ON EXACT SOLUTIONS OF THE COUPLED KLEIN-GORDON-SCHRÖDINGER AND THE COMPLEX COUPLED KDV EQUATIONS USING MAPPING METHOD

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Abstract. The mapping method is used by the aid of symbolic computation system Mathematica for constructing exact solutions to the coupled Klein-Gordon-Schrödinger equations and to the complex coupled KdV equations. The solutions obtained in this paper include Jacobi elliptic solutions, combined Jacobi elliptic solutions, rational solutions, triangular solutions, soliton solutions and combined soliton solutions.

1. Introduction

In the last past decades, directly seeking for exact solutions of nonlinear partial differential equations (NLPDEs) has become one of central theme of perpetual interest in Mathematical Physics. The existence of computer systems like Maple or Mathematica has greatly facilitated this task. There has been a great amount of activity aiming to find powerful methods for obtaining such solutions. We can cite, Bäcklund transformation [1], Darboux transformation [2], Jacobi elliptic function method [3], tanh-function method [4], sine-cosine function method [5,6], homogenous balance method [7], modified tanh-function method [8], Jacobi function expansion method [9,10]. Very recently, a unified method called the mapping method has been developed to obtain Jacobi elliptic functions, solitons and periodic functions to some NLPDEs [11-13]. Compared with the previous methods mentioned above, the mapping method gives more general solutions without much extra effort and without repetitive calculations. For example, in previous methods, Jacobi elliptic functions in terms of sn(ξ), cn(ξ), dn(ξ) may be obtained by applying sn-expansion, cn-expansion, dn-expansion. Thus many similar repetitive calculations have to be made. What is remarkable about the mapping method is that it allows one to find both Jacobi elliptic functions, triangular functions and soliton using the same and unique procedure. More importantly, the method permits the classification of the solutions depending on three parameters.

In this paper, we would like to apply the mapping method to the coupled Klein-Gordon-Schrödinger equations and to the complex coupled KdV equations to seek their explicit exact solutions. The key idea in this generalized indirect method, is to use the solutions of an auxiliary ordinary differential equation (the first kind Jacobi equation containing three parameters) to construct abundant families of Jacobi elliptic function solutions for the above mentioned equation. The new auxiliary ode reads [12]

$$(\frac{d\varphi}{d\xi})^2 = r + a\varphi^2(\xi) + b/(2)\varphi^4(\xi), \qquad (\mathbf{1})$$

where r, a and b are constants. We shall construct the exact solutions the coupled Klein-Gordon-Schrödinger equations and to the complex coupled KdV equations by using the following solutions of Eq.(1)

$$\varphi(\xi)=\begin{cases} sn\,\xi;\ cd\,\xi=cn\,\xi/dn\,\xi & : \ r=1,\ \ a=-(m^2+1),\ \ b=2m^2;\\ cn\,\xi & : \ r=1-m^2,\ \ a=2m^2-1,\ \ b=-2m^2;\\ dn\,\xi & : \ r=m^2-1,\ \ a=2-m^2,\ \ b=-2;\\ ns\,\xi=1/sn\,\xi;\ dc\,\xi=dn\,\xi/cn\,\xi & : \ r=m^2,\ \ a=-(m^2+1),\ \ b=2;\\ nc\,\xi=1/cn\,\xi & : \ r=-m^2,\ \ a=2m^2-1,\ \ b=2(1-m^2);\\ nd\,\xi=1/dn\,\xi & : \ r=-1,\ \ a=2-m^2,\ \ b=2(m^2-1);\\ cs\,\xi=cn\,\xi/sn\,\xi & : \ r=1-m^2,\ \ a=2-m^2,\ \ b=2;\\ sc\,\xi=sn\,\xi/cn\,\xi & : \ r=1,\ \ a=2-m^2,\ \ b=2(1-m^2);\\ sd\,\xi=sn\,\xi/dn\,\xi & : \ r=1,\ \ a=2m^2-1,\ \ b=2m^2(m^2-1);\\ ds\,\xi=dn\,\xi/sn\,\xi & : \ r=m^2(m^2-1),\ \ a=2m^2-1,\ \ b=2;\\ ns\,\xi\pm cs\,\xi & : \ r=1/4,\ a=(1-2m^2)/2,\ \ b=1/2;\\ nc\,\xi\pm sc\,\xi & : \ r=(1-m^2)/4,\ a=(1+m^2)/2,\ \ b=(1-m^2)/2;\\ ns\,\xi+ds\,\xi & : \ r=m^4/4,\ a=(m^2-2)/2,\ \ b=1/2;\\ sn\,\xi\pm i\ cn\,\xi & : \ r=m^2/4,\ \ a=(m^2-2)/2,\ \ b=m^2/2,\ \ i^2=-1;\\ C/\xi & : \ r=0,\ \ a=0\ \ b=2, \end{cases} \qquad (\mathbf{2})$$

where C is a constant. The Jacobi elliptic functions $sn\,\xi=sn\,(\xi|m)$, $cn\,\xi=cn\,(\xi|m)$, and $dn\,\xi=dn\,(\xi|m)$, where $m(0<m<1)$ is the modulus of elliptic function, are double periodic and possess properties of triangular functions, namely

$$sn^2\xi+cn^2\xi=1,\ \ dn^2\,\xi+m^2sn^2\,\xi=1,\ \ \frac{d}{d\xi}sn\,\xi=cn\,\xi dn\,\xi,$$

$$\frac{d}{d\xi}cn\,\xi=-sn\,\xi dn\,\xi,\quad \frac{d}{d\xi}dn\,\xi=-m^2cn\,\xi sn\,\xi. \qquad (\mathbf{3})$$

In addition, we see that other solutions are obtained from solutions (2) in case of degeneracies:
When m$\rightarrow$ 0, the Jacobi elliptic functions degenerate to the trigonometric functions solutions of the given NLPDE i.e.

$sn\,\xi\rightarrow\ \sin\,\xi,\ \ cn\,\xi\rightarrow\ \cos\,\xi,\ \ dn\,\xi\rightarrow\ 1.$ **(4)**

When m$\rightarrow$ 1, the Jacobi elliptic functions degenerate to the solitonic solutions of the given NLPDE i.e.

$sn\,\xi\rightarrow\ \tanh\,\xi,\ \ cn\,\xi\rightarrow\ sech\,\xi,\ \ dn\,\xi\rightarrow\ sech\,\xi.$ **(5)**

Detailed explanations about Jacobi elliptic functions can be found in Refs.[14,15]. The other 9 Jacobi functions are all expressible in terms of $sn\,\xi,\ cn\,\xi,\ dn\,\xi$.

