ON RATIONAL MAPS BETWEEN K3 SURFACES

by

V.V. Nikulin

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

Steklov Mathematical Institute
ul. Vavilova 42
Moscow 117966, GSP–1
USSR

MPI/90-23

ON RATIONAL MAPS BETWEEN K3 SURFACES

V.V.Nikulin

To the memory of Constantin Carathéodory

§ 1. Introduction

Here, a K3 surface is a non-singular projective algebraic surface X over complex numbers field $\mathbb{C}$ with the trivial space of the regular 1-dimensional differential forms: $\Omega^1[X]=0$, and the trivial sheaf of the regular 2-dimensional differential forms: $\Omega^2_X\cong\mathcal{O}_X$, where the $\mathcal{O}_X$ is the sheaf of regular functions on X. The last condition is equivalent to the existence of a regular non-zero 2-dimensional differential form ω_X which has not zeros on X.

Thanks to global Torelli theorem due to I.I.Piateckii-Shapiro and I.R.Shafarevich [PSh-Sh], we know very much about isomorphisms between K3 surfaces over the complex numbers field $\mathbb{C}$. Two K3 surfaces are isomorphic iff there periods are isomorphic.

Recently, I.R.Shafarevich posed an analogous question about rational maps between K3 surfaces: How can one know using periods when does a rational map between two K3 surfaces exist? A description of rational maps between K3 surfaces is interesting maybe from the view-point of the Arithmetic of K3 surfaces.

Let X be an algebraic K3 surface (over $\mathbb{C}$), let $H_X=H^2(X,\mathbb{Z})$, and let S_X and T_X be respectively the lattices of cohomology classes of algebraic and transcendental cycles on the surface X. By definition, T_X is the orthogonal complement to S_X in H_X with respect to the intersection pairing. Here and in what follows "lattice" means a "non-degenerate symmetric bilinear form over $\mathbb{Z}$". Hodge decomposition of $H_X\otimes\mathbb{C}$ induces a Hodge decomposition of $T_X\otimes\mathbb{C}$.

It is defined by one-dimensional linear subspace $H^{2,0}(X)\subset T_X\otimes\mathbb{C}$. I.R.Shafarevich posed the following

Question 1.1. Is it true that a rational map between K3 surfaces X and Y (i.e. an inclusion over $\mathbb{C}$ of the fields $\mathbb{C}(Y)\subset\mathbb{C}(X)$ of rational functions) exists iff there exist a positive $\lambda\in\mathbb{Q}$ and an isomorphism $\varphi:T_Y\otimes\mathbb{Q}\longrightarrow T_X\otimes\mathbb{Q}$ such that $\varphi(x\cdot y)=\lambda(x\cdot y)$ for any $x,\ y\in T_Y\otimes\mathbb{Q}$ (or φ is a similarity of quadratic forms over $\mathbb{Q}$), and $\varphi(H^{2,0}(Y))=H^{2,0}(X)$?

Let $\gamma:X\dashrightarrow Y$ be a rational map between K3 surfaces. Then a resolution of indefinite points of γ gives a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, dashed, "\gamma"] & & Y \end{tikzcd}$$

where Z is a non-minimal non-singular projective K3 surface, α is a birational morphism and β is a morphism. It gives the inclusion $\gamma^*=(\alpha^*)^{-1}\beta^*:T_Y(d)\longrightarrow T_X$ of the lattices of a finite index for which $\gamma^*(H^{2,0}(Y))=H^{2,0}(X)$ (γ^* preservers periods). Here d is the degree of γ and M(d) is the lattice obtaining multiplying by d of the form of the lattice M. The inclusion γ^* does not depend of a choice of Z, α and β, and is the invariant of the rational map γ. Let $d=d'm^2$, where d' and m are the positive integers and d' is square-free. Then γ^* gives a canonical chain of inclusions

$$T_Y(d')\longleftarrow mT_Y(d')=T_Y(d'm^2)\xrightarrow{\gamma^*}T_X$$

of lattices of finite index. Here, we use the following notations: for $m\in\mathbb{Q}$ the mM denotes the sublattice (or the overlattice) of the lattice M which is $mM=\{mv\,|\,v\in M\}$ with the form which is the restriction on mM of the form of the lattice M. (We use the notation M^m to denote the orthogonal sum of m exemplary of the lattice M.) We canonically (by the obvious way) identify sublattice $mT_Y(d')$ of the lattice $T_Y(d')$ and the lattice $T_Y(d'm^2)$. This chain gives the isomorphism $\overline{\gamma^*}:T_Y(d')\otimes\mathbb{Q} \longrightarrow T_X\otimes\mathbb{Q}$ of forms over $\mathbb{Q}$, which we call the modification corresponding to the rational map γ. At first, the lattice T_X is replaced on some sublattice $T_X'\subset T_X$ (e.g., $T_X'=\gamma^*(T_Y(d'm^2))$ or $\overline{\gamma^*}(T_Y(d'))\cap T_X)$, then T_X' is replaced on some overlattice $T_Y(d')$, and then $T_Y(d')$ is replaced on the lattice T_Y by dividing the form on d'.

We want to discuss here the following question similar to the question 1.1.

Question 1.2. Let X and Y be K3 surfaces, d' be a square-free positive integer and $\varphi:T_Y(d')\otimes\mathbb{Q} \longrightarrow T_X\otimes\mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (i.g., φ is an abstract modification of the lattices T_X and T_Y) and $\varphi(H^{2,0}(Y))=H^{2,0}(X)$. Is it true, that then there esists a rational map f:X--->Y such that $\varphi = \overline{f^*}$?

We say that an abstract modification φ above is trivial for a prime p iff $p\nmid d'$ and φ induces an isomorphism $\varphi_p:T_Y(d')\otimes\mathbb{Z}_p \longrightarrow T_X\otimes\mathbb{Z}_p$ of p-adic lattices. It is sufficient to prove the conjecture 1.1 for every prime p only, i.e., for modifications φ, which are nontrivial for one prime p only (One can deduce this from the epimorphicity of the Torelli map for K3 surfaces [Ku] and the following arithmetical fact: a primitive embedding of a lattice S into an unimodular indefinite lattice L exists iff for an every prime p, a primitive embedding of the lattice $S\otimes\mathbb{Z}_p$ into $L\otimes\mathbb{Z}_p$ exists.)

The basic result of the paper is to show that the answer to the Question 1.2 is positive if p=2 and rk T_X=rk $T_Y\le 5$.

Theorem 1.3. Let X and Y be algebraic K3 surfaces with rk T_X=rk $T_Y \le 5$, and $\varphi:T_Y(d)\otimes\mathbb{Q} \longrightarrow T_X\otimes\mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (i.e., φ is an abstract modification of the lattices T_X and T_Y) for which $\varphi(H^{2,0}(Y))=H^{2,0}(X)$, $d|2$, and φ induces an isomorphism $\varphi_p:T_Y(d)\otimes\mathbb{Z}_p \longrightarrow T_X\otimes\mathbb{Z}_p$ of p-adic lattices for any $p\neq2$.

Then there exists a sequence $X=X_1, X_2,\dots,X_{n+1}=Y$ of K3 surfaces and rational maps $f_i:X_i- - ->X_{i+1}$ of degree 2 such that the rational map $f=f_n\cdot\ldots\cdot f_2\cdot f_1$ induces the modification φ, i.e., $\varphi=\overline{f}*$.

See the proof of the theorem 3.1 below.

The proof of the theorem is based on two our old papers [N2] and [N3]. If $h:X- - ->Y$ is a rational map of degree 2 between K3 surfaces, then the Galois involution ι of this map is a symplectic involution of the surface X, i.e., ι acts trivially in the space $H^{2,0}(X)=\Omega^2[X]$ of regular 2-forms of X. The map h is the composition of the quotient map $X \longrightarrow X/\{id,\ \iota\}$ and the minimal resolution of singularities $Y \longrightarrow X/\{id,\ \iota\}$. So, to set up the rational map of degree 2 of K3 surface X in other K3 surface, one should find a symplectic involution on X. In [N2] symplectic involutions (and, more generally, finite abelian symplectic groups) of K3 surfaces were described very completely, see § 2. To investigate modifications under sequence of involutions of K3 surfaces, we use discriminant form technique developed in [N3]. Of cause, constantly, we use global Torelli theorem for K3 surfaces [PSh-Sh]. We should say that results of [N2] and [N3] we have mentioned above were used already by D.R.Morrison in [Mo] to prove that for K3 surface X with rk $T_X\le3$ a rational map of degree 2 in Kummer K3 surface exists (to prove this fact, he used also results of [N1] about the characterization of Kummer surfaces). But, to prove theorem 1.3, the more careful analysis than in [Mo] is required.

We want to remark, that also we prove the Theorem 2.2.7 below which gives an effective criterion for the preserving periods modification over 2 of transcendental periods of arbitrary K3 surfaces would be defined by a composition of degree two rational maps between the K3 surfaces. We deduce the Theorem 1.3 from this Theorem 2.2.7.

From the Theorem 1.3 and the characterization of Kummer surfaces in [N1], see also [Mo], we obtain the following theorem which was proved by I.R.Shafarevich and the author together.

Theorem 1.4 (V.V.Nikulin and I.R.Shafarevich). Let X and Y be algebraic K3 surfaces. Suppose that for all odd prime p there are primitive embeddings of p-adic lattices:

$$T_X\otimes\mathbb{Z}_p \subset U^3\otimes\mathbb{Z}_p \quad \text{and} \quad T_Y\otimes\mathbb{Z}_p \subset U^3\otimes\mathbb{Z}_p;$$

and for p=2 there are embeddings of the quadratic forms over the field $\mathbb{Q}_2$:

$$T_X\otimes\mathbb{Q}_2 \subset U^3\otimes\mathbb{Q}_2 \quad \text{and} \quad T_Y\otimes\mathbb{Q}_2 \subset U^3\otimes\mathbb{Q}_2.$$

Here U is an even unimodular lattice of the signature (1, 1). (Rougthly speaking, X and Y have the transcendental lattices of the Abelian surfaces over $\mathbb{Z}_p$ for any $p\neq 2$ and over $\mathbb{Q}_2$.)

Then the answer to the question 1.2 is positive for the K3 surfaces X and Y. ≻

The proof of the theorem 1.3 shows that some success in the investigation of rational maps between K3 surfaces is connected with a construction of some concrete rational maps between K3 surfaces (similar to maps of degree 2, which we use here). They should play the same role as the factorization of Abelian surfaces by the points of order p. Every rational map between Abelian surfaces is a composition of such rational maps and of an automorphism.

See some further remarks to the Theorems 1.3 and 1.4 in § 4.

At last, we would like to mention some results related with rational maps between K3 surfaces. In the situation of the question 1.2 (or 1.1), the cycle $Z_\varphi \in (T_X \otimes T_Y) \otimes \mathbb{Q}$ corresponding to φ belongs to $H^{2,2}(X \times Y, \mathbb{Q})$. Suppose that $d'=1$. I.R.Shafarevich posed the following conjecture [Sh], which is a particular case of the Hodge conjecture: the cycle Z_φ is algebraic. This conjecture is proved now if rk $T_X \le 17$, and more generally, if the lattice S_X represents zero (or X has a pencil of elliptic curves). See [Shi-I] for rk $T_X=2$, [Mo] for rk $T_X \le 3$, [Mu] for rk $T_X \le 11$, and [N4] for the case when the lattice S_X represents zero. Thus, this more weak conjecture is proved in much more generality now.

