parity of the partition $V = V_1 \sqcup \ldots \sqcup V_h$ is the parity of this permutation. We denote it by $\varepsilon(V_1, \ldots, V_h)$. If P is a subset of V then we denote $\varepsilon(P, V \setminus P)$ simply by $\varepsilon(P)$ (when it is no doubt about V). As usual, $|P|$ denotes the number of elements of P.

In the sequel we'll use also the following notations. Let V be a subset of vertices of Δ. We denote by $X(V)$ the exterior product in $\Lambda(\mathcal{G})$ of the elements x_s corresponding to vertices of V (written in the increasing order). For example, if $V = \{v_1, v_3, v_4\}$ then $X(V) = x_1 \wedge x_3 \wedge x_4$. Analogously, we denote by $C(E)$ the exterior product of basic commutators corresponding to an ordered set E of directed edges. In particular, for any tree $T \in \Delta$ (rather for the set of edges of T) the element $C(T) = c_1(T) \wedge \ldots \wedge c_{s-1}(T)$ is defined. The typical basic element of $\Lambda(\mathcal{G})$ is consequently $C(E) \wedge X(V)$.

Finitely we denote by $x(V)$ the product of generators $x(v)$, $v \in V$, *in the group* G written in the *decreasing* order (for V in the above example, $x(V) = x_4x_3x_1$).

3.4. Let's return to a resolution $\mathbf{Z}G \otimes \Lambda^*\mathcal{G}$ with the differential $d = d_0 + \ldots + d_r$ constructed in the theorem 1. In our notations the application of d_0 can be written as follows:

$$d_0(1 \otimes C(E) \wedge X(V)) = \sum_{e \in E} (-1)^{\varepsilon(e)}(c(e) - 1) \otimes C(E \setminus e) \wedge X(V)$$

(elements of the type $1 \otimes C(E) \wedge X(V)$ represent a basis of free G-module $\mathbf{Z}G \otimes \Lambda^*\mathcal{G}$.

Let's describe $d_1, \ldots, d_r$. Recall that it is enough to determine these G-morphisms only on the elements of the type $1 \otimes X(V)$.

Proposition 5. In the above notations

$$d_1(1 \otimes X(V)) = \sum_{v \in V} (-1)^{\varepsilon(v)}(x(v) - 1) \otimes X(V \setminus v).$$

If $s \geq 2$ then

$$d_s(1 \otimes X(V)) = \sum_{\substack{P \in V,\, |V|=s \\ T \in \max(P)}} (-1)^{\varepsilon(P)+\varepsilon(T)+s-1} x(P)u(T) \otimes C(T) \wedge X(V \setminus P).$$

In particular, if $V = \{v_{s_1}, \ldots, v_{s_p}\}$

$$d_2(1 \otimes X(V)) = \sum_{P=\{v_{s_i}, v_{s_j}\}} (-1)^{i+j} x_{s_j} x_{s_i} \otimes c_{s_i s_j} \wedge X(V \setminus P).$$

Proof. Roughly speaking, d_s removes step by step all subsets P of s vertices from V and inserts instead $s-1$ commutators "spanning" a maximal tree in $\Gamma(P)$ (each tree contributes a summand). We can also write

$$d_s(1 \otimes X(V)) = \sum_{P \in V,\, |P|=s} (-1)^{\varepsilon(P)} d_s(1 \otimes X(P)) \wedge X(V \setminus P). \quad (15)$$

Our aim is to check for $s \geq 2$ the equality

$$d_0 d_s(1 \otimes X(V)) = -\sum_{i=1}^{s-1} d_i d_{s-i}(1 \otimes X(V)). \quad (16)$$

In view of (15) it is not hard to see that we may restrict ourselves to the case $|V| = s$ and moreover, $V = \{v_1, \ldots, v_s\}$ (it may be done by direct computations but one should be careful with signs).

