Euclidean simplices generating discrete reflection groups

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Introduction

Let P be a convex polytope in spherical space $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$ or hyperbolic space $\mathbb{H}^n$. Consider the group G_P generated by the reflections with respect to the facets of P. We call G_P a *reflection group generated by* P. The problem we consider in this paper is to list polytopes generating discrete reflection groups.

The answer is known only for some combinatorial types of polytopes. Already in 1873, Schwarz [10] listed spherical triangles generating discrete groups. In 1998, E. Klimenko and M. Sakuma [9] solved the problem for hyperbolic triangles. In [2], [4], [3], [5] the problem was solved for hyperbolic quadrilaterals, compact hyperbolic pyramids and triangular prisms, hyperbolic simplices, and Lambert cubes in $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{H}^3$. In [6] the problem was solved for spherical simplices.

In this paper, we use the method of [6] to classify Euclidean simplices generating discrete reflection groups.

1 Preliminaries

A convex polytope in $\mathbb{S}^n$, $\mathbb{E}^n$ or $\mathbb{H}^n$ is called a *Coxeter polytope* if all its dihedral angles are integer parts of π. The group G_F generated by the reflections in the facets of any Coxeter polytope F is discrete, and F is a fundamental domain of G_F.

On the other hand, any discrete group G generated by reflections coincides with G_F for some Coxeter polytope F. If $G = G_P$ for some non-Coxeter polytope P, then P consists of several copies of F, and any two copies containing a facet in common are symmetric with respect to this facet.

Spherical reflection groups. Let G be a reflection group acting on $\mathbb{S}^n$. Suppose that G acts on $\mathbb{S}^n$ discretely, i.e. G is a finite group. Then G is generated by the reflections with respect to the facets of some spherical Coxeter polytope . It is shown by H. S. M. Coxeter [1] that any spherical Coxeter polytope containing no pair of antipodal points of $\mathbb{S}^n$ is a simplex.

To describe Coxeter polytopes we use *Coxeter diagrams*. A Coxeter diagram of a Coxeter polytope F is a graph whose nodes v_i correspond to the facets Π_i of F. Nodes v_i and v_j are joined by a $(k-2)$-fold edge if the dihedral angle formed up by Π_i and Π_j equals $\frac{\pi}{k}$ (if Π_i is orthogonal to Π_j, v_i and v_j are disjoint). Indecomposable spherical and Euclidean Coxeter simplices were classified by Coxeter [1]. The list of their Coxeter diagrams is represented in Table 1.

Euclidean reflection groups. Let G be a discrete reflection group acting on $\mathbb{E}^n$. Let F be a fundamental chamber of G. As it is shown in [1], F is a direct product of several simplices and simplicial cones.

Suppose that G is generated by a simplex (the simplex may not be a Coxeter simplex). Then G is an indecomposable group and the fundamental chamber of G is compact. In this case the fundamental chamber of G is one of the simplices $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$ and $\widetilde{G}_2$ (see Table 1).

See [11] for more information about discrete reflection groups.

We use the notation A_n, $\widetilde{A}_n$, B_n and so on for Coxeter simplices as well as for the groups generated by these simplices. Finite root systems are denoted by A_n, B_n and so on. See [8] for background on root systems.

Let P be a simplex generating discrete reflection group (P may not be a Coxeter simplex). Clearly, in this case all the dihedral angles of P are of the type $\frac{\pi m}{k}$. Hence, for any simplex generating discrete reflection group we can construct the following *generalized Coxeter diagram*: the nodes v_i of the diagram correspond to the facets Π_i of P; the nodes v_i and v_j are joined by a $(k-2)$-fold edge that is decomposed into m parts if the dihedral angle formed up by Π_i and Π_j equals to $\frac{\pi m}{k}$.

2 Families of simplices

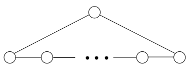
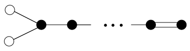
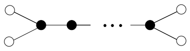
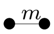
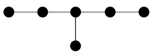
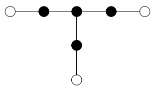
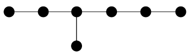
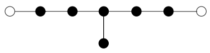
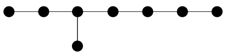
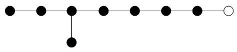
Spherical simplices generating discrete reflection groups. Let $\mathbb{E}^n$ be n-dimensional Euclidean space, and S^{n-1} be the unit sphere centered at the origin. Any hyperplane of S^{n-1} is a section of S^{n-1} by some hyperplane of $\mathbb{E}^n$ containing the origin. Hence, any $(n-1)$-dimensional simplex in S^{n-1} is an intersection of S^{n-1} with an interior of some cone with n facets in $\mathbb{E}^n$ centered at the origin.

So, suppose that P is a spherical $(n-1)$-dimensional simplex. Then we can define P by unit outward normal vectors $f_1, ..., f_n$ to the facets of the corresponding cone.

