Functional-analytic approach to Riemann - Roch theorems

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Introduction. The main purpose of the various theorems of Riemann - Roch type is to extract the topological information contained in a holomorphic vector bundle on a complex manifold, or, more generally, in a coherent sheaf on a complex-analytic set. A typical theorem of this type has two main parts. First, one constructs for any coherent sheaf $\mathcal{L}$, the Chern character $ch(\mathcal{L})$ with values in a suitable space of differential forms. Second, one investigates the functorial properties of this correspondence, particularily if it commutes with the operation of proper product of complex spaces. This last property is expressed by the standard equality

$$f_*((Todd\,M)\cdot ch(\mathcal{L})) = (Todd\,N)\cdot ch(f_!\mathcal{L})$$

where $f : M \to N$ is a morphism of complex manifolds, $Todd\,M$ is the Todd class of the manifold M , $\mathcal{L}$ is a coherent sheaf on M , f_* is the operation of the direct image on differential forms (i.e. integration along the fibre of f), anf $f_!$ is the operation of the direct image of coherent sheaves.

There seem to be two main approaches to theorems of this type: the purely algebro-geometric one in the case of algebraic varieties, and the approach via differential operators, in the case of vector bundles on regular manifolds.

The purpose of this work is to explain an alternative approach, wich has its roots in operator theory. In contrast to the theorems obtained by the use of differential operators, our method works for coherent sheaves over any complex space, in particular a singular complex space.

The usual method of working with coherent sheaves — projective resolutions — has some obvious disadvantages. First, such a resolution exists, in general, only in the algebraic category. Next, it is difficult to take a canonical choice of a projective resolution. The construction of the Chern character by the use of such a resolution is also very complicated; this is illustrated e.g. by the work [A-LJ].

The main idea of our approach is to construct for any coherent sheaf its canonical globally defined infinite-dimensional free resolution. This simplifies drastically the situation.

By infinite-dimensional free resolution we mean the following: a complex, consisting of Frechet spaces X_i , and differentials $\alpha_i(z) : X_i \to X_{i+1}$, depending holomorphically on the variables z, such that the corresponding complex $\mathcal{O}^{X_\bullet}$ of sheaves of germs of its holomorphic sections is quasiisomorphic to the given coherent sheaf $\mathcal{L}$. (Such a complex is necessarily pointwise Fredholm, i.e. has at most finite-dimensional homology.)

The construction of the infinite-dimensional free resolution uses the Cech complex corresponding to given coherent sheaf, and has its origins in the theory of Toeplitz operators. The final result of the construction, however, does not involve operator-theoretic notions and can be briefly formulated in the following way:

Main tool. *There exists an exact functor, attaching to any coherent sheaf its infinite-dimensional free resolution.*

The functoriality of the resolution is a crucial property; it enables us to extend the Riemann-Roch theorems obtained via this resolution to the case of the higher K-functors.

The main point in the theorems of Riemann-Roch type is the commutation of the Chern character with the operation of the direct image under proper morphisms of complex spaces. For this, we construct a topological homotopy of complexes between the resolution of the given coherent sheaf and the resolution of its direct image. This topological homotopy is functorial in the sheaf.

The infinite-dimensional free resolution constructed in this way immediately defines a Riemann - Roch functor with values in the topological K-theory.

Furthermore, this resolution is crucial for our construction of the Chern character of a coherent sheaf with values in a suitable space of differential forms. For this, one developes in the infinite-dimensional context, namely for parametrized Fredholm complexes of Frechet spaces, the analogues of the main ingredients of the theory of characteristic classes for vector bundles, such as the trace, connection, and curvature. The construction of the corresponding objects is again, in some sense, functorial, and compatible with the homotopy of the direct image. It allows us to obtain analogues in the singular case and for higher K-functors of some theorems known in the regular case, in particular of the Hermitian Riemann–Roch Theorem of Bismut-Köhler [B-K].

The content of the paper is as follows: in the first section we give a brief account of the construction of the infinite-dimensional free resolution, given in [L1].

The second section contains the definitions of the trace and the Chern character of a holomorphic Fredholm complex of Frechet spaces. This enables us to construct the Chern character of a coherent sheaf with values in the Hodge cohomology, and its extension to higher K-functors.

The present construction of the Chern character has some flexibility due to the fact that it includes an arbitrary choice of a so-called essential homotopy of the complex. A choice of an essential homotopy gives us an explicit differential form representing this Chern character. In section 3 we associate to any scalar product a particular essential homotopy for which our construction yields a singular analogue of the Hermitian Riemann-Roch theorem, proved in the regular case by Bismut and Köhler [B-K]. As in the preceeding section, the results include: the construction of Hermitian Riemann-Roch morphisms for higher K-functors, and the independence of the Chern characters constructed in this way from the choice of the scalar product.

Finally,, in section 4 we consider the case of a linear bundle. We show that for a particular essential homotopy our construction yields a Chern character with values both in the symmetric differential forms and in the Chow ring. These Chern characters are the main ingredients in the Aritmetic Riemann-Roch theorem conjectured by Bismut-Gillet-Soulé [S]. The present approach should give further, more general results in this direction. The purpose of the paper is to announce the results, and to outline the main ideas of their proofs with some details. There are several gaps which are still to be filled, especially in the construction of the higher Hermitian Chern characters in section 3.

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1. Construction of the resolution. The classical Riemann - Roch - Hirzebruch theorem counts the Euler characteristics of the $\overline{\partial}$-complex $\left(\overline{\partial}, \Omega^{0,\bullet}(M,E)\right)$ on the manifold M and vector bundle E. In the case when M is singular complex space, and instead of vector bundle E we take a coherent sheaf $\mathcal{L}$ on M, the $\overline{\partial}$-complex is no more defined, but one can replace it by the Cech complex $(\delta, C^\bullet(M,\mathcal{U},\mathcal{L}))$, consisting on the alternating cochains of the Stein covering $\mathcal{U}$ of M with coefficients in $\mathcal{L}$.

1.1. Euristic remarks. In the Atiyah - Singer proof of the Riemann - Roch - Hirzebruch theorem, the relevant operator-theoretic object (so-called Fredholm module), is given not only by the complex $\left(\overline{\partial}, \Omega^{0,\bullet}(M,E)\right)$, but also by representation of the algebra $C_0(M)$ of continuous functions on M with finite support, at any stage of this complex. The latter is defined simply by the operators M_f of the multiplication by the continuous function f.

One could try to find a similar Fredholm module connected with the Cech complex. To construct the corresponding representations, one could look (in the case of regular M) to the standard bicomplex, connecting the $\overline{\partial}$ and δ - complexes. This suggests the following construction: one should take the complex $(\delta, C_h^\bullet(M,\mathcal{U},\mathcal{L}))$, composed of the Hilbert spaces $\Gamma_{(2)}(U_\alpha,\mathcal{L})$ of square-integrable sections of $\mathcal{L}$. The representation of $C_0(M)$ should be given by the collection of Toeplitz operators T_f , $f \in C_0(M)$, acting in $\Gamma_{(2)}(U_\alpha\mathcal{L})$ by the formula $T_f = P \circ M_f$, where P is the orthogonal projection from $L^2(U_\alpha,\mathcal{L})$ to $\Gamma_{(2)}(U_\alpha,\mathcal{L})$.

This construction would be sufficient for the proof of Riemann - Roch type theorem, provided that the algebras of Toeplitz operators in all the spaces $\Gamma_{(2)}(U_\alpha,\mathcal{L})$ commute modulo compact operators. However, this question is very difficult to be answered even in the regular case. Fortunatelly, the use of Toeplitz operators can be avoided at all.

To explain the idea, consider the Euclidean space $\mathbb{C}^n$, a pseudoconvex domain $U \subset \mathbb{C}^n$, and a coherent sheaf $\mathcal{L}$ on U. Let $\Gamma_{(2)}(U,\mathcal{L})$ be, as above, the space of square-integrable sections of $\mathcal{L}$ on U (under some Riemamian metrics), and denote by $(T_1,\ldots,T_n)$ the operators of the multiplication by the coordinate functions of $\mathbb{C}^n$, acting in the space $\Gamma_{(2)}(U,\mathcal{L})$.

Definition 1.1.1. The main role in what follows will play the *parametrized Koszul complex* of the operators $(T_1, \ldots, T_n)$, i.e. the Koszul complex[1] of the operators $(T_1 - z_1 I, \ldots, T_n - z_n I)$, considered as a complex holomorphically depending on the parameter $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. We shall denote this complex by $K^h_\bullet(U, \mathcal{L})(z)$, and let $K_\bullet(U, \mathcal{L})(z)$ be the analogous complex constructed for the operators above acting in the nuclear Frechet space $\Gamma(U, \mathcal{L})$ of *all* holomorphic sections of $\mathcal{L}$ over U. The next assertions, which can be proved on a standard way (see [L2]), show the connection between the Koszul complex given above, the algebra of Toeplitz operators, and the sheaf $\mathcal{L}$:

Proposition 1.1.2. *The complexes $K^h_\bullet(U, \mathcal{L})(z)$ and $K_\bullet(U, \mathcal{L})(z)$ are Fredholm for $z \in \mathbb{C}^n \backslash F$, where $F := bU \bigcap supp(\mathcal{L})$. The complexes $\mathcal{O}K^h_\bullet(U, \mathcal{L})(z)$, resp. $\mathcal{O}K_\bullet(U, \mathcal{L})(z)$, of sheaves of germs of holomorphic sections of these complexes are quasiisomorphic by a natural quasiisomorphism (we will call it* evaluation map*) to the sheaf $\mathcal{L}|_U$;*

$$\mathcal{O}^{K_\bullet} \to \mathcal{L}|_U \to 0$$

Proposition 1.1.3. *Suppose that the Toeplitz operators on $\Gamma_{(2)}(U, \mathcal{L})$ commute modulo compact operators. Then the class of $K^\bullet_h(U, \mathcal{L})(z)$ in the group $K^0(\mathbb{C}^n \backslash F)$ coincides with the Alexander dual to the element of $K_1(F)$, determined by the algebra of Toeplitz operators.*

Remark 1.1.4. Combining the propositions above, one obtains a very short proof of (a generalization of) the Boutet de Monvel's index theorem for Toeplitz operators (and therefore, of the Atiyah - Singer index theorem). Indeed, prop. 1.1.2 proves that the complex $K^\bullet_h(U, \mathcal{L})(z)$ carries the index class for the algebra of Toeplitz operators, and prop. 1.1.3 shows that this class can be calculated by an arbitrary locally free resolution of $\mathcal{L}$. In particular, if E is a holomorphic vector bundle on the strongly pseudoconvex domain V on the complex manifold M , then, taking an embedding $e : M \to \mathbb{C}^n$, a domain $U \subset \mathbb{C}^n$ such that $U \bigcap e(M) = V$, and $\mathcal{L} := e_*(E)$, one obtains the usual Boutet de Monvel's index formula.