Consider the left-hand side of (16). It is equal to

$$\sum_{\substack{T\in\max(V)\\ e\in T^1}} (-1)^{\varepsilon(T)+\varepsilon(e,T^1\setminus e)+s-1} x(V)u(T)(c(e)-1)\otimes C(T\setminus e).$$

What happens to a tree T when a fixed edge e is taken off? Evidently, T just splits into 2 disjoint subtrees T_1 and T_2 (in order to fix indices of trees we'll always assume that the maximal vertex $v_s\in T_2$). Suppose that T_1 has i vertices, then T_2 has exactly $s-i$ vertices and both T_1, T_2 define unique subsets P_1, P_2 of its vertices.

Consider now the right-hand side of (16). In the full expression of it we can find 2 summands corresponding to the pair T_1, T_2. One of them is equal to

$$\begin{gathered}(-1)^{\varepsilon(T_2)+\varepsilon(T_1)+\varepsilon(P_2)+s-i} w(T_2)w(T_1)\otimes C(T_2)\wedge C(T_1) = \\ = (-1)^{\varepsilon(T_2)+\varepsilon(T_1)+\varepsilon(P_2)+s-i+(s-i-1)(i-1)} w(T_2)w(T_1)\otimes C(T_1)\wedge C(T_2)\end{gathered}$$

(if a tree Q has only one vertex v then we denote by $w(Q)$ the element $x(v)-1\in \mathbf{Z}G$ else $w(Q)=x(P)u(Q)\in G$), the second

$$(-1)^{\varepsilon(T_1)+\varepsilon(T_2)+\varepsilon(P_1)+i} w(T_1)w(T_2)\otimes C(T_1)\wedge C(T_2).$$

Let

$$\mu=\mu(T_1,T_2)=\varepsilon(T_1)+\varepsilon(T_2)+\varepsilon(P_1)+i.$$

It follows from the definition of signs that

$$\varepsilon(T_2)+\varepsilon(T_1)+\varepsilon(P_2)+s-i+(s-i-1)(i-1)\equiv \mu-1 \pmod 2.$$

Thus, the equality (16) turns into

$$\begin{aligned}&\sum_{\substack{T\in\max(V)\\ e\in T^1}} (-1)^{\varepsilon(T)+\varepsilon(e,T^1\setminus e)+s-1} w(T)(c(e)-1)\otimes C(T\setminus e) = \\ &\quad = \sum_{T_1,T_2} (-1)^{\mu}(w(T_1)w(T_2)-w(T_2)w(T_1))\otimes C(T_1)\wedge C(T_2). \qquad (17)\end{aligned}$$

For a fixed pair of disjoint trees T_1, T_2 we denote by $\mathcal{E}_+(T_1,T_2)$ (resp. $\mathcal{E}_-(T_1,T_2)$) the set of directed edges going from T_1 to T_2 (resp. from T_2 to T_1) and

$$\mathcal{E}(T_1,T_2)=\mathcal{E}_+(T_1,T_2)\sqcup\mathcal{E}_-(T_1,T_2).$$

If $e \in \mathcal{E}(T_1, T_2)$ we denote by $T_1 \#_e T_2$ the tree obtained from T_1 and T_2 by joining the edge e to them. Now, the equality (17) is the consequence of the equalities (for each pair T_1, T_2)

$$\sum_{T=T_1\#_e T_2} (-1)^{\varepsilon(T)+\varepsilon(e,T^1\setminus e)+s-1} w(T_1\#_e T_2)(c(e)-1) \otimes C(T\setminus e) =$$
$$= (-1)^{\mu}(w(T_1)w(T_2) - w(T_2)w(T_1)) \otimes C(T_1) \wedge C(T_2). \tag{18}$$

We are going to obtain a simpler form of (18). First of all, the element

$$(-1)^{\varepsilon(T)} C(T) = (-1)^{\varepsilon(T)} c_1(T) \wedge \ldots \wedge c_{s-1}(T)$$

is independent from the order of the commutators in the exterior product provided the determinant $\delta = \det(e_0, e_1(T), \ldots, e_{r-1}(T))$ which is responsible for the sign of the tree $T = T_1 \#_e T_2$ is computed in accordance with this order (if you make the transposition of two commutators in the exterior product you have to make a transposition of corresponding lines in the determinant, and you have 2 changes of sign). Thus we may assume that $c_1(T) = c(e)$ (i.e. the second line of δ is related to e) and that $C(T \setminus e) = C(T_1) \wedge C(T_2)$.