Denote by $\Pi_1, ..., \Pi_n$ the facets of P. The hyperplanes containing $\Pi_1, ..., \Pi_n$ decompose S^{n-1} into 2^n simplices P_1, ..., P_{2^n} encoded by n-tuples of vectors $\{\pm f_1, ..., \pm f_n\}$. In this paper the set of simplices P_1,...,P_{2^n} is called a *family*. Each of the simplices P_1,...,P_{2^n} generates the same reflection group as P does. Thus, we can study the families instead of studying of simplices themselves. (In fact, any family contains at most 2^{n-1} simplices up to isometry: the simplex $\{f_1, ..., f_n\}$ is always congruent to $\{-f_1, ..., -f_n\}$).

Let P be a simplex generating a discrete reflection group G_P. Clearly, the dihedral angles of P are rational numbers multiplied by π. Moreover, if G is an indecomposable spherical reflection group distinct from $G_2^{(m)}$, H_3 and H_4, then any dihedral angle of P is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$, where $k = 2, 3$ or 4 (cf. Table 1).

Table 1: Coxeter diagrams. Connected elliptic and parabolic Coxeter diagrams are listed in left and right columns respectively. Special nodes are marked.

$\mathbf{A_n}$ $(n \geq 1)$		$\widetilde{\mathbf{A}}_\mathbf{1}$	
		$\widetilde{\mathbf{A}}_\mathbf{n}$ $(n \geq 2)$	
$\mathbf{B_n = C_n}$ $(n \geq 2)$		$\widetilde{\mathbf{B}}_\mathbf{n}$ $(n \geq 3)$	
		$\widetilde{\mathbf{C}}_\mathbf{n}$ $(n \geq 2)$	
$\mathbf{D_n}$ $(n \geq 4)$		$\widetilde{\mathbf{D}}_\mathbf{n}$ $(n \geq 4)$	
$\mathbf{G_2^{(m)}}$		$\widetilde{\mathbf{G}}_\mathbf{2}$	
$\mathbf{F_4}$		$\widetilde{\mathbf{F}}_\mathbf{4}$	
$\mathbf{E_6}$		$\widetilde{\mathbf{E}}_\mathbf{6}$	
$\mathbf{E_7}$		$\widetilde{\mathbf{E}}_\mathbf{7}$	
$\mathbf{E_8}$		$\widetilde{\mathbf{E}}_\mathbf{8}$	
$\mathbf{H_3}$			
$\mathbf{H_4}$			

Analogously to Coxeter simplices, any simplex whose dihedral angles equal either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$ can be represented by the following *family diagram*: the nodes v_i correspond to the facets of P; the nodes v_i and v_j are joined by a $(k-2)$-fold edge if the angle formed up by Π_i and Π_j is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$ (if Π_i is orthogonal to Π_j, v_i and v_j are disjoint).

Note, that if P is a Coxeter simplex then the diagram corresponding to P is a Coxeter diagram of P.

Any two simplices in one family have the same family diagram. So, we can assign to a family the diagram of any simplex contained in the family. We call this diagram a *family diagram*. It is proved in [6] (Lemma 2) that any graph having no edge of multiplicity greater than two is a family diagram for at most one family of simplices. The same statement for the dihedral group $G_2 = G_2^{(6)}$ is evident.

Euclidean simplices generating discrete reflection groups. Let P be a simplex in $\mathbb{E}^n$ generating discrete reflection group G_P. Let $\Pi_0, ..., \Pi_n$ be the facets of P, and $f_0, ..., f_n$ be the outward unit normal vectors to $\Pi_0, ..., \Pi_n$. The hyperplanes containing $\Pi_0, ..., \Pi_n$ decompose $\mathbb{E}^n$ into $2^{n+1}-1$ domains P_1,...,$P_{2^{n+1}-1}$ encoded by the vectors $\{\pm f_0, ..., \pm f_n\}$. All these domains except the initial simplex P are non-compact. Simplex P is encoded by the vectors $\{f_0, ..., f_n\}$. Each of the domains P_i generates the same reflection group as P does.

By a *family of Euclidean simplices* we call a set of simplices having the same set of vectors $\{\pm f_0, ..., \pm f_n\}$. Note, that in the Euclidean case any two simplices contained in one family are mutually similar (moreover, any two of these simplices are homothetic).

We classify Euclidean simplices generating discrete reflection groups up to similarity.

Furthermore, note that any dihedral angle of P is either $\frac{\pi}{k}$ or $\frac{\pi(k-1)}{k}$, where $k = 2, 3, 4$ or 6 (see Table 1). Hence, we can define *family diagrams* for Euclidean simplices in the same way as for the spherical ones.

Lemma 1. *Let Φ and Ψ be two different families of Euclidean simplices generating discrete reflection groups. Then their family diagrams are distinct.*

The proof of the lemma follows the proof of Lemma 2 from paper [6].

3 Special vertices

In this section we prove several auxiliary facts.

A hyperplane α is called a *mirror* of the group G if G contains a reflection with respect to α.