1.2. Main construction. (see [L1]). Now, one can "replace" the Toeplitz operators with the corresponding Koszul complexes. One can use now the spaces $\Gamma(U_\alpha, \mathcal{L})$ instead of $\Gamma_{(2)}(U_\alpha, \mathcal{L})$, and there is no need of the use of hard analytic technics. The construction proceeds as follows:

Let M be a complex space, regularily embedded[2] in the complex manifold $\tilde{M}$. Take a covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $\tilde{M}$ by contractible pseudoconvex domains. Suppose that each domain $U_\alpha := U_{i_1} \bigcap \ldots \bigcap U_{i_k}$, where $\alpha = \{i_1, \ldots, i_k\} \subset I$, has a fixed coordinate system, and therefore a fixed parametrized Koszul complex $K^\bullet(U_\alpha, \mathcal{L})(z)$, defined as in 1.1.1. For any $\alpha \subset I$, one can transfer the complex $K^\bullet(U_\alpha, \mathcal{L})(z)$ onto U_α. It can be extended on the whole $\tilde{M}$ as a smooth complex, exact outside of U_α ; we will denote it by the same symbol. Put

[1] The Koszul complex of n commuting endomorphisms of a linear space can be defined as the total complex of the n-cube diagramm formed by these endomorphisms.

[2] The assumption of the embeddability into a complex manifold is not necessary. It is shown in [L1] that each complex space possess an embedding in an almost complex manifold, and such an embedding is sufficiently good for the construction which follows.

$$X_{\bullet,p}(z) = \bigoplus_{|\alpha|=p} K^{\bullet}(U_\alpha, \mathcal{L})(z)$$

Then this complex is an infinite-dimensional free resolution on $\tilde{M}$ of the sheaf $\mathcal{C}^p(\mathcal{U}, \mathcal{L})$ (i.e. the complex of sheaves of its holomorphic sections is quasiisomorphic to the latter sheaf). The family of complexes $X_{\bullet,p}(z)$ does not form a bicomplex. Nevertheless, one can define for them some substitute of the total complex of a bicomplex.Indeed, a simple algebraic reasoning shows that there exist "correcting" maps

$$r_{q,p,n}(z) : X_{q,p}(z) \to X_{q-n,p+n+1}(z)$$

such that the total complex, assembled by all $X_{\bullet,p}(z)$, with differentials determined by the differentials of these complexes and by the correcting maps $r_{q,p,n}(z)$, is indeed a complex (i.e. the product of two consecutive differentials is zero) and is quasiisomorphic via the evaluation map to the complex of sheaves $\mathcal{C}^{\bullet}(\mathcal{U}, \mathcal{L})$. The only non-zero entries of the correcting maps are acting from $\Gamma(U_\alpha, \mathcal{L})$ to $\Gamma(U_\beta, \mathcal{L})$ for $\alpha \subset \beta$; they are depending holomorphically on the parameter near U_α , and are smooth far from it. Moreover, these maps depend only on the covering $\mathcal{U}$ and the choosen coordinate systems on its elements. One easily can see that any two sets of correcting maps are linearily homotopic.

In [L1] the complex,constructed above, is denoted by $KC_{\bullet}(M,\mathcal{U},\mathcal{L})(z)$. Here, for the sake of brevity, we will denote it by $X_{\bullet}^{\mathcal{L}}(z)$, or by $X_{\bullet}^{\mathcal{U},\mathcal{L}}(z)$, or by $X_{\bullet}^{M,\mathcal{U},\mathcal{L}}(z)$. Let us note some of its properties:

Proposition 1.2.1. *The complex $X_{\bullet}^{\mathcal{L}}(z)$ satisfies the following statements:*

1/ $X_{\bullet}^{\mathcal{L}}(z)$ is a smooth complex of nuclear Frechet spaces. Locally it splits to a direct sum of a holomorphic complex and smooth exact complex. The sheaf of holomorphic sections of the former being quasiisomorphic to the sheaf $\mathcal{L}$. Roughly speaking, there exists a quasiisomorphic epimorphism of complexes of sheaves on $\tilde{M}$:

$$\mathcal{O}^{X_\bullet} \to \mathcal{C}_{\bullet}(\mathcal{U}, \mathcal{L}) \to 0$$

2/ $X_{\bullet}^{\mathcal{L}}(z)$ is an exact functor of the sheaf $\mathcal{L}$; to any morphism of coherent sheaves $\varphi : \mathcal{L} \to \mathcal{M}$ there corresponds a canonical constant *morphism of complexes*[3] *$\varphi_{\bullet}^{X} : X_{\bullet}^{\mathcal{L}}(z) \to X_{\bullet}^{\mathcal{M}}(z)$ (In particular, this shows that the construction above immediately extends to perfect complexes of Frechet sheaves).*

3/ Let us adopt the convention: when speaking on the covering $\mathcal{U}$, we will assume that its definition includes also the choosen coordinate systems on its elements, as well as the correcting maps $r_{q,p,n}(z)$. So, we will write $\mathcal{U}_1 \subset \mathcal{U}_2$, if $\mathcal{U}_2$ contains all the elements of $\mathcal{U}_1$, and coordinate systems and correcting maps for these elements in $\mathcal{U}_2$ are the same as in $\mathcal{U}_1$.

Then, if $\mathcal{U}_1 \subset \mathcal{U}_2$, then there exists a canonical constant quasiisomorphic monomorphism of complexes $X_{\bullet}^{\mathcal{U}_1,\mathcal{L}}(z) \to X_{\bullet}^{\mathcal{U}_2,\mathcal{L}}(z)$.

[3]By morphism of complexes, we mean any homomorphism, preserving the grading and commuting with the differentials.

4/ For any vector bundle E on $\tilde{M}$, the complex $E \otimes X_\bullet^{\mathcal{L}}(z)$ is naturally (with respect to $\mathcal{L}$) quasiisomorphic to the complex $X_\bullet^{E\otimes\mathcal{L}}(z)$.

1.3. Behavior under proper maps. Let $U \subset \mathbb{C}^n$, $V \subset \mathbb{C}^m$ be Stein domains with coordinate functions $z_1, \ldots, z_n$, resp. $w_1, \ldots, w_m$, and $\mathcal{L}$ be a coherent sheaf on $U \times V$. Let f be the projection of $U \times V$ on U. Let $T_1, \ldots, T_n$, resp. $S_1, \ldots, S_m$ be the operators of multiplication by the coordinate functions in $\Gamma(U \times V, \mathcal{L})$.

Denote by $K_\bullet^t(U \times V, \mathcal{L})(z, w)$, $t \in \mathbb{C}$, the Koszul complex in $\mathbb{C}^{n+m}$ of the operators $T_1 - z_1 I, \ldots, T_n - z_n I$, $t.S_1 - w_1, \ldots, t.S_m - w_m$, acting in the space $\Gamma(U \times V, \mathcal{L})$. For $t = 1$ we obtain the complex $K_\bullet(U \times V, \mathcal{L})(z, w)$ and for $t = 0$ - the complex $i_! K_\bullet(U, f_* \mathcal{L})(w)$, where i is the coordinate embedding of $\mathbb{C}^n$ in $\mathbb{C}^{n+m}$, and $i_!$ is the corresponding Koszul - Thom complex; the complexes $K_\bullet^t(U \times V, \mathcal{L})(z, w)$ form a continuous family of complexes, joining it, and therefore realize a (topological) homotopy between it.

So the good functorial properties of the Koszul complexes permit us to construct the corresponding homotopy globally. Take a proper morphism $f : M \to N$ of complex spaces, and regular embeddings $\varrho_M : M \to \tilde{M}$, $\varrho_N : N \to \tilde{N}$. Then $(\varrho_M, f \circ \varrho_N) : M \to \tilde{M} \times \tilde{N}$ is again a regular embedding, and the projectiom $\tilde{M} \times \tilde{N} \to \tilde{N}$ agrees with the map f. Take coverings $\mathcal{U}$, resp. $\mathcal{V}$, of $\tilde{M}$, resp. $\tilde{N}$, in the above sense, i.e. together with the coordinate systems and the correcting maps. Then $\mathcal{U} \times \mathcal{V}$ determines a covering of $\tilde{M} \times \tilde{N}$.

Take a smooth embedding $i : \tilde{M} \to \mathbb{R}^{2N} = \mathbb{C}^N$ such that the normal bundle to $i\left(\tilde{M}\right)$ has a complex structure. Denote by $\widehat{X}_\bullet^{\mathcal{U}\times\mathcal{V},\mathcal{L}}(z, w)$ the Koszul - Thom transformation of the complex $X_\bullet^{\mathcal{U}\times\mathcal{V},\mathcal{L}}(z, w)$ inder the embedding $i \times I_{\tilde{N}}$; using the trivialisations of the normal bundle, it can be represented as a smooth complex of Frechet spaces on $\mathbb{C}^N \times \tilde{N}$, "infinitesimally holomorphic" near the subset $i\left(\tilde{M}\right) \times \tilde{N}$.

Now, including the parameter $t \in \mathbb{C}$, we obtain a complex on $\mathbb{C}^N \times \tilde{N} \times \mathbb{C}$, which will be denoted by $X_\bullet^{f,\mathcal{L}}(z, w, t)$.

This complex is supported on the subspace $t.i\left(\tilde{M}\right) \times \tilde{N}$ of $\mathbb{C}^N \times \tilde{N}$; its restriction for $t = 1$ coincides with $\widehat{X}_\bullet^{\mathcal{U}\times\mathcal{V},\mathcal{L}}(z, w)$ given above, and its restriction to $t = 0$ is equal to the Koszul - Thom transformation of the complex $X_\bullet^{N,\mathcal{V},f_!\mathcal{L}}(w)$ under the embedding $\tilde{N} \rightsquigarrow \tilde{N} \times \{0\} \subset \tilde{N} \times \mathbb{C}^N$. Roughly speaking, so-defined complex determines a canonical topological homotopy between (some Koszul-Thom transformations of) the complexes $f_* X_\bullet^{M,\mathcal{L}}(z)$ and $X_\bullet^{N,f_!\mathcal{L}}(z)$.