Secondly, using relations in G we have

$$x(P_1)x(P_2) = x(V) \prod_{e\in\mathcal{E}_+(T_1,T_2)} c(e); \quad x(P_2)x(P_1) = x(V) \prod_{e\in\mathcal{E}_-(T_1,T_2)} c(e).$$

Then (18) follows from the equality in $\mathbf{Z}G$

$$\sum_{T=T_1\#_e T_2} (-1)^{\varepsilon(T)+s-1+\mu} u(T_1\#_e T_2)(c(e)-1) =$$
$$= u(T_1)u(T_2) \prod_{e\in\mathcal{E}_+(T_1,T_2)} c(e) - u(T_1)u(T_2) \prod_{e\in\mathcal{E}_-(T_1,T_2)} c(e). \tag{19}$$

Lemma 3. If in the equality (19) the edge e belongs to the set $\mathcal{E}_+(T_1, T_2)$ then $(-1)^{\varepsilon(T)+s-1+\mu} = 1$, otherwise $(-1)^{\varepsilon(T)+s-1+\mu} = -1$.
Proof. Recall that $\mu = \varepsilon(T_1) + \varepsilon(T_2) + \varepsilon(P_1) + i$. Let $e = e_{jk}$. Then

$$\det(v_j, e_{jk}, e_1(T_1), \ldots, e_1(T_2), \ldots) =$$
$$= \det(v_j, v_j + e_{jk}, e_1(T_1), \ldots, e_1(T_2), \ldots) =$$
$$= \det(v_j, v_k, e_1(T_1), \ldots, e_1(T_2), \ldots) = (-1)^{\varepsilon(T)}.$$

Applying the Laplace theorem for the decomposition of determinant along the fixed columns corresponding to vertices of T_1 (there is only one non-trivial minor) we obtain that

$$(-1)^{\varepsilon(T)} = (-1)^{\mu+s-1},\ if\ v_j \in T_1,\ v_k \in T_2.$$

The remaining case is treated analogously. ■

Now we can rewrite (19) in the following manner

$$\sum_{e\in\mathcal{E}(T_1,T_2)} (-1)^{\tilde{e}} u(T_1\#_e T_2)(c(e)-1) =$$
$$= u(T_1)u(T_2) \prod_{e\in\mathcal{E}_+(T_1,T_2)} c(e) - u(T_1)u(T_2) \prod_{e\in\mathcal{E}_-(T_1,T_2)} c(e). \qquad (20)$$

where $\tilde{e} = 0$ if $e \in \mathcal{E}_+(T_1, T_2)$ and $\tilde{e} = 1$ otherwise.

Note that by virtue of the definition of $u(T)$ each label $u(T_1\#_e T_2)$ may be represented as $u(T_1)u(T_2)z(e)$ where $z(e)$ is the product of commutators corresponding only to the edges $f \in \mathcal{E}(T_1, T_2)$. In other words, $u(T_1\#_e T_2)$ is always "divisible" by $u(T_1)u(T_2)$. Cancelling in (20) the common factor $u(T_1)u(T_2)$ finitely we have to check the equality

$$\sum_{e\in\mathcal{E}(T_1,T_2)} (-1)^{\tilde{e}} z(e)(c(e)-1) = \prod_{e\in\mathcal{E}_+(T_1,T_2)} c(e) - \prod_{e\in\mathcal{E}_-(T_1,T_2)} c(e). \quad (21)$$

Ordering of the set $\mathcal{E}(T_1, T_2)$. Let $e, f \in \mathcal{E}(T_1, T_2)$ and $Z = Z(e, f)$ be the uniquely defined circuit obtained from T_1 and T_2 by joining e, f. We'll say that $e \prec f$ if walking along positive orientation of Z we occur in the tree T_2 after passing through the edge e. Otherwise we'll say that $e \succ f$.

