GENERAL TYPE OF THE MODULI SPACES OF K3 SURFACES

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In this talk at the Mathematische Arbeitstagung 2005 I am presenting my joint results with **Klaus Hulek** (Hannover) and **Gregory Sankaran** (Bath) on the Kodaira dimension of the moduli of $K3$ surfaces.

A moduli space of polarized $K3$ surfaces can be identified with the quotient of a classical hermitian domain of type IV by some arithmetic group. The general set-up for the problem is the following. Let L be an even integral lattice with a quadratic form of signature $(2,n)$,

$$\mathcal{D}(L) = \{z \in \mathbb{P}(L \otimes \mathbb{C}) \,:\, z \cdot z = 0,\ z \cdot \bar{z} > 0\}^+$$

be an n-dimensional Hermitian domain ($+$ denotes one of two connected components), $O(L)^+$ be the index 2 subgroup of the integral orthogonal group $O(L)$ preserving $\mathcal{D}(L)$. The arithmetic group in the question is $\Gamma_L = \{\gamma \in O(L)^+ : \gamma|_{L^*/L} = \mathrm{id}\}$ where L^* is the dual lattice of L. We are interested in the geometric properties of the arithmetic quotient $\Gamma_L \setminus \mathcal{D}(L)$.

K3-surfaces. A compact complex surface S is called $K3$ surface if S is simply connected and there exists a holomorphic 2-form $\omega_S \in H(S,\Omega^2)$ without zeros. For example, a smooth quartic in $\mathbb{P}^3(\mathbb{C})$ is a K3 surface.

The second cohomology group $H^2(S,\mathbb{Z})$ with the intersection pairing is an even unimodular lattice of signature $(3,19)$, i.e.,

$$H^2(S,\mathbb{Z}) \cong L_{K3} = 3U + 2E_8(-1), \quad \text{where} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a hyperbolic plane. The nowhere zero 2-form $\mathbb{C}\omega_S$ considered as a subspace of $L_{K3} \otimes \mathbb{C}$ is the period of S. The Torelli-type theorem proved by Piatetskii-Shapiro and Shafarevich in 1971 claims that the isomorphism class of S is uniquely determined by its period. The moduli of all polarized algebraic K3 surfaces is a union of 19-dimensional irreducible algebraic varieties and one picks out a component by fixing the degree.

A polarized K3 surface of degree 2d is a pair (S,H) consisting of a $K3$ surface S and a primitive pseudo-ample divisor H on S of degree $H^2 = 2d > 0$. If h is the corresponding vector in the lattice L_{K3} then its orthogonal complement

$$(h)^{\perp}_{L_{K3}} \cong L_{2d} = 2U + 2E_8(-1) + < -2d >$$

is a lattice of signature $(2,19)$.

Moduli of polarized K3 surfaces. The 2-form ω_S of (S,H) determines a point of $\mathcal{D}(L_{2d})$ modulo the group $\Gamma_{2d} = \Gamma(L_{2d})$. According to the global Torelli theorem of [P-SS71] and the surjectivity of the periodic map $\mathcal{F}_{2d} = \Gamma_{2d} \setminus \mathcal{D}(L_{2d})$ is *the coarse moduli space of polarized $K3$ surfaces of degree $2d$.* (For $2d = 4$ we get the moduli space of quartics.)

By the result of Baily and Borel $\Gamma_{2d} \setminus \mathcal{D}(L_{2d})$ is a quasi-projective variety. One of the fundamental problems is to determine its birational type. For very small d ($2d = 4, 6, 8$) a K3 surface of degree $2d$ is a complete intersection in $\mathbb{P}^{d+1}$ and the moduli $\mathcal{F}_{2d}$ were classically known. Mukai considered some other polarizations and proved

(*Mukai 87, 89, 96*): The moduli spaces $\mathcal{F}_{2d}$ are unirational for $1 \le d \le 11$ and $d = 17, 19$.

In the other direction Kondo [K93] and Gritsenko [G94] showed

(*Kondo 93*): For sufficiently big primes $p >> 0$ the moduli space $\mathcal{F}_{2p^2}$ is of general type; (No effective bound for primes p is known.)

(*Gritsenko 94*): Let $\Gamma_{2d}(q)$ be the intersection of Γ_{2d} with the principal congruence subgroup of level q. Then $\Gamma_{2d}(q) \setminus \mathcal{D}(L_{2d})$ is of general type for any d if $q \ge 3$.