The homotopy $X_\bullet^{f,\mathcal{L}}(z, w, t)$ has properties similar to these of $X_\bullet^{\mathcal{L}}(z)$:

Proposition 1.3.1. *$X_\bullet^{f,\mathcal{L}}(z, w, t)$ has the following properties:*

1′/ The complex of sheaves of sheaves of germs of holomorphic sections of $X_\bullet^{f,\mathcal{L}}(z, w, t)$ is quasi-isomorphic to the direct image of the sheaf $\mathcal{L}$ under the mapping $z \in \tilde{M} \rightsquigarrow (f(z), t \cdot i(z), t) \in \tilde{N} \times \mathbb{C}^N \times \mathbb{C}$ (its specialisation on $t = 1$ and $t = 0$ coincides with $\mathcal{L}$ and $f_!\mathcal{L}$ respectively).

2/ , 3/ , 4/ are the same as in 1.2.1.

1.4. Technical problems. Let us note two types of technical difficulties, appearing in the process of work with the complex constructed above:

1/ The spaces X_n , forming the complex $X_\bullet^{\mathcal{L}}(z)$, are not Hilbert. They are nuclear Frechet spaces. However, scaling the domains of the covering $\mathcal{U}$, one can include these spaces in a nuclear scale of nuclear Frechet spaces. More precisely, we obtain a scale $X_\bullet^{\tau,\mathcal{L}}(z)$ with a real parameter τ , such that for $\tau < \tau'$ the morphism of complexes $X_\bullet^{\tau,\mathcal{L}}(z) \to X_\bullet^{\tau',\mathcal{L}}(z)$ is a quasiisomorphism, and the operators $X_n^{\tau,\mathcal{L}} \to X_n^{\tau',\mathcal{L}}$ are nuclear embeddings of Frechet spaces. Some functional-analytic technics (see [L1], part 1) show that one can operate with such a scale of complexes in the same way as with complexes of Hilbert spaces.

2/ The complex $X_\bullet^{\mathcal{L}}(z)$ is not analytic; roughly, it divides into an analytic part and exact smooth part. If we denote the differentials of this complex by $\alpha_\bullet(z)$, then this means that the morphisms of complexes $\bar{\partial}\alpha_\bullet(z)$, $\partial\bar{\partial}\alpha_\bullet(z)$, ... are canonically homotopic to zero. This will be essential in the rest of the paper, and will allow us to neglect the "non-analytic" part.

In the algebraic case, i.e. when M is a quasiprojective variety, and the covering is affine, the constructions above can be performed without "smooth part"; however, the spaces involved become more complicated from the functional - analytic point of view.

To fix the ideas, we, perhaps oversimplifying, will speak of the complex $X_\bullet^{\mathcal{L}}(z)$ as of a holomorphic complex of Hilbert spaces.

1.5. Riemann - Roch theorem in the sense of Baum - Fulton - Macpherson. The work [B-F-M 2] gives a construction in the algebraic category (i.e. under the assumption that any coherent sheaf on M has a projective resolution on the ambient projective space) of a natural transformation of functors $\alpha_M : K_0^{alg}(M) \to K_0^{top}(M)$ from the Grothendieck group $K_0^{alg}(M)$ of the category of all coherent sheaves on the complex space M, to the corresponding topological K-group $K_0^{top}(M) := K^0\left(\tilde{M}, \tilde{M}\backslash M\right)$, commuting with the proper maps of complex spaces.

Our construction proves this theorem in the complex-analytic case. Indeed, the complex $X_\bullet^{\mathcal{L}}(z)$ considered as a continious Fredholm complex of Frechet spaces, determines an element $\alpha_M([\mathcal{L}])$ of the topological K-group $K^0\left(\tilde{M}, \tilde{M}\backslash M\right)$, and the topological homotopy $X_\bullet^{f,\mathcal{L}}(z,w,t)$ constructed above proves the equality $f_*\alpha_M([\mathcal{L}]) = \alpha_M([f_!\mathcal{L}])$. Moreover, since the constructions are functorial with respect to $\mathcal{L}$, then, applying it on the classifying space of the category of coherent sheaves (in the sense of Quillen or Waldhausen), one obtains:

Proposition 1.5.1. *On the category of coherent sheaves on complex sets there exist Riemann - Roch transformations on higher K-functors :*

$$\alpha_M^i : K_i^{alg}(M) \to K_i^{top}(M)$$

commuting with the proper direct images.

Similar, but more complicated, construction, gives a proof in the analytic category of the Riemann-Roch theorem for higher bivariant K-functors in the sense of Fulton - Macpherson.

Composing the homomorphism α_M^i above with the topological Chern character, one obtains a Riemann - Roch theorem with values in de Rham homology. In the rest of the paper we show that one can retrieve from the construction above some more precise Riemann - Roch theorems.

2. Riemann - Roch theorem in Hodge cohomology. We propose here a construction of the Riemann - Roch theorem for coherent sheaves on the complex space M with values in the Hodge cohomology $\bigoplus H^p_M\left(\tilde{M}, \Omega^{p,0}\left(\tilde{M}\right)\right)$. A theorem of this type in the analytic category was proved first by mainly combinatorial methods in series of papers by O'Brian -Toledo-Tong in the 80-s. Our approach, which can be applied for higher K-functors also, yields more exact theorems of the Riemann - Roch type.

In the paper [A-LJ] , Angeniol and Lejeune-Jalabert show that the characteristic classes of the coherent sheaves can be expressed by the derivatives of the differentials of its locally free resolutions.

The construction in the above paper (involving the Illusie's trace for the endomorphism of a perfect complex) can be made explicite only in some particular cases. The definition of the derivatives of differentials of a complex need coordinate frames, which exist only locally, and the operators of change of the basis make the formulas very complicate.

The situation seems to be different when using the infinite-dimensional free resolution constructed above. In the present section we will define, in the spirit of [A-LJ], the Chern character of a coherent sheaf $\mathcal{L}$ by the use of the parametrized complex of Frechet spaces $X^{\mathcal{L}}_\bullet(z)$. As we noted, for the sake of simplicity we will consider $X^{\mathcal{L}}_\bullet(z)$ as a holomorphic complex of Hilbert spaces.

2.1. Trace of a closed endomorphism of holomorphic Fredholm complex. Let $X_\bullet(z) = \{X_i, \alpha_i(z)\}$ be a holomorphically depending on the parameter $z \in \tilde{M}$ pointwise Fredholm complex of Hilbert spaces, defined on the complex manifold $\tilde{M}$. Denote by $Hom_p(X_\bullet, X_\bullet)$ the space of all endomorphisms of $X_\bullet(z)$ of order p ,i.e. of all sets of (bounded and linear) operators $F_\bullet = \{F_i : X_i \to X_{i+p}\}_{i \in \mathbb{Z}}$. The standard differential of this complex is defined as the commutator $[\cdot, \alpha_\bullet(z)]$ with the differential $\alpha_\bullet(z)$ of the initial complex. The vector-function $F_\bullet(z)$ with values in the space of the p-endomorphisms $Hom_p(X_\bullet, X_\bullet)$ will be called *closed*, if it is annihilated by this differential, i.e. if $[F_\bullet(z), \alpha_\bullet(z)] = 0$.

Denote by $Hom^f_\bullet(X_\bullet, X_\bullet)$ the subcomplex of $Hom_\bullet(X_\bullet, X_\bullet)$, consisting on the operators with a finite-dimensional image. The assumption that $X_\bullet(z)$ is Fredholm implies

Lemma 2.1.1. *The embedding $Hom^f_\bullet(X_\bullet, X_\bullet) \to Hom_\bullet(X_\bullet, X_\bullet)$ is a quasiisomorphism.*

This quasiisomorphism can be noncanonically inverted. We will call the given set of operators $S_\bullet(z) = \{S_i(z) : X_i \to X_{i-1}\}$ an *essential homotopy* for the complex $X_\bullet(z)$, if all the operators $[S(z), \alpha(z)] - I$ are finite - dimensional, and an *exact homotopy*, if all $[S(z), \alpha(z)] - I$ are zero operators (the latter may happen only if $X_\bullet(z)$ is an exact complex).

Now, if $F(z)$ is closed, and $S_\bullet(z)$ is an essential homotopy, then the homomorphism $F(z) - [S(z) \circ F(z), \alpha(z)]$ is finite-dimensional and homological to $F(z)$; so, any essential homotopy defines a map from $Hom_\bullet(X_\bullet, X_\bullet)$ to $Hom^f_\bullet(X_\bullet, X_\bullet)$.

Denote by $\Omega^{0,p} Hom_q(X_\bullet, X_\bullet)$ the sheaf of germs of differential forms on $\tilde{M}$ of degree $(0,p)$ with smooth sections of $Hom_q(X_\bullet, X_\bullet)$ as a coefficients. These sheaves form a bicomplex $\Omega^{0,\bullet} Hom_\bullet(X_\bullet, X_\bullet)$ with first and second differentials $\bar{\partial}$ and $[\cdot, \alpha(z)]$ respectively. Since $\alpha(z)$ is supposed holomorphic, these differentials commute. Denote the total complex of this bicomplex by $\Omega Hom_\bullet(X_\bullet, X_\bullet)$. Then the lemma above implies that the embedding $\Omega Hom^f_\bullet(X_\bullet, X_\bullet) \to \Omega Hom_\bullet(X_\bullet, X_\bullet)$ is a quasiisomorphism also.

In this case, as well as above, the choice of essential homotopy $S_\bullet(z)$ for the complex $X_\bullet(z)$ enables us to find for any closed section $F_\bullet(z)$ of $\Omega Hom_\bullet(X_\bullet, X_\bullet)$ a closed section $f(z)$ of

$\Omega Hom^f_\bullet(X_\bullet, X_\bullet)$ homological to $F_\bullet(z)$. The procedure of finding f from F is the well-known diagram-chase, or zig-zag, on the stages of the bicomplex $\Omega^{0,\bullet} Hom_\bullet(X_\bullet, X_\bullet)$.