Lemma 2. *Let P be a simplex in $\mathbb{E}^n$ generating discrete reflection group G_P. Then no mirror of G_P decomposing P is parallel to a facet of P.*

Proof. Let $\Pi_0, ..., \Pi_n$ be the facets of P and let V_0 be the vertex opposite to Π_0. Suppose that there exists a mirror of G_P that is parallel to Π_0 and decomposes P. Since G_P is discrete, there exist only finitely many of such mirrors. Let m be one of them closest to V_0.

Denote by h the homothety with center V_0 taking Π_0 to m. Denote by r_i the reflection with respect to Π_i $(i = 0, ..., n)$, and by r the reflection with respect to m. Since $r \in G_P$, $r = r_{i_1}...r_{i_l}$ for some l. From the other hand, $r = hr_0h^{-1}$.

Consider the reflection $hrh^{-1} = (hr_{i_1}h^{-1})...(hr_{i_l}h^{-1})$. Since $hr_ih^{-1} \in G_P$ for any $i = 0, ..., n$, we have $hrh^{-1} \in G_P$. Furthermore, hrh^{-1} is a reflection with respect to some hyperplane m' parallel to m. Moreover, m' decomposes P and goes closer to V_0 than m. This contradicts to the choice of m.

□

Lemma 3. *Let P be a simplex generating discrete group G_P. Let V be a vertex of P and $\Pi_1, ..., \Pi_n$ be the facets of P containing V. Then the stabilizer $Fix(V, G_P)$ of V in G_P coincides with the group G generated by the reflections with respect to $\Pi_1, ..., \Pi_n$.*

Proof. Suppose that the lemma is false. Then there exists a non-empty set M of simplices for which the statement of the lemma is broken. We may assume that $P \in M$ is a simplex minimal by the inclusion. Let V be a vertex of P for which the statement of the lemma is false. Then among the dihedral angles formed up by $\Pi_1, ..., \Pi_n$ there exists a dihedral angle $\pi\frac{k}{m}$, where $1 < k < m$ are mutually co-prime integers. Suppose that this angle is formed up by Π_1 and Π_n.

Consider a mirror Π of G such that Π contains $\Pi_1 \cap \Pi_n$, and the angle formed up by Π and Π_1 is equal to $\frac{\pi}{m}$. This mirror decomposes P into two simplices P_1 and P_2. Let P_1 be the simplex having a facet Π_1. Denote by G_{P_1} the group generated by P_1. Clearly, $G_{P_1} = G_P$. Furthermore, the group generated by the reflections with respect to the facets $\Pi, \Pi_1, ..., \Pi_{n-1}$ coincides with the group generated by the reflections with respect to the facets $\Pi_1, ..., \Pi_n$. From the other hand, the stabilizer $Fix(V, G_{P_1})$ of V in G_{P_1} coincides with $Fix(V, G_P)$. By the assumption, $Fix(V, G_P)$ does not coincide with the group generated by the reflections with respect to $\Pi_1, ..., \Pi_n$. Hence, $Fix(V, G_{P_1})$ differs from the group generated by the reflections with respect to $\Pi, \Pi_1, ..., \Pi_{n-1}$. Thus, the statement of the lemma is broken for P_1. This contradicts to the assumption that $P \in M$ is the minimal simplex, and the lemma is proved.

□

Let P be any Euclidean polytope generating discrete reflection group. Suppose that there exists a vertex V of P such that the stabilizer of V contains a linear part of any element of G_P. We call such a vertex a *special vertex* of P. It is known (see [8], Ch. 6) that any Euclidean Coxeter simplex has at least one special vertex.

Lemma 4. *Let P be a simplex generating discrete reflection group G_P. Then P has at list one special vertex.*

Proof. Let F be a fundamental polytope of G_P contained in P. Since G_P is indecomposable, F is a simplex. Any fundamental polytope of discrete reflection group is Coxeter polytope, thus F is a Coxeter simplex. Let V be a special vertex of F.

Suppose that F is either an inner point of P or an inner point of some face of P. Let Π be any facet of P not containing V. Let m be the mirror of G_P parallel to Π and containing V. Then m cuts P, that contradicts to Lemma 2.

Thus, V is a vertex of P. Let $\Pi_1, ..., \Pi_n$ be the facets of P containing V. By Lemma 3, the reflections with respect to $\Pi_1, ..., \Pi_n$ generate the stabilizer $Fix(V, G_P)$ of V in G_P. Thus, V is a special vertex of P.

□

4 Simplices generating given reflection group

Let G be an indecomposable Euclidean reflection group. Let P be a simplex generating G, and $f_0, f_1, ..., f_n$ be the vectors orthogonal to the facets of P. By Lemma 4, there exists a special vertex V of P. Let $\Pi_0, ..., \Pi_n$ be facets of P, such that Π_i is orthogonal to f_i for $i = 0, \dots, n$. We may assume that V is contained in $\Pi_1, ..., \Pi_n$.