In this talk I would like present the following new result

Main Theorem (Gritsenko, Hulek, Sankaran). *Let $d \ge 67$ be square-free ($d \ne 69, 77$). Then the moduli space $\mathcal{F}_{2d}$ of polarized $K3$ surfaces of degree $2d$ is of general type.*

Branch locus. We shall construct pluricanonical forms by means of modular forms. There might be three types of possible obstruction to this. They are the boundary of $\mathcal{F}_{2d}$ in its compactification, non-canonical singularities arising from fixed loci of the group action, and the ramification locus of the projection $\mathcal{D}(L_{2d}) \to \mathcal{F}_{2d}$. We show that only the third obstruction is in fact essential.

Theorem 1. *1) The ramification locus of the projection $\mathcal{D}(L_{2d}) \to \mathcal{F}_{2d}$ is defined by reflections σ_r such that $r^2 = -2$ or $r^2 = -2d$.*

2) For any (-2)-vector r we have $r^{\perp}_{L_{2d}} \cong K_{2d}$ or $\cong M_{2d}$ where

$$K_{2d} = U + 2E_8(-1) + <2> + <-2d>, \qquad M_{2d} = U + 2E_8(-1) + \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-d}{2} \end{pmatrix}$$

if $d = 1 \mod 4$.

3) For any $(-2d)$-vector r the determinant of $r^{\perp}_{L_{2d}}$ does not depend on d.

Toroidal compactification. As was proved by Kondo in [K93] the spaces $\mathcal{F}_{2d}$ only have canonical singularities. The toroidal compactification is not unique. Choosing possible refinements of a suitable fan we can ensure that the toroidal construction does not contribute to the singularities and that all singularities are finite quotient singularities. In fact we obtain the following

Theorem 2. *Let d be cube-free. There exists a toroidal compactification $\mathcal{F}^{tor}_{2d}$ of $\mathcal{F}_{2d}$ such that holds:*

(i) $\mathcal{F}^{tor}_{2d}$ has only canonical singularities;

(ii) For each boundary component B there is no branch divisor of π_B contained in $\mathcal{F}^{tor}_{2d} \setminus \mathcal{F}_{2d}$.

To show that $\mathcal{F}_{2d}^{tor}$ is of general type (i.e., that its Kodaira dimension is equal to 19) we have to prove the following asymptotic

$$\dim H^0(\mathcal{F}_{2d}^{tor}, \Omega^{\otimes k}) = O(k^{19})$$

for the dimension of the k-fold pluricanonical forms on $\mathcal{F}_{2d}^{tor}$. Let $\mathcal{F}_{2d}^{o}$ be the open part of $\mathcal{F}_{2d}$ such that the projection π is unramified over $\mathcal{F}_{2d}^{o}$. For any Γ_{2d}-modular form F of weight $19k$ we can define $F(z)(dz)^k \in H^0(\mathcal{F}_{2d}^{o}, \Omega^{\otimes k})$ where dz is the standard volume element of $\mathcal{D}(L_{2d})$. (One can even consider the last description as a definition of modular forms.) According to Theorem 2, $F(z)(dz)^k$ can be extended to $\mathcal{F}_{2d}^{tor}$ if

(1) $F(z)$ is zero of order at least k on the boundary (Tai's criterion);
(2) $F(z)$ is zero of order at least k on the ramification locus.

To estimate the last obstruction we use the Mumford-Hirzebruch proportionality principle.

The Mumford-Hirzebruch proportionality principle ([H58], [M77]) gives us a major term of the dimension of the space of cusp forms. Let L be of signature $(2, n)$ and $\Gamma \subset O(L)$ be an arithmetic group: then

$$\dim S_k(\Gamma) = \frac{2}{n!}\mathrm{vol}_{MH}(\Gamma)k^n + O(k^{n-1}).$$

The constant $\mathrm{vol}_{MH}(\Gamma)$, which we call the Mumford–Hirzebruch volume, is the ratio of the volume of the fundamental domain by the volume of the compact dual manifold $D^c(L) \cong SO(n+2)/SO(2) \times SO(n)$. Both volume forms should coincide in a common base point defined by a maximal compact subgroup of $O(2, n)$. If Γ acts freely on $D(L)$ then according to the proportionality principle

$$\mathrm{vol}_{MH}(\Gamma) = \frac{\mathrm{vol}(\Gamma \setminus D(L))}{\mathrm{vol}(D^c(L))} = \frac{e(\Gamma \setminus D(L))}{e(D^c(L))} = \chi(\Gamma \setminus D(L)).$$

Therefore the calculation of $\mathrm{vol}_{MH}(\Gamma)$ is equivalent to the explicit determination of the Euler–Poincare measure of the group Γ. We can solve this question using the Siegel theory of indefinite quadratic forms.