2. New exact solutions of S-K-S equations

$$i\frac{\partial}{\partial t}\Psi + \Delta\Psi + \rho\Phi\Psi = 0,$$
$$\frac{\partial^2}{\partial t^2}\Phi - \Delta\Phi + \mu^2\Phi - \rho|\Psi|^2 = 0, \qquad (\mathbf{6})$$

which describes a system of conserved scalar nucleons interacting with neutral scalar mesons. Here, Ψ represents a complex scalar neucleon field and Φ a real scalar meson field. The real constant μ and ρ describe the mass of a meson and a couple constant respectively, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Fukuda and Tsutsumi have studied (6) using the Galerkin method [16] . Their global solutions, asymptotic behavior and stability have been done in [17-22]. Some exact solutions of these equations have been found very recently by Darwish and Fan [23]. We will present here many new solutions which have not been obtained in [23]. In order to obtain the travelling wave solutions of S-K-S equations, we make the following transformation [23]

$$\Psi = U(\xi)\ exp(i\eta), \quad \Phi = V(\xi), \qquad (\mathbf{7})$$

where $\xi = x + cy + dz + et, \quad \eta = px + qy + Rz + st$ and reduce the system (6) the following odes

$$(1 + c^2 + d^2)U'' - (s + p^2 + q^2 + R^2)U + \rho UV = 0,$$
$$(e^2 - 1 - c^2 - d^2)V'' + \mu^2 V - \rho U^2 = 0, \qquad (\mathbf{8})$$
$$e + 2(p + qc + dR) = 0.$$

The third equation of (29) in [23] should be replaced by $e + 2(p + qc + dr) = 0$. The solution of (8) can be expressed in the form

$$U(\xi) = \sum_{i=0}^{n} a_i\varphi^i(\xi), \quad V(\xi) = \sum_{i=0}^{l} b_i\varphi^i(\xi). \qquad (\mathbf{9})$$

n and l are positive integers that can be determined by balancing the highest order of dispersive terms U'' and V'' with UV and U^2 respectively. Thus we obtain n=l=2. Therefore we may choose the following ansatz

$$U(\xi) = a_0 + a_1\varphi(\xi) + a_2\varphi^2(\xi), \quad V(\xi) = b_0 + b_1\varphi(\xi) + b_2\varphi^2(\xi), \qquad (\mathbf{10})$$

where φ satisfies (1). Substituting (10) into (8) along with Eq.(1) and using Mathematica yields a system of equations of powers φ. Setting the coefficients of φ^i in the system of equations to zero, we can deduce the following set of algebraic polynomials with respect to unknowns $a_0,\ a_1,\ a_2,\ b_0\ b_1$ and b_2 namely:

$$\begin{gathered}
-(p^2+q^2+R^2+s)a_0+2r(1+c^2+d^2)a_2+\rho a_0b_0=0,\\
-(p^2+q^2+R^2+s)a_1+a(1+c^2+d^2)a_1+\rho(a_1b_0+a_0b_1)=0,\\
-(p^2+q^2+R^2+s)a_2+4a(1+c^2+d^2)a_2+\rho(a_2b_0+a_1b_1+a_0b_2)=0,\\
b(1+c^2+d^2)a_1+\rho(a_2b_1+a_1b_2)=0,\\
3b(1+c^2+d^2)a_2+\rho a_2b_2=0,\\
-2r(1+c^2+d^2-e^2)b_2+\mu^2b_0-\rho a_0^2=0,\\
-a(1+c^2+d^2-e^2)b_1+\mu^2b_1-2\rho a_0a_1=0,\\
-4a(1+c^2+d^2-e^2)b_2+\mu^2b_2-\rho(a_1^2+2a_0a_2)=0,\\
b(1+c^2+d^2-e^2)b_1+2\rho a_2a_1=0,\\
3b(1+c^2+d^2-e^2)b_2+\rho a_2^2=0.\\
e+2(p+qc+dR)=0.
\end{gathered}\tag{11}$$

We find the following a solution from the above mentioned system, namely
$a_1=0,\quad b_1=0,\quad a_0=\pm\frac{\sqrt{1+c^2+d^2}}{2\rho\sqrt{1+c^2+d^2-e^2}}[4a(1+c^2+d^2-e^2)-\mu^2],\quad b_0=\frac{(1+c^2+d^2)}{4\rho\mu^2(1+c^2+d^2-e^2)}[8(1+c^2+d^2-e^2)^2(2a^2-3br)-8a(1+c^2+d^2-e^2)\mu^2+\mu^4],\ \ a_2=\pm\,3b\frac{\sqrt{1+c^2+d^2}\sqrt{1+c^2+d^2-e^2}}{\rho},\quad b_2=\frac{-3b(1+c^2+d^2)}{\rho},\quad s=\frac{1}{\mu^2}[2(1+c^2+d^2-e^2)(2a^2-3br)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)},\quad R=-\frac{e+2p+2qc}{2d},$ **(12)**
where p, q, r, a, b are arbitrary constants, and c, d, e are also arbitrary constants but they satisfy the following relation
$(1+c^2+d^2-e^2)^2=\frac{\mu^4}{8(2a^2-3br)},\quad 1+c^2+d^2-e^2\ >0,$ **(13)**
Since $0<m<1$, we should mentioned that the Jacobi elliptic function and the combined Jacobi elliptic function in (2) are such that $2a^2-3br\ >0$.
Hence, the solution of Eq.(6) reads
$\Psi=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\left[\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4a(1+c^2+d^2-e^2)-\mu^2]+3b\sqrt{1+c^2+d^2-e^2}\ \varphi^2(\xi)\right]\ exp(i\eta)$
$\Phi=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[8(1+c^2+d^2-e^2)^2(2a^2-3br)-8a(1+c^2+d^2-e^2)\mu^2+\mu^4]-3b\ \varphi^2(\xi)\},$ **(14.a)**
where
$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[2(1+c^2+d^2-e^2)(2a^2-3br)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{8(2a^2-3br)},\quad 1+c^2+d^2-e^2\ >0.$ **(14.b)**

Thanks to the arbitrariness of r, a and b, by choosing them properly into the ode (1) such that the corresponding solution $\varphi(\xi)$ of this equation is one of Jacobi elliptic functions, or combined Jacobi elliptic functions, or rational functions in relation (2). We may have multiple kinds of solutions of Eq.(6). In the following , we consider several cases

Case 1. r=1, a=-$(1+m^2)$, $b=2m^2$

$\Psi_1=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(m^2+1)(1+c^2+d^2-e^2)-\mu^2]+$
$6m^2\sqrt{1+c^2+d^2-e^2}\ sn^2(\xi)\}\ exp(i\eta)$
$\Phi_1=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)+8(m^2+1)(1+c^2+d^2-e^2)\mu^2+$
$\mu^4]-6m^2\ sn^2(\xi)\},$ **(15.a)**
$\Psi_2=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(m^2+1)(1+c^2+d^2-e^2)-\mu^2]+6m^2\sqrt{1+c^2+d^2-e^2}\ cd^2(\xi)\}\ exp(i\eta)$
$\Phi_2=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)+8(m^2+1)(1+c^2+d^2-e^2)\mu^2+$
$\mu^4]-6m^2\ cd^2(\xi)\},$ **(15.b)**
where
$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+$
$1))-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2>$
0 **(15.c)**

Case 2. $r=1-m^2\ \ a=2m^2-1\ \ b=-2m^2$
$\Psi_3=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2m^2-1)(1+c^2+d^2-e^2)-\mu^2]-6m^2\sqrt{1+c^2+d^2-e^2}\ cn^2(\xi)\}exp(i\eta)$
$\Phi_3=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2m^2-1)(1+c^2+d^2-$
$e^2)\mu^2+\mu^4]+6m^2\ cn^2(\xi)\},$ **(16.a)**
where
$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+$
$1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2>$
$0.$ **(16.b)**