The stabilizer $Fix(V, G)$ of V in G is a Weyl group of some finite root system Δ. Suitably normalizing the vectors $f_0, f_1, ..., f_n$, we may assume that these vectors belong to Δ. Since V is a special vertex of P, there exists a mirror of G through V parallel to Π_0. By Lemma 3, suitably normalizing f_0 we may assume that f_0 is contained in Δ. Note, that $f_0, f_1, ..., f_n$ are linearly dependent vectors, however any n of these vectors are linearly independent.

Now we are able to find all families of simplices generating given group G.

Let F be a fundamental simplex of G, and let U be a special vertex of F. Denote by W the stabilizer $Fix(U, G)$, and let Δ be the corresponding root system. Let $v_0, ..., v_n$ be any set containing $n + 1$ vectors of Δ, such that any n of these vectors are linearly independent. Let $\Pi_1, ..., \Pi_n$ be the mirrors of G containing U and orthogonal to $v_1, ..., v_n$. Let Π_0 be a mirror of G closest to U, orthogonal to v_0 and not containing U. Then the mirrors $\Pi_0, \Pi_1, ..., \Pi_n$ are the facets of some simplex P generating a finite index subgroup of G. If, in addition, the reflections with respect to $\Pi_1, ..., \Pi_n$ generate W, then some simplex similar to P generates G (the only exception is the group $W = B_n = C_n$; in this case we can obtain either $\widetilde{B}_n$ or $\widetilde{C}_n$ depending on the vector v_0, see Section 4.2).

In other word, to classify families generating G it is sufficient to follow the algorithm:

1) Find all linearly independent systems $f_1, ..., f_n$ in Δ, generating W (we say that $f_1, ..., f_n$ generate W, if the reflections with respect to the hyperplanes through the origin orthogonal to these vectors generate W).

2) For each system $f_1, ..., f_n$ obtained on the previous step, add a vector $f_0 \in \Delta$ in such a way that any n of vectors $f_0, f_1, ..., f_n$ are linearly independent. This vector f_0 should be chosen by all possible ways.

3) If $W = B_n = C_n$, examine which of the groups $\widetilde{B}_n$ and $\widetilde{C}_n$ is generated by $f_0, f_1, ..., f_n$.

4) Among the obtained systems one should find the systems corresponding to different families, i.e. families having different family diagrams.

The order of vectors in the systems is not important for us. The Weyl group acts on $\mathbb{E}^n$, hence, it acts on $(n+1)$-tuples of vectors. We do not differ $(n+1)$-tuples equivalent with respect to this action.

4.1 Simplices generating $\widetilde{A}_n$

Let P be a simplex generating the group $\widetilde{A}_n$. We may assume that the vectors $f_0, f_1, ..., f_n$ belong to the root system $\mathrm{A}_n = \{\pm(h_i - h_j)\}, 0 \le i < j \le n$, where $h_0, ..., h_n$ is a standard basis of $\mathbb{E}^{n+1}$).

For any simplex $P = \{f_0, f_1, ...f_n\}$ generating $\widetilde{A}_n$ we construct the following *graph* $\Gamma(P)$: the nodes $v_0, ..., v_n$ of Σ correspond to the vectors $h_0, ...h_n$; the nodes v_i and v_j are joined by an edge if one of vectors $(h_i - h_j)$ and $-(h_i - h_j)$ belongs to the set $\{f_0, ..., f_n\}$. Clearly, two simplices from one family have the same graph.

Theorem 1. *There exists a unique family of simplices generating $\widetilde{A}_n$. This family consists of Coxeter simplices $\widetilde{A}_n$.*

Proof. Let $P = \{f_0, f_1, ...f_n\}$ be a simplex generating $\widetilde{A}_n$ and $\Gamma(P)$ be the corresponding graph. Then $\Gamma(P)$ contains exactly $n+1$ nodes and the same number of edges. Since any n vectors contained in the set $\{f_0, ..., f_n\}$ are linearly independent, $\Gamma(P)$ has no cycles containing less than $n+1$ nodes. Hence, $\Gamma(P)$ is a cycle with $n+1$ nodes.

Note, that the graph $\Gamma(P)$ determines the family diagram of P. In more details, the nodes of the family diagram correspond to the edges of $\Gamma(P)$, two nodes are joined if the corresponding edges of $\Gamma(P)$ have a common node. Hence, in case of $\widetilde{A}_n$ the family diagram is a cycle, too. By Lemma 1, the family of simplices is completely determined by a family diagram. Thus, P is a Coxeter simplex $\widetilde{A}_n$.

□

4.2 Simplices generating $\widetilde{B}_n$ and $\widetilde{C}_n$

Let P be a simplex generating the group $\widetilde{B}_n$ or $\widetilde{C}_n$. We may assume that the vectors $f_0, f_1, ..., f_n$ belong to the root system $\mathrm{B}_n = \{\pm h_i,\ \pm h_i \pm h_j\}, 1 \le i < j \le n$, where $h_1, ..., h_n$ is a standard basis of $\mathbb{E}^n$).