Let's show that this relation is an order relation on the set $\mathcal{E}(T_1, T_2)$, i.e. if $e \prec f$, $f \prec h$ then $e \prec h$. Suppose that $h \prec f$. Consider the union of all edges from the circuits $Z(e, f)$, $Z(f, h)$ and $Z(e, h)$. It is a graph $\Phi = \Phi(e, f, h)$ whose vertices have degree 2 except two vertices of degree 3. Moreover, Φ contains no separating edges. Any edge b of Φ belongs exactly to 2 of the above 3 circuits. If $e \prec f$, $f \prec h$, $h \prec e$, then on succesive walk in positive direction of 3 circuits, we pass through the edge b twice: one time in accordance with its direction and second time in opposite one. Let b be the senior edge of Φ (in the sense of the order for the edges of the simplex Δ). Obviously, b is the senior edge in both circuits to which it belongs. Since we choose positive orientation of a circuit according to the direction of its senior edge then we obtain a contradiction, i.e. $e \prec h$.

Thus, we order all edges of $\mathcal{E}(T_1, T_2)$. Write them down in the increasing order: $e_1 \prec \ldots \prec e_t$. We introduce the following notations ($e \in \mathcal{E}(T_1, T_2)$):

$$z_-(e) = \prod_{\substack{f \succ e \\ f \in \mathcal{E}_-(T_1,T_2)}} c(f); \quad z_+(e) = \prod_{\substack{f \prec e \\ f \in \mathcal{E}_+(T_1,T_2)}} c(f).$$

In the left-hand side of (21) we have $z(e) = z_-(e) z_+(e)$. In fact, if e is fixed (say, $e \in \mathcal{E}_+(T_1, T_2)$), then $c(f)$ is a factor of $z(e)$ in two cases, namely, if either $f \succ e$, $f \in \mathcal{E}_-(T_1, T_2)$ or $f \succ e$, $f \in \mathcal{E}_+(T_1, T_2)$. In both cases f is positive relative to the tree $T = T_1 \#_e T_2$. We will call such a decomposition of $z(e)$ into 2 factors polar.

The proof of the equality (21) is based on the following

Lemma 4. Let

$$S(p) = \sum_{i=1}^{p} (-1)^{\tilde{e}_i} (z(e_i) c(e_i) - z(e_i)), \; p = 1, \ldots, t.$$

If $e_p \in \mathcal{E}_+(T_1, T_2)$ then

$$S(p) = - \prod_{e \in \mathcal{E}_-(T_1,T_2)} c(e) + z(e_p) c(e_p),$$

else

$$S(p) = - \prod_{e \in \mathcal{E}_-(T_1,T_2)} c(e) + z(e_p),$$

Proof. We proceed by induction on p. If $p = 1$ then we have $z(e_1) = z_-(e_1)$. Suppose that we have proved the lemma for some p, and $e_p \in \mathcal{E}_+(T_1, T_2)$ (the remaining case can be treated in a similar way). If $e_{p+1} \in \mathcal{E}_+(T_1, T_2)$ then

$$S(p+1) = - \prod_{e \in \mathcal{E}_-(T_1,T_2)} c(e) + z(e_p) c(e_p) - z(e_{p+1}) + z(e_{p+1}) c(e_{p+1}),$$