As usual, for any element $f = \{f_i : X_i \to X_i\}_{i\in\mathbb{Z}} \in Hom^f_0(X_\bullet, X_\bullet)$ of order zero one defines the trace (or supertrace) of f by the formula $tr(f) := \sum(-1)^i tr(f_i)$. It is easy to see that if f is of the form $f = [G, \alpha(z)]$, then $tr\, f = 0$. So one can formulate

Definition 2.1.2. Let $f(z)$ be any section of $\Omega Hom^f_\bullet(X_\bullet, X_\bullet)$, and $f_{p,0}(z)$ be its component in $\Omega^{0,p} Hom^f_0(X_\bullet, X_\bullet)$. Then one defines $tr\, f(z) \in \Omega^{0,p}\left(\tilde{M}\right)$ as the $(0,p)$ - differential form $tr\, f_{p,0}(z)$.

Lemma 2.1.3. *The trace defined above is a morphism of the complex of sheaves* $\Omega Hom^f_\bullet(X_\bullet, X_\bullet)$ *into the Dolbeaux complex of sheaves* $\Omega^{0,\bullet}\left(\tilde{M}\right)$.

Definition 2.1.4. Let $F(z)$ be a closed endomorphism of $X_\bullet(z)$ of order p, depending holomorphically on z. Then it determines a closed section of the complex $\Omega Hom_\bullet(X_\bullet, X_\bullet)$ also. Let $f(z)$ be any section of $\Omega Hom^f_\bullet(X_\bullet, X_\bullet)$, homological to $F(z)$. Then one defines the trace $tr\, F(z)$ of $F(z)$ as the class of $tr\, f(z)$ in the Dolbeaux cohomology group $H^{0,p}\left(\tilde{M}, \mathcal{O}_{\tilde{M}}\right)$.

Remark 2.1.5. In the definition above only the $\overline{\partial}$-cohomological class of the form $tr\, F(z)$ is determined. However, if we fix the essential homotopy $S_\bullet(z)$ for the complex $X_\bullet(z)$, then we obtain a concrete choice for $f(z)$, and therefore a concrete differential form representing this class. To emphasize the dependence on $S_\bullet(z)$, we will denote it by $tr_S F(z)$ (note that this form depends linearly on $F_\bullet(z)$). The diagram chase procedure of finding f from F can be described as follows: the element $F(z)$ is of bidegree $(0,p)$; multiplying by S , and then applying $\overline{\partial}$, we obtain an element of bidegree $(1, p-1)$, and so on, until we reach the degree $(p,0)$, where we find the element $f_{p,0}(z)$ involved in the definition of the trace.

Let us note also that if $S_\bullet(z)$ is an exact homotopy, then $tr_S F(z)$ is zero.

2.2. Trace with values in local cohomology. Suppose that the set of the points z , such that $X_\bullet(z)$ is not exact, is contained in the complex set $M \subset \tilde{M}$. Then one can define a modification of the trace above with values in the local cohomology $H^{0,p}_M\left(\tilde{M}, \mathcal{O}_{\tilde{M}}\right)$. Indeed, any element of $H^{0,p}_M\left(\tilde{M}, \mathcal{O}_{\tilde{M}}\right)$ can be represented by a pair $(\omega, \tilde{\omega})$, where $\omega \in \Omega^{0,p}\left(\tilde{M}\right)$, $\tilde{\omega} \in \Omega^{0,p-1}\left(\tilde{M}\backslash M\right)$, $\overline{\partial}\omega = 0$, $\overline{\partial}\tilde{\omega} = \omega$. Alternative representation: imposing some growth conditions on $\tilde{\omega}$ near M , one can extend $\tilde{\omega}$ as a current on the whole $\tilde{M}$ and then consider the form $\omega - \overline{\partial}\tilde{\omega}$; this is a differential form of the type $(0,p)$ with currents as coefficients, concentrated on M (we will denote the space of all such forms by $\Omega^{0,p}_M\left(\tilde{M}\right)$) , which represents the same element of $H^{0,p}_M\left(\tilde{M}, \mathcal{O}_{\tilde{M}}\right)$.

Now, take $f(z)$ as above, and let $\tilde{f}(z)$ be a section of the complex $\Omega Hom^f_\bullet(X_\bullet, X_\bullet)$ on $\tilde{M}\backslash M$, such that the image of $\tilde{f}(z)$ under the differential of this complex is equal to $f(z)$. Then one can take $\omega = tr\, f_{p,0}(z)$, and $\tilde{\omega} = tr\, \tilde{f}_{p-1,0}(z)$. The pair $(\omega, \tilde{\omega})$ represents $tr\, F(z)$.

To take a concrete representative for the local trace, one must choose an essential homotopy $S_\bullet(z)$ for $X^{\mathcal{L}}_\bullet(z)$ on $\tilde{M}$, and an exact homotopy $\tilde{S}_\bullet(z)$ on $\tilde{M}\backslash M$, such that $\tilde{S}_\bullet(z) - S_\bullet(z)$ is finite-dimensional.

2.3. Traces of a higher order. We will need

Lemma 2.3.1. *Suppose that for any non-empty subset $I = (i_0, \dots, i_k) \subset \{0, \dots, n\}$ one has a fixed essential homotopy $S_I(z)$ of the complex $X_\bullet(z)$. Let $F(z)$ be a closed holomorphic endomorphism of $X_\bullet(z)$ of order p , $p \geq n$. Then there exist differential forms $\omega_I \in \Omega^{0,p-k}\left(\tilde{M}\right)$, such that:*

1/ for any $i \in \{0, \dots, n\}$ one has $\omega_{\{i\}} = tr_{S_i} F(z)$

2/ and for any $I = (i_1, \dots, i_k) \subset \{1, \dots, n\}$ one has:

$$\overline{\partial}\omega_I = \sum_{j=1}^{k} (-1)^j \omega_{I_j} \ ., \quad \text{where } I_j := I \setminus \{i_j\}$$

Proof. Denote by $C_\bullet = \{C_p, A_p(z)\}_p$ (or, when necessary, by $C_\bullet^{\mathcal{L}}$) the cone of the embedding $\Omega Hom_\bullet^f(X_\bullet, X_\bullet) \to \Omega Hom_\bullet(X_\bullet, X_\bullet)$. Then $C_\bullet$ is an exact complex of sheaves, and any essential homotopy S of $X_\bullet(z)$ defines an uniquely determined exact homotopy of $C_\bullet$; we will denote it again by $S(\cdot)$. We will define the trace on $C_\bullet$ as a superposition of the canonical epimorphism $C_\bullet \to \Omega Hom_\bullet^f(X_\bullet, X_\bullet)$ with the trace $\Omega Hom_\bullet^f(X_\bullet, X_\bullet) \to \Omega^{0,\bullet}\left(\tilde{M}\right)$ defined above.

Now let $F_\bullet(z)$ be a closed holomorphic endomorphism of $X_\bullet(z)$ of degree p. Then $F_\bullet(z)$ defines a closed (i.e. annihilated by the differential $A_p(z)$) section of C_p. Denote $F_\bullet^i(z) := S_{\{i\}}(F)$. We have $A_{p-1}(z)F_\bullet^i(z) = F_\bullet(z)$. For any $i < j$ the element $G_\bullet^{i,j} := F_\bullet^i - F_\bullet^j$ is closed, and one defines $F_\bullet^{i,j} := S_{i,j}(G_\bullet^{i,j})$; then $A_{p-2}(z)F_\bullet^{i,j} = G_\bullet^{i,j}$. Again, for any triple $i < j < k$ the section $G_\bullet^{i,j,k} := F_\bullet^{i,k} - F_\bullet^{i,j} - F_\bullet^{j,k}$ is closed, and its image by $S_{i,j,k}$ is denoted by $F_\bullet^{i,j,k}$. Using an induction on $|I|$, one can construct for any I with $|I| = k$ a section $F_\bullet^I(z)$ of C_{p-k} such that

$$A_{p-k}(z)F_\bullet^I(z) = \sum_{j=1}^{k} (-1)^j F_\bullet^{I_j}(z)$$

Then the differential forms $\omega_I := tr\, F_\bullet^I(z)$ will satisfy the requirements of the proposition.

Second proof. One can give an alternative, rather geometric, proof of the above proposition. Take a slightly different version of it; let $S_0(z), \dots, S_n(z)$ be essential homotopy for the holomorphic Fredholm complex $X_\bullet(z)$ on $\tilde{M}$. Let σ be an n-dimensional simplex, and $t_1, \dots, t_n$ be linear coordinates on σ. Consider the complex $X_\bullet(z)$ as a complex defined on $\sigma \times \tilde{M}$ with differentials not depending on t. Then one can take an analog of the bicomplex $\Omega^{0,\bullet} Hom_\bullet(X_\bullet, X_\bullet)$, replacing the second differential $\overline{\partial}_z$ with $\overline{\partial}_z - d_t$ (obviously it will commute with the first differential $[\cdot, \alpha_\bullet(z)]$).

Since all the essential homotopy of a given Fredholm complex form an affine set, one can define a family of homotopy $S_\bullet(z,t)$, $t \in \sigma$, such that its values on the vertexes of σ will coincide with the given homotopy $S_0(z), \dots, S_n(z)$.

Calculating $tr\, F(z)$ in the bicomplex $\Omega^{0,\bullet} Hom_\bullet(X_\bullet, X_\bullet)$ defined on $\sigma \times \tilde{M}$, by the use of homotopy $S_\bullet(z,t)$, one obtains a differential form $\omega(z,t)$ which can be decomposed as

$$\omega(z,t) = \sum_{i=0}^{n} \omega_i(z,t)$$

where $\omega_i(z,t)$ is its homogenious part of degree i with rapport to dt. Since ω is $\left(\overline{\partial}_z - d_t\right)$ - closed, then we have $d_t\omega_i = \overline{\partial}_z\omega_{i+1}$. For any set $I = (i_0,\ldots,i_k) \subset \{0,\ldots,n\}$ denote by σ_I the corresponding subsimplex of σ. One can define the forms ω_I , involving only z-differentials, by the equality $\omega_k|_{\sigma_I} = \omega_I \wedge dt_{i_0} \wedge \ldots \wedge dt_{i_k}$; then, applying Stokes formula to the equality above for $i = n-1$, and making the obvious computations, we obtain $\overline{\partial}_z\,\omega_\sigma = \sum(-1)^i\omega_{b_i\sigma}$, where b_i is the i-th face map.