For any simplex $P = \{f_0, f_1, ...f_n\}$ generating $\widetilde{B}_n$ or $\widetilde{C}_n$ we construct the following *graph* $\Gamma(P)$: the nodes $v_1, ..., v_n$ of Σ correspond to the vectors $h_1, ...h_n$; the nodes v_i and v_j are joined by an edge if one of vectors $\pm(h_i - h_j)$ and $\pm(h_i + h_j)$ belongs to the set $\{f_0, ..., f_n\}$; the node v_i is *marked* if $\{f_0, f_1, ..., f_n\}$

contains $\pm h_i$. If $\{f_0, f_1, ..., f_n\}$ contains both $\pm(h_i + h_j)$ and $\pm(h_i - h_j)$, the nodes v_i and v_j are joined by two edges.

Since the system of vectors $f_0, f_1, ..., f_n$ is indecomposable, $\Gamma(P)$ is connected. Clearly, $\Gamma(P)$ contains at least one marked node (otherwise f_i belongs to D_n for any i, where D_n is embedded in B_n as a set of long roots).

Lemma 5. *Let P be a simplex generating $\widetilde{B}_n$ or $\widetilde{C}_n$.*
1) If $\Gamma(P)$ contains more than one marked node then P is a Coxeter simplex $\widetilde{C}_n$.
2) If $\Gamma(P)$ contains a unique marked node then $\Gamma(P)$ is one of the graphs shown in the left column of Table 2.

Proof. Suppose that $\Gamma(P)$ contains more than one marked node. Then $\Gamma(P)$ contains a path from one marked node to another. Consider such a path that does not intersect itself. The vectors corresponding to the edges and marked nodes of this path are linearly dependent. Hence, the path contains all edges of $\Gamma(P)$, and $\Gamma(P)$ is the graph shown in Fig. 1. In this case $P = \widetilde{C}_n$.

Now suppose that $\Gamma(P)$ contains a unique marked node. Since the number of edges of $\Gamma(P)$ equals n and the number of nodes equals $(n+1) - 1 = n$, $\Gamma(P)$ contains a unique cycle (it may consist of two nodes). Since any n vectors are linearly independent, if the cycle contains less than n edges then it does not contain the marked node. Hence, $\Gamma(P)$ have a subgraph shown in the left column of Table 2. From the other hand, this subgraph corresponds to linearly dependent system of vectors. Hence, this subgraph coincides with $\Gamma(P)$.

□

Figure 1: Graph $\Gamma(\widetilde{C}_n)$.

Theorem 2. *There exists exactly $n - 1$ families of simplices generating the group $\widetilde{B}_n$. Family diagrams and generalized Coxeter diagrams of these simplices are shown in Table 2.*

Proof. By the second statement of Lemma 5, any simplex P generating $\widetilde{B}_n$ corresponds to a graph $\Gamma(P)$ shown in the left column of Table 2.

Let us show that any graph Γ shown in the left column of Table 2 corresponds to a simplex generating $\widetilde{B}_n$. It is easy to see that any n of vectors $f_0, f_1, ..., f_n$ are linearly independent and all these vectors are linearly dependent. Hence, Γ corresponds to some simplex P in $\mathbb{E}^n$. The group generated by P is a maximal rank indecomposable subgroup of $\widetilde{B}_n$, and the fundamental simplex of this subgroup has a dihedral angle equal to $\frac{\pi}{4}$. By [7], P generates either $\widetilde{B}_n$ or $\widetilde{C}_n$.

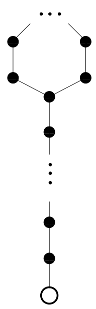
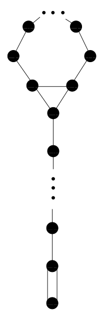
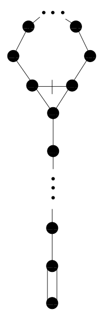
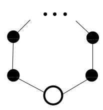
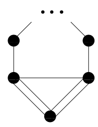
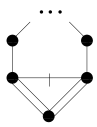
Let φ and ψ be two dihedral angles equal to $\frac{\pi}{4}$ formed up by mirrors of G_P. Then there exists an element γ of G_P such that $\gamma(\varphi) = \psi$. The group $\widetilde{C}_n$ contains two equivalency classes of such dihedral angles. Hence, $G_P \neq \widetilde{C}_n$ and P generates $\widetilde{B}_n$.

Furthermore, let us show that each graph shown in the left column of the Table 2 corresponds to a unique family of simplices generating $\widetilde{B}_n$. Indeed, the family diagram of the family containing P can be easily recovered from $\Gamma(P)$: all but one nodes of the diagram correspond to the edges of $\Gamma(P)$, two nodes are adjacent if the corresponding edges have a common point; the rest node corresponds to the marked node of $\Gamma(P)$, this node is joined by a 2-fold edge with all the nodes that correspond to edges of $\Gamma(P)$ incident to the marked node. Thus, any graph shown in the left column of Table 2 corresponds to a unique family diagram, and, by Lemma 1, it corresponds to a unique family. An explicit calculation shows that the generalized Coxeter diagram of simplex generating $\widetilde{B}_n$ is one of the diagrams shown in the right column of Table 2.