Theorem 3. *For any even lattice L of signature $(2, n)$ containing at least one hyperbolic plane the following formula holds:*

$$\mathrm{vol}_{MH}(O(L)) = 2 \cdot |\det L|^{(n+3)/2} \cdot \prod_p \alpha_p(L)^{-1} \cdot \prod_{k=1}^{n+2} \pi^{-k/2}\Gamma(\frac{k}{2})$$

where $\alpha_p(S)$ are the local densities of the quadratic form L.

Corollary 1. **$K3$-modular forms**. According to the above formulae we get that for $d > 1$

$$\dim S_k(\Gamma_{2d}) = \frac{2^{-9}}{19!}\frac{|B_2 \cdot B_4 \cdot \ldots \cdot B_{20}|}{20!!} \cdot d^{10} \prod_{p|d}(1 + p^{-10})\, k^{19} + O(k^{18}).$$

where B_{2m} is the Bernoulli number.

Corollary 2. The modular forms on branch divisors. Let us consider the lattice K_{2d} of signature $(2,18)$ for Theorem 1. We put $d = d_0 t^2$, where d_0 is a positive square-free, D is the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{d})$, χ_D is the corresponding quadratic character. It follows that

$$\dim S_k(\Gamma(K_{2d})) = \frac{F_2(d)}{18!}\,\frac{B_2\cdot B_4\cdot\ldots\cdot B_{18}}{18!!}\cdot\frac{B_{10,\chi_D}}{10}\,t^{19}\prod_{p|2t}(1-\chi_D(p)p^{-10})k^{18}+O(k^{17})$$

where $B_{k,\chi_D} = -k\cdot L(1-k,\chi_D)$ is the Bernoulli number with respect to the character χ_D and

$$F_2(d) = \begin{cases} 2^{10} & \text{if } d\equiv 1 \pmod 4 \text{ and } \ d>1 \\ 2^{-9} & \text{if } d\equiv 2,\ 3 \pmod 4). \end{cases}$$

A similar formula is valid for the lattice M_{2d}. The next step is

Theorem 4. *The cusp obstruction is not essential.*

To proof this theorem we use cusp forms of small (<19) weights. They do exist according to [G94]. For $d>36$ we have a cusp form F_{11} of weight 11 with respect to $\Gamma_{2d}\cap SO(L_{2d})$. Then for an even k and for any cusp form $F\in S_{8k}(\Gamma_{2d})$ of weight $8k$ the modular form $G = F_{11}^k F\in S_{19k}^{(k+1)}(\Gamma_{2d})$ vanishes of order at least $k+1$ on the boundary (compare with [GS96]). Moreover one can show that any form of odd weight automatically vanishes on the $(-2d)$-part of the ramification locus. Now let D be a component of the (-2)-ramification divisor. We need that G vanishes of order k along D. Restriction to D gives us an exact sequence

$$0\to S_k(\Gamma_{2d})(-nD)\to S_k(\Gamma_{2d})\to S_{k+n}(\Gamma_M)$$

where $S_k(\Gamma_{2d})(-nD)$ is the space of all forms in $S_k(\Gamma_{2d})$ which vanish of order n and M is one of the lattices of signature $(2,18)$ of Theorem 1. It follows that the obstruction to extending forms $G(dz)^k$ lies in a space

$$B = \bigoplus_{n=0}^{k-1} S_{8k+n}(\Gamma_M).$$

It now remains to estimate the dimension of B for each of the (finitely many) components of the ramification locus. This obstruction and the dimension of $S_{8k}(\Gamma_{2d})$ have different asymptotic behavior according to the Corollaries to Theorem 3. Therefore we have got the following

Theorem 5. *If d is sufficiently large and cube-free then $\mathcal{F}_{2d}$ is of general type. Moreover our method is effective and it gives an effective bound for d. For instance, let p, q be primes such that $p>481$ and $q>10^6$. Then $\mathcal{F}_{2p^2}$ and $\mathcal{F}_{2q}$ are of general type.*

This method based on the Mumford-Hirzebruch proportionality principle and on the existence of cusp forms of small weights is effective and we can apply it to many quotient spaces of different dimensions. To improve the result about the moduli of $K3$ surfaces it would be better to have a cusp form of a small weight vanishing on the (-2)-part of the ramification locus. We can in fact construct such a cusp form using the pull-back of the Borcherds function Φ_{12}.