The construction above can be applied to any simplicial complex with fixed essential homotopy at any vertex; taking the baricentric subdivision of a simplex, we obtain the lemma.

Remark 2.3.2. If in the second proof one chooses $S_\bullet(z,t)$ to be a linear function of t , then one will obtain the same forms ω_I as in the first proof. However, in the section 3 we will need some different choice of $S_\bullet(z,t)$, corresponding to the variation of the scalar product.

2.4. Chern character of a Fredholm complex. Take an analytic pointwise Fredholm complex $X_\bullet(z) = \{X_i, \alpha_i(z)\}$ as above, and put $(d\alpha(z))^p = d\alpha(z)\circ\ldots\circ d\alpha(z)$ - a product of p factors. Then, differentiating the equality $\alpha(z)\circ\alpha(z) = 0$, one obtains that $(d\alpha(z))^k$ commutes (in the graded sense) with the differential $\alpha(z)$ and therefore is a closed holomorphic section of the complex $\Omega Hom_\bullet\left(X_\bullet, X_\bullet \otimes \Omega^{p,0}\left(\tilde{M}\right)\right)$. The tensor factor $\Omega^{p,0}\left(\tilde{M}\right)$ does not change seriously the situation described above;at least, one can take locally a p-tuple of holomorphic vector fields $l_1\ldots l_p$ and consider $d_{l_1}\alpha\circ\ldots\circ d_{l_p}\alpha$ as a section of $\Omega Hom_\bullet(X_\bullet, X_\bullet)$. Taking account of the linearity of the trace, one obtain $tr\ (d\alpha(z))^p$ as an element of $H^p\left(\tilde{M}, \Omega^{p,0}\left(\tilde{M}\right)\right)$.

Definition 2.4.1. One defines the p-th Newton class of the complex $X_\bullet(z)$ by the formula $\nu^p(X_\bullet) := tr\ (d\alpha(z))^p \in H^p\left(\tilde{M}, \Omega^{p,0}\left(\tilde{M}\right)\right)$. Using the local construction, it can be taken in $H^p_M\left(\tilde{M}, \Omega^{p,0}\left(\tilde{M}\right)\right)$. The Chern character of $X_\bullet(z)$ with values in Hodge cohomology is defined by

$$ch\,(X_\bullet) = \sum_{p\geq 0}\frac{1}{p!}tr\ (d\alpha(z))^p$$

(If the homotopy S is fixed, we will write it as $ch_S\,(X_\bullet)$.)

This construction can be interpreted in the terms of theory of superconnections, developed by Quillen; indeed, if one consider the superconnection $D := d + \alpha(z)$, then its curvature is $D^2 = d\alpha(z)$, and the construction above fits in the standard scheme.

One can represent the same differential form as the "essential part" of the Chern character of an usual connection. Fix the essential homotopy S , and consider the connection $D_1 = d - S(z)\circ d\alpha(z)$. Let us denote by $(\omega)_{(p,q)}$ the component of the differential form ω lying in the space $\Omega^{p,q}$.

Lemma 2.4.2. *Under the notations above for any natural n one has*

$$\left(tr_S\left(D_1^2\right)^n\right)_{(p,q)} = \begin{cases} tr_S\ (d\alpha)^n & \text{if } p = q = n \\ 0 & \text{if } p < q \end{cases}$$

The things can be made more explicit for a good choice of the homotopy S. Take a decomposition $X_\bullet(z) = E_\bullet(z) \bigoplus L_\bullet(z)$ as a sum of finite-dimensional subcomplex $E_\bullet(z)$ and exact subcomplex $L_\bullet(z)$, smootly depending on $z \in \tilde{M}$. Then for any i the vector bundle $E_i(z)$ can be considered as a finite-dimensional subbundle of the trivial infinite-dimensional bundle with fiber X_i. One can define an essential homotopy $S_\bullet(z)$ such that $S|_E = 0$ and $S|_L$ is an exact homotopy for $L_\bullet(z)$. Then $[S, \alpha] = I - P$, where $P = P_\bullet(z)$ is the projection from $X_\bullet$ to $E_\bullet$ parallel to $L_\bullet$. Define the connection D_i on the bundle $E_i(z)$ by the formula $D_i\xi := P_i(z) \circ d\xi(z)$, where $\xi(z)$ is a section of $E_i(z)$.

Lemma 2.4.3. *Under the choice of the homotopy S given above for any natural n one has*

$$\left(\sum_i (-1)^i tr \left(D_i^2\right)^n\right)_{(p,q)} = \begin{cases} tr_S\,(d\alpha)^n & \text{if } p = q = n \\ 0 & \text{if } p < q \end{cases}$$

2.5. Chern character of a coherent sheaf. The definition above, applied to the resolution $X_\bullet^{\mathcal{L}}(z)$, gives the Chern character $ch(\mathcal{L})$ of a coherent sheaf $\mathcal{L}$ on M. Namely, denote

$$\tau_{\tilde{M}}(\mathcal{L}) := ch\left(X_\bullet^{\mathcal{L}}(z)\right) \quad , \quad ch(\mathcal{L}) := \left(Todd\,\tilde{M}\right)^{-1} \cdot \tau_{\tilde{M}}(\mathcal{L})$$

The next assertion shows that $ch(\mathcal{L})$ does not depend on the choice of the embedding $\varrho : M \to \tilde{M}$.

Lemma 2.5.1. *If $\tilde{M} \to \tilde{N}$ is a smooth embedding of complex manifolds with a normal bundle E, then*

$$\tau_{\tilde{N}}(\mathcal{L}) = (Todd\,E) \cdot \tau_{\tilde{M}}(\mathcal{L})$$

Indeed, if one take the coordinate system on $\tilde{N}$ such that the intersection of $\tilde{M}$ with any chart on $\tilde{N}$ is a coordinate subspace of this chart, then the complex $X_\bullet^{N,\mathcal{U},\mathcal{L}}(z)$ will coincide with the Koszul - Thom transform of the complex $X_\bullet^{M,\mathcal{U},\mathcal{L}}(z)$ with respect to the normal bundle E.

Remark 2.5.2. The assertion of the lemma is still valid if $\tilde{N}$ is smooth (non-complex) manifold and the normal bundle E possess a complex structure. This fact is important in the proof of the covariance of the Chern character under proper maps.

The construction of the Chern character can be extended to the higher K-groups. We will describe this first in the simplest case corresponding to the Chern character for the functor K_1^{alg}. Take an exact sequence of sheaves $\mathcal{E} : 0 \to \mathcal{L}, \to \mathcal{L} \to \mathcal{L}'' \to 0$, and the corresponding exact sequence of complexes $0 \to X_\bullet^{\mathcal{L}'}(z) \to X_\bullet^{\mathcal{L}}(z) \to X_\bullet^{\mathcal{L}''}(z) \to 0$. Suppose we have fixed the homotopy S' , S , S'' for each of these complexes. Lifting S'' (which does not change the traces), one can consider $S' \bigoplus S''$ as a homotopy for $X_\bullet^{\mathcal{L}}(z)$. Then, for the construction above, one finds an element $ch(\mathcal{E}) \in \bigoplus \Omega^{p,p-1}\left(\tilde{M}\right)$ such that

$$\overline{\partial} ch(\mathcal{E}) = ch_S(\mathcal{L}) - ch_{S' \bigoplus S''}(\mathcal{L}) = ch_S(\mathcal{L}) - ch_{S'}(\mathcal{L}') - ch_{S''}(\mathcal{L}'')$$

In the general case, denote by *coh* M the category of all coherent sheaves on M, and recall that by the Waldhausen definition (see [W]) the higher K-group $K_k^{alg}(M)$ can be defined as $k+1$

-th homotopy group of the classifying space of the simplicial category [4] which we will denote by $W.coh\, M$. The k- simplexes in this category are given by the filtered objects $\sigma : \mathcal{L}_1 \subset \ldots \subset \mathcal{L}_k$. We will write $\mathcal{L}_\sigma := \mathcal{L}_k$.

To construct the higher Chern characters $ch(\sigma)$, or rather $\tau(\sigma)$, one can use an induction on k as in lemma 2.3.1. Suppose that for any such a k-simplex σ we have fixed an essential homotopy S_σ for the complex $X_\bullet^{\mathcal{L}_\sigma}(z)$.

Next, if $\sigma_1 \subset \sigma_2$, then $\mathcal{L}_{\sigma_1}$ is a factor-sheaf of a subsheaf of $\mathcal{L}_{\sigma_2}$. Then we will fix an embedding $C_\bullet^{\mathcal{L}_{\sigma_1}} \to C_\bullet^{\mathcal{L}_{\sigma_2}}$ including extension by zero (on the stage of subsheaf) and lifting (on the stage of factor-sheaf). It is important to note that the lifting does not change the trace of the finite-dimensional component.

Fix the natural p ; then for any coherent sheaf $\mathcal{L}$ one has an closed endomorphism $F_\bullet^{\mathcal{L}}(z) := (d\alpha_\bullet(z))^p$ of the complex $X_\bullet^{\mathcal{L}}(z)$. The property 2 in 1.2.1 shows that for any morphism of sheaves $\varphi : \mathcal{L} \to \mathcal{M}$ there is an equality

$$F_\bullet^{\mathcal{L}}(z) \circ \varphi_\bullet^X = \varphi_\bullet^X \circ F_\bullet^{\mathcal{M}}(z)$$

Therefore, the pair $\left(X_\bullet^{\mathcal{L}}(z), F_\bullet^{\mathcal{L}}(z)\right)$ determines an exact functor from the category of coherent sheaves to the category of Fredholm complexes endowed with a closed holomorphic endomorphism of degree p.