The Theorem 1.3 was inspired several our discussions with I.R.Shafarevich (by his initiative) on the rational maps problem for K3 surfaces. The Theorem 1.4 was deduces by I.R.Shafarevich and the author together. These theorems would not be appeared without Shafarevich's interest to this subject. We are very grateful to I.R.Shafarevich for his interest and support to this paper.

Notations for lattices and quadratic forms. Following to [N3], we will use the following definitions and notations connected with lattices and quadratic forms.

We denote as $x \cdot y$ the value of the form of the lattice M for a pair x, $y \in M$, and $x^2 = x \cdot x$.

The lattice M is called <u>even</u> iff x^2 is even for any $x \in M$.

The <u>discriminant group</u> $\mathfrak{A}_M$ of a lattice M is the $\mathfrak{A}_M = M^*/M$, where

$M^*=\mathrm{Hom}(M, \mathbb{Z})$.

The discriminant bilinear form b_M of a lattice M is the symmetric bilinear pairing $b_M:\mathfrak{A}_M\times\mathfrak{A}_M \longrightarrow \mathbb{Q}/\mathbb{Z}$, where $b_M(x^*+M, y^*+M)=x^*\cdot y^*+\mathbb{Z}$, x^*, $y^* \in M^*$. Here we extend linearly the bilinear form of M on the M^*. The form b_M is degenerate.

For an even lattice M the discriminant quadratic form $q_M:\mathfrak{A}_M \longrightarrow \mathbb{Q}/2\mathbb{Z}$ is defined as $q_M(x^*+M)=(x^*)^2+2\mathbb{Z}$ for $x^*\in M^*$. The quadratic form q_M has the bilinear form b_M.

The symbol $\oplus$ denotes the orthogonal sum of lattices and bilinear and quadratic forms.

The symbol $(A)^{\perp}_{B}$ denotes the orthogonal complement to A in B.

The discriminant form of a lattice M is the orthogonal sum of its p-components (the restrictions of the form on the p-components of the group $\mathfrak{A}_M$), which are defined by the discriminant forms of the p-adic lattices $M_p=M\otimes\mathbb{Z}_p$.

Every p-adic lattice is an orthogonal sum of the following elementary p-adic lattices: the lattice $K_\theta^{(p)}(p^k)$ of the rank 1 has the matrix $\langle\theta p^k\rangle$, $\theta\in\mathbb{Z}_p^*$; the 2-adic lattice $U^{(2)}(2^k)$ of the rank 2 has the matrix

$$\left\langle\begin{pmatrix} 0 & 2^k \\ 2^k & 0 \end{pmatrix}\right\rangle ;$$

the 2-adic lattice $V^{(2)}(2^k)$ of the rank 2 has the matrix

$$\left\langle\begin{pmatrix} 2^{k+1} & 2^k \\ 2^k & 2^{k+1} \end{pmatrix}\right\rangle .$$

The discriminant quadratic forms of the p-adic lattices $K_\theta^{(p)}(p^k)$, $U^{(2)}(2^k)$ and $V^{(2)}(2^k)$, $k\ge 1$, are denoted as $q_\theta^{(p)}(p^k)$, $u_+^{(2)}(2^k)$, $v_+^{(2)}(2^k)$ respectively. Their bilinear forms are denoted as $b_\theta^{(p)}(p^k)$, $u_-^{(2)}(2^k)$, $v_-^{(2)}(2^k)$ respectively.

In this article we consider only even lattices and even 2-adic lattices. Thus, here, the term "discriminant form" denotes every time discriminant quadratic form.

For an finite abelian group $\mathfrak{A}$ the symbol $\ell(\mathfrak{A})$ denotes the minimal number of generators of $\mathfrak{A}$. For a form q on a finite abelian group $\mathfrak{A}$ we denote $\mathfrak{A}_q=\mathfrak{A}$ and $\ell(q)=\ell(\mathfrak{A})$.

The discriminant discr(S) of a lattice S is the determinant of the matrix of S in some basis. A lattice S is called unimodular iff discr S is invertible. The lattice U is an even unimodular lattice of the signature (1, 1). It is unique up to isomorphism. The lattice E_8 is an even unimodular lattice of the signature (0, 8). It is unique up to isomorphisms too. The signature $(t_{(+)}, t_{(-)}, t_{(0)})$ of a quadratic form over $\mathbb{R}$ is the number of its positive, negative and zero squares. We don't show the number $t_{(0)}$ if the form is non-degenerate.

The invariants of a lattice S is a triplet $(t_{(+)}, t_{(-)}, q)$, where the $(t_{(+)}, t_{(-)})$ is a signature of the S and $q\cong q_S$, where q_S is the discriminant form of the S. These invariants are equivalent to the genus of the S.

An embedding $N\subset M$ of lattices is called primitive iff the quotient-module M/N is a free module.

§ 2. Compositions of degree 2 rational maps between K3 surfaces.

Following to [N2] (see [Mo] also), we will give here basic constructions connected with symplectic involutions of K3 surfaces.

2.1. Let X be a K3 surface and let ι be a symplectic involution of X. The following results are contained in [N2].

Let

$$S_{\iota}=\{\ x\in H_X \mid \iota(x)=-x\ \},$$

and

$$T^{\iota}=\{\ x\in H_X \mid \iota(x)=x\ \}.$$

The lattice S_{ι} is a negative-definite lattice of the rk $S_{\iota}=8$, the discriminant group $\mathfrak{A}_{S_{\iota}}\cong(\mathbb{Z}/2\mathbb{Z})^8$, and S_{ι} has not elements δ with square $\delta^2=-2$. By the classification of the definite unimodular lattices of rank ≤ 8 (see [Se], for example), $S_{\iota}=E_8(2)$. The lattice S_{ι} is the primitive sublattice of the lattice S_X The lattice S_X is a primitive sublattice of the lattice $H_X=H^2(X, \mathbb{Z})$ also. Thus, we have a sequence of primitive embeddings of lattices:

$$S_{\iota}\subset S_X\subset H_X\ . \qquad (2.1)$$

The lattice H_X is an even unimodular lattice of the signature (3, 19). It follows (see [Se], for example) that $H_X\cong U^3\oplus E_8{}^2$ The lattice S_{ι} has the unique (up to isomorphism) primitive embedding into the lattice H_X. It follows that $T^{\iota}=(S_{\iota})^{\perp}_{H_X}\cong U^3\oplus E_8(2)$. By (2.1), then $T_X=(S_X)^{\perp}_{H_X}$ is a primitive sublattice of T^{ι}, and we have a sequence of primitive embeddings of lattices:

$$T_X\subset T^{\iota}\subset H_X\ . \qquad (2.2)$$

Vice versa, suppose we have a primitive embedding $S\subset S_X$ of lattices, where $S\cong E_8(2)$. Then there exists $w\in W^{(2)}(S_X)$, such that $w(S)=S_{\iota}$ for some symplectic involution ι of the X. Here $W^{(2)}(S_X)$ is the group generated by all reflections with respect to elements $\delta\in S_X$ with the square $\delta^2=-2$.

The symplectic involution ι has precisely 8 fixed points, and the local action of ι in these points is the multiplication on -1. It follows, that $X/\{id,\ \iota\}$ has precisely 8 singular points of the type $\mathbb{A}_1$, which are the images under the quotient morphism $\pi:X\longrightarrow X/\{id,\ \iota\}$ of the fixed points. Let $\sigma:Y\longrightarrow X/\{id,\ \iota\}$ be the minimal resolution. The pre-images σ^{-1} of the singular points of $X/\{id, \iota\}$ are non-singular rational curves $\Gamma_1,\dots, \Gamma_8$ of Y with divisor classes $e_1,\dots, e_8$, which generate the primitive negative-definite sublattice

$$Q_\iota = [e_1,\dots,e_8, (e_1+\dots+e_8)/2] \qquad (2.3)$$

with the form $e_i\cdot e_j=-2\delta_{ij}$, of the lattice S_Y. So, we have the sequence of primitive embeddings of lattices:

$$Q_\iota\subset S_Y\subset H_Y. \qquad (2.4)$$

It follows that the discriminant group $\mathfrak{A}_{Q_\iota}\cong(\mathbb{Z}/2\mathbb{Z})^6$, and the discriminant form $q_{Q_\iota}\cong u_+^{(2)}(2)^3$. Let $R^\iota=(Q_\iota)^\perp_{H_Y}$. By (2.4), we have the sequence

$$T_Y\subset R^\iota\subset H_X \qquad (2.5)$$

of the primitive embeddings of the lattices. The lattices Q_ι and R^ι are the orthogonal complements one to another in the even unimodular lattice H_X. It follows [N3] that $q_{R^\iota}\cong-q_{Q_\iota}\cong-u_+^{(2)}(2)^3\cong u_+^{(2)}(2)^3$, the lattice Q_ι has the unique up to isomorphism primitive embedding in H_Y, and $R^\iota\cong U^3\oplus Q_\iota$.

Let $\tau=\sigma^{-1}\pi:X\dashrightarrow Y$ be the corresponding rational map of the degree 2. This map gives the embedding of the lattices

$$\tau^*:R^\iota(2) \longrightarrow T^\iota, \qquad (2.6)$$

which has the obvious property:

$$\tau^*(H^{2,0}(Y))=H^{2,0}(X).$$

A lattice (or an 2-adic lattice) F is called <u>2-elementary</u> iff the discriminant group $\mathfrak{A}_F\cong(\mathbb{Z}/2\mathbb{Z})^a$. For 2-elementary lattices the following duality takes place: To a 2-elementary lattice F, the 2-elementary lattice $F^\times=F^*(2)$ is corresponding, and the canonical embedding $F \subset F^*$ gives the canonical embedding

$$F(2) \subset F^*(2) = F^{\times}, \qquad (2.7)$$

and we have the following duality property:

$$(F^{\times})^{\times} = (F^*(2))^*(2) = F. \qquad (2.8)$$

The fundamental fact is that the embedding (2.6) is extended to the isomorphism (this extension is obviously unique) of the lattices:

$$\tau^*: R^{\iota}(2) \subset (R^{\iota})^{\times} \cong T^{\iota}, \qquad (2.9)$$

where the embedding $R^{\iota}(2) \subset (R^{\iota})^{\times}$ is the canonical embedding (2.7). Thus, by (2.7) and (2.9) we have the following canonical isomorphisms of the lattices:

$$\tau^*: R^{\iota}(2) \cong (T^{\iota})^{\times}(2) = (T^{\iota})^*(4) = 2(T^{\iota})^* \subset T^{\iota}. \qquad (2.10)$$

By (2.2), (2.5), (2.6), and (2.10), we have the following isomorphism, which describes the modification corresponding to the rational map $\tau: X \dashrightarrow Y$:

$$\tau^*: T_Y(2) \cong (T_X \otimes \mathbb{Q}) \cap (T^{\iota})^{\times}(2) = 2((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*) \subset T^{\iota}. \qquad (2.11)$$

2.2. Here, we want to deduce from the properties 2.1 some general statements connected with K3 surfaces with symplectic involutions. It will be useful in what follows.