Case 3. $r=m^2-1\ \ a=2-m^2\ \ b=-2$
$\Psi_4=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2-m^2)(1+c^2+d^2-e^2)-\mu^2]-6\sqrt{1+c^2+d^2-e^2}\ dn^2(\xi)\}\ exp(i\eta)$
$\Phi_4=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2-m^2)(1+c^2+d^2-$
$e^2)\mu^2+\mu^4]+6\ dn^2(\xi)\},$ **(17.a)**
where
$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+$
$1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2>$
$0.$ **(17.b)**

Case 4. $r = m^2 \;\; a = -m^2 - 1 \;\; b = 2$

$\Psi_5 = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(m^2+1)(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\; ns^2(\xi)\}\; exp(i\eta)$

$\Phi_5 = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)+8(m^2+1)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\; ns^2(\xi)\},$ **(18.a)**

$\Psi_6 = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(m^2+1)(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\; dc^2(\xi)\}\; exp(i\eta)$

$\Phi_6 = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)+8(m^2+1)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\; dc^2(\xi)\},$ **(18.b)**

where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[4(1 + c^2 + d^2 - e^2)(m^4 - m^2 + 1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{16(m^4-m^2+1)}, \quad 1+c^2+d^2-e^2 > 0.$ **(18.c)**

Case 5. $r = -m^2 \;\; a = 2m^2 - 1 \;\; b = 2 - 2m^2$

$\Psi_7 = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2m^2 - 1)(1 + c^2 + d^2 - e^2) - \mu^2] + 6(1 - m^2)\sqrt{1 + c^2 + d^2 - e^2}\; nc^2(\xi)\}\; exp(i\eta)$

$\Phi_7 = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2m^2-1)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6(1-m^2)\; nc^2(\xi)\},$ **(19.a)**

where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[4(1 + c^2 + d^2 - e^2)(m^4 - m^2 + 1) - (p^2 + q^2 + R^2)\mu^2] + \frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1 + c^2 + d^2 - e^2)^2 = \frac{\mu^4}{16(m^4-m^2+1)}, \quad 1 + c^2 + d^2 - e^2 > 0.$ **(19.b)**

Case 6. $r = -1 \;\; a = 2 - m^2 \;\; b = 2m^2 - 2$

$\Psi_8 = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2 - m^2)(1 + c^2 + d^2 - e^2) - \mu^2] - 6(1 - m^2)\sqrt{1 + c^2 + d^2 - e^2}\; nd^2(\xi)\}\; exp(i\eta)$

$\Phi_8 = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1))-8(2-m^2)(1+c^2+d^2-e^2)\mu^2+\mu^4]+6(1-m^2)\; nd^2(\xi)\},$ **(20.a)**

where

$\xi = x+cy+dz+et, \quad \eta = px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{16(m^4-m^2+1)}, \quad 1+c^2+d^2-e^2 > 0.$ **(20.b)**

Case 7. $r = 1 - m^2 \;\; a = 2 - m^2 \;\; b = 2$

$\Psi_9 = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2-m^2)(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\; cs^2(\xi)\}\; exp(i\eta)$

$\Phi_9 = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2-m^2)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\; cs^2(\xi)\},$ **(21.a)**

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[4(1 + c^2 + d^2 - e^2)(m^4 - m^2 + 1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{16(m^4-m^2+1)}, \quad 1+c^2+d^2-e^2 >$

$0.$ (**21**.b)

Case 8. $r=1 \;\; a=2-m^2 \;\; b=2-2m^2$

$\Psi_{10}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2-m^2)(1+c^2+d^2-e^2)-\mu^2]+ 6(1-m^2)\sqrt{1+c^2+d^2-e^2}\; sc^2(\xi)\}\; exp(i\eta)$

$\Phi_{10}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2-m^2)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6(1-m^2)\; sc^2(\xi)\},$ (**22**.a)

$\xi=x+cy+dz+et,\;\; \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2\;>0.$ (**22**.b)

Case 9. $r=1 \;\; a=2m^2-1 \;\; b=2m^4-2m^2$

$\Psi_{11}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2m^2-1)(1+c^2+d^2-e^2)-\mu^2]- 6m^2(1-m^2)\sqrt{1+c^2+d^2-e^2}\; sd^2(\xi)\}\; exp(i\eta)$

$\Phi_{11}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2m^2-1)(1+c^2+d^2-e^2)\mu^2+\mu^4]+6m^2(1-m^2)\; sd^2(\xi)\},$ (**23**.a)

$\xi=x+cy+dz+et,\;\; \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2\;>0.$ (**23**.b)

Case 10. $r=m^4-m^2 \;\; a=2m^2-1 \;\; b=2$

$\Psi_{12}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(2m^2-1)(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\; ds^2(\xi)\}\; exp(i\eta)$

$\Phi_{12}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2(m^4-m^2+1)-8(2m^2-1)(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\; ds^2(\xi)\},$ (**24**.a)

$\xi=x+cy+dz+et,\;\; \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)(m^4-m^2+1)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16(m^4-m^2+1)},\quad 1+c^2+d^2-e^2\;>0.$ (**24**.b)

Case 11. $r=\frac{1}{4} \;\; a=\frac{1-2m^2}{2} \;\; b=\frac{1}{2}$

$\Psi_{13}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(1-2m^2)(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[ns(\xi)\pm cs(\xi)]^2\}\; exp(i\eta)$

$\Phi_{13}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2(16m^4-16m^2+1)-4(1-2m^2)(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3}{2}\;[ns(\xi)\pm cs(\xi)^2]^2\},$ (**25**.a)

where

$\xi=x+cy+dz+et,\;\; \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)(16m^4-16m^2+1)}{4}-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{(16m^4-16m^2+1)},\quad 1+c^2+d^2-e^2\;>0.$ (**25**.b)

Case 12. $r=\frac{1-m^2}{4} \;\; a=\frac{1+m^2}{2} \;\; b=\frac{1-m^2}{2}$

$\Psi_{14}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(1+m^2)(1+c^2+d^2-e^2)-\mu^2]+\frac{3(1-m^2)}{2}\sqrt{1+c^2+d^2-e^2}[nc(\xi)\pm sc(\xi)]^2\}\; exp(i\eta)$

$\Phi_{14}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2(m^4+14m^2+1)-4(1+m^2)(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3(1-m^2)}{2}\;[nc(\xi)\pm sc(\xi)]^2\},$ (**26**.a) where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)(m^4+14m^2+1)}{4} - (p^2 + q^2 + R^2)\mu^2] + \frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{(m^4+14m^2+1)}, \quad 1+c^2+d^2-e^2 > 0.$ (**26**.b)

Case 13. $r = \frac{m^4}{4} \quad a = \frac{m^2-2}{2} \quad b = \frac{1}{2}$

$\Psi_{15} = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(m^2-2)(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[ns(\xi)+ds(\xi)]^2\}\ exp(i\eta)$