Thus, families of simplices generating $\widetilde{B}_n$ are in one-to-one correspondence with graphs shown in the left column of Table 2. To find the number of these families note, that $\Gamma(P)$ is uniquely determined by the number of edges in the cycle. The latter is any integer number N satisfying $2 \leq N \leq n$.

$\square$

Table 2: Simplices generating $\widetilde{B}_n$.

$\Gamma(P)$	Family diagram	Generalised Coxeter diagram
		
		

As a corollary of Theorem 2 and the first statement of Lemma 5 we obtain the following theorem:

Theorem 3. *There exists a unique family of simplices generating $\widetilde{C}_n$. This family consists of Coxeter simplices $\widetilde{C}_n$.*

4.3 Simplices generating $\widetilde{D}_n$

Let P be a simplex generating the group $\widetilde{D}_n$. We may assume that the vectors $f_0, f_1, ..., f_n$ belong to the root system $\mathrm{D}_n = \{\pm h_i \pm h_j)\}, 1 \leq i < j \leq n$, where $h_1, ..., h_n$ is a standard basis of $\mathbb{E}^n$).

For any simplex $P = \{f_0, f_1, ...f_n\}$ generating $\widetilde{D}_n$ we construct the following *graph* $\Gamma(P)$: the nodes $v_1, ..., v_n$ of Σ correspond to the vectors $h_1, ...h_n$; the nodes v_i and v_j are joined by an edge if one of vectors $\pm(h_i - h_j)$ and $\pm(h_i + h_j)$ belongs to the set $\{f_0, ..., f_n\}$; if $\{f_0, f_1, ..., f_n\}$ contains both $\pm(h_i + h_j)$ and $\pm(h_i - h_j)$ then v_i and v_j are joined by two edges.

The system of vectors $f_0, f_1, ..., f_n$ is indecomposable, hence, $\Gamma(P)$ is connected. Since $\Gamma(P)$ has n nodes and $n+1$ edges, $\Gamma(P)$ contains at least two cycles C_1 and C_2.

Suppose that C_1 and C_2 have no common nodes. Since the graph containing two disjoint cycles corresponds to a linearly dependent system of vectors, $\Gamma(P)$ is one of the graphs shown in the left column of Table 3. If C_1 and C_2 have a unique common node then $\Gamma(P)$ is the graph shown at the bottom of left column of Table 3.

Lemma 6. *Suppose that $\Gamma(P)$ contains two cycles having at least two common nodes. Then the system of vectors $f_0, f_1, ..., f_n$ contains n linearly dependent vectors.*

Proof. Consider the graph $\Gamma(P)$ colored by the following way: edges corresponding to vectors $\pm(h_i + h_j)$ are red, and the rest edges, i.e. the edges corresponding to $\pm(h_i - h_j)$, are black. Note that substituting the vector h_i by $-h_i$ we change the color of all edges incident to v_i. Thus, preserving the vectors $f_0, f_1, ..., f_n$, we can make all but one edge of a given cycle black (the rest edge is either red or black). Vectors $h_1 - h_2, h_2 - h_3, ..., h_k - h_1$ are linearly dependent. Hence, each cycle containing an even number of red edges corresponds to a system of linearly dependent vectors.

Consider common nodes of two cycles contained in $\Gamma(P)$. There are at least three paths L_1, L_2 and L_3 joining these nodes in $\Gamma(P)$. Denote by $c(L_i)$ the number of red edges in L_i. Then $c(L_i) + c(L_j)$ is even for some $i \neq j$, $i, j = 1, 2, 3$. We assume that $c(L_1) + c(L_2)$ is even. Then the cycle $C = L_1 \cup L_2$ contains even number of red edges, so, it corresponds to some linearly dependent vectors. Since some edges of $\Gamma(P)$ do not belong to C, the number of these linearly dependent vectors is less than $n+1$. The contradiction proves the lemma.

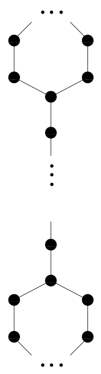
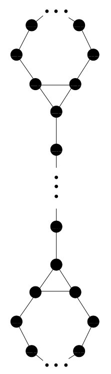
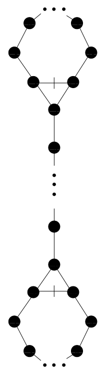
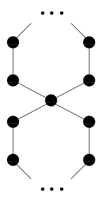
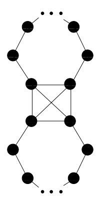
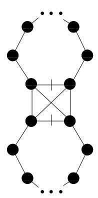
□

Corollary 1. *$\Gamma(P)$ contains exactly two cycles and coincides with one of graphs shown in the left column of Table 3.*

Thus, we obtain the following theorem:

Theorem 4. *The group $\widetilde{D}_n$ is generated by exactly $\frac{1}{4}n(n-2)$ families of simplices if n is even, and by exactly $\frac{1}{4}(n-1)^2$ families if n is odd. Family diagrams and generalized Coxeter diagrams of these simplices are presented in Table 3.*

Table 3: Simplices generating $\widetilde{D}_n$.