Suppose that we are in the situation described above, i.e. that for any coherent sheaf $\mathcal{L}$ one has an functoriarily depending on $\mathcal{L}$ closed endomorphism $F_\bullet^{\mathcal{L}}(z)$ of the complex $X_\bullet^{\mathcal{L}}(z)$. Using the inductive construction of lemma 2.3.1, we obtain

Proposition 2.5.3. *One can attach to any k-simplex σ a section $F_\bullet^\sigma(z)$ of $C_{p-k}^{\mathcal{L}_\sigma}$ such that:*
1/ if $\sigma = \{\mathcal{L}\}$, then $F_\bullet^\sigma(z)$ coincides with the endomorphism $F_\bullet^{\mathcal{L}_\sigma}(z)$ fixed above, and
2/ for any k-simplex σ one has

$$A_{p-k}^{\mathcal{L}_\sigma}\left(F_\bullet^\sigma(z)\right) = \sum_{i=0}^{k} (-1)^i\, F_\bullet^{b_i(\sigma)}(z)$$

where for $i = 0$ and $i = k$ in the right hand side one takes the liftings, described above.

Taking the trace morphism $tr : C_\bullet^{\mathcal{L}_\sigma} \to \Omega^{0,\bullet}\left(\tilde{M}\right)$, one can define $tr_\sigma(F) := tr\ (F_\sigma(z)) \in \Omega^{p,p-k+1}$ and, finally

$$\tau_{\tilde{M}}(\sigma) := \sum_{p\geq k} \frac{1}{p!}\, tr_\sigma\left((d\alpha)^p\right) \quad , \quad ch(\sigma) := \left(Todd\, \tilde{M}\right)^{-1} \cdot \tau_{\tilde{M}}(\sigma)$$

Summarizing, we obtain

Proposition 2.5.4. *There exists a mapping $\sigma \rightsquigarrow ch(\sigma)$ attaching to any k-simplex $\sigma \in W_k coh\, M$ a differential form $ch(\sigma) \in \bigoplus_{p\geq k-1} \Omega^{p,p-k+1}\left(\tilde{M}\right)$ such that for any σ one has:*

$$\overline{\partial}\, ch(\sigma) = \sum_{i=0}^{k} (-1)^i\, ch\left(b_i(\sigma)\right)$$

[4] Usually denoted by $S.coh\, M$.

where $b_i(\sigma)$ denotes the i-th face map for σ.

The mapping, stated in the proposition, defines the *higher Riemann - Roch functors:*

$$ch_k : K_k^{alg}(M) \to \bigoplus_{p\geq k} H^{p-k}\left(\tilde{M}, \Omega^{p,0}\left(\tilde{M}\right)\right)$$

Indeed, the mapping $\sigma \rightsquigarrow ch(\sigma)$ above can be extended by linearity to a mapping $\pi \rightsquigarrow ch(\pi)$ defined for any simplicial complex $\pi \subset W.coh\, M$ and satisfying

$$\bar{\partial} ch(\pi) = ch(b\pi)$$

with $b\pi$ denoting the simplicial boundary of π. Taking π to run over the generators of the $k+1$ - th homotopy group of $W.coh\, M$, one obtains the group homomorphism ch_k.

It is easy to describe how the Chern character, defined above, depends on the choice of the homotopy S_σ:

Proposition 2.5.5. *Suppose one has fixed two different choosing maps $\sigma \rightsquigarrow S'_\sigma$, $\sigma \rightsquigarrow S''_\sigma$, and let $ch'(\sigma)$, $ch''(\sigma)$ be the corresponding Chern characters. Then there exists a map $\sigma \rightsquigarrow R(\sigma)$, $W_k coh\, M \to \bigoplus_{p\geq k} \Omega^{p,p-k}\left(\tilde{M}\right)$ such that for any k-simplex σ one has*

$$\bar{\partial} R(\sigma) = ch'(\sigma) - ch''(\sigma) - \sum (-1)^i R(b_i(\sigma))$$

The proof uses the lemma 2.3.1 in the same way as in the construction of $ch(\sigma)$ above. Indeed, consider the simplicial space $W.coh\, M \times [0,1]$ with chosen homotopy S'_σ for $\sigma \subset W.coh\, M \times \{0\}$ and S''_σ for $\sigma \subset W.coh\, M \times \{1\}$, and extend this choice up to a choice of homotopy for any simplex of the whole simplicial space $W.coh\, M \times [0,1]$. Now, if we denote by $R(\sigma)$ the differential form corresponding to the simplicial complex $\sigma \times [0,1]$, then the equality of the lemma follows from the construction.

2.6. Functoriality under the direct image. We will use the construction from section 1.3. Let $f : M \to N$ be a proper morphism of complex spaces, and $M \to \tilde{M}$, $N \to \tilde{N}$ be regular embeddings. Embedding $\tilde{M}$ into $\mathbb{C}^N$ and then in $\tilde{N} \times \mathbb{C}^N$ in the way described there, one obtains from 2.5.1

$$\tau_{\tilde{N}\times\mathbb{C}^N}(\mathcal{L}) = \left(Todd\, \tilde{M}\right)^{-1} \cdot \left(Todd\, \tilde{N}\right) \cdot \tau_{\tilde{M}}(\mathcal{L})$$

We have to prove the equality $f_* \tau_{\tilde{N}\times\mathbb{C}^N}(\mathcal{L}) = \tau_{\tilde{N}}(f_! \mathcal{L})$. This can be done using the topological homotopy $X_\bullet^{f,\mathcal{L}}(z,w,t)$ on $\mathbb{C}^N \times \tilde{N} \times \mathbb{C}$. Let us recall that in our case the projection $\mathbb{C}^N \times \tilde{N}$ extends the map f , and therefore the action of the functor f_* on the differential forms on $\mathbb{C}^N \times \tilde{N}$ can be defined as an integration with respect of this projection, i.e. along the z - coordinate, $z \in \mathbb{C}^N$.

For the proof, chose the essential homotopy $S_\bullet(z,w,t)$ of the complex $X_\bullet^{f,\mathcal{L}}(z,w,t)$, which is an exact homotopy for this complex outside a subset which is compact with respect of z , and with respect to t. Consider the endomorphisms $(d\alpha)^p$ of this complex. Here $\alpha(z,w,t)$ is the differential of the complex $X_\bullet^{f,\mathcal{L}}(z,w,t)$ above, and its derivatives in the expression $d\alpha(z,w,t)$ are taken with respect of all the variables except t. Since $\alpha(z,w,t)$ depends holomorphically on $t \in \mathbb{C}$, then one can take the analog of the bicomplex $\Omega^{0,\bullet} Hom_\bullet \left(X_\bullet^{f,\mathcal{L}}, X_\bullet^{f,\mathcal{L}}\right)$, with second differential equal to $\overline{\partial}_z + \overline{\partial}_w + \overline{\partial}_t$. Denote by $tr\ (d\alpha)^p$ the trace of $(d\alpha)^p$, obtained with the help of the homotopy $S_\bullet(z,w,t)$ in this bicomplex. Summing up, one defines the corresponding Chern character, which obviously can be written in the form

$$ch\left(X_\bullet^{f,\mathcal{L}}(z,w,t)\right) := \sum \frac{1}{p!} tr\ (d\alpha)^p = \omega(z,w,t) = \omega_1(z,w,t) + \omega_2(z,w,t) \wedge d\overline{t}$$

where ω_1 and ω_2 do not contain differentials with respect to t. Note that the form $\omega(z,w,t)$ has compact support with respect to z and t.

The fact that $\omega(z,w,t)$ is closed with respect to $\overline{\partial}_z + \overline{\partial}_w + \overline{\partial}_t$ implies that $\omega_1 \in \bigoplus \Omega^{(p,p)}$ is $\overline{\partial}_{z,w}$ - closed, and $\overline{\partial}_{z,w}\omega_2 = \frac{d}{dt}\omega_1$. It is easy to see that $\omega(z,w,1)$ represents $\tau_{\tilde{N}\times\mathbb{C}^N}(\mathcal{L})$, and $\omega(z,w,0)$ represents $\tau_{\tilde{N}}(f_!\mathcal{L})$. Taking the standard solution of the $\overline{\partial}$ problem on the t-plane, we obtain:

$$\omega_1(z,w,1) - \omega_1(z,w,0) = \left(\overline{\partial}_z + \overline{\partial}_w\right)\left(-\frac{1}{2\pi i}\int_{\mathbb{C}} \omega_2(z,w,t)\left(\frac{1}{t} - \frac{1}{t-1}\right) dt \wedge d\overline{t}\right)$$

Integrating this equality along the z-coordinates, one obtains:

$$f_*\tau_{\tilde{N}\times\mathbb{C}^N}(\mathcal{L}) - \tau_{\tilde{N}}(f_!\mathcal{L}) = \overline{\partial}_w T(\mathcal{L})$$

where we denoted by $T(\mathcal{L})$ the integral with respect to z of the differential form, included into brackets in the preceeding equality.

The same construction applies to the higher K-groups[5]. For this, choose for any simplex $\sigma \in W_k coh\, M$ essential homotopies $S_\sigma^1(z,w)$ for $X_\bullet^{\mathcal{L}_\sigma}(z,w)$, and $S_\sigma^0(z,w)$ for $X_\bullet^{f_!\mathcal{L}_\sigma}(z,w)$. It can be extended up to a homotopy $S_\sigma^t(z,w)$ for the complex $X_\bullet^{f,\mathcal{L}_\sigma}(z,w,t)$.