The polar decomposition yields $z(e_p) c(e_p) = z(e_{p+1})$, i.e. step of induction is done. The case $e_{p+1} \in \mathcal{E}_-(T_1, T_2)$ is left to the reader. ∎

In any case, if $p = t$ we get just the equality (21), i.e. the proposition 5 is proved. **Q.E.D.**

4 An isomorphism between H_nG and $H_n\mathcal{G}$

We continue the previous notations. In the section 3 we have constructed an explicit resolution of $\mathbf{Z}$ over $\mathbf{Z}G$. Applying to it the trivialisation functor $\mathbf{Z}\otimes_{\mathbf{Z}G}-$ we obtain the chain complex $(\Lambda^*\mathcal{G}, d)$ with the differential $d = -d_2 + d_3 \ldots + (-1)^{r-1}d_r$ (here we especially alternate the signs). The d_s is defined by the following properties:

1. $d_s(C(E)\wedge X(V)) = (-1)^{|E|}C(E)\wedge d_s(X(V))$;
2. $d_s(X(V)) = \sum\limits_{\substack{P\subset V,\,|P|=s\\ T\in\max(P)}} (-1)^{\varepsilon(P)+\varepsilon(T)}C(T)\wedge X(V\setminus P)$.

The n-th homology group of this complex is isomorphic to H_nG.

The homology groups $H_n\mathcal{G}$ can be computed by means of the complex $(\Lambda^*\mathcal{G}, \partial)$ where

$$\partial(z_1\wedge\ldots\wedge z_n) = \sum_{i<j}(-1)^{i+j}[z_i, z_j]\wedge z_1\ldots\widehat{z_i}\ldots\widehat{z_j}\ldots\wedge z_n.$$

It is not hard to see that $\partial = -d_2$ in the above notations.

The main goal of this section is to prove the existence of an isomorphism between H_nG and $H_n\mathcal{G}$. It is sufficient to construct an automorphism $F:\Lambda^*\mathcal{G}\to\Lambda^*\mathcal{G}$ of the graded $\mathbf{Z}$-module $\Lambda^*\mathcal{G}$ for which the equality $dF = F\partial$ holds.

Note that $\Lambda(\mathcal{G})$ is actually a bi-graded algebra:

$$\Lambda(\mathcal{G}) = \bigoplus_{q,p\geq 0}\Lambda^{qp}(\mathcal{G}),$$

where the elements of the type $C(E)\wedge X(V)$, $|E| = q$, $|V| = p$, form a $\mathbf{Z}$-basis of $\Lambda^{qp}(\mathcal{G})$. Evidently, the morphism d_s belongs to the group $Homgr^{s-1,-s}(\Lambda(\mathcal{G}),\Lambda(\mathcal{G}))$.

We'll look for F of the type $F = 1 - f_1 + \ldots + (-1)^r f_r$ where 1 stands for the identity map and $f_k\in Homgr^{k,-k}(\Lambda(\mathcal{G}),\Lambda(\mathcal{G}))$, i.e. f_k maps $\Lambda^{qp}(\mathcal{G})$ into $\Lambda^{q+k,p-k}(\mathcal{G})$. Such an F is always an automorphism since $F = 1+N$ and N is nilpotent ($N^{r+1} = 0$).

Now we have to solve the equation

$$(-d_2+\ldots+(-1)^{r-1}d_r)(1-f_1+\ldots+(-1)^r f_r) = (1-f_1+\ldots+(-1)^r f_r)(-d_2).$$

Since it is graded it splits into the following system

$$\left\{\begin{array}{lcl} d_3 & = & f_1 d_2 - d_2 f_1 \\ \ldots & & \\ d_k + d_{k-1} f_1 + \ldots + d_3 f_{k-3} & = & f_{k-2} d_2 - d_2 f_{k-2} \\ \ldots & & \\ d_r + d_{r-1} f_1 + \ldots + d_3 f_{r-3} & = & f_{r-2} d_2 - d_2 f_{r-2}. \end{array}\right. \tag{22}$$

Two equations in which f_{r-1}, f_r appear are of the kind $0 = 0$, i.e. for fixed r f_{r-1} and f_r are irrelevant.