$\Phi_{15} = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2(m^4-16m^2+16)-4(m^2-2)(1+c^2+d^2-e^2)\mu^2+\mu^4] - \frac{3}{2}\ [ns(\xi)+ds(\xi)]^2\},$ (**27**.a)

where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)(m^4-16m^2+16)}{4} - (p^2 + q^2 + R^2)\mu^2] + \frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{(m^4-16m^2+16)}, \quad 1+c^2+d^2-e^2 > 0.$ (**27**.b)

Case 14 $r = \frac{m^2}{4} \quad a = \frac{m^2-2}{2} \quad b = \frac{m^2}{2}$

$\Psi_{16} = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(m^2-2)(1+c^2+d^2-e^2)-\mu^2]+\frac{3m^2}{2}\sqrt{1+c^2+d^2-e^2}[sn(\xi)\pm i\ cn(\xi)]^2\}\ exp(i\eta)$

$\Phi_{16} = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2(m^4-16m^2+16)-4(m^2-2)(1+c^2+d^2-e^2)\mu^2+\mu^4] - \frac{3m^2}{2}\ [sn(\xi)\pm i\ cn(\xi)]^2\},$ (**28**.a)

where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)(m^4-16m^2+16)}{4} - (p^2 + q^2 + R^2)\mu^2] + \frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{(m^4-16m^2+16)}, \quad 1+c^2+d^2-e^2 > 0.$ (**28**.b)

Case 15 $r = 0 \quad a = 0 \quad b = 2$

There is no rational solution for this case.

Remark 1

When m$\rightarrow$ 0, these solutions (13)-(26) degenerate as triangular function solutions of Eq.(6), namely

Case 16. $r = 0 \quad a = -1 \quad b = 2$

$\Psi_{17} = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\ csc^2(\xi)\}\ exp(i\eta)$

$\Phi_{17} = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2 + 8(1+c^2+d^2-e^2)\mu^2 + \mu^4] - 6\ csc^2(\xi)\},$ (**29**.a)

$\Psi_{18} = \pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-4(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\ sec^2(\xi)\}\ exp(i\eta)$

$\Phi_{18} = \frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2 + 8(1+c^2+d^2-e^2)\mu^2 + \mu^4] - 6\ sec^2(\xi)\},$ (**29**.b)

where

$\xi = x + cy + dz + et, \quad \eta = px + qy - \frac{e+2p+2qc}{2d}z + \{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2) - (p^2 + q^2 + R^2)\mu^2] + \frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t, \quad (1+c^2+d^2-e^2)^2 = \frac{\mu^4}{16}, \quad 1+c^2+d^2-e^2 > 0.$ (**29**.c)

Case 17. $r=1\ \ a=2\ \ b=2$

$\Psi_{19}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[8(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\cot^2(\xi)\}\ exp(i\eta)$

$\Phi_{19}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2-16(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\cot^2(\xi)\},$ (**30**.**a**)

$\Psi_{20}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[8(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\tan^2(\xi)\}\ exp(i\eta)$

$\Phi_{20}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2-16(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\tan^2(\xi)\},$ (**30**.**b**)

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16},\quad 1+c^2+d^2-e^2\ >0.$ (**30**.**c**)

Case 18. $r=\frac{1}{4}\ \ a=\frac{1}{2}\ \ b=\frac{1}{2}$

$\Psi_{21}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[csc(\xi)\pm\cot(\xi)]^2\}\ exp(i\eta)$

$\Phi_{21}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2-4(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3}{2}\ [csc(\xi)\pm\cot(\xi)]^2\},$ (**31**.**a**)

$\Psi_{22}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[2(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[sec(\xi)\pm\tan(\xi)]^2\}\ exp(i\eta)$

$\Phi_{22}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2-4(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3}{2}\ [sec(\xi)\pm\tan(\xi)]^2\},$ (**31**.**b**)

where

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)}{4}-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\mu^4,\quad 1+c^2+d^2-e^2\ >0.$ (**31**.**c**)

Remark 2

When m→ 1, these solutions (15)-(28) degenerate as soliton solutions of Eqs.(6), namely

Case 19. r=1, a=-2, $b=2$

$\Psi_{23}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-8(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\tanh^2(\xi)\}\ exp(i\eta)$

$\Phi_{23}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2+16(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\tanh^2(\xi)\},$ (**32**.**a**)

where

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16},\quad 1+c^2+d^2-e^2\ >0.$ (**32**.**b**)

Case 20. $r=0\ \ a=1\ \ b=-2$

$\Psi_{24}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(1+c^2+d^2-e^2)-\mu^2]-6\sqrt{1+c^2+d^2-e^2}\ sech^2(\xi)\}exp(i\eta)$

$\Phi_{24}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2-8(1+c^2+d^2-e^2)\mu^2+\mu^4]+6\ sech^2(\xi)\},$ **(33.a)**

where

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16},\quad 1+c^2+d^2-e^2>0.$ **(33.b)**

Case 21. $r=1\ \ a=-2\ \ b=2$

$\Psi_{25}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-8(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\coth^2(\xi)\}\ exp(i\eta)$

$\Phi_{25}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2+16(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\coth^2(\xi)\},$ **(34.a)**

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16},\quad 1+c^2+d^2-e^2>0.$ **(34.b)**

Case 22. $r=0\ \ a=1\ \ b=2$

$\Psi_{26}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[4(1+c^2+d^2-e^2)-\mu^2]+6\sqrt{1+c^2+d^2-e^2}\ csch^2(\xi)\}\ exp(i\eta)$

$\Phi_{26}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[16(1+c^2+d^2-e^2)^2-8(1+c^2+d^2-e^2)\mu^2+\mu^4]-6\ csch^2(\xi)\},$ **(35.a)**

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[4(1+c^2+d^2-e^2)-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\frac{\mu^4}{16},\quad 1+c^2+d^2-e^2>0.$ **(35.b)**

Case 23. $r=\frac{1}{4}\ \ a=-\frac{1}{2}\ \ b=\frac{1}{2}$

$\Psi_{27}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-2(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[\coth(\xi)\pm csch(\xi)]^2\}\ exp(i\eta)$

$\Phi_{27}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2+4(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3}{2}\ [\coth(\xi)\pm csch(\xi)]^2\},$ **(36.a)**

$\Psi_{28}=\pm\frac{\sqrt{1+c^2+d^2}}{\rho}\{\frac{1}{2\sqrt{1+c^2+d^2-e^2}}[-2(1+c^2+d^2-e^2)-\mu^2]+\frac{3}{2}\sqrt{1+c^2+d^2-e^2}[\tanh(\xi)\pm i\ sech(\xi)]^2\}\ exp(i\eta)$

$\Phi_{28}=\frac{1+c^2+d^2}{\rho}\{\frac{1}{4\mu^2(1+c^2+d^2-e^2)}[(1+c^2+d^2-e^2)^2+4(1+c^2+d^2-e^2)\mu^2+\mu^4]-\frac{3}{2}\ [\tanh(\xi)\pm i\ sech(\xi)]^2\},$ **(36.b)**

where

$\xi=x+cy+dz+et,\quad \eta=px+qy-\frac{e+2p+2qc}{2d}z+\{\frac{1}{\mu^2}[\frac{(1+c^2+d^2-e^2)}{4}-(p^2+q^2+R^2)\mu^2]+\frac{3(1+c^2+d^2)\mu^2}{4(1+c^2+d^2-e^2)}\}t,\quad (1+c^2+d^2-e^2)^2=\mu^4,\quad 1+c^2+d^2-e^2>0.$ **(36.c)**