2.2.1. Let us consider the following general situation, connected with lattices. Suppose we have an even unimodular lattice L and two primitive sublattices $T \subset L$, $Q \subset L$ which are orthogonal one to another: $T \perp Q$. Let $[T \oplus Q]$ be the primitive sublattice in L generated by $T \oplus Q$. Then the subgroup

$$\Gamma_{[T \oplus Q]} = [T \oplus Q]/(T \oplus Q) \subset \mathfrak{A}_T \oplus \mathfrak{A}_Q$$

is an isotropic subgroup with respect to quadratic form $q_T \oplus q_Q$, and

$\Gamma_{[T\oplus Q]}\cap(\mathfrak{A}_T\oplus 0)=\Gamma_{[T\oplus Q]}\cap(0\oplus\mathfrak{A}_Q)=0\oplus 0$. Let π_T and π_Q be the projections in $\mathfrak{A}_T$ and $\mathfrak{A}_Q$ respectively. Let

$$\mathfrak{H}=\pi_T(\Gamma_{[T\oplus Q]})\subset\mathfrak{A}_T$$

be the subgroup of $\mathfrak{A}_T$. Then we have the inclusion

$$\xi:\mathfrak{H}\longrightarrow\mathfrak{A}_Q$$

of the groups, where $\xi=\pi_Q(\pi_T)^{-1}$, and ξ gives the inclusion of the quadratic forms:

$$\xi:q_T|\mathfrak{H}\longrightarrow -q_Q.$$

We would like to express the overlattice $T\subset((Q)^{\perp}_{L})^{*}\cap(T\otimes\mathbb{Q})$ of a finite index of the T using the subgroup $\mathfrak{H}$.

Lemma 2.2.1. $(((Q)^{\perp}_{L})^{*}\cap(T\otimes\mathbb{Q}))/T = \mathfrak{H} \subset \mathfrak{A}_T$.

<u>Proof.</u> Let $P=(T\oplus Q)^{\perp}_{L}$. Then $T\oplus P\oplus Q\subset L$ is a sublattice of a finite index. For a sublattice $F\subset L$, we denote by $[F]$ a primitive sublattice $[F]=L\cap(F\otimes\mathbb{Q})$ of L generated by F. We have the subgroups

$$\Gamma_L=L/(T\oplus P\oplus Q)\subset\mathfrak{A}_T\oplus\mathfrak{A}_P\oplus\mathfrak{A}_Q,$$

$$\Gamma_{[T\oplus P]}=[T\oplus P]/(T\oplus P)\subset\mathfrak{A}_T\oplus\mathfrak{A}_P\subset\mathfrak{A}_T\oplus\mathfrak{A}_P\oplus\mathfrak{A}_Q\ ,$$

$$\Gamma_{[T\oplus Q]}=[T\oplus Q]/(T\oplus Q)\subset\mathfrak{A}_T\oplus\mathfrak{A}_Q\subset\mathfrak{A}_T\oplus\mathfrak{A}_P\oplus\mathfrak{A}_Q\ .$$

Here we identify $\mathfrak{A}_T=\mathfrak{A}_T\oplus 0\oplus 0$, $\mathfrak{A}_P=0\oplus\mathfrak{A}_P\oplus 0$, $\mathfrak{A}_Q=0\oplus 0\oplus\mathfrak{A}_Q$. Let π_T, π_P, π_Q be the corresponding projections in $\mathfrak{A}_T$, $\mathfrak{A}_P$, $\mathfrak{A}_Q$ respectively. The subgroups Γ_L, $\Gamma_{[T\oplus P]}$, and $\Gamma_{[T\oplus Q]}$ are obviously isotropic with respect to the form $q_T\oplus q_P\oplus q_Q$.

It follows, that we have to prove that

$$([T\oplus P]^{*}/(T\oplus P))\cap\mathfrak{A}_T = \pi_T(\Gamma_{[T\oplus Q]}).$$

The lattice L is unimodular. It follows that $([T\oplus P]^{*}/(T\oplus P))=(\pi_T\oplus\pi_P)(\Gamma_L)$. Thus, we have to prove that

$$(\pi_T\oplus\pi_P)(\Gamma_L)\cap\mathfrak{A}_T=\pi_T(\Gamma_{[T\oplus Q]}).$$

This is equivalent to $\Gamma_L\cap(\mathfrak{A}_T\oplus 0\oplus\mathfrak{A}_Q)=\Gamma_{[T\oplus Q]}$. This evidently follows from the fact that the $[T\oplus Q]$ is a primitive sublattice of the L. >

2.2.2. Now, let us consider the case of the section 2.1 above when K3 surface X has a symplectic involution ι, and specify the situation of the section 2.2.1 to the case $L=H_X$, $T=T_X$, $Q=S^{\iota}$.

The primitive sublattice $M=[T_X\oplus S_\iota]$ in H_X, which is generated by the sublattice $T_X\oplus S_\iota$ of the lattice H_X, is defined by the inclusion of the forms

$$\xi: q_{T_X}|\mathfrak{H} \longrightarrow -q_{S_\iota}=u_+(2)^4, \qquad (2.12)$$

where $\mathfrak{H}$ is a subgroup of the discriminant group $\mathfrak{A}_{T_X}$. It is defined by the graphic $\Gamma_\xi=[T_X\oplus S_\iota]/(T_X\oplus S_\iota)\subset\mathfrak{A}_{T_X}\oplus\mathfrak{A}_{S_\iota}$ of the ξ, which is an isotropic subgroup of the form $q_{T_X}\oplus q_{S_\iota}$ in $\mathfrak{A}_{T_X}\oplus\mathfrak{A}_{S_\iota}$. The discriminant form

$$q_M=q_{T_X}\oplus q_{S_\iota}|((\Gamma_\xi)^{\perp}_{q_{T_X}\oplus q_{S_\iota}}/\Gamma_\xi). \qquad (2.13)$$

By (2.12), the $\mathfrak{H}\cong(\mathbb{Z}/2\mathbb{Z})^\alpha$ is a 2-elementary group, $\alpha\le 8$, and also $\Gamma_\xi\cong(\mathbb{Z}/2\mathbb{Z})^\alpha$. Let $x_1,\dots, x_\alpha$ be a basis of Γ_ξ. By the inclusion (2.12), there exist a basis $x_1,\dots, x_\alpha$ of the isotropic group Γ_ξ and elements $y_1,\dots,y_\alpha$ of the form q_{S_ι} such that we have with respect to the form $q_{T_X}\oplus q_{S_\iota}$: $[x_i,y_i]\perp[x_j,y_j]$ if $i\ne j$, and $[x_i,y_i]\cong u_+^{(2)}(2)$. It follows that

$$q_M\cong q_{T_X}\oplus u_+^{(2)}(2)^{4-\alpha} \quad \text{if } \alpha\le 4; \qquad (2.14)$$

and

$$q_{T_X}\cong q'_{T_X}\oplus u_+^{(2)}(2)^{\alpha-4} \text{ and } q_M\cong q'_{T_X} \text{ if } \alpha > 4. \qquad (2.15)$$

We used here the fact that the orthogonal term $u_{+}^{(2)}(2)$ is splitting off uniquely up to isomorphism from a finite quadratic form. It follows that

$$\ell(q_{M_p})=\ell(q_{(T_X)_p}), \text{ if } p\neq 2; \qquad (2.16)$$

and

$$\ell(q_{M_2})=\ell(q_{(T_X)_2})+8-2\alpha, \text{ if } p=2. \qquad (2.17)$$

Obviously,

$$\mathrm{rk}\ M = \mathrm{rk}\ T_X + 8. \qquad (2.18)$$

The following conditions are sufficient and necessary for the existence of a primitive embedding of an even lattice with invariants $(t_{(+)},t_{(-)},q)$ into an indefinite even unimodular lattice with signature $(l_{(+)},l_{(-)})$:

$$t_{(+)}\le l_{(+)},\ t_{(-)}\le l_{(-)}; \qquad (2.19)$$

$$t_{(+)}+t_{(-)}+\ell(q)\le l_{(+)}+l_{(-)}; \qquad (2.20)$$

$$(-1)^{l_{(+)}-t_{(+)}}|\mathfrak{A}_q| \equiv \mathrm{discr}\ K(q_p) \bmod (\mathbb{Z}_p^*)^2 \qquad (2.21)$$

for all odd prime p for which $t_{(+)}+t_{(-)}+\ell(q_p)=l_{(+)}+l_{(-)}$;

$$|\mathfrak{A}_q| \equiv \pm\mathrm{discr}\ K(q_2) \bmod (\mathbb{Z}_2^*)^2 \qquad (2.22)$$

if $t_{(+)}+t_{(-)}+\ell(q_2)=l_{(+)}+l_{(-)}$ and $q_2\not\cong q_\theta^{(2)}(2)\oplus q_2'$. Here $K(q_p)$ is a p-adic lattice with the discriminant form q_p and $\mathrm{rk}\ K(q_p)=\ell(\mathfrak{A}_{q_p})$ (the form $K(q_p)$ is unique up to isomorphism). See [N3, theorem 1.12.2].

By (2.14) - (2.22), the following conditions are sufficient and necessary for the existence of a primitive embedding of the lattice M corresponding to the isomorphism ξ into the lattice H_X:

$$\operatorname{rk} T_X + \ell(q_{(T_X)_p}) \le 14 \qquad (2.23)$$

for all odd prime p, and

$$|\mathfrak{A}_{T_X}| \equiv -\operatorname{discr} K(q_{(T_X)_p}) \bmod (\mathbb{Z}_p^*)^2 \qquad (2.24)$$

for all odd prime p for which $\operatorname{rk} T_X + \ell(q_{(T_X)_p}) = 14$;

$$\alpha \ge (\operatorname{rk} T_X + \ell(q_{(T_X)_2}))/2 - 3, \qquad (2.25)$$

and

$$|\mathfrak{A}_{T_X}| \equiv \pm\operatorname{discr} K(q_{(T_X)_2}) \bmod (\mathbb{Z}_2^*)^2 \qquad (2.26)$$

if $\alpha = (\operatorname{rk} T_X + \ell(q_{(T_X)_2}))/2 - 3$ and $q_{(T_X)_2} \not\cong q_{\vartheta}^{(2)}(2) \oplus q'$.

The conditions (2.25), (2.26) and the strong inequalities

$$\operatorname{rk} T_X + \ell(q_{(T_X)_p}) < 14 \qquad (2.27)$$

for all odd prime p are sufficient for the existence of a primitive embedding of the lattice M into the lattice H_X.

By the Lemma 2.2.1,

$$((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*)/T_X = \mathfrak{H}, \qquad (2.28)$$

that defines the lattice $(T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*$. By (2.28) and (2.11) we get

Lemma 2.2.2. The $\tau^*(T_Y(2)) \subset T_X$ is defined by the following:

$$\tau^*(T_Y(2)) = 2((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*) \subset T_X \subset (T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*,$$

and

$$((T_X \otimes \mathbb{Q}) \cap (T^{\iota})^*)/T_X = \mathfrak{H} \subset \mathfrak{A}_{T_X}.$$

2.2.3. We can repeat results of **2.2.2** to obtain similar results for the K3 surface Y which has a rational map of the degree two $\tau: X - - \to Y$ of a K3 surface X, defined by a symplectic involution ι

of the X. Here we apply results of the **2.2.1** to $L=H_Y$, $T=T_Y$, and $Q=Q_\iota$.