$\Gamma(P)$	Family diagram	Generalised Coxeter diagram
		
		

Proof. By Cor. 1, a simplex P generating $\widetilde{D}_n$ corresponds to a graph $\Gamma(P)$ shown in the left column of Table 3.

Let us show that any graph Γ shown in the left column of Table 3 corresponds to a simplex generating $\widetilde{D}_n$. For each cycle of Γ choose an edge v_iv_j and put the vector $h_i - h_j$ in correspondence with this edge (in other words, suppose these edges to be red). For all other edges v_kv_l take vectors $h_k + h_l$. It is easy to see that any n of these vectors are linearly independent and all these vectors are linearly dependent. Hence, Γ corresponds to some simplex P in $\mathbb{E}^n$. By [7], the group $\widetilde{D}_n$ has no finite index indecomposable subgroups different from $\widetilde{D}_n$. Therefore, P generates $\widetilde{D}_n$.

Now, show that each graph Γ shown in the left column of Table 3 corresponds to a unique family of simplices generating $\widetilde{D}_n$. Indeed, a family diagram of the family containing P can be recovered from $\Gamma(P)$: nodes of diagram correspond to edges of Γ, two nodes are adjacent if the corresponding edges have a common point. Therefore, the family does not depend on the choice of initial edges. Moreover, in the beginning of the procedure we could make red not one but several edges: the family diagram would not be changed. Hence, family diagram of simplices generating $\widetilde{D}_n$ are in one-to-one correspondence with graphs shown in the left column of Table 3. By Lemma 1, each family diagram corresponds to a unique family. Thus, families of simplices generating $\widetilde{D}_n$ are in one-to-one correspondence with graphs shown in the left column of Table 3. An explicit calculation shows that generalized Coxeter diagram of simplex generating $\widetilde{D}_n$ is one of diagrams shown in the right column of Table 3.

To find the number of families note, that $\Gamma(P)$ is uniquely determined by the numbers N_1 and N_2 of edges in two cycles. Numbers N_1 and N_2 are any two integers satisfying $N_1 + N_2 \le n+1$ and $2 \le N_1 \le N_2 \le n$. The number of such pairs (N_1, N_2) equals either $\frac{1}{4}n(n-2)$ or $\frac{1}{4}(n-1)^2$ if n is even or odd respectively.

$\square$

4.4 Simplices generating other groups

In Sections 4.1–4.3 we have described all families generating the groups $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$ and $\widetilde{D}_n$. Now, we are left to classify families generating finitely many of other indecomposable Euclidean reflection groups. Namely, we a left with the groups $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, and $\widetilde{G}_2$. We can find the complete answer following the algorithm contained in the beginning of Section 4. As the result we obtain the lists which are rather large: there exist

17 families of simplices generating $\widetilde{E}_6$,
142 families of simplices generating $\widetilde{E}_7$,
1736 families of simplices generating $\widetilde{E}_8$,
11 families of simplices generating $\widetilde{F}_4$,
and 2 families of simplices generating $\widetilde{G}_2$.

Appendix contains the complete list of these families.

Appendix

Appendix contains the list of families generating $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, and $\widetilde{G}_2$. Families generating E_6, E_7 and E_8 are encoded in the following way. For a family $\pm f_0, \pm f_1, ..., \pm f_n$ construct a symmetrical matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 2|(f_i, f_j)|$. This is a doubled unsigned Gram matrix of the system $f_0, f_1, ..., f_n$. The upper triangle of G^+ is filled up by 0 and 1. Let p be a decimal number which is equal to the binary number $\overline{g_{1,2}g_{1,3}\dots g_{1,n}g_{2,3}\dots g_{2,n}\dots g_{i,i+1}\dots g_{i,n}\dots g_{n-1,n}}$. The number p depends on the ordering of vectors $f_0, f_1, ...f_n$. We choose p the smallest possible.

Families generating $\widetilde{E}_6$

515583 128885 104443 104126 104021 104011 64447 40671 39646 39583 39573 35838 35695 35629 35622 35131 35128

Families generating $\widetilde{E}_7$

33554431 33554375 33148285 33148159 16576223 16574174 16574039 16540255 15062365 15062363 8288107 8287019 7273471 7273323 7273255 7273249 6813566 6813547 6813536 6813517 6746927 6746917 6746893 6740393 6709105 6709103 6708972 6706915 6706849 6706529 6483890 6483689 6483675 6483674 6482603 6482587 6482586 6480625 6480411 6480410 6466539 6466515 6466502 6462463 6462076 6462033 6462031 6462029 6462028 3112959 3112885 3112884 2579953 2578873 2359295 2359289 2358204 2358201 2358199 2358196 2358193 2353855 2353852 2353846 2353845 2353843 2352764 2352719 2325497 2325429 2325425 2322165 2321148 2321078 2321074 2320093 2320092 2320044 2320022 2320020 2319949 2318901 2318861 2305919 2305916 2305909 2305885 2305876 2305868 2304831 2304821 2304790 2304780 2301661 2301659 2301656 2301596 2301594 2301593 2301591 2301590 2301588 2301587 2301586 2301584 2177023 2177012 2177006 2177002 2176996 2176993 2176983 2176982 2176958 2176940 2176938 2176934 2176928 2176910 2175854 2175852 2175847 2175844 2175823 2175822 2175820 2175813 2175812 2166783 2166638 2166635 2166634 2166572 2166567 2166563 2166562 2166079 2166074 2166073 2166059 2166057 2166056