All the above solutions are quite new and different from the ones reported in [21]

3. Exact solutions of complex coupled KdV equations

The complex coupled KdV equations read

$$\begin{aligned} u_t &= \frac{1}{2}u_{xxx} - 3uu_x + 3(|v|^2)_x, \\ v_t &= -v_{xxx} + 3uv_x. \end{aligned} \tag{37}$$

These equations were obtained, when Wu et al [24] considered a $4\times\ 4$ matrix spectral problem with three potentials. (37) can degenerate to the well known KdV equation and the Hirota-Satsuma coupled equation when v=0 and Im(v)=0 respectively [25,26]. some exact solutions of system (37) were obtained by Fan and Chao [27] and by Hon and Fan [28]. We will apply the mapping method for constructing new exact solutions to the system (37). After making the transformation $u(x,t) = U(\xi),\ \ v(x,t) = e^{i\theta}V(\xi)$, where $\xi = x+ct,\ \ \theta = px+qt+R$, system (37) becomes the following system of ordinary differential equations

$$\begin{aligned} &cU' - \frac{1}{2}U''' + 3UU' - 3(V^2)' = 0, \\ &cV' + V''' - 3p^2V' - 3UV' = 0, \\ &(q-p^3)V + 3pV'' - 3pUV = 0. \end{aligned} \tag{38}$$

The third equation of system (16) in [28] should be replaced by $(q-p^3)V+3pV''-3pUV = 0$. The balance procedure gives n=l=2. Then we assume the following form of solutions

$$U(\xi) = a_0+a_1\varphi(\xi)+a_2\varphi^2(\xi), \quad V(\xi) = b_0+b_1\varphi(\xi)+b_2\varphi^2(\xi), \tag{39}$$

where φ satisfies (1). Substituting (39) into (38) along with Eq.(1) and using Mathematica yields a system of equations of powers φ. Setting the coefficients of φ^i in the system of equations to zero, we can deduce the following set of algebraic polynomials with respect to unknowns $a_0,\ a_1,\ a_2,\ b_0\ b_1\ and\ b_2$ namely:

$$\begin{aligned} &b_1(c-3p^2+a-3a_0) = 0, \\ &-3a_1b_1 + 2b_2(c-3p^2+4a-3a_0) = 0, \\ &3(bb_1 - a_2b_1 - 2a_1b_2) = 0. \\ &6b_2(2b-a_2) = 0, \\ &(-2c+a-6a_0)a_1 + 12b_0b_1 = 0. \\ &2(-3a_1^2 - 2ca_2 + 4aa_2 - 6a_0a_2 + 6b_1^2 + 12b_0b_2) = 0, \\ &3(ba_1 - 6a_1a_2 + 12b_1b_2) = 0, \\ &12(ba_2 - a_2^2 + 2b_2^2) = 0, \\ &b_0(q-p^3-3pa_0) + 6prb_2 = 0, \end{aligned} \tag{40}$$

$$-3pa_1b_0+b_1(q-p^3+3ap-3pa_0)=0,$$
$$-3p(a_2b_0+a_1b_1)+b_2(q-p^3+12ap-3pa_0),$$
$$3p(bb_1-a_2b_1-a_1b_2)=0,$$
$$3pb_2(3b-a_2)=0.$$

Solvng these equations, two kinds of solutions have been found

$a_1=0,\quad b_1=0,\quad p=0,\quad q=0,\quad a_0=(4a+c)/3,\quad b_0=\pm 2(a+c)/3,\quad a_2=2b,\quad b_2=\pm b,$ **(41.a)**

and

$a_0=(a+c-3p^2)/3,\quad b_0=0,\quad a_1=0,\quad b_1=\pm\sqrt{b(2c-a-3p^2)}/\sqrt{3},\quad a_2=b,\quad b_2=0,\quad q=p(c{-}2a{-}2p^2).$ **(41.b)**

Therefore the solution of (37) is given by

$$u=\frac{4a+c}{3}+2b\varphi^2(x+ct),$$
$$v=\pm[\frac{2}{3}(a+c)+b\varphi^2(x+ct)]e^{iR}, \tag{42.1}$$

and

$$u'=\frac{a+c-3p^2}{3}+b\varphi^2(x+ct),$$
$$v'=\pm\sqrt{\frac{b(2c-a-3p^2)}{3}}\varphi(x+ct)e^{i[px-p(2p^2+2a-c)t+R]}, \tag{42.2}$$

where p, c, R, a, b and r are arbitrary constants which should be chosen such that $b(2c-a-3p^2)\ >0$. As r, a and b are arbitrary constants, we may choose appropriate values for these parameters in order to obtain Jacobi elliptic function, combined Jacobi elliptic function and rational function solutions from relation (2). Therefore according to (2) abundant families of explicit and exact travelling wave solutions are found as follows for the complex coupled KdV equation (37).

Case .1 $r=1\ \ a=-(1+m^2)\ \ b=2m^2$

$$u_1=\frac{-4(1+m^2)+c}{3}+4m^2\ sn^2(x+ct),$$
$$v_1=\pm\left[\frac{2}{3}[-(1+m^2)+c]+2m^2\ sn^2(x+ct)\right]e^{iR}, \tag{43.1}$$

and

$$u_1'=\frac{-(1+m^2)+c-3p^2}{3}+2m^2\ sn^2(x+ct),$$
$$v_1'=\pm\sqrt{\frac{2m^2[2c+(1+m^2)-3p^2]}{3}}\ sn(x+ct)e^{i[px-p(2p^2-2(1+m^2)-c)t+R]}, \tag{43.2}$$

$$u_2 = \frac{-4(1+m^2)+c}{3} + 4m^2\ cd^2(x+ct),$$

$$v_2 = \pm\left[\frac{2}{3}[-(1+m^2)+c] + 2m^2\ cd^2(x+ct)\right] e^{iR}, \qquad (\mathbf{44.1})$$

and

$$u_2' = \frac{-(1+m^2)+c-3p^2}{3} + 2m^2\ cd^2(x+ct),$$

$$v_2' = \pm\sqrt{\frac{2m^2[2c+(1+m^2)-3p^2]}{3}}\ cd(x+ct)e^{i[px-p(2p^2-2(1+m^2)-c)t+R]}, \qquad (\mathbf{44.2})$$

Case 2. $r = 1-m^2 \ \ a = (2m^2-1) \ \ b = -2m^2$

$$u_3 = \frac{4(2m^2-1)+c}{3} - 4m^2\ cn^2(x+ct),$$

$$v_3 = \pm\left[\frac{2}{3}((2m^2-1)+c) - 2m^2\ cn^2(x+ct)\right] e^{iR}, \qquad (\mathbf{45.1})$$

and

$$u_3' = \frac{(2m^2-1)+c-3p^2}{3} - 2m^2\ cn^2(x+ct),$$

$$v_3' = \pm\sqrt{\frac{-2m^2[2c-(2m^2-1)-3p^2]}{3}}\ cn(x+ct)e^{i[px-p(2p^2+2(2m^2-1)-c)t+R]}, \qquad (\mathbf{45.2})$$