Let

$$Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})) = \bigoplus_{q,p} Homgr^{q,p}(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})).$$

Considering the summands with $q + p \equiv 0 \pmod 2$ and the summands with $q + p \equiv 1 \pmod 2$ we obtain a decomposition

$$Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})) = Homgr_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})) \oplus Homgr_1(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})),$$

i.e. $Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$ is a superalgebra. As usual, we call the corresponding morphisms even or odd. For example, all d_s are odd and f_k are even. Moreover, we introduce a differential D on the graded $\mathbf{Z}$-module $Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$ of total degree (-1) by the formula $Df = d_2 f - (-1)^{\tilde{f}} f d_2$. Note that the left-hand sides of (22) are boundaries relative D.

To get a solution of (22) we impose some restrictions on f_k.

Definition 1. A morphism $f \in Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$ is said to be a morphism of the type $(0, q, p)$ if

1. $f \in Homgr^{p,-p}(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})) \subset Homgr_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$;
2. $f(\Lambda^{st}(\mathcal{G})) = 0$ if $s < q$ or $t < p$;
3. for any E,V $\quad f(C(E) \wedge X(V)) =$

$$= \sum_{\substack{B \subset E \\ |B| = q}} \sum_{\substack{R \subset V \\ |R| = p}} (-1)^{\mu} C(E \setminus B) \wedge f(C(B) \wedge X(R)) \wedge X(V \setminus R).$$

where $\mu = \varepsilon(E \setminus B) + \varepsilon(R)$.

Definition 2. A morphism $f \in Homgr(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$ is said to be a morphism of the type $(1, q, p)$ if

1. $f \in Homgr^{p-1,-p}(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})) \subset Homgr_1(\Lambda(\mathcal{G}), \Lambda(\mathcal{G}))$;

2. $f(\Lambda^{st}(\mathcal{G})) = 0$ if $s < q$ or $t < p$;

3. for any E,V $\quad f(C(E) \wedge X(V)) =$

$$= (-1)^{|E|-q} \sum_{\substack{B \subset E \\ |B|=q}} \sum_{\substack{R \subset V \\ |R|=p}} (-1)^{\mu} C(E \setminus B) \wedge f(C(B) \wedge X(R)) \wedge X(V \setminus R).$$

(μ as above).

In other words, the morphisms of the type $(0, q, p)$ (of the type $(1, q, p)$) are even (resp. odd) and are completely determined by their values on $\Lambda^{qp}(\mathcal{G})$. In the sequel this property will be called shortly *the principle of localisation.* For example, d_s is of the type $(1, 0, s)$ (see the beginning of this section).

The connection between these two types of morphisms is depicted in the following lemma.

Lemma 5. If f is a morphism of the type $(0, q, p)$ then $-Df = fd_2 - d_2 f$ is a morphism of the type $(1, q-1, p+2)$.

Proof. It can be checked by direct computations. ■

Let's go back to solving the system (22). Consider a simpler system

$$\begin{cases} d_3 &= f^{(1)}_{(1)} d_2 - d_2 f^{(1)}_{(1)} &= -Df^{(1)}_{(1)} \\ \dots & & \\ d_r &= f^{(1)}_{(r-2)} d_2 - d_2 f^{(1)}_{(r-2)} &= -Df^{(1)}_{(r-2)} \end{cases} \tag{23}$$