Families generating $\widetilde{E}_8$

16642998271 16575889407 16575348735 8036015852 8036015839 8036015609 8027492222 8027491834 8023230399 8023229951 8023229630 7834757047 7834756850 7834756844 7834756831 7834756727 7834756687 7834756660 7834756615 3758096383 3758096377 3749707775 3749707770 3749572602 3749572601 3749572351 3749572334 3749572333 3749572331 3749572319 3745310463 3745310445 3745310335 3745310327 3745310302 3743179391 3743179383 3743179327 3743179295 3611160286 3611159190 3611158111 3611158095 3611158094 3611157143 3611157014 3611156999 3548144143 3548143711 3548143710 3543645815 3543645809 3543613052 3543613049 3543612031 3543612028 3543612025 3543612020 3543612017 3543611997 3543611995 3543477119 3543477079 3543477023 3523012220 3523012217 3523012215 3523012209 3523012191 3523012188 3523012175 3523012164 3522978431 3522978420 3522978391 3522978380 3522978375 3522978367 3522978335 3522978319 3522978311 3522741887 3522741881 3522741876 3522741873 3522741847 3522741375 3522741372 3522741369 3514759039 3514759030 3514759029 3514759022 3514759021 3514759019 3514725247 3514725238 3514725237 3514725223 3514725220 3514725217 3514725183 3514725166 3514725165 3514725163 3514725159 3514725153 3514463009 3514454878 3514454877 3514454872 3514454871 3514454865 3514454863 3514454860 3514454815 3514454810 3514454809 3514454798 3514454797

3514454795	3514454554	3514454335	3514454330	3514454329	3514454303
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3514355995	3514353535	3514353526	3514353525	3514353511	3514353508
3514353505	3514352767	3514352758	3514352757	3514352749	3514352747
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1404517228	1404501995	1404499819	1404498785	1404498731	1404492745
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1402894113	1402893409	1402893349	1402893347	1402893091	1402892385
1402892323	1402453806	1402402667	1402402603	1402402595	1402402187
1402402123	1402401633	1402401571	1402401569	1402401153	1402401089
1402394211	1402394155	1402394147	1402394145	1402393899	1402388259
1402386347	1402386219	1402369893	1402369891	1402369889	1402368865
1402368803	1402361681	1402361673	1402361667	1402361665	1402361601
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1370190505	1370190501	1370190475	1370190379	1370190369	1370190345
1370190255	1370190123	1370190113	1370189995	1370189985	1370189961
1369112535	1369108216	1369103847	1369103846	1369103843	1369103842
1369102917	1369101979	1369101731	1369101730	1369101687	1369036784
1369034085	1369034084	1369034081	1369034080	1368973285	1368973093
1368972197	1368971105	1368968677	1368968676	1368968673	1368968672
1368968481	1368967521	1368967520	1368966565	1368966564	1368966517
1368932727	1368127459	1368127438	1368127115	1368126950	1368126892
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332881633 332881404 332881397 332880545 332880316 332880309 332872644

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299622391	299622131	299622119	299622103	299621879	299596788	299596760
299596752	299596748	299589556	299589296	299589284	299589268	299588604
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Families generating $\widetilde{F}_4$

Families generating F_4 are encoded in the following way. For a family determined by $\pm f_0, \pm f_1, ..., \pm f_4$ construct a symmetric matrix $G^+ = \{g_{i,j}\}$, where $g_{i,j} = 0$ if f_i is orthogonal to f_j, $g_{i,j} = 1$ if $\angle f_i f_j = \frac{\pi}{3}$ or $\frac{2\pi}{3}$, $g_{i,j} = 2$ if $\angle f_i f_j = \frac{\pi}{4}$ or $\frac{3\pi}{4}$. Let p be a decimal number which is equal to the base three number $\overline{g_{1,2}g_{1,3}...g_{1,n}g_{2,3}...g_{2,n}...g_{i,i+1}...g_{i,n}...g_{n-1,n}}$. The number p depends on the numbering of vectors $f_1, ...f_n$. We choose p the smallest possible. Then p depends only on the family of simplices.

32560 4100 3232 3220 1213 1159 1153 1124 1064 827 825

Families generating $\widetilde{G}_2$

The group G_2 can be generated either by triangle with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6})$ or by triangle with angles $(\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6})$.

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