The primitive sublattice $M=[T_Y\oplus Q_\iota]$ in H_Y, which is generated by the sublattice $T_Y\oplus Q_\iota$ of the lattice H_Y, is defined by the inclusion of the forms

$$\xi: q_{T_Y}|\mathfrak{H} \longrightarrow -q_{Q_\iota}=u_+(2)^3, \qquad (2.29)$$

where $\mathfrak{H}$ is a subgroup of the discriminant group $\mathfrak{A}_{T_Y}$. It is defined by the graphic $\Gamma_\xi=[T_Y\oplus Q_\iota]/(T_Y\oplus Q_\iota)\subset\mathfrak{A}_{T_Y}\oplus\mathfrak{A}_{Q_\iota}$ of the ξ, which is an isotropic subgroup of the form $q_{T_Y}\oplus q_{Q_\iota}$ in $\mathfrak{A}_{T_Y}\oplus\mathfrak{A}_{Q_\iota}$. The discriminant form

$$q_M=(q_{T_Y}\oplus q_{Q_\iota})|((\Gamma_\xi)^{\perp}_{q_{T_Y}\oplus q_{Q_\iota}}/\Gamma_\xi). \qquad (2.30)$$

By (2.29), $\mathfrak{H}\cong(\mathbb{Z}/2\mathbb{Z})^\beta$ is a 2-elementary group, $\beta\le6$, and also $\Gamma_\xi\cong(\mathbb{Z}/2\mathbb{Z})^\beta$. Similarly to the case **2.2.2**, we get:

$$q_M\cong q_{T_Y}\oplus u_+^{(2)}(2)^{3-\beta}, \text{ if } \beta\le 3; \qquad (2.31)$$

and

$$q_{T_Y}\cong q'_{T_Y}\oplus u_+^{(2)}(2)^{\beta-3} \text{ and } q_M\cong q'_{T_Y} \text{ if } \beta>3. \qquad (2.32)$$

It follows that

$$\ell(q_{M_p})=\ell(q_{(T_Y)_p}), \text{ if } p\ne2; \qquad (2.33)$$

and

$$\ell(q_{M_2})=\ell(q_{(T_Y)_2})+6-2\beta, \text{ if } p=2. \qquad (2.34)$$

Obviously,

$$\mathrm{rk}\, M = \mathrm{rk}\, T_Y + 8. \qquad (2.35)$$

By (2.19) - (2.22) and (2.31) - (2.35), the following conditions are sufficient are necessary for the existence of a primitive embedding of the lattice M corresponding to the inclusion

ξ into the lattice H_Y:

$$\mathrm{rk}\, T_Y + \ell(q_{(T_Y)_p}) \le 14 \qquad (2.36)$$

for all odd prime p , and

$$|\mathfrak{A}_{T_Y}| \equiv -\mathrm{discr}\, K(q_{(T_Y)_p}) \bmod (\mathbb{Z}_p^*)^2 \qquad (2.37)$$

for all odd prime p for which $\mathrm{rk}\, T_Y + \ell(q_{(T_Y)_p}) = 14$;

$$\beta \ge (\mathrm{rk}\, T_Y + \ell(q_{(T_Y)_2}))/2 - 4, \qquad (2.38)$$

and

$$|\mathfrak{A}_{T_Y}| \equiv \pm\mathrm{discr}\, K(q_{(T_Y)_2}) \bmod (\mathbb{Z}_2^*)^2, \qquad (2.39)$$

if $\beta = (\mathrm{rk}\, T_Y + \ell(q_{(T_Y)_2}))/2 - 4$ and $q_{(T_Y)_2} \not\cong q_\vartheta^{(2)}(2) \oplus q'$.

The conditions (2.38), (2.39) and the strong inequalities

$$\mathrm{rk}\, T_Y + \ell(q_{(T_Y)_p}) < 14 \qquad (2.40)$$

for all odd prime p are sufficient for the existence of a primitive embedding of the lattice M into the lattice H_Y.

2.2.4. Let X be a K3 surface. The pair $(T_X, H^{2,0}(X) \subset T_X \otimes \mathbb{C})$ is called <u>the transcendental periods</u> of the X. For two K3 surfaces X a Y, <u>an isomorphism of their transcendental periods</u> is an isomorphism $\varphi: T_X \cong T_Y$ of the lattices such that $(\varphi \otimes \mathbb{C})(H^{2,0}(X)) = H^{2,0}(Y)$. We say that a K3 surface X <u>is defined by its transcendental periods</u> iff every K3 surface X' with the transcendental periods isomorphic to that of X is isomorphic to X.

Lemma 2.2.3. Let Z be an algebraic K3 surface (over $\mathbb{C}$) which either has a symplectic involution or has a rational map of the degree 2 $\tau: X - - \to Z$ of a K3 surfaces X.

Then Z is defined by its transcendental periods, and for any K3 surface Z' and an isomorphism $\varphi:T_{Z'}\cong T_Z$ of the transcendental periods, $\varphi=f^*$ for some isomorphism $f:Z\cong Z'$ of the surfaces.

<u>Proof.</u> Suppose that K3 surface X has a symplectic involution ι and let $\varphi:T_X\longrightarrow T_{X'}$ be an isomorphism of the periods for K3 surface X'.

From the analog of Witt's theorem [N2], [N3], it follows that a primitive embedding of an even lattice K into an even unimodular lattice L is unique up to isomorphisms (for every two embeddings $i:K\subset L$, $i':K\subset L$ we have $i'=gi$ for an automorphism g of L) if the conditions a), b), c) below take place:

a) the lattice $(K)^{\perp}_{L}$ is indefinite;

b) $\operatorname{rk} K + \ell(\mathfrak{A}_{K_p}) \le \operatorname{rk} L - 2$ for all prime $p\neq 2$;

c) either $\operatorname{rk} K + \ell(\mathfrak{A}_{K_2}) \le \operatorname{rk} L - 2$ or $q_{K_2}\cong q_{K_2}{}'\oplus u_+^{(2)}(2)$.

By (2.15), (2.23), and (2.25), the conditions a), b) and c) above hold for the primitive embedding $T_X\subset H_X$. It follows that the primitive embedding $T_X\subset H_X$ is unique up to isomorphism. Thus, the isomorphism $\varphi:T_X\longrightarrow T_{X'}$ of the lattices has an extension $\Phi:H_X\longrightarrow H_{X'}$.

Let for a K3 surface Z

$$V(Z)=\{\ x\in S_Z\otimes\mathbb{R}\ \mid\ x^2>0\ \}$$

and let $V^+(Z)$ be a half cone of the $V(Z)$ which contains a polarization of the Z.

Suppose that $\Phi(V^+(X))=V^+(X')$. Then, there exists an element $w\in W^{(2)}(X)$ such that $\Phi w(h_X)=h_{X'}$ for polarizations h_X and $h_{X'}$ of X and X'. The w is trivial in T_X. From the global Torelli theorem [PSh-Sh], it follows that an isomorphism $f:X'\longrightarrow X$ exists such that

$f^* = \Phi w$. It follows that $f^*|T_X = \varphi$.

Suppose that $\Phi(V^+(X)) = -V^+(X')$. In this case, let us find an automorphism Ψ of the lattice H_X such that $\Psi|T_X = id_{T_X}$ and $\Psi(V^+(X)) = -V^+(X)$. Then we can replace Φ by $\Psi\Phi$ to reduce the case to the previews one.

The discriminant form $q_{S_X} \cong -q_{T_X}$ because $S_X = (T_X)^{\perp}_{H_X}$ and the S_X is primitive in H_X. From this fact and (2.15), (2.23), (2.25), it follows that

$$\mathrm{rk}\, S_X \geq \ell(\mathfrak{A}_{(S_X)_p}) + 8 \qquad (2.41)$$

for all odd $p \neq 2$, and

$$\mathrm{rk}\, S_X \geq \ell(\mathfrak{A}_{(S_X)_2}) + 16 - 2\alpha, \qquad (2.42)$$

where $\alpha \leq 8$. By (2.15),

$$q_{(S_X)_2} = u_+^{(2)}(2) \oplus q', \text{ if } \alpha \geq 5. \qquad (2.43)$$

It follows (see [Kn] and [N3, theorem 1.13.2]) that a lattice with the same invariants $(t_{(+)}, t_{(-)}, q)$ as the lattice S_X is unique up to isomorphisms. From this fact and the criterion of the existence of an even lattice with given invariants $(t_{(+)}, t_{(-)}, q)$ (see [N3, theorem 1.10.1]), it follows that

$$S_X = S_1 \oplus S_2, \text{ where } S_1 \cong U \text{ or } S_1 \cong U(2).$$

For the lattice S_1 the discriminant group $\mathfrak{A}_{S_1} \cong (\mathbb{Z}/2\mathbb{Z})^a$, $a=0$ or 2, is a 2-elementary group. It follows that there exists the automorphism Ψ of the H_X which is the $(-id)$ in S_1 and which is identical in $(S_1)^{\perp}_{H_X}$. The Ψ gives an automorphism which we look for.

In the case when $Z=Y$ has a rational map of the degree two

$$X - - \to Y$$

of the K3 surface X, the proof is the same if one uses 2.2.3. $\blacktriangleright$

The (2.11) and the Lemma 2.2.2 show that the modification defined by a rational map of the degree two $\tau:X- - ->Y$ of K3 surfaces is defined by a primitive embedding $T_X\subset T^{\iota}$ of the lattices where $T^{\iota}\cong U^3\oplus E_8(2)$. The Lemma below shows that every such embedding is possible and reduces the problem of the description of modifications to a purely arithmetic one.

Let us denote $T\cong T^{\iota}\cong U^3\oplus E_8(2)$.

Lemma 2.2.4. Let X be a K3 surface and $T_X\subset T\cong U^3\oplus E_8(2)$ a primitive embedding of lattices.

Then there exists a symplectic involution ι of X such that for the corresponding rational map of the degree two $\tau:X- - ->Y$ of K3 surfaces

$$\tau^*T_Y(2)=2(T^*\cap(T_X\otimes\mathbb{Q}))\subset T_X.$$

<u>Proof.</u> In fact, in the proof of the Lemma 2.2.3, we have shown that a primitive embedding $T_X\longrightarrow H_X$ of the lattices is unique up to isomorphisms, if a primitive embedding $T_X\subset T$ exists. It follows that an extension $T\subset H_X$ of the natural primitive embedding $T_X\subset H_X$ exists, where an embedding $T\subset H_X$ is also primitive. The lattice T is 2-elementary. It follows that the involution ϑ of the lattice H_X exists, which is identical in the lattice T and is the multiplication by (-1) in the lattice $S=(T)^{\perp}_X$. The $q_S\cong-q_T\cong u_+^{(2)}(2)^4$, rk S = 8. Then the lattice $S\cong S_1(2)$ where the lattice S_1 is an even lattice. Particularly, the lattice S has not elements with the square (-2). It follows [N2], that there exists $w\in W^{(2)}(S_X)$ such that $w\vartheta w^{-1}=\iota^*$ for a symplectic involution ι of the X. The automorphism w gives the isomorphism $w:T\longrightarrow T^{\iota}$ of the lattices which is identical in the lattice T_X. It follows that for the rational map corresponding to ι of the degree two $\tau:X- - ->Y$ of K3

surfaces we have (see (2.11)) that

$$\tau^*T_Y(2)=2((T^{\iota})^*\cap(T_X\otimes\mathbb{Q}))=2(T^*\cap(T_X\otimes\mathbb{Q})). \quad \triangleright$$

By the results above, we get

Theorem 2.2.5. Let X be an algebraic K3 surface.