Pull-back of the Borcherds function Φ_{12}. The Borcherds function Φ_{12} is the denominator function of the fake monster Lie algebra. It is a modular form of (singular) weight 12

$$\Phi_{12} : \mathcal{D}(L_{2,26}) \to \mathbb{C}, \qquad L_{2,26} = 2U + 3E_8(-1)$$

with respect to the group $\Gamma(L_{2,26})$ (see [B95]). Its divisor is the union of all rational quadratic (Heegner or Humbert) divisors defined by (-2)-vectors in $L_{2,26}$. The pull-back of this function gives us very many interesting automorphic forms (see [B95, pp. 200–201], [GN98, pp. 257–258]). In the context of the moduli of $K3$-surfaces this construction was used in [BKPS98] and in [K99]. We summarize their results in a suitable form.

Let be $l \in E_8(-1)$ with $l^2 = -2d$. The choice of l determines an embedding of L_{2d} into $L_{2,26}$ as well as an embedding of the domain $\mathcal{D}(L_{2d})$ into $\mathcal{D}(L_{2,26})$. We put $R_l = \{r \in E_8(-1) : r^2 = -2,\ r \cdot l = 0\}$, $N_l = |R_l|$. Then ([BKPS98]) the function

$$F_l = \left. \frac{\Phi_{12}(z)}{\prod_{\{\pm r\} \in R_l} (z \cdot r)} \right|_{\mathcal{D}(L_{2d})} \in M_{k+\frac{N_l}{2}}(\Gamma_{2d})$$

is a non-trivial modular form of weight $k + \frac{N_l}{2}$ vanishing on all (-2)-divisors of $\mathcal{D}(L_{2d})$. Moreover ([K99]) this is a cusp form if d is square free and $N_l > 0$.

Therefore the main point for us is for which $2d > 0$ there exists a vector $l \in E_8$ such that $l^2 = 2d$ and l is orthogonal to at least two and at most 12 roots.

Theorem 6. *Such a vector l in E_8 does exist if*

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d), \tag{N}$$

where $N_L(2d)$ denotes the number of representations of $2d$ by a lattice L.

The meaning of the coefficients in (N) is the following. The root system E_8 contains a root system of type E_7 (with $2 \cdot 63$ roots) and a bouquet of 28 root systems A_2 centered in A_1 which is orthogonal to E_7.

The inequality (N) is not valid only for a finite number of d because its sides have different asymptotic $O(d^{5/2})$ and $O(d^2)$ respectively. We can get exact formulae for the theta-series in the right hand side in terms of some Eisenstein series of weight 3. As for the left hand side, we note that the number $N_{E_7}(2d)$ is the Fourier coefficient $e_{4,1}(d,0)$ of the Jacobi-Eisenstein series $E_{4,1}(\tau, z)$. According to the result of Eichler and Zagier (see [EZ85])

$$e_{4,1}(d,0) = (\textit{Simple const}) \cdot (2d)^{5/2} \cdot L_{4d}^{(Z)}(3)$$

where $L_{4d}^{(Z)}(s)$ is Zagier's generalization of the L-function of quadratic field.

The last d for which inequality (N) is not valid is 143. But $143 = 1^2+5^2+6^2+9^2$ and this representation induces a vector in the sublattice $4A_1 \subset E_8$ which represents $2 \cdot 143$ and is orthogonal exactly to twelve roots in E_8. A similar representation exists for many others $d < 143$.

The obstruction coming from the $(-2d)$-part of the ramification locus (see Theorem 1) is very small. For example for a prime p it is not essential for $p > 11$. Taking this into account we finish the proof of the main theorem.

Conclusions. 1. In this talk we give a preliminary version of the main theorem. We are going to improve this result in the near future.

2. The condition "to be cube free" in Theorem 2 is rather technical. The same result might be well true for any d.

3. We hope to prove the cuspidality of the pull-back of the Borcherds form without restriction "to be square free" on d.

4. We are planning to finish this project with two short lists of polarizations for which the moduli of polarized $K3$ surfaces of degree $2d$ might be of non-general type or might be unirational (uniruled).

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