Since d_k is of the type $(1, 0, k)$, by lemma 5 one should look for $f^{(1)}_{(k-2)}$ among the morphisms of the type $(0, 1, k-2)$. Let $|V| = k$. Then

$$d_k(X(V)) = \sum_{T \in \max(V)} (-1)^{\varepsilon(T)} C(T).$$

On the other hand,

$$-Df^{(1)}_{(k-2)}(X(V)) = f^{(1)}_{(k-2)} d_2(X(V))$$

since $f^{(1)}_{(k-2)}(\Lambda^{0k}(\mathcal{G})) \equiv 0$ by localisation principle. Again in accordance with it the following equality is sufficient to determine $f^{(1)}_{(k-2)}$:

$$f^{(1)}_{(k-2)} d_2(X(V)) =$$
$$= \sum_{\substack{L \subset V \\ |L|=2}} (-1)^{\varepsilon(L)} f^{(1)}_{(k-2)}(C(L) \wedge X(V \setminus L)) = \sum_{T \in \max(V)} (-1)^{\varepsilon(T)} C(T).$$

Elements of the type $C(L)\wedge X(V)=c_{ij}\wedge X(P)$, $|P|=k-2$ form a basis of $\Lambda^{1,k-2}(\mathcal{G})$. We define $f^{(1)}_{(k-2)}$ on these basic elements as follows. If $\{v_i,v_j\}\cap P\neq\emptyset$ then $f^{(1)}_{(k-2)}(c_{ij}\wedge X(P))=0$. For the other basic elements (they are of the type $C(L)\wedge X(V\setminus L))$, where $L=\{v_i,v_j\}$, $V=L\sqcup P$) the value of $f^{(1)}_{(k-2)}(C(L)\wedge X(V\setminus L)\in\Lambda^{k-1,0}(\mathcal{G})$ may be arbitrary provided the only condition is satisfied (one condition for every fixed V):

$$\sum_{\substack{L\subset V\\ |L|=2}}(-1)^{\varepsilon(L)}f^{(1)}_{(k-2)}(C(L)\wedge X(V\setminus L))=\sum_{T\in\max(V)}(-1)^{\varepsilon(T)}C(T).$$

4.2. Solution of the system (22). Let's fix a solution of the system (23), i.e. a collection of morphisms $(f^{(1)}_{(1)},\ldots,f^{(1)}_{(r-2)})$. It turns out that a solution of (22) i.e. a collection $(f_1,\ldots,f_{r-2})$, may be obtained in the following manner:

1.

$$f_p=\sum_{\substack{t_1>\ldots>t_n\geq 1\\ q_1t_1+\ldots+q_nt_n=p,\,q_i\geq 1}}f^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)},\quad p=1,\ldots,r-2.$$

2. Morphisms $f^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)}$ are of the type $(0,q,p)$ where $q=\sum\limits_{i=1}^{n}q_i$ and are defined inductively according to the localisation principle ($|E|=q-1$, $|V|=p$) by the formula:

$$f^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)}(C(E)\wedge c\wedge X(V))=$$
$$\sum_{i=1}^{n}\sum_{\substack{R\subset V\\ |V\setminus R|=t_i}}(-1)^{\nu_i}f^{(q_1,\ldots,q_i-1,\ldots,q_n)}_{(t_1,\ldots,t_i,\ldots,t_n)}(C(E)\wedge X(R))\wedge f^{(1)}_{(t_i)}(c\wedge X(V\setminus R)).$$

Here $\nu_i=p-t_i+\varepsilon(R)$. If $q_i=1$ for some i then we omit in the right-hand sum both indices $q_i-1=0$ and t_i.

Note that one can parametrise the summands of f_p by all Young diagrams on p cells (having q_1 lines of length t_1,..., q_n lines of length t_n). The special role of the last commutator in the inductive definition of $f=f^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)}$ is quite conventional (like a decomposition of a determinant along the last column). In fact, f always possesses the following property:

$$f(c_1\wedge\ldots\wedge c_q\wedge X(V))=(-1)^{sign(\pi)}f(c_{\pi(1)}\wedge\ldots\wedge c_{\pi(q)}\wedge X(V)),$$

where $\pi \in Sym_q$.