If X has a rational map of the degree two $\tau:X- - ->Y$ in a K3 surface Y then the following condition (i) holds:

(i) $\mathrm{rk}\ T_X + \ell(q_{(T_X)_p}) \le 14$ for all odd prime p, and $|\mathfrak{A}_{T_X}| \equiv -\mathrm{discr}\ K(q_{(T_X)_p}) \bmod (\mathbb{Z}_p^*)^2$ for all odd prime p for which $\mathrm{rk}\ T_X + \ell(q_{(T_X)_p}) = 14$;

If the condition (i) holds, then there is the bijection between modifications $\tau^*:T_Y(2)\longrightarrow T_X$ corresponding to rational maps of degree two $\tau:X- - ->Y$ between K3 surfaces X and Y, and pairs $(\mathfrak{H}, \vartheta)$ defined below.

Here $\mathfrak{H}\cong(\mathbb{Z}/2\mathbb{Z})^{\alpha}$ is 2-elementary subgroup $\mathfrak{H}\subset\mathfrak{A}_{(T_X)_2}$ such that the condition (ii) below holds.

(ii) There exists an embedding $\xi:q_{T_X}|\mathfrak{H}\longrightarrow u_+^{(2)}(2)^4$ of the finite quadratic forms, and

$$\alpha \ge (\mathrm{rk}\ T_X + \ell(q_{(T_X)_2}))/2 - 3,$$

and

$$|\mathfrak{A}_{T_X}| \equiv \pm\mathrm{discr}\ K(q_{(T_X)_2}) \bmod (\mathbb{Z}_2^*)^2$$

if $\alpha = (\mathrm{rk}\ T_X + \ell(q_{(T_X)_2}))/2 - 3$ and $q_{(T_X)_2}\ne q_{\vartheta}^{(2)}(2)\oplus q'$.

For the lattice $T_X\subset\tilde{\mathfrak{H}}\subset T_X{}^*$ defined by the equality $\tilde{\mathfrak{H}}/T_X=\mathfrak{H}$, the ϑ is an isomorphism of the lattices

$$\vartheta:T_Y(2)\xrightarrow{\sim}2\tilde{\mathfrak{H}}\subset T_X,$$

such that $\vartheta(H^{2,0}(Y))=H^{2,0}(X)$. For the $\mathfrak{H}$ satisfying the condition (ii) there exists a K3 surface Y and an isomorphism ϑ with these properties.

The $\vartheta = \tau^*$ for a rational map $\tau:X- - -> Y$ of the degree two.

Proof. We leave the reader to deduce it from the Lemmas above.>

2.2.5. Let us define the composition of modifications which will correspond to the composition of rational maps.

Let T_1, T_2, T_3 be lattices and $\varphi_1:T_1(d_1)\otimes\mathbb{Q}\longrightarrow T_2\otimes\mathbb{Q}$, $T_2(d_2)\otimes\mathbb{Q}\longrightarrow T_3\otimes\mathbb{Q}$ be isomorphisms of symmetric bilinear forms over $\mathbb{Q}$, where d_1, d_2 are square-free positive integers. In other words, we have two abstract modifications of the lattices T_1, T_2, T_3. Let $d_1d_2=m^2(d_1d_2)'$ where m and $(d_1d_2)'$ are the integers and $(d_1d_2)'$ is square free. Then the sequence of inclusions of lattices

$$T_1((d_1d_2)')=(1/m)T_1(d_1d_2)\supset T_1(d_1d_2)$$

is defined. It gives the identification of the forms over $\mathbb{Q}$

$$T_1((d_1d_2)')\otimes\mathbb{Q}=(1/m)T_1(d_1d_2)\otimes\mathbb{Q}=T_1(d_1d_2)\otimes\mathbb{Q},$$

and the isomorphism $\overline{\varphi_2\varphi_1}$ of the forms

$$\overline{\varphi_2\varphi_1}:T_1((d_1d_2)')\otimes\mathbb{Q}=(1/m)T_1(d_1d_2)\otimes\mathbb{Q}=T_1(d_1d_2)\otimes\mathbb{Q}\xrightarrow{\varphi_1}T_2(d_2)\otimes\mathbb{Q}\xrightarrow{\varphi_2}T_3\otimes\mathbb{Q}$$

which is called the composition of the modifications φ_1, φ_2.

Suppose that $f_1:X_1- - ->X_2$, $f_2:X_2- - ->X_3$ are two rational maps between algebraic surfaces. Then the modification $\overline{f_2f_1}^*$ corresponding to the composition f_1f_2 of the rational maps is obviously the composition of the modifications $\overline{f_1}^*$, $\overline{f_2}^*$.

2.2.6. Using the results above, we want to describe modifications corresponding to rational maps $f:X- - ->Y$ between K3 surfaces X and Y which are compositions $f=f_n\cdot\ldots\cdot f_1$ of rational maps f_1, f_2, ..., f_n of the degree two. A composition of any two rational maps of this type is a rational map of this type also. Thus, these rational maps define the <u>category</u> $\mathcal{K}$ of the rational maps.

Lemma 2.2.6. Let $f:X\text{---}>Y$ be a rational map between K3 surfaces X and Y, which is a composition $f=f_n\cdot\ldots\cdot f_1$ of the rational maps of the degree two $f_1:X_1=X\text{---}>X_2$, ..., $f_n:X_n\text{---}>X_{n+1}=Y$ between the non-singular algebraic surfaces $X_1,\ldots, X_{n+1}$ (i.e. $f\in\mathcal{K}$).

Then the minimal models of the surfaces $X_1,\ldots, X_{n+1}$ are K3 surfaces. So, we can choose birationally the surfaces $X_1,\ldots, X_{n+1}$ being K3 surfaces.

<u>Proof.</u> Rational maps $f_1,\ldots, f_n$ give the isomorphisms

$$H^{2,0}(X)=H^{2,0}(X_1)\cong H^{2,0}(X_2)\cong\ldots\cong H^{2,0}(X_{n+1})\cong H^{2,0}(Y),$$

because $H^{2,0}(X)\cong{}^{2,0}(Y)\cong\mathbb{C}$. It follows that Galois involutions $\iota_1,\ldots, \iota_n$ of the maps $f_1,\ldots, f_n$ are trivial in the spaces

$$H^{2,0}(X)=H^{2,0}(X_1)\cong H^{2,0}(X_2)\cong\ldots\cong H^{2,0}(X_{n+1})\cong H^{2,0}(Y).$$

Then the involution ι_1 is a symplectic involution of the K3 surface $X_1=X$. Let Y be the minimal resolution of the singularities of $X/\{id, \iota\}$. We know (see [N2] and also **2.1**) that the surface Y is a K3 surface. The surface X_2 is birationally isomorphic to the surface Y, and its minimal model is a K3 surface. Thus, we can suppose that $X_2=Y$ is a K3 surface. In such a way, we obtain the proof using the induction. ➤

Using the Theorem 2.2.5 and the Lemma 2.2.6, we obtain the following description of the modifications corresponding to rational maps from the category $\mathcal{K}$ between K3 surfaces.

Theorem 2.2.7. Let X be an algebraic K3 surface.

If X has a rational map $f:X\text{- - ->}Y$ in a K3 surface Y which is a composition of rational maps of the degree two, and deg $f > 1$, then the condition (i) of the theorem 2.2.5 holds for T_X.

Let for T_X the condition (i) of the theorem 2.2.5 holds, a positive integer $d|2$, and Y is a K3 surface.

Then modifications $\overline{f^*}:T_Y(d)\otimes\mathbb{Q}\xrightarrow{\sim}T_X\otimes\mathbb{Q}$ corresponding to rational maps $f:X- - ->Y$ which are compositions $f=f_n\cdots f_1$ of rational maps $f_1,\dots,f_n$ of the degree two ($d=1$ if n is even, and $d=2$ if n is odd) are defined by sequences $(T_1,\mathfrak{H}_1)$, $(T_2,\mathfrak{H}_2),\dots,$ $(T_n,\mathfrak{H}_n)$ of pairs and by the isomorphisms ϑ defined below. An every such sequence and an every ϑ are possible.

Here, T_i, $i=1,\dots, n$, are sublattices of the maximal rank in the form $T_X\otimes\mathbb{Q}$ for i odd, and in the form $T_X(1/2)\otimes\mathbb{Q}$ for i even. Here $\mathfrak{H}_i\cong(\mathbb{Z}/2\mathbb{Z})^{\alpha_i}$ is a 2-elementary subgroup $\mathfrak{H}_i\subset\mathfrak{A}_{T_i}$. The lattices T_i are defined by the induction. The sublattice $T_1=T_X\subset T_X\otimes\mathbb{Q}$. For $1\le i\le n$ the sublattice $T_{i+1}(2)=2\tilde{\mathfrak{H}}_i\subset T_i$, where $\tilde{\mathfrak{H}}_i/T_i=\mathfrak{H}_i$. It gives the inclusion $T_{i+1}\subset T_X(1/2)\otimes\mathbb{Q}$ if i is odd, and the inclusion $T_{i+1}\subset T_X(1/4)\otimes\mathbb{Q}=(1/2)T_X\otimes\mathbb{Q}=T_X\otimes\mathbb{Q}$, if i is even. For the every pair $(T_i,\mathfrak{H}_i)$, $1\le i\le n$, the condition (ii) of the Theorem 2.2.5 should be true (one should replace in the condition the T_X by T_i, and $\mathfrak{H}$ by $\mathfrak{H}_i$).

The $\vartheta:T_Y\longrightarrow T_{n+1}$ is an isomorphism of the lattices which induces the isomorphism of the periods, i.e. $\vartheta(H^{2,0}(Y))=H^{2,0}(X)\subset T_X\otimes\mathbb{C}$. For the sequence $(T_1, \mathfrak{H}_1)$, $(T_2, \mathfrak{H}_2),\dots,$ $(T_n, \mathfrak{H}_n)$ satisfying the condition above there exists such K3 surface Y and an isomorphism ϑ.

The modification $\overline{f^*}$ defining by the sequence and the ϑ is the composition of the ϑ and of the inclusion of the sublattice $T_{n+1}\subset T_X\otimes\mathbb{Q}$ for n even and $T_{n+1}\subset T_X(1/2)\otimes\mathbb{Q}$ for n odd under multiplication of the forms by d=2 for n odd.

Proof. The Theorem follows from the Theorem 2.2.5 using compositions of rational maps and modifications above (it is more difficult to formulate this theorem then to deduce it from the Theorem 2.2.5). >

Remark 2.2.8. From the theorem 2.2.7, we obtain the following sequence of sublattices of the form $T_X \otimes \mathbb{Q}$:

$$T_1 \supset T_2(2) \subset T_3 \supset T_4(2) \subset \ldots \quad \text{in the } T_X \otimes \mathbb{Q},$$

where $(1/2)T_{i+1}(2)/T_i = \mathfrak{H}_i$ for all odd i, and $T_{i+1}(1/2)/T_i = \mathfrak{H}_i$ for i even.

The theorem 2.2.7 reduces the description of modifications corresponding to rational maps between K3 surfaces from the category $\mathcal{K}$ to the purely algebraic problem. We will use the Theorem 2.2.7 for the proof of the basic Theorem 3.1 of the paper (Theorem 1.3. of the Introduction) in the following paragraph.

§ 3. Rational maps between K3 surfaces with the transcendental lattice of the rank ≤ 5.

Here we prove the basic theorems (the Theorems 1.3 and 1.4 of the Introduction) of the paper.