Now, performing to the simplicial category of all such complexes on $\tilde{N}\times\mathbb{C}^N$ the construction of the higher Chern character from the proposition 2.5.4, one obtains a set of differential forms

$$\omega_\sigma(z,w,t) = \omega_\sigma^1(z,w,t) + \omega_\sigma^2(z,w,t) \wedge d\overline{t}$$

$$\omega_\sigma^1 \in \bigoplus_{p\geq k-1} \Omega^{p,p-k+1}\left(\tilde{N}\times\mathbb{C}^N\right),\ \omega_\sigma^2 \in \bigoplus_{p\geq k}\Omega^{p,p-k}\left(\tilde{N}\times\mathbb{C}^N\right)$$

The equality $\left(\overline{\partial}_{z,w} + \overline{\partial}_t\right)\omega_\sigma = \sum(-1)^i \omega_{b_i(\sigma)}$ translates into:

$$\overline{\partial}_{z,w}\omega_\sigma^1 = \sum(-1)^i\omega_{b_i(\sigma)}^1$$

[5] Since the operation of direct image maps the category of coherent sheaves into the category of perfect complexes, all the constructions must be done in the Waldhausen classifying space of the latter category, whose definition is slightly more complicated. Here, we will not pay attention to this.

$$\frac{d}{d\bar{t}}\omega^1_\sigma - \overline{\partial}_{z,w}\omega^2_\sigma = \sum(-1)^i\omega^2_{b_i(\sigma)}$$

The first equation shows that $\omega^1_\sigma(z,w,1) = \tau_{\tilde{N}\times\mathbb{C}^N}(\sigma)$, and $\omega^1_\sigma(z,w,0) = \tau_{\tilde{N}}(f_!\sigma)$. Denote by $T(\sigma)$ the integral along the z-coordinate of the differential form

$$-\frac{1}{2\pi i}\int\limits_{\mathbb{C}} \omega^2_\sigma(z,w,t)\left(\frac{1}{t}-\frac{1}{t-1}\right) dt\wedge d\bar{t}$$

The integral along z of the restrictions of the form $\omega^1_\sigma(z,w,t)$ on $t=1$ and $t=0$ are equal to $f_*\tau_{\tilde{N}\times\mathbb{C}^N}(\sigma)$ and to $\tau_{\tilde{N}}(f_!\sigma)$ respectively. Taking the solution of the $\overline{\partial}_t$ - problem as above, one obtains:

Proposition 2.6.1. *There exists a mapping $\sigma \rightsquigarrow T(\sigma)$ attaching to any k-simplex from $W_k coh\, M$ a differential forms $T(\sigma)\in\bigoplus\Omega^{p,p-k-1}\left(\tilde{N}\right)$, such that*

$$f_*\tau_{\tilde{N}\times\mathbb{C}^N}(\sigma) - \tau_{\tilde{N}}(f_!\sigma) = \overline{\partial}T(\sigma) + \sum_{i=0}^{k}(-1)^i T(b_i(\sigma))$$

Corollary 2.6.2. *The Riemann - Roch morphisms ch_k , defined in the preceeding section, commute with the direct image.*

Remark 2.6.3. One can prove in the same way as in the proposition 2.5.5 that the form $T(\sigma)$ depends on the choice of the homotopy $S^t_\sigma(z,w)$ (with fixed $S^0_\sigma(z,w)$ and $S^1_\sigma(z,w)$) only up to a canonically choosen $\overline{\partial}$-exact form.

3. Hermitian Riemann - Roch theorem. To formulate the results above in a more precise and natural form, we endow the spaces forming the complex $X^{\mathcal{L}}_\bullet(z)$ with inner products. This permits, first, to make a canonical choice of the homotopy $S_\bullet(z)$, and second, roughly speaking, to replace everywhere $\overline{\partial}$ with $\partial\overline{\partial}$.

Let $\mathcal{L}$ is a coherent sheaf endowed with an Hermitian metrics. More precisely, suppose that the spaces $\Gamma(U,\mathcal{L})$, forming the components of the complex $X^{\mathcal{L}}_\bullet(z)$, can be represented as limits of the spaces of the type $\Gamma_{(2)}(U,\mathcal{L})$ of square-integrable sections of $\mathcal{L}$ with a fixed inner (scalar) products. This can be done, in particular, when $\mathcal{L}$ is a coherent sheaf endowed with an Hermitian metrics, and the inner products are determined by the given metrics. Then one has a canonical choice of the homotopy $S(z)$ such that its images are orthogonal to the kernels of $\alpha(z)$ (let call it "orthogonal homotopy").

The construction of the orthogonal homotopy can be described as follows. Let $\Delta_\bullet(z) := [\alpha_\bullet(z),\alpha^*_\bullet(z)] = \{\Delta_i(z)\}_i$ be the Laplacian of the complex $X^{\mathcal{L}}_\bullet(z)$. If this complex is exact, one can define the homotopy $\tilde{S}_\bullet(z)$ inductively:

$$\tilde{S}_i(z) := \alpha^*_{i-1}(z)\circ\left(I-\tilde{S}_{i+1}(z)\circ\alpha_i(z)\right)\circ(\Delta_i(z))^{-1}$$

In the case of Fredholm complexes, in general, the orthogonal homotopy can be defined only locally; namely, fix a point $z_0\in\tilde{M}$, and take $\varepsilon>0$, not belonging to the spectrum $sp\ \Delta_\bullet(z_0)$ of the Laplacian, such that the intersection of the interval $[0,\varepsilon]$ with $sp\ \Delta_\bullet(z_0)$ contains only eugenvalues of finite multiplicity. For z in a sufficiently small nelghborhood of z_0 , denote by $E^\varepsilon_\bullet(z)$ the subcomplex of $X^{\mathcal{L}}_\bullet(z)$ consisting of the spectral subspaces of $\Delta_\bullet(z)$ corresponding to $[0,\varepsilon]$, and by $L^\varepsilon_\bullet(z)$ its orthogonal complements, which consists of spectral subspaces for $[\varepsilon,\infty)$ and is a subcomplex also. Let $\Delta^\varepsilon_\bullet(z)$ the restriction of $\Delta_\bullet(z)$ on $L^\varepsilon_\bullet(z)$.

Now one can define in this neighborhood the essential homotopy $S^\varepsilon_\bullet(z)$ by the formula above with $\Delta_\bullet(z)$ replaced by $\Delta^\varepsilon_\bullet(z)$. On $\tilde{M}\backslash M$ (which we suppose to be non-empty) one can take the exact homotopy $\tilde{S}_\bullet(z)$ as above.

The representatives of the Newton classes in $\Omega^{p,p}_M\left(\tilde{M},\right)$, constructed by the means of $S^\varepsilon_\bullet(z)$ and $\tilde{S}_\bullet(z)$, does not depend on ε and therefore are defined globally; indeed, if we take $0<\varepsilon<\varepsilon'$ satisfying the conditions above, then the orthogonal complement of $E^\varepsilon_\bullet(z)$ in $E^{\varepsilon'}_\bullet(z)$ is an exact subcomplex which is invariant with respect to $\tilde{S}_\bullet(z)$, and therefore its contribution to the characteristic classes is killed.

Proposition 3.1.1. *The Chern character form $ch_S\left(X^{\mathcal{L}}_\bullet(z)\right)$, constructed by the use of the orthogonal homotopy given above, is Hermitian-symmetric and ∂ and $\overline{\partial}$-closed.*

Proof. We will use the proposition 2.4.3. Since the decompositions $X_i = E^\varepsilon_i(z)\bigoplus L^\varepsilon_i(z)$ are orthogonal, then the connections $D_i := P_{E_i}\circ d$ are preserving the induced Hermitian metrics on $E^\varepsilon_i(z)$ and therefore $ch\left(E^\varepsilon_i, D_i\right)$ are symmetric differential forms. Then proposition 2.4.3 shows that

$$ch_S\left(X^{\mathcal{L}}_\bullet(z)\right) = \sum(-1)^i\, ch\left(E^\varepsilon_i, D_i\right)$$

which proves the assertion.

In other words, for any coherens sheaf $\mathcal{L}$ one has

$$\tau_{\tilde{M}}(\mathcal{L}) \in \bigoplus_{p\geq 0}\mathcal{A}^{p,p}_M\left(\tilde{M}\right)$$

where $\mathcal{A}^{p,p}_M\left(\tilde{M}\right)$ is the space of the Hermitian-symmetric differential forms of bidegree (p,p) with coefficients currents concentrated on M , which are both ∂- and $\overline{\partial}$- closed.

Choosing a symmetric differential form, representing $\left(Todd\,\tilde{M}\right)^{-1}$, one obtains the Chern character $ch(\mathcal{L}) := \left(Todd\,\tilde{M}\right)^{-1}\cdot\tau_{\tilde{M}}(\mathcal{L})$ (we will call it *Hermitian Chern character*) as a differential form, belonging to the same space $\bigoplus_{p\geq 0}\mathcal{A}^{p,p}_M\left(\tilde{M}\right)$.

In order to extend the definition to the higher K-functors, we will need the following analog of lemma 2.3.1:

Lemma 3.1.2. *Let $\mathcal{L}$ be a coherent sheaf on M and suppose that for $i = 0, \ldots, n$ one has an inner product h_i on the components of the complex $X_\bullet^{\mathcal{L}}(z)$. Then for any non-empty subset $I = \{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$ there exists a differential form $\omega_I \in \mathcal{A}_M^{p-k,p-k}\left(\tilde{M}\right)$, such that*

1/ $\omega_{\{i\}}$ coincides with the Hermitian Chern character $ch_{h_i}\left(X_\bullet^{\mathcal{L}}(z)\right)$, constructed with respect of the inner product h_i, and

2/ for any I

$$\partial\overline{\partial}\omega_I = \sum_{j=1}^{k}(-1)^j\omega_{I_j} \quad , \quad \text{where } I_j := I\setminus\{i_j\}$$

Sketch of the proof. One can use arguments similar to that of the second proof of 2.3.1. All the inner products in a Hilbert space form a contractible set. Therefore, one can define a continuous family of inner products h_t, where t runs over the n-dimensional simplex σ, such that its values at the vertexes of σ will coincide with the given metrics $h_0, \ldots, h_n$. Let $S_\bullet(z,t)$ be the orthogonal homotopy for the complex $X_\bullet^{\mathcal{L}}(z)$. Then, by lemma 2.4.3, one reduces to the case of hermitian vector bundles E_i.

When σ is an 1-simplex, then one can use the Bott - Chern construction ([B-C], prop. 3.15) which construct explicitely $ch(\sigma)$ corresponding to the homotopy between the connections induced by the deformation of the induced Hermitian metrics on the bundles $E_i(z)$. For the higher dimensions, one needs to prove some higher-dimensional analog of the Bott-Chern formula.

Having the proposition above, one can proceed for the definition of the higher Chern characters in the same way as in the section 2.5. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a fixed covering of $\tilde{M}$ as above. Denote by $hcohM$ the category consisting of all the coherent sheaves on M, together with fixed "inner products" on the spaces $\Gamma(U_\alpha, \mathcal{L})$, $\alpha \subset I$; that means, a presentation of all such spaces as inverse limits of Hilbert spaces.

Proposition 3.1.3. *There exists a mapping $\sigma \rightsquigarrow ch(\sigma)$ attaching to any k-simplex $\sigma \in W_k hcoh\, M$ a differential form $ch(\sigma) \in \bigoplus_{p\geq k-1} \mathcal{A}_M^{p-k+1,p-k+1}\left(\tilde{M}\right)$ such that*

a/ If the filtration determining σ consists on a single sheaf $\mathcal{L}$, then $ch(\sigma)$ coincides with the Hermitian Chern character $ch(\mathcal{L})$ constructed above.

b/For any simplex σ one has:

$$\partial\overline{\partial}\, ch(\sigma) = \sum_{i=0}^{k}(-1)^i\, ch\left(b_i(\sigma)\right)$$

where $b_i(\sigma)$ denotes the i-th face map.