Lemma 6. The following equality is satisfied in $Homgr(\Lambda(\mathcal{G}), (\Lambda(\mathcal{G}));$

$$-Df^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)} = \sum_{i=1}^{n} d_{t_i+2} f^{(q_1,\ldots,q_i-1,\ldots,q_n)}_{(t_1,\ldots,t_i,\ldots,t_n)}. \tag{24}$$

Proof. It is not hard to show that d_k is a morphism of the type $(1, q, p+k)$ if f is of the type $(0, q, p)$. Consequently, both sides of (24) are of the type $(1, q-1, p+2)$ due to lemma 5. According to the localisation principle one should verify (24) only on the basic elements of $\Lambda^{q-1,p+2}(\mathcal{G})$.

Let $C(E) \wedge X(V)$ be a basic element of $\Lambda^{q-1,p+2}(\mathcal{G})$. Let's denote by f the morphism $f^{(q_1,\ldots,q_n)}_{(t_1,\ldots,t_n)}$, by $f^{(i)}$ - $f^{(q_1,\ldots,q_i-1,\ldots,q_n)}_{(t_1,\ldots,t_i,\ldots,t_n)}$.

On the one hand,

$$\begin{aligned}
&-Df(C(E) \wedge X(V)) = fd_2(C(E) \wedge X(V)) = \\
&= \sum_{L\subset V,\, |L|=2} (-1)^{|E|+\varepsilon(L)} f(C(E) \wedge C(L) \wedge X(V \setminus L)) = \\
&\sum_{i=1}^{n} \sum_{\substack{L\subset V\\ |L|=2}} \sum_{\substack{R\subset V\setminus L\\ |R|=p-t_i}} (-1)^{\lambda_i} f^{(i)}(C(E) \wedge X(R)) \wedge f^{(1)}_{(t_i)}(C(L) \wedge X(V \setminus L \setminus R)) \\
&\qquad = \sum_{i=1}^{n} \sum_{\substack{R\subset V\\ |R|=p-t_i}} (-1)^{\mu_i} f^{(i)}(C(E) \wedge X(R)) \wedge f^{(1)}_{(t_i)} d_2(X(V \setminus R)) = \\
&\qquad\quad = \sum_{i=1}^{n} \sum_{\substack{R\subset V\\ |R|=p-t_i}} (-1)^{\mu_i} f^{(i)}(C(E) \wedge X(R)) \wedge d_{t_i+2}(X(V \setminus R))
\end{aligned}$$

where $\lambda_i = |E| + p - t_i + \varepsilon(L) + \varepsilon(R, V \setminus L \setminus R)$, $\mu_i = |E| + p - t_i + \varepsilon(R, V \setminus R)$.

On the other hand,

$$\begin{aligned}
&\sum_{i=1}^{n} d_{t_i+2} f^{(i)}(C(E) \wedge X(V)) = \\
&\qquad = \sum_{i=1}^{n} \sum_{\substack{R\subset V\\ |R|=p-t_i}} (-1)^{\varepsilon(R,V\setminus R)} d_{t_i+2} f^{(i)}(C(E) \wedge X(R)) \wedge X(V \setminus R) = \\
&\qquad\quad = \sum_{i=1}^{n} \sum_{\substack{R\subset V\\ |R|=p-t_i}} (-1)^{\mu_i} f^{(i)}(C(E) \wedge X(R)) \wedge d_{t_i+2}(X(V \setminus R))
\end{aligned}$$

(μ_i is the same as above). Proof is completed. ■

If we put the expression for f_p into the right-hand side of the p-th equation of the system (22), apply the lemma 6 and then collect in the left-hand side of the same equation all summands with the fixed factor d_k, $k = 3, \ldots, p+2$, we just obtain the equality

$$d_{p+2} + d_{p+1} f_1 + \ldots + d_3 f_{p-1} = -D f_p.$$

Thus, the system (22) is solved and we have an isomorphism $H_n G \cong H_n \mathcal{G}$.

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