Theorem 3.1. Let X and Y be algebraic K3 surfaces with rk T_X=rk $T_Y \le 5$, and $\varphi: T_Y(d) \otimes \mathbb{Q} \longrightarrow T_X \otimes \mathbb{Q}$ be an isomorphism of quadratic forms over $\mathbb{Q}$ (i.e., φ is an abstract modification of the lattices T_X and T_Y) for which $\varphi(H^{2,0}(Y)) = H^{2,0}(X)$, $d|2$, and φ induces an isomorphism $\varphi_p: T_Y(d) \otimes \mathbb{Z}_p \longrightarrow T_X \otimes \mathbb{Z}_p$ of p-adic lattices for any prime $p \ne 2$.

Then there exists a sequence $X=X_1, X_2, \ldots, X_{n+1}=Y$ of K3 surfaces and rational maps $f_i: X_i - - \to X_{i+1}$ of degree 2 such that the rational map $f=f_n\cdot\ldots\cdot f_2\cdot f_1$ induces the modification φ, i.e., $\varphi=\overline{f}*$.

Proof. We divide it on several steps.

3.1. We denote $T=T_X$ and $\tilde{T}=\varphi(T_Y)\subset T\otimes\mathbb{Q}(1/d)$. Using the Theorem 2.2.7 and the Remark 2.2.8, one should find a sequence of the $\mathbb{Z}$-sublattices of the form $T\otimes\mathbb{Q}$:

$$T=T_1\supset T_2(2)\subset T_3\supset\ldots T_{n+1}(d)=\tilde{T}(d), \qquad (3.1)$$

where n is odd if d=2, and n is even if d=1, such that the conditions of the Theorem 2.2.7 hold. A sequence which satisfy the conditions of the Theorem 2.2.7 is called further an acceptable.

By the condition of the Theorem 3.1, $T\otimes\mathbb{Z}_p=\tilde{T}\otimes\mathbb{Z}_p$ for any odd prime p. According to the Theorem 2.2.7, quotient modules of the modules of the sequence (3.1) should be 2-groups. Thus, one should find the sequence (3.1) over ring $\mathbb{Z}_2$ only. One has the obvious inequality $\ell(\mathfrak{A}_{(T_X)_p})\le \mathrm{rk}\ T_X\le 5$ for every p. Then $\ell(\mathfrak{A}_{(T_X)_p})+\mathrm{rk}\ T_X<14$. Thus, the condition (i) of the Theorem 2.2.7 is true, and for a construction of the sequence (3.1) we should satisfy to the condition (ii) of the Theorem 2.2.7 only.

3.2. At first, for rk $T\le 5$, we will construct an acceptable sequence $T=T_1,\ldots,T_{m+1}=T'$ of lattices such that m is odd and $T'=2T(1/2)\subset T\otimes\mathbb{Q}(1/2)$. Thus, the lattice, $T'\cong T(2)$. We consider the most difficult cases rk T =4 and 5.

Let rk T = 4.

Let (over $\mathbb{Z}_2$) $T=S_1\oplus S_2\oplus R(2)$ where S_1, S_2 are lattices of the rank 1, and R is an even lattice of the rank 2. Let $\{\zeta_1\}$ be a bases of the S_1, $\{\zeta_2\}$ a basis of the S_2, and $\{\zeta_3,\ \zeta_4\}$ a basis of the lattice R(2). Let us prove that the following sequence of lattices is acceptable:

$$T_1=[\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4],\qquad T_2=[2\zeta_1,\ 2\zeta_2,\ \zeta_3,\ \zeta_4](1/2),$$

$$T_3=[\zeta_1,\ 2\zeta_2,\ \zeta_3,\ \zeta_4],\qquad T_4=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4](1/2).$$

In this case the subgroup $\mathfrak{H}_1=[\zeta_1,\ \zeta_2,\ \zeta_3/2,\ \zeta_4/2]/[\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4]$, and, evidently, there exists an embedding of the forms $q_{T_1}|\mathfrak{H}_1 \longrightarrow u_+^{(2)}(2)^4$. We have: $\alpha_1=2>1\ge(\mathrm{rk}\ T_1+\ell(q_{T_1}))/2-3$ since $4=\mathrm{rk}\ T_1 \ge \ell(q_{T_1})$. It proves the condition (ii) of the Theorem 2.2.7 for the pair $(T_1,\mathfrak{H}_1)$. The lattice $T_2=S_1(2)\oplus S_2(2)\oplus R$, and $\alpha_2=1$. In this case $\mathfrak{H}_2=[\zeta_1,\ 2\zeta_2,\ \zeta_3,\ \zeta_4]/[2\zeta_1,\ 2\zeta_2,\ \zeta_3,\ \zeta_4]$, and evidently an embedding $q_{T_2}|\mathfrak{H}_2 \longrightarrow u_+^{(2)}(2)^4$ of the forms exists. Since the lattice T_2 is even then either R is unimodular or $\ell(\mathfrak{A}_R)=2$. If the lattice R is unimodular, then $\alpha_2=1>(\mathrm{rk}\ T_2+\ell(q_{T_2}))/2-3$. If R is not unimodular, then we have the equality $\alpha_2=1=(\mathrm{rk}\ T_2+\ell(q_{T_2}))/2-3$. And we should prove the congruence (where we consider the lattice T_2 as a lattice over $\mathbb{Z}$):

$$|\mathfrak{A}_{T_2}|\equiv\pm\mathrm{discr}\ K(q_{(T_2)_2})\ \mathrm{mod}(\mathbb{Z}_2{*})^2.$$

In this case, $K(q_{(T_2)_2})\cong(T_2)_2=T_2\otimes\mathbb{Z}_2$, and this congruence holds because

$$\mathrm{discr}\ T_2=\pm|\mathfrak{A}_{T_2}|.$$

for the lattice T_2 over $\mathbb{Z}$. The $\alpha_3=1$, and the proof of the condition (ii) for $(T_3,\ \mathfrak{H}_3)$ is the same.

The same proof of the condition (ii) should be produced in all cases which we consider below. We will leave this procedures to the reader.

Now, suppose that the lattice T has not a representation of the type above. From the decomposition of 2-adic lattices in an orthogonal sum of lattices of the rank 1 and 2, one obtains that it is possible only in the following two cases which we consider at once.

The case $T=R_1(2^m)\oplus R_2(2^n)$, where R_1, R_2 are an even unimodular lattices of the rank two, $m\geq 0$, $n\geq 0$. Let $\{\zeta_1, \zeta_2\}$ be a basis of the lattice $R_1(2^m)$ and $\{\zeta_1, \zeta_2\}$ a basis of the $R_2(2^m)$. If $m=n=0$ then the sequence of lattices

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4], \quad T_2=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2)$$

is acceptable. Suppose that $n\geq 1$. Then the following sequence of the lattices is acceptable:

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4], \quad T_2=[2\zeta_1, 2\zeta_2, \zeta_3, 2\zeta_4](1/2),$$
$$T_3=[2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4], \quad T_4=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2).$$

The case $T=S_1\oplus S_2\oplus R$, where S_1, S_2 are even lattices of the rank one, and R is an unimodular lattice of the rank two. If one of the lattices $S_1(1/2)$, $S_2(1/2)$ is not even, then the following sequence of the lattices is acceptable:

$$T_1=T, \quad T_2=2T(1/2).$$

Now suppose that the lattice $S_2(1/2)$ is even. Let $\{\zeta_1\}$ be a basis of the S_1, $\{\zeta_2\}$ be a basis of the S_2, and $\{\zeta_3,\zeta_4\}$ be a basis of the lattice R. Then the following sequence is acceptable:

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4], \quad T_2=[2\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4](1/2),$$
$$T_3=[\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4], \quad T_4=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4](1/2).$$

Let $\operatorname{rk} T = 5$.

Suppose that $T=S_1\oplus S_2\oplus S_3\oplus S_4\oplus S_5$, where $\operatorname{rk} S_i=1$, and the lattices $S_4(1/2)$ and $S_5(1/2)$ are even. Let $\{\zeta_i\}$ be a basis of the S_i. Then the following sequence of lattices is acceptable:

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5], \ T_2=[2\zeta_1, 2\zeta_2, 2\zeta_3, \zeta_4, \zeta_5](1/2),$$

$T_3=[2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_4=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5](1/2)$,
$T_5=[\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4, \zeta_5]$, $T_6=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$.

Let $S=S_1\oplus S_2\oplus S_3\oplus R$, where S_1, S_2, S_3 are lattices of the rank 1, rk R = 2, and the lattices $S_3(1/2)$ and $R(1/2)$ are even. Let $\{\zeta_1\}$ be a basis of the S_1, $\{\zeta_2\}$ be a basis of the S_2, $\{\zeta_4\}$ be a basis of the S_3, and $\{\zeta_4, \zeta_5\}$ be a basis of the R. In this case the following sequence of lattices is acceptable:

$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_2=[2\zeta_1, 2\zeta_2, 2\zeta_3, \zeta_4, \zeta_5](1/2)$,
$T_3=[2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_4=[2\zeta_1, 2\zeta_2, \zeta_3, 2\zeta_4, 2\zeta_5](1/2)$,
$T_5=[2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_6=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$.

Now suppose that the lattice T has not representations of the types above. Then, only, the following cases are possible. We consider them at once.

The case $T=S\oplus R_1(2^m)\oplus R_2(2^n)$, $m\geq 0$, $n\geq 0$, where rk S=1 and R_1, R_2 are even unimodular lattices of the rank 2. Let $\{\zeta_1\}$ be a basis of the S, $\{\zeta_2, \zeta_3\}$ of the $R_1(2^m)$, $\{\zeta_4, \zeta_5\}$ of the $R_2(2^n)$. Suppose that $m\geq 1$. Then we obtain the following acceptable sequence:

$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_2=[2\zeta_1, \zeta_2, \zeta_3, 2\zeta_4, 2\zeta_5](1/2)$,
$T_3=[2\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_4=[2\zeta_1, \zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$,
$T_5=[2\zeta_1, \zeta_2, \zeta_3, \zeta_4, 2\zeta_5]$, $T_6=[2\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$.

Suppose that m=n=0. If the lattice S(1/2) is not even, then we obtain the following acceptable sequence:

$$T_1=T,\ T_2=2T(1/2).$$

If the lattice S(1/2) is even, then the following sequence is acceptable:

$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5]$, $T_2=[\zeta_1, 2\zeta_2, 2\zeta_3, 2\zeta_4, 2\zeta_5](1/2)$,

$$T_3=[\zeta_1,\ \zeta_2,\ \zeta_3,\ 2\zeta_4,\ 2\zeta_5],\qquad T_4=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ 2\zeta_5](1/2).$$

The case $T=R\oplus S_1\oplus S_2\oplus S_3$, where R is an even unimodular lattice of the rank 2, and S_1, S_2, S_3 are lattices of the rank one. The case, when all lattices $S_1(1/2)$, $S_2(1/2)$, $S_3(1/2)$ are not even is reduced to the previous case, because then $S_1\oplus S_2\oplus S_3=R'(2)\oplus S'$, where R' is an even unimodular lattice of the rank 2 and S' is a lattice of the rank 1. Thus, we can suppose that the lattice $S_3(1/2)$ is even. Let $\{\zeta_1,\ \zeta_2\}$ be a basis of the R, $\{\zeta_3\}$ of the S_1, $\{\zeta_4\}$ of the S_2, and $\{\zeta_5\}$ of the S_5. Suppose that one of the lattices $S_1(1/2)$ or $S_2(1/2)$ is not even. In this case, we have the following acceptable sequence:

$$T_1=[\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_5],\qquad T_2=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ \zeta_5](1/2),$$

$$T_3=[\zeta_1,\ \zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ \zeta_5],\qquad T_4=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ 2\zeta_5](1/2).$$

Suppose now that the lattice $S_2(1/2)$ is even (together with the lattice $S_3(1/2)$). Then the following sequence is acceptable:

$$T_1=[\zeta_1,\ \zeta_2,\ \zeta_3,\ \zeta_4,\ \zeta_5],\qquad T_2=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ \zeta_4,\ \zeta_5](1/2),$$

$$T_3=[\zeta_1,\ \zeta_2,\ 2\zeta_3,\ \zeta_4,\ \zeta_5],\qquad T_4=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ \zeta_5](1/2),$$

$$T_5=[\zeta_1,\ \zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ \zeta_5],\qquad T_6=[2\zeta_1,\ 2\zeta_2,\ 2\zeta_3,\ 2\zeta_4,\ 2\zeta_5](1/2).$$

It finishes the proof of the statement.