One can show that the definition does not depend essentially on the choice of the inner products. Namely, suppose that we have two functors from *coh M* to *hcoh M* inducing the identity functor on sheaves; that means, two ways of choice of inner products for any sheaf. Let $ch'(\sigma)$, $ch''(\sigma)$ be the corresponding Chern characters. Then we have

Proposition 3.1.4. *In the conditions above there exists a map* $\sigma \rightsquigarrow R(\sigma)$, $W_k coh\, M \to \bigoplus_{p\geq k} \Omega_M^{p-k,p-k}\left(\tilde{M}\right)$ *such that*

$$\partial\overline{\partial}R(\sigma) = ch'(\sigma) - ch''(\sigma) + \sum (-1)^i R(b_i(\sigma))$$

The covariance of the Hermitian Chern character with respect to the proper morphisms can be proved in the same way as in the preceeding section. We will sketch it for the case of K_0 , i.e. for the functor $\mathcal{L} \rightsquigarrow ch(\mathcal{L})$. In this case, one consider again the complex $X_\bullet^{f,\mathcal{L}}(z,w,t)$ on $\mathbb{C}^N \times \tilde{N} \times \mathbb{C}$. The scalar products on the complex $X_\bullet^{\mathcal{L}}(z)$ determine the scalar products on $X_\bullet^{f,\mathcal{L}}(z,w,t)$ (and the scalar products on $X_\bullet^{f_!\mathcal{L}}(w)$ on $\tilde{N}$).

The construction is similar to that from 2.6,but this time, unlike 2.6, we will include in the definition of the exterior derivatives $d\alpha(z,w,t)$ of the differential $\alpha(z,w,t)$ of the complex $X_\bullet^{f,\mathcal{L}}(z,w,t)$ the differentiation with respect to all variables - z , w and t. Denote by $\omega(z,w,t) \in \bigoplus \mathcal{A}^{p,p}\left(\tilde{N} \times \mathbb{C}^N\right)$ the Hermitian Chern character of this complex, calculated in the same way as in 2.6. If one denotes by $\omega_1(z,w,t)$ the part of $\omega(z,w,t)$, not containing dt and $d\bar{t}$, and by $\omega_2(z,w,t) \wedge dt \wedge d\bar{t}$ - the part containg both dt and $d\bar{t}$, then the ∂ and $\overline{\partial}$-closedness of the differential form $\omega(z,w,t)$ implies

$$\frac{d}{dt}\frac{d}{d\bar{t}}\omega_1(z,w,t) = \partial_{z,w}\overline{\partial}_{z,w}\omega_2(z,w,t)$$

and one has to solve the corresponding $\partial\overline{\partial}$ problem on the t - plane. One obtains:

$$\omega^1(z,w,1) - \omega^1(z,w,0) = \partial_{z,w}\overline{\partial}_{z,w}\left(-\frac{1}{2\pi i}\int_{\mathbb{C}} \omega^2(z,w,t)\log\left(\frac{|t|}{|t-1|}\right)^2 dt\wedge d\bar{t}\right)$$

On the other hand, it is easy to see from the construction that $\omega^1(z,w,1)$ and $\omega^1(z,w,0)$ represent the Hermitian Chern characters of the complexes $X_\bullet^{\mathcal{L}}(z,w)$ and $X_\bullet^{f_!\mathcal{L}}(w)$ correspondingly. As above, one integrates along the z-coordinates and denotes by $T(\mathcal{L})$ the integral from the differential form included in the brackets in the right hand side.

Performing an analogous construction for the higher K-functors, one obtains:

Proposition 3.1.5. *There exists a mapping* $\sigma \rightsquigarrow T(\sigma)$ *attaching to any k-simplex from* $W_k hcoh\, M$ *a differential form* $T(\sigma) \in \bigoplus_{p\geq k} \mathcal{A}_N^{p-k,p-k}\left(\tilde{N}\right)$ *, such that*

$$f_*\tau_{\tilde{N}\times\mathbb{C}^N}(\sigma) - \tau_{\tilde{N}}(f_!\sigma) = \partial\overline{\partial}T(\sigma) + \sum_{i=0}^{k}(-1)^i T(b_i(\sigma))$$

The forms $T(\sigma)$ depend on the choice of the inner products on $X_\bullet^{\mathcal{L}}(z)$ in some canonical way; namely, one has:

Proposition 3.1.6. *Suppose that, as in* 3.1.4, *we have two different choices of inner products on the objects of the category W.coh M , and let $T'(\sigma)$, $T''(\sigma)$ be the corresponding torsion forms constructed in the assertion above. Then one can attach to any simplex σ a differential form $L(\sigma) \in \bigoplus_{p\geq k+1} \mathcal{A}_N^{p-k-1,p-k-1}\left(\tilde{N}\right)$ such that*

$$\partial\bar{\partial}\, L(\sigma) = T'(\sigma) - T''(\sigma) - \sum (-1)^i L\left(b_i(\sigma)\right)$$

Suppose that we have an Hermitian metrics on the coherent sheaf $\mathcal{L}$, and one takes the scalar products on $X_\bullet^{\mathcal{L},\mathcal{U}}(z)$ induced by this metrics.Then it must be noted that the Hermitian Chern character, constructed on this way, will depend on the covering $\mathcal{U}$; this happens because the Hermitian metrics on $\mathcal{L}$ induced by its representation as a homology sheaf of the complex of Hilbert spaces $X_\bullet^{\mathcal{L}}(z)$, will not coincide with the original Hermitian structure on $\mathcal{L}$ (it will be somehow averaged along the elements of $\mathcal{U}$). In order to obtain the Chern character for the original metrics, one has to prove the following:

Conjecture 3.1.7. *When the diameter of the covering $\mathcal{U}$ tends to zero, then the corresponding Hermitian Chern characters of $X_\bullet^{\mathcal{L},\mathcal{U}}(z)$ converge some limit, coinciding in the regular case with the Chern character of the given metrics.*

4. Riemann - Roch functor with values in the Chow ring. We will briefly describe how the approach developed above can be applied, under suitable choice of homotopy, to the Riemann - Roch functor with values in the Chow ring. It seems important for us that the Chern character with values in the Chow ring, and the Hermitian Chern character constructed in the section 3 (which are the main components of the "Aritmetic Riemann - Roch theorem", conjectured and partially proved in the regular case by Bismut - Gillet - Soule), can be treated in a parallel way.

As we noted, in the construction above one can obtain the representatives of the Chern character as differential forms with currents as coefficients. So the problem is: how to choose $S_\bullet(z)$ (perhaps a generalised functions of z) such that the constucted differential form representing the Chern character is a sum of terms of the type δ_Z , which denotes the current of integration over the regular part of the cycle Z.

We will show on a particular case of a linear bundle that the construction of such a representatives can be connected with the Hermitian Chern character, considered in the last paragraph.

Take the free infinite dimensional resolution $X_\bullet^E(z)$ of a linear bundle E: namely, suppose that the homology sheaf of $X_\bullet^E(z)$ in the degree zero is isomorphic to E , and others are trivial. Let $s(z) : \mathbb{C} \to E$ be a section of this bundle, having a regular zero set which will be denoted by Z. The map s can be lifted up to a smooth map $\overline{s}(z) : \mathbb{C} \to ker\,\alpha_0(z)$. One can choose $\overline{s}(z)$ such that its image is orthogonal to the image of $\alpha_{-1}(z)$.

The map $\overline{s}(z)$ can be considered as a morphism of complexes $\mathbb{C}_0 \to X_\bullet^E(z)$, where $\mathbb{C}_0$ denotes the complex with $\mathbb{C}$ at stage zero and zero elsewhere. Denote by $\overline{X}_\bullet^E(z)$ the cone of this morphism: then obviously $X_\bullet^E(z)$ and $\overline{X}_\bullet^E(z)$ have the same Chern character. Take the orthogonal homotopy $S_\bullet(z)$ for $X_\bullet^E(z)$, constructed in the last paragraph; then, the map $\overline{s}^{-1}$, together with $S_\bullet(z)$, determines an exact homotopy $\overline{S}_\bullet(z)$ for $\overline{X}_\bullet^E(z)$ on $M\backslash Z$. Its extension as an operator-valued current on the whole M can be considered as an essential homotopy for this complex. Applying the construction form section 2, one obtains that the Chern character in this case will coincide with the fundamental current δ_Z of the cycle Z:

$$ch_{\overline{S}}\left(\overline{X}_\bullet^E(z)\right) = \delta_Z$$

Both $S_\bullet(z)$ and $\overline{S}_\bullet(z)$ are orthogonal on $M\backslash Z$, and its difference is finite-dimensional. Using the constructions from the section 3, one shows:

Proposition 4.1.1. *There exists a smooth differential form φ_E on $M\backslash Z$, satisfying*

$$\partial\overline{\partial}\varphi_E = ch_S\left(X_\bullet^E(z)\right) - ch_{\overline{S}}\left(\overline{X}_\bullet^E(z)\right) = ch_S\left(X_\bullet^E(z)\right) - \delta_Z$$

Remark. The differential form φ_E is a Green form for Z, and the pair (Z, φ_E) , is the aritmetic Chern character of E in the sense of Bismut-Gillet-Soule (see [S]).

To deal with the general case, one should lift the whole thing on the Grassmanian. The construction of the Chern character in the Chow ring for a complex of vector bundles is given by the Macpherson Grassmannian - graph construction (see [B-F-M1]); for our purposes it will be better to follow the variant of it, constructed by Iversen [I]. The same construction of the Grassmannian should be performed for the infinite-dimensional free complexes appearing in the our construction; so the complex should be canonically lifted on the Grassmannian, where its homology sheaves should have sufficiently many sections, and then one should use, as above, the essential homotopy determined by these sections. One may conjecture that, in this way, one should define the aritmetic Chern character of the complex $X_\bullet^{\mathcal{L}}(z)$, and, using the homotopy $X_\bullet^{\mathcal{L}}(z, w, t)$, should be able to prove its covariance under proper maps.

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