Case 3. $r = m^2-1 \ \ a = (2-m^2) \ \ b = -2$

$$u_4 = \frac{4(2-m^2)+c}{3} - 4\ dn^2(x+ct),$$

$$v_4 = \pm\left[\frac{2}{3}((2-m^2)+c) - 2\ dn^2(x+ct)\right] e^{iR}, \qquad (\mathbf{46.1})$$

and

$$u_4' = \frac{(2-m^2)+c-3p^2}{3} - 2\ dn^2(x+ct),$$

$$v_4' = \pm\sqrt{\frac{-2[2c-(2-m^2)-3p^2]}{3}}\ dn(x+ct)e^{i[px-p(2p^2+2(2-m^2)-c)t+R]}, \qquad (\mathbf{46.2})$$

Case 4. $r = m^2 \ \ a = -(1+m^2) \ \ b = 2$

$$u_5 = \frac{-4(1+m^2)+c}{3} + 4\ ns^2(x+ct),$$

$$v_5 = \pm\left[\frac{2}{3}(-(1+m^2)+c) + 2\ ns^2(x+ct)\right] e^{iR}, \qquad (\mathbf{47.1})$$

and

$$u_5' = \frac{-(1+m^2)+c-3p^2}{3} + 2\ ns^2(x+ct),$$

$$v_5' = \pm\sqrt{\frac{2[2c+(1+m^2)-3p^2]}{3}}\ ns(x+ct)e^{i[px-p(2p^2-2(1+m^2)-c)t+R]}, \qquad (\mathbf{47.2})$$

$$u_6 = \frac{-4(1+m^2)+c}{3} + 4\, dc^2(x+ct),$$
$$v_6 = \pm\left[\frac{2}{3}(-(1+m^2)+c) + 2\, dc^2(x+ct)\right] e^{iR}, \qquad (\mathbf{48.1})$$

and

$$u_6' = \frac{-(1+m^2)+c-3p^2}{3} + 2\, dc^2(x+ct),$$
$$v_6' = \pm\sqrt{\frac{2[2c+(1+m^2)-3p^2]}{3}}\, dc(x+ct)e^{i[px-p(2p^2-2(1+m^2)-c)t+R]}, \qquad (\mathbf{48.2})$$

Case 5. $r=-m^2 \;\; a=(2m^2-1) \;\; b=2(1-m^2)$

$$u_7 = \frac{4(2m^2-1)+c}{3} + 4(1-m^2)\, nc^2(x+ct),$$
$$v_7 = \pm\left[\frac{2}{3}((2m^2-1)+c) + 2(1-m^2)\, nc^2(x+ct)\right] e^{iR}, \qquad (\mathbf{49.1})$$

and

$$u_7' = \frac{(2m^2-1)+c-3p^2}{3} + 2(1-m^2)\, nc^2(x+ct),$$
$$v_7' = \pm\sqrt{\frac{2(1-m^2)[2c-(2m^2-1)-3p^2]}{3}}\, nc(x+ct)e^{i[px-p(2p^2+2(2m^2-1)-c)t+R]}, \qquad (\mathbf{49.2})$$

Case 6. $r=-1 \;\; a=(2-m^2) \;\; b=2(m^2-1)$

$$u_8 = \frac{4(2-m^2)+c}{3} + 4(m^2-1)\, nd^2(x+ct),$$
$$v_8 = \pm\left[\frac{2}{3}((2-m^2)+c) + 2(m^2-1)\, nd^2(x+ct)\right] e^{iR}, \qquad (\mathbf{50.1})$$

and

$$u_8' = \frac{(2-m^2)+c-3p^2}{3} + 2(m^2-1)\, nd^2(x+ct),$$
$$v_8' = \pm\sqrt{\frac{2(m^2-1)[2c-(2-m^2)-3p^2]}{3}}\, nd(x+ct)e^{i[px-p(2p^2+2(2-m^2)-c)t+R]}, \qquad (\mathbf{50.2})$$

Case 7. $r=1-m^2 \;\; a=(2-m^2) \;\; b=2$

$$u_9 = \frac{4(2-m^2)+c}{3} + 4\, cs^2(x+ct),$$
$$v_9 = \pm\left[\frac{2}{3}((2-m^2)+c) + 2\, cs^2(x+ct)\right] e^{iR}, \qquad (\mathbf{51.1})$$

and

$$u_9' = \frac{(2-m^2)+c-3p^2}{3} + 2\, cs^2(x+ct),$$
$$v_9' = \pm\sqrt{\frac{2[2c-(2-m^2)-3p^2]}{3}}\, cs(x+ct)e^{i[px-p(2p^2+2(2-m^2)-c)t+R]}, \qquad (\mathbf{51.2})$$

Case 8. $r = 1 \;\; a = (2 - m^2) \;\; b = 2(1 - m^2)$

$$u_{10} = \frac{4(2 - m^2) + c}{3} + 4(1 - m^2)\; sc^2(x + ct),$$
$$v_{10} = \pm\left[\frac{2}{3}((2 - m^2) + c) + 2(1 - m^2)\; sc^2(x + ct)\right] e^{iR}, \qquad (\mathbf{52.1})$$

and

$$u'_{10} = \frac{(2 - m^2) + c - 3p^2}{3} + 2(1 - m^2)\; sc^2(x + ct),$$
$$v'_{10} = \pm\sqrt{\frac{2(1 - m^2)[2c - (2 - m^2) - 3p^2]}{3}}\; sc(x + ct) e^{i[px - p(2p^2 + 2(2 - m^2) - c)t + R]}, \qquad (\mathbf{52.2})$$

Case 9. $r = 1 \;\; a = (2m^2 - 1) \;\; b = 2m^2(m^2 - 1)$

$$u_{11} = \frac{4(2m^2 - 1) + c}{3} + 4m^2(m^2 - 1)\; sd^2(x + ct),$$
$$v_{11} = \pm\left[\frac{2}{3}((2m^2 - 1) + c) + 2m^2(m^2 - 1)\; sd^2(x + ct)\right] e^{iR}, \qquad (\mathbf{53.1})$$

and

$$u'_{11} = \frac{(2m^2 - 1) + c - 3p^2}{3} + 2m^2(m^2 - 1)\; sd^2(x + ct),$$
$$v'_{11} = \pm\sqrt{\frac{2m^2(m^2 - 1)[2c - (2m^2 - 1) - 3p^2]}{3}}\; sd(x + ct) e^{i[px - p(2p^2 + 2(2m^2 - 1) - c)t + R]}, \qquad (\mathbf{53.2})$$