3.3. Here, for a lattice T of $\mathrm{rk}\ T \le 5$ and with an even lattice $T(1/2)$, we will construct an acceptable sequence $T=T_1,\ldots,\ T_m=T''$ of lattices such that m is odd and $T''=T(1/2)\subset T\otimes\mathbb{Q}(1/2)$.

Suppose that $\mathrm{rk}\ T \le 4$. Then the following sequence is acceptable:

$$T_1=T,\ T_2=T(1/2).$$

Suppose that $\mathrm{rk}\ T = 5$.

Let $T=R_1(2)\oplus R_2(2)\oplus S(4)$, where the lattices R_1, R_2, S are even and rk R_1=rk R_2=2, rk S=1. Let $\{\zeta_1, \zeta_2\}$ be a basis of the $R_1(2)$, $\{\zeta_3, \zeta_4\}$ be a basis of the $R_2(2)$, and $\{\zeta_5\}$ of the $S(4)$. Then the following sequence is acceptable:

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5], \quad T_2=[2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2),$$
$$T_3=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5/2], \quad T_4=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2).$$

Let $T=R_1(2)\oplus R_2(4)\oplus S(2)$ where the lattices R_1, R_2, S are even. Let $\{\zeta_1, \zeta_2\}$ be a basis of the $R_1(2)$, $\{\zeta_3, \zeta_4\}$ be a basis of the $R_2(4)$, and $\{\zeta_5\}$ be a basis of the $S(2)$. Then the following sequence is acceptable:

$$T_1=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5], \quad T_2=[2\zeta_1, 2\zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2),$$
$$T_3=[\zeta_1, \zeta_2, \zeta_3/2, \zeta_4/2, \zeta_5], \quad T_4=[\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5](1/2).$$

Now suppose that the lattice T has not representations of the type above. Then $T=R_1(2)\oplus R_2(2)\oplus S(4)$, where R_1, R_2 are even unimodular lattices and rk R_1 = rk R_2 = 2, S is an odd unimodular lattice and rk S = 1. Then the following sequence is acceptable:

$$T_1=T, \quad T_2=T(1/2).$$

It finishes the proof of the statement.

3.4. Here we will finish the proof of the Theorem. We consider the most difficult case rk T_X = rk T = 5.

Let us reduce the case d=2 to the case d=1. Using 3.2, we can find an acceptable sequence $T=T_1,\dots, T_m$, such that $T_m=2T(1/2)$. In the case d=2 both lattices T_m and $\tilde{T}$ are contained in the one form $T\otimes\mathbb{Q}(1/2)$. It is sufficient to find an acceptable sequence for $T=T_m$ and $\tilde{T}$ where both lattices are contained in the one form $T\otimes\mathbb{Q}(1/2)$. Thus, we have deal with the case d=1 now.

Now suppose that d=1. Then both lattices T and $\tilde{T}$ are lattices of the one quadratic form $T\otimes\mathbb{Q}$. Let $S=T\cap\tilde{T}$. Thus, we have the following sequence of inclusions of the lattices of the form $T\otimes\mathbb{Q}$:

$$T \supset S \subset \tilde{T}.$$

Using results 3.2, we can find an acceptable sequence

$$T=T_1, \ldots, T_{2m}=2T\subset T\otimes\mathbb{Q}.$$

Using results 3.3, we can find an acceptable sequence

$$2\tilde{T}=S_1, \ldots, S_{2n}=\tilde{T}\subset T\otimes\mathbb{Q}.$$

Thus, it is sufficient to find an acceptable sequence with the first term $2T$ and with the final term $2\tilde{T}$. The lattices $2T$ and $2\tilde{T}$ are more convenient because the lattice $2T\cong T(4)$ and the lattice $2\tilde{T}\cong\tilde{T}(4)$ where T and $\tilde{T}$ are even lattices.

Thus, it is sufficient to find an acceptable sequence for the lattices $T\cong T'(4)$ and $\tilde{T}\cong\tilde{T}'(4)$ where T' and $\tilde{T}'$ are even lattices. Further, we suppose that it is true.

The quotient group T/S is a finite abelian 2-group. It follows that there exists a sequence of sublattices of the form $T\otimes\mathbb{Q}$:

$$T=S_1\supset S_2\supset\ldots\supset S_a=S,$$

for which $S_i/S_{i+1}\cong\mathbb{Z}/2\mathbb{Z}$, $i=1,\ldots a-1$. Let S_i' be a sublattice of $T\otimes\mathbb{Q}$ which satisfies the condition:

$$S_i \supset S_{i+1} \supset S_i' \supset 2S_i, \text{ and } S_i'/2S_i\cong(\mathbb{Z}/2\mathbb{Z})^2.$$

Then, evidently

$$S_{i+1}/S_i'\cong(\mathbb{Z}/2\mathbb{Z})^2.$$

Let us show that the sequence of the lattices

$$S_i,\ S_i'(1/2),\ S_{i+1}$$

is acceptable.

The lattice $S_i=M(4)$ where M is an even lattice (since it is true for the lattice T and $S_i\subset T$). Then, the sublattice S_i' is constructed from the subgroup $\mathfrak{H}=(1/2)S_i'/S_i\subset\mathfrak{A}_{S_i}$, $\mathfrak{H}\cong(\mathbb{Z}/2\mathbb{Z})^2$ and $q_{S_i}|\mathfrak{H}=0$. It follows that there exists an embedding of the forms:

$$q_{S_i}|\mathfrak{H}\longrightarrow u_+(2)^4.$$

We have: $\ell(\mathfrak{A}_{(S_i)_2})=5$ because $S_i=M(4)$ where M is a lattice. So, we have the equality: $2=(\mathrm{rk}\ S_i+\ell(\mathfrak{A}_{(S_i)_2})/2-3$. Thus, we should prove the congruence for the lattice S_i over $\mathbb{Z}$:

$$|\mathfrak{A}_{S_i}|\equiv\pm\mathrm{discr}\ K(q_{(S_i)_2})\ \mathrm{mod}(\mathbb{Z}_2{}^*)^2.$$

Since $S_i=S_i'(4)$, in this case $K(q_{(S_i)_2})\cong S_i\otimes\mathbb{Z}_2$. It follows that $\mathrm{discr}\ S_i=\pm|\mathfrak{A}_{S_i}|$, and the condition (ii) of the Theorem 2.2.8 is true.

The lattice $S_i'(1/2)\subset S_i(1/2)\subset T(1/2)=T'(2)$, where T' is an even lattice. Using this fact, in the same way as above, one proves that the sequence of the lattices $S_i'(1/2)$, S_{i+1} is acceptable. The corresponding to this sequence subgroup $\mathfrak{H}$ of the discriminant group of the lattice $S_i'(1/2)$ is $\mathfrak{H}=S_{i+1}(1/2)/S_i'(1/2)\cong(\mathbb{Z}/2\mathbb{Z})^2$.

In such a way, we obtain an acceptable sequence of sublattices of $T\otimes\mathbb{Q}$:

$$T=S_1\supset S_1'(2)\subset S_2\supset\dots\subset S_{a-1}\supset S_{a-1}'(2)\subset S_a=S$$

The quotient group $\tilde{T}/S$ is a finite abelian 2-group also. Then we can find a sequence of sublattices of the form $T\otimes\mathbb{Q}$:

$$S=P_1\subset P_2\subset\dots\subset P_{b-1}\subset P_b=\tilde{T}$$

with $P_{i+1}/P_i\cong\mathbb{Z}/2\mathbb{Z}$, $1\le i\le b-1$. Let P_i' be a sublattice of the form $T\otimes\mathbb{Q}$ which satisfy the condition:

$$2P_{i+1}\subset P_i'\subset P_i \text{ and } P_i/P_i'\cong(\mathbb{Z}/2\mathbb{Z})^2.$$

Let us show that the sequence of lattices

$$P_i,\ P_i'(1/2),\ P_{i+1}$$

is acceptable.

The lattice $P_i=M(4)$ where M is an even lattice, since it holds for $\tilde{T}$, and P_i is a sublattice of the $\tilde{T}$. Then the lattice $P_i'(1/2)$ is constructed from the subgroup $\mathfrak{H}=(1/2)P_i'/P_i\subset\mathfrak{A}_{P_i}$, $\mathfrak{H}\cong(\mathbb{Z}/2\mathbb{Z})^3$ and $q_{P_i}|\mathfrak{H}=0$. It follows that there exists an embedding of the forms:

$$q_{P_i}|\mathfrak{H}\longrightarrow u_+^{(2)}(2)^4.$$

Since rk $P_i=5$, then we have the strong inequality:

$$3 > (\mathrm{rk}\ P_i+\ell(\mathfrak{A}_{(P_i)_2})/2 - 3 = 2.$$

It proves the condition (ii) of the Theorem 2.2.7, and the sequence of lattices P_i, $P_i'(1/2)$ is acceptable.

The lattice $P_i'(1/2) \subset P_i(1/2)\cong M(2)$, where the lattice M is even. Using this fact, in the same way as above, one proves that the sequence of the lattices $P_i'(1/2)$, P_{i+1} is acceptable. The corresponding to this sequence subgroup $\mathfrak{H}$ of the discriminant group of the lattice $P_i'(1/2)$ is $\mathfrak{H}=P_{i+1}(1/2)/P_i'(1/2)\cong(\mathbb{Z}/2)^3$.

In such a way, we obtain an acceptable sequence of the lattices of the form $T\otimes\mathbb{Q}$:

$$S=P_1\supset P_1'(2)\subset P_2\supset\ldots\subset P_{b-1}\supset P_{b-1}'(2)\subset P_b=\tilde{T}.$$

This finishes the proof of the Theorem. $\triangleright$

From the theorem 3.1 and the theory of Kummer surfaces, we obtain the following theorem (Theorem 1.3 of the Introduction). This theorem was proved by I.R.Shafarevich and the author together.

Theorem 3.2. (V.V.Nikulin and I.R.Shafarevich). Let X and Y be algebraic K3 surfaces. Suppose that for all odd prime p there are primitive embeddings of p-adic lattices:

$$T_X\otimes\mathbb{Z}_p\subset U^3\otimes\mathbb{Z}_p \quad\text{and}\quad T_Y\otimes\mathbb{Z}_p\subset U^3\otimes\mathbb{Z}_p;$$

and for p=2 there are embeddings of the quadratic forms over the field $\mathbb{Q}_2$:

$$T_X\otimes\mathbb{Q}_2\subset U^3\otimes\mathbb{Q}_2 \quad\text{and}\quad T_Y\otimes\mathbb{Q}_2\subset U^3\otimes\mathbb{Q}_2.$$

Let for the positive square-free integer d we have an isomorphism $\varphi:T_Y(d)\otimes\mathbb{Q}\longrightarrow T_X\otimes\mathbb{Q}$ of quadratic forms over $\mathbb{Q}$ (an abstract modification) and $\varphi(H^{2,0}(Y))=H^{2,0}(X)$.

Then there exists a rational map $f:X\longrightarrow Y$ such that $\varphi=\overline{f^*}$.

<u>Proof.</u> One can see very easy that for any odd prime p we have an isomorphism: $U\otimes\mathbb{Z}_p\cong U(2)\otimes\mathbb{Z}_p$, and that $U\otimes\mathbb{Q}_2\cong U(2)\otimes\mathbb{Q}_2$. It follows, that for any odd prime p there are primitive embeddings

$$T_X\otimes\mathbb{Z}_p\subset U(2)^3\otimes\mathbb{Z}_p \quad\text{and}\quad T_Y\otimes\mathbb{Z}_p\subset U(2)^3\otimes\mathbb{Z}_p$$

and

$$T_X\otimes\mathbb{Q}_2\subset U(2)^3\otimes\mathbb{Q}_2 \quad\text{and}\quad T_Y\otimes\mathbb{Q}_2\subset U(2)^3\otimes\mathbb{Q}_2.$$

The lattice $U(2)^3$ is unique in its genus (it follows from the classification of the unimodular lattices). Then, there exist embeddings of the lattices $T_X\subset U(2)^3$ and $T_Y\subset U(2)^3$ such that these embeddings are primitive over all odd prime p. Let T_1 be the primitive sublattice of $U(2)^3$, generated by T_X, and T_2 be the primitive sublattice of $U(2)^3$ generated by T_Y. We have the natural identifications $T_X\otimes\mathbb{Q}=T_1\otimes\mathbb{Q}$ and $T_Y\otimes\mathbb{Q}=T_2\otimes\mathbb{Q}$ of the quadratic forms over $\mathbb{Q}$ such that for all odd prime p we have $T_X\otimes\mathbb{Z}_p=T_1\otimes\mathbb{Z}_p$ and $T_Y\otimes\mathbb{Z}_p=T_2\otimes\mathbb{Z}_p$ under the identifications. Surfaces X and Y are algebraic. It follows that $\operatorname{rk} T_X=\operatorname{rk} T_Y\le 5$ since there are embeddings $T_X\subset U(2)^3$ and $T_Y\subset U(2)^3$. From the prove of the theorem 3.1, it follows that there are K3 surfaces X_1 and Y_1, and rational maps $g_1:X\dashrightarrow X_1$ and

$g_2:Y_1 \dashrightarrow Y$, which are compositions of the rational maps of the degree two, and isomorphisms of the lattices $\vartheta_1:T_{X_1}\cong T_1$ and $\vartheta_2:T_2\cong T_{Y_1}$ such that $\overline{g_1{}^*}= \vartheta_1\otimes\mathbb{Q}$ and $\overline{g_2{}^*} =\vartheta_2\otimes\mathbb{Q}$ under the identifications above of the quadratic forms over $\mathbb{Q}$: $T_X\otimes\mathbb{Q}=T_1\otimes\mathbb{Q}$ and $T_Y\otimes\mathbb{Q}=T_2\otimes\mathbb{Q}$. Under the identifications, the preserving periods modification $\varphi:T_Y(d_1)\otimes\mathbb{Q}\cong T_X\otimes\mathbb{Q}$ defines the preserving periods modification

$$\varphi_1=(\vartheta_1\otimes\mathbb{Q})^{-1}\cdot\varphi\cdot(\vartheta_2\otimes\mathbb{Q})^{-1}:T_{Y_1}(d_1)\otimes\mathbb{Q} \cong T_{X_1}\otimes\mathbb{Q}.$$

The lattices $T_{X_1}\cong T_1$ and $T_{Y_1}\cong T_2$ have primitive embeddings into the lattice $U(2)^3$. It follows from the criterion of [N1] for K3 surface to be Kummer surface and [N3] (see [Mo]) that both K3 surfaces X_1 and Y_1 are Kummer surfaces. We recall that if A is an Abelian surface and ι is a multiplication by -1 on the A, then the minimal resolution Z of singularities of the surface $A/\{1, -1\}$ is called Kummer surface. This surface is an algebraic K3 surface. It is not difficult to prove that the statement of the theorem is true for the Abelian surfaces and homomorphisms of Abelian surfaces. The transcendental lattices of Z and A are naturally identified: $T_Z=T_A(2)$, and under this identification $H^{2,0}(Z)=H^{2,0}(A)$. It follows that the theorem is true for Kummer surfaces (an every homomorphism between Abelian surfaces gives the rational map of the corresponding Kummer surfaces and the corresponding modification of their transcendental periods). Thus, there exists a rational map $h:X_1 \dashrightarrow Y_1$, and $\overline{h^*}=\varphi_1$. Then the rational map $g_2\cdot h\cdot g_1:X \dashrightarrow Y$ gives the modification φ. >

Remark 3.3. It is very easy to reformulate the conditions of the theorem 3.2 using discriminant forms:

$$\mathrm{rk}\ T_X + \ell(q_{(T_X)_p}) \le 6$$

for all odd prime p, and

$$|\mathfrak{A}_{T_X}| \equiv -\mathrm{discr}\ K(q_{(T_X)_p}) \bmod (\mathbb{Z}_p^*)^2$$

for all odd prime p for which $\mathrm{rk}\ T_X + \ell(q_{(T_X)_p}) = 6$;

$$\mathrm{rk}\ T_X + \ell(\tilde{q}_{(T_X)_2}) \le 6,$$

and

$$|\mathfrak{A}_{T_X}| \equiv \pm\ \mathrm{discr}\ K(\tilde{q}_{(T_X)_2})$$

if $\mathrm{rk}\ T_X + \ell(\tilde{q}_{(T_X)_2}) = 6$ and $\tilde{q}_{(T_X)_2} \not\cong q_\vartheta^{(2)}(2) \oplus q'$. (Here $\tilde{q}_{(T_X)_2}$ is the discriminant form of a maximal even overlattice of the lattice $T_X \otimes \mathbb{Z}_2$).

Remark 3.4. The condition of the Theorem 3.2 holds if $\mathrm{rk}\ T_X = \mathrm{rk}\ T_Y \le 3$. Thus, in this case the theorem 3.2 is true.

§ 4. Several remarks.

We want to give here several remarks about results above.

4.1. The Theorem 3.1 (or the Theorem 1.3 of the Introduction) is not true for $\mathrm{rk}\ T_X = 6$. If $(T_X)_2 = T_X \otimes \mathbb{Z}_2 \cong V^{(2)}(1)^3$, then the condition (ii) of the Theorem 2.2.5 does not hold. Thus, the surface X has not rational maps of the degree two into other K3 surfaces, and the Theorem 3.1 is not true for the surface X and any other K3 surface Y (for example for Y=X).

4.2. Let us remark that an every abstract modification $\varphi: T_1(d) \otimes \mathbb{Q} \longrightarrow T_2 \otimes \mathbb{Q}$ of the lattices defines the inverse modification $\varphi^{-1}: T_2(d) \otimes \mathbb{Q} \longrightarrow T_1 \otimes \mathbb{Q}$. Their composition (in the sense of 2.2.5) $\varphi^{-1} \cdot \varphi : T_1 \otimes \mathbb{Q} \longrightarrow T_1 \otimes \mathbb{Q}$ should be the identical map. Thus, a rational map f:X--->Y of surfaces gives also an inverse modification $\overline{f^*}^{-1}:T_X(d_1)\otimes\mathbb{Q} \longrightarrow T_Y\otimes\mathbb{Q}$.

For the rk T_X = rk T_Y = 6 we obtain the following variant of the Theorem 3.1: An every abstract modification $\varphi:T_X(d)\otimes\mathbb{Q} \longrightarrow T_Y\otimes\mathbb{Q}$ satisfying to the conditions of the theorem 3.1 is a composition of the modifications corresponding to rational maps of the degree two between K3 surfaces and of their inverse. The proof of the statement is similar to the proof of the theorem 3.1.

4.3. For the rk T_X=7 the statement above is not true. There are K3 surfaces with rk T_X=7 such that for the lattice T_X the condition (i) of the theorem 2.2.5 does not hold. This K3 surface has not symplectic involutions and has not rational maps of the degree two Z--->X of a K3 surface Z.

4.4. Results of the paper show that it is very important in the questions 1.1 and 1.2 constructing some examples of rational maps between K3 surfaces. Here we used rational maps of the degree two between K3 surfaces and rational maps between Kummer surfaces which are induced by the homomorphisms between Abelian surfaces. All other rational maps between K3 surfaces in this paper were compositions of these rational maps.

It would be very interesting to describe rational maps f:X--->Y of the degree 3 between K3 surfaces. If f is Galois map then f is defined by the action of the abelian symplectic group of the order 3 on the surface X, and all these actions and the corresponding quotient maps f are described in [N2]. In this case, rk T_X=rk T_Y $\le$ $\le$ 10, and these maps are very rare. But a description of the non-normal rational maps f of the degree 3 is unknown now.

We don't know examples of rational maps f:X--->Y of degree > 1 between general (with rk S_X = rk S_Y = 1) K3 surfaces X and Y.

References

[Kn] M. Kneser, Klassenzahlen indefiniter quadratischer Formen in drei oder merh Veranderlichen, Arch. Math. (Basel) **7** (1956), 323-332.

[Ku] Vik.S. Kulikov, Degenerations of K3 surfaces and Enriques surfaces, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 1008-1042 = Math. USSR Izvestiya. **11** (1977),957-989.

[Mo] D.R. Morrison, On K3 surfaces with large Picard number, Invent. Math. **75** (1984), 105-121.

[Mu] Sh. Mukai, On the moduli space of bundles on K3 surfaces, I, Proc. Symposium on Vector Bundles, Tata Institute (1984)

[N1] V.V. Nikulin, On Kummer surfaces, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), 278-293 = Math.USSR Izvestiya., **9** (1975), 261-275.

[N2] V.V. Nikulin, Finite groups of automorphisms of Kälerian surfaces of type K3, Trudy Mosk. Mat. Ob. **38** (1979), 75-137 = Trans. Moscow Math. Soc. **38** (1980), 71-135.

[N3] V.V. Nikulin, Integral symmetric bilinear forms and some of their geometrical applications, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), 111-177 = Math.USSR Izvestiya., **14** (1980), 103-167.

[N4] V.V.Nikulin, On correspondences between K3 surfaces. Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), 402-411 = Math. USSR Izvestiya., **30** (1988), 375-383.

[PSh-Sh] I.I. Piateckii-Shapiro and I.R. Shafarevich, A Torelli theorem for algebraic surfaces of type K3, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 530-572 = Math.USSR Izvestiya., **5**

(1971), 547-587.

[Se] J.-P. Serre, Cours d'arithmétique, Presses Univ. France, Paris, 1970.

[Sh] I.R. Shafarevich, Lé théorème de Torelli pour les surfaces algébriques de type K3, Proc. Internat. Congr. Math. (Nice, 1970), Vol. 1, Gautier-Villars, Paris, 1971, 413-417.

[Shi-I] T. Shioda and H. Inose, On singular K3 surfaces, Complex Analysis and Algebraic Geometry: Papers Dedicated to K.Kodaira, Iwanami Shoten, Tokyo and Cambridge University Press, Cambridge, 1977, 119-136.

Steklov Mathematical Institute,
ul. Vavilova 42,
Moscow 117966, GSP-1, USSR