Case 10. $r = m^2(m^2 - 1) \;\; a = (2m^2 - 1) \;\; b = 2$

$$u_{12} = \frac{4(2m^2 - 1) + c}{3} + 4\; ds^2(x + ct),$$
$$v_{12} = \pm\left[\frac{2}{3}((2m^2 - 1) + c) + 2\; ds^2(x + ct)\right] e^{iR}, \qquad (\mathbf{54.1})$$

and

$$u'_{12} = \frac{(2m^2 - 1) + c - 3p^2}{3} + 2\; ds^2(x + ct),$$
$$v'_{12} = \pm\sqrt{\frac{2[2c - (2m^2 - 1) - 3p^2]}{3}}\; ds(x + ct) e^{i[px - p(2p^2 + 2(2m^2 - 1) - c)t + R]}, \qquad (\mathbf{54.2})$$

Case 11. $r = 1/4 \;\; a = (1/2 - m^2) \;\; b = 1/2$

$$u_{13} = \frac{4(1/2 - m^2) + c}{3} + [ns(x + ct) \pm cs(x + ct)]^2,$$
$$v_{13} = \pm\left[\frac{2}{3}((1/2 - m^2) + c) + \frac{1}{2}[ns(x + ct) \pm cs(x + ct)]^2\right] e^{iR}, \qquad (\mathbf{55.1})$$

and

$$u'_{13} = \frac{(1/2 - m^2) + c - 3p^2}{3} + \frac{1}{2}[ns(x+ct) \pm cs(x+ct)]^2,$$

$$v'_{13} = \pm\sqrt{\frac{[2c - (1/2 - m^2) - 3p^2]}{6}}[ns(x+ct) \pm cs(x+ct)]e^{i[px - p(2p^2 + 2(1/2 - m^2) - c)t + R]}, \quad \textbf{(55.2)}$$

Case 12. $r = (1-m^2)/4 \;\; a = (1+m^2)/2 \;\; b = (1-m^2)/2$

$$u_{14} = \frac{2(1+m^2) + c}{3} + (1-m^2)[nc(x+ct) \pm sc(x+ct)]^2,$$

$$v_{14} = \pm\left[\frac{2}{3}((1+m^2)/2 + c) + \frac{1-m^2}{2}[nc(x+ct) \pm sc(x+ct)]^2\right] e^{iR}, \quad \textbf{(56.1)}$$

and

$$u'_{14} = \frac{(1+m^2)/2 + c - 3p^2}{3} + \frac{1-m^2}{2}[nc(x+ct) \pm sc(x+ct)]^2,$$

$$v'_{14} = \pm\sqrt{\frac{(1-m^2)[2c - (1+m^2)/2 - 3p^2]}{6}}[nc(x+ct) \pm sc(x+ct)]e^{i[px - p(2p^2 + (1+m^2) - c)t + R]}, \quad \textbf{(56.2)}$$

Case 13. $r = m^4/4 \;\; a = (m^2-2)/2 \;\; b = 1/2$

$$u_{15} = \frac{2(m^2-2) + c}{3} + [ns(x+ct) + ds(x+ct)]^2,$$

$$v_{15} = \pm\left[\frac{2}{3}((m^2-2)/2 + c) + \frac{1}{2}[ns(x+ct) + ds(x+ct)]^2\right] e^{iR}, \quad \textbf{(57.1)}$$

and

$$u'_{15} = \frac{(m^2-2)/2 + c - 3p^2}{3} + \frac{1}{2}[ns(x+ct) + ds(x+ct)]^2,$$

$$v'_{15} = \pm\sqrt{\frac{[2c - (m^2-2)/2 - 3p^2]}{6}}[ns(x+ct) + ds(x+ct)]e^{i[px - p(2p^2 + (m^2-2) - c)t + R]}, \quad \textbf{(57.2)}$$

Case 14. $r = m^2/4 \;\; a = (m^2-2)/2 \;\; b = m^2/2$

$$u_{16} = \frac{2(m^2-2) + c}{3} + m^2[sn(x+ct) \pm i\; cn(x+ct)]^2,$$

$$v_{16} = \pm\left[\frac{2}{3}((m^2-2)/2 + c) + \frac{m^2}{2}[sn(x+ct) \pm i\; cn(x+ct)]^2\right] e^{iR}, \quad \textbf{(58.1)}$$

and

$$u'_{16} = \frac{(m^2-2)/2 + c - 3p^2}{3} + \frac{m^2}{2}[sn(x+ct) \pm i\; cn(x+ct)]^2,$$

$$v'_{16} = \pm\sqrt{\frac{m^2[2c - (m^2-2)/2 - 3p^2]}{6}}[sn(x+ct) \pm i\; cn(x+ct)]e^{i[px - p(2p^2 + (m^2-2) - c)t + R]}, \quad \textbf{(58.2)}$$

Case 15. $r = 0 \;\; a = 0 \;\; b = 2$

$$u_{17} = \frac{c}{3} + \frac{4C^2}{(x+ct)^2},$$
$$v_{17} = \pm \left[\frac{2}{3}c + 2\frac{C^2}{(x+ct)^2}\right] e^{iR}, \qquad (\mathbf{59.1})$$

and

$$u'_{17} = \frac{c - 3p^2}{3} + 2\frac{C^2}{(x+ct)^2},$$
$$v'_{17} = \pm\sqrt{\frac{2[2c-3p^2]}{3}} \frac{C}{(x+ct)} e^{i[px - p(2p^2-c)t + R]}, \qquad (\mathbf{59.2})$$

Remark 3

When m$\rightarrow 0$, the solutions (43)-(58) degenerate as triangular function namely

Case 16. $r = 0 \;\; a = -1 \;\; b = 2$

$$u_{18} = \frac{-4+c}{3} + 4\ csc^2(x+ct),$$
$$v_{18} = \pm \left[\frac{2}{3}(-1+c) + 2\ csc^2(x+ct)\right] e^{iR}, \qquad (\mathbf{60.1})$$

and

$$u'_{18} = \frac{-1+c-3p^2}{3} + 2\ csc^2(x+ct),$$
$$v'_{18} = \pm\sqrt{\frac{2[2c+1-3p^2]}{3}}\ csc(x+ct) e^{i[px - p(2p^2-2-c)t + R]}, \qquad (\mathbf{60.2})$$
$$u_{19} = \frac{-4+c}{3} + 4\ sec^2(x+ct),$$
$$v_{19} = \pm \left[\frac{2}{3}(-1+c) + 2\ sec^2(x+ct)\right] e^{iR}, \qquad (\mathbf{61.1})$$

and

$$u'_{19} = \frac{-1+c-3p^2}{3} + 2\ sec^2(x+ct),$$
$$v'_{19} = \pm\sqrt{\frac{2[2c+1-3p^2]}{3}}\ sec(x+ct) e^{i[px - p(2p^2-2-c)t + R]}, \qquad (\mathbf{61.2})$$

Case 17. $r = 1 \;\; a = 2 \;\; b = 2$

$$u_{20} = \frac{8+c}{3} + 4\cot^2(x+ct),$$
$$v_{20} = \pm \left[\frac{2}{3}(2+c) + 2\cot^2(x+ct)\right] e^{iR}, \qquad (\mathbf{62.1})$$

and

$$u'_{20} = \frac{2+c-3p^2}{3} + 2\cot^2(x+ct),$$

$$v'_{20} = \pm\sqrt{\frac{2[2c-2-3p^2]}{3}}\cot(x+ct)e^{i[px-p(2p^2+4-c)t+R]}, \quad (\mathbf{62.2})$$

$$u_{21} = \frac{8+c}{3} + 4\tan^2(x+ct),$$

$$v_{21} = \pm\left[\frac{2}{3}(2+c) + 2\tan^2(x+ct)\right]e^{iR}, \quad (\mathbf{63.1})$$

and

$$u'_{21} = \frac{2+c-3p^2}{3} + 2\tan^2(x+ct),$$

$$v'_{21} = \pm\sqrt{\frac{2[2c-2-3p^2]}{3}}\tan(x+ct)e^{i[px-p(2p^2+4-c)t+R]}, \quad (\mathbf{63.2})$$

Case 18. $r = 1/4 \;\; a = 1/2 \;\; b = 1/2$

$$u_{22} = \frac{2+c}{3} + [csc(x+ct) \pm \cot(x+ct)]^2,$$

$$v_{22} = \pm\left[\frac{2}{3}(1/2+c) + \frac{1}{2}[csc(x+ct) \pm \cot(x+ct)]^2\right]e^{iR}, \quad (\mathbf{64.1})$$

and

$$u'_{22} = \frac{1/2+c-3p^2}{3} + \frac{1}{2}[csc(x+ct) \pm \cot(x+ct)]^2,$$

$$v'_{22} = \pm\sqrt{\frac{[2c-1/2-3p^2]}{6}}[csc(x+ct) \pm \cot(x+ct)]e^{i[px-p(2p^2+1-c)t+R]}, \quad (\mathbf{64.2})$$

$$u_{23} = \frac{2+c}{3} + [sec(x+ct) \pm \tan(x+ct)]^2,$$

$$v_{23} = \pm\left[\frac{2}{3}(1/2+c) + \frac{1}{2}[sec(x+ct) \pm \tan(x+ct)]^2\right]e^{iR}, \quad (\mathbf{65.1})$$

and

$$u'_{23} = \frac{1/2+c-3p^2}{3} + \frac{1}{2}[sec(x+ct) \pm \tan(x+ct)]^2,$$

$$v'_{23} = \pm\sqrt{\frac{[2c-1/2-3p^2]}{6}}[sec(x+ct) \pm \tan(x+ct)]e^{i[px-p(2p^2+1-c)t+R]}, \quad (\mathbf{65.2})$$

Remark 4

When m$\rightarrow$ 1, the solutions (52) degenerate as soliton, combined soliton solutions , namely

Case 19. $r = 1 \;\; a = -2 \;\; b = 2$

$$u_{24} = \frac{-8+c}{3} + 4\tanh^2(x+ct),$$

$$v_{24} = \pm\left[\frac{2}{3}[-2+c] + 2\tanh^2(x+ct)\right]e^{iR}, \quad (\mathbf{66.1})$$

and

$$u'_{24} = \frac{-2 + c - 3p^2}{3} + 2\tanh^2(x + ct),$$
$$v'_{24} = \pm\sqrt{\frac{2[2c + 2 - 3p^2]}{3}}\tanh(x + ct)e^{i[px - p(2p^2-4-c)t+R]}, \qquad (\mathbf{66.2})$$

Case 20. $r = 0 \;\; a = 1 \;\; b = -2$

$$u_{25} = \frac{4 + c}{3} - 4\; sech^2(x + ct),$$
$$v_{25} = \pm\left[\frac{2}{3}((1 + c) - 2\; sech^2(x + ct)\right] e^{iR}, \qquad (\mathbf{67.1})$$

and

$$u'_{25} = \frac{1 + c - 3p^2}{3} - 2\; sech^2(x + ct),$$
$$v'_{25} = \pm\sqrt{\frac{-2[2c - 1 - 3p^2]}{3}}\; sech(x + ct)e^{i[px - p(2p^2+2-c)t+R]}, \qquad (\mathbf{67.2})$$

Case 21. $r = 1 \;\; a = -2 \;\; b = 2$

$$u_{26} = \frac{-8 + c}{3} + 4\coth^2(x + ct),$$
$$v_{26} = \pm\left[\frac{2}{3}(-2 + c) + 2\coth^2(x + ct)\right] e^{iR}, \qquad (\mathbf{68.1})$$

and

$$u'_{26} = \frac{-2 + c - 3p^2}{3} + 2\coth^2(x + ct),$$
$$v'_{26} = \pm\sqrt{\frac{2[2c + 2 - 3p^2]}{3}}\coth(x + ct)e^{i[px - p(2p^2-4-c)t+R]}, \qquad (\mathbf{68.2})$$

Case 22. $r = 0 \;\; a = 1 \;\; b = 2$

$$u_{27} = \frac{4 + c}{3} + 4\; csch^2(x + ct),$$
$$v_{27} = \pm\left[\frac{2}{3}(1 + c) + 2\; csch^2(x + ct)\right] e^{iR}, \qquad (\mathbf{69.1})$$

and

$$u'_{27} = \frac{1 + c - 3p^2}{3} + 2\; csch^2(x + ct),$$
$$v'_{27} = \pm\sqrt{\frac{2[2c - 1 - 3p^2]}{3}}\; csch(x + ct)e^{i[px - p(2p^2+2-c)t+R]}, \qquad (\mathbf{69.2})$$

Case 23. $r = 1/4 \;\; a = -1/2 \;\; b = 1/2$

$$u_{28} = \frac{-2 + c}{3} + [\coth(x + ct) \pm csch(x + ct)]^2,$$

$$v_{28} = \pm \left[\frac{2}{3}(-1/2+c) + \frac{1}{2}[\coth(x+ct) \pm csch(x+ct)]^2\right] e^{iR}, \qquad \mathbf{(70.1)}$$

and

$$u'_{28} = \frac{-1/2+c-3p^2}{3} + \frac{1}{2}[\coth(x+ct) \pm csch(x+ct)]^2,$$

$$v'_{28} = \pm\sqrt{\frac{[2c+1/2-3p^2]}{6}}[\coth(x+ct) \pm csch(x+ct)]e^{i[px-p(2p^2-1-c)t+R]}, \qquad \mathbf{(70.2)}$$

$$u_{29} = \frac{-2+c}{3} + [\tanh(x+ct) \pm i\ sech(x+ct)]^2,$$

$$v_{29} = \pm \left[\frac{2}{3}(-1/2+c) + \frac{1}{2}[\tanh(x+ct) \pm i\ sech(x+ct)]^2\right] e^{iR}, \qquad \mathbf{(71.1)}$$

and

$$u'_{29} = \frac{-1/2+c-3p^2}{3} + \frac{1}{2}[\tanh(x+ct) \pm i\ sech(x+ct)]^2,$$

$$v'_{29} = \pm\sqrt{\frac{[2c+1/2-3p^2]}{6}}[\tanh(x+ct) \pm i\ sech(x+ct)]e^{i[px-p(2p^2-1-c)t+R]}, \qquad \mathbf{(71.2)}$$

4. CONCLUSION

Based on the mapping method and the use of symbolic computation system Mathematica, we construct explicit, rich and new solutions such as Jacobi elliptic function, combined Jacobi elliptic function, rational, triangular function, soliton and combined soliton solutions to the coupled Klein-Gordon-Schrödinger equations and to the complex coupled KdV equations.

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