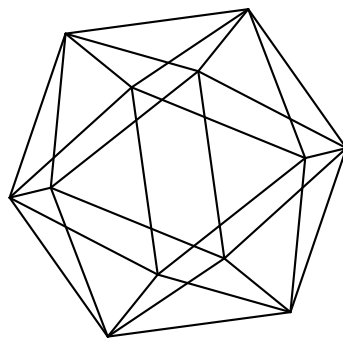


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Homotopy quotients and comodules of supercommutative Hopf algebras

by

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HOMOTOPY QUOTIENTS AND COMODULES OF SUPERCOMMUTATIVE HOPF ALGEBRAS

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ABSTRACT. We study induced model structures on Frobenius categories. In particular we consider the case where \mathcal{C} is the category of comodules of a supercommutative Hopf algebra A over a field k . Given a graded Hopf algebra quotient $A \rightarrow B$ satisfying some finiteness conditions, the Frobenius tensor category \mathcal{D} of graded B -comodules with its stable model structure induces a monoidal model structure on \mathcal{C} . We consider the corresponding homotopy quotient $\gamma : \mathcal{C} \rightarrow Ho\mathcal{C}$ and the induced quotient $\mathcal{T} \rightarrow Ho\mathcal{T}$ for the tensor category \mathcal{T} of finite dimensional A -comodules. Under some mild conditions we prove vanishing and finiteness theorems for morphisms in $Ho\mathcal{T}$. We apply these results in the $Rep(GL(m|n))$ -case and study its homotopy category $Ho\mathcal{T}$.

1. Introduction	1
Part 1. Induced model structures on categories of comodules	
2. Background on model categories	9
3. The stable module category of a Frobenius category	11
4. Induced model structure	13
5. The homotopy category $Ho\mathcal{C}$	19
6. Comodules and supercommutative Hopf algebras	20
7. The monoidal model structure	26
8. Clean decompositions	30
9. Construction of cofibrant replacements	32
10. Minimal models	38
11. Categories with weights and vanishing theorems	42
12. Isogenies	45
Part 2. The homotopy category associated to $GL(m n)$	
13. Cofibrant replacements and an explicit description of $Ho\mathcal{T}$	47
14. The degree filtration and cofibrant replacements	55
15. Restriction and DS -cohomology	62
16. Semisimple quotients	65
17. The case $GL(m 1)$: Morphisms and cofibrant replacements	67
18. The case $GL(m 1)$: Semisimple quotients	71
References	74

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1. INTRODUCTION

1.1. Model structures for representations of supergroups. In this article we study a way to construct model structures on an abelian Frobenius category \mathcal{C} . The constructions are generalizations of a model structure naturally appearing in the representation theory of the general linear supergroup $GL(m|n)$. In the supergroup case the model structure gives an abstract way to think about resolutions by Kac modules. The associated homotopy category (in the sense of Quillen) in turn is interesting since its monoidal structure can be seen as an approximation of that of $Rep(GL(m|n))$. So before we wade into technical matters, let us consider the $GL(m|n)$ -case first.

We work in this case over an algebraically closed field of characteristic 0. The supergroup $GL(m|n)$ contains the parabolic subgroup $P(m|n)^+$ of upper triangular block matrices. An irreducible representation of weight λ of the even subgroup $G_0 = GL(m) \times GL(n)$ can be trivially extended to $P(m|n)^+$ and then induced to $GL(m|n)$. This parabolic induction yields the so-called Kac modules

$$V(\lambda) = Ind_{P(m|n)^+}^{GL(m|n)} L_P(\lambda),$$

the universal highest weight modules in $\mathcal{T}_{m|n} = Rep(GL(m|n))$. They are the standard modules in the highest weight category $\mathcal{T}_{m|n}$. The twisted dual $V(\lambda)^*$ (see section 13) (also called anti Kac module) is then the corresponding costandard module. The full subcategory \mathcal{T}_+ of representations with a filtration by Kac modules (simply called Kac objects) and the full subcategory \mathcal{T}_- of representations with a filtration by anti Kac modules (called anti Kac objects) are orthogonal in the sense that

$$Ext^1(\mathcal{T}_+, \mathcal{T}_-) = 0$$

Furthermore $\mathcal{T}_+ \cap \mathcal{T}_- = Proj$ (every tilting module is projective).

It can easily be shown that a module M is in \mathcal{T}_- if and only if its restriction to $P(m|n)^+$ is projective, i.e. zero in the stable category of $Rep(P(m|n)^+)$ (or equivalently injective, since projectives and injectives coincide in $Rep(P(m|n)^+)$ and also $Rep(GL(m|n))$). The stable category itself can be realized as the homotopy category of a model category. Roughly speaking a *model category* is a category with three classes \mathcal{L} (the *cofibrations*), \mathcal{R} (the *fibrations*) and \mathcal{W} (the *weak equivalences*) of morphisms enjoying various lifting properties. Its homotopy category is then the localization by the class of weak equivalences. The categories $Rep(P(m|n)^+)$ and $Rep(GL(m|n))$ are related by the faithful restriction functor Res , its left adjoint $Coind$ and its right adjoint Ind . It can be easily seen that $Coind \cong Ind$ and hence both are exact. This situation remains true if we allow all algebraic representations of $GL(m|n)$ (those corresponding to arbitrary comodules of $k[GL(m|n)]$), not just the finite dimensional ones. Indeed every algebraic representation is an inductive limit of finite dimensional ones, and hence the algebraic representations can simply be identified

with the ind-category $Rep(GL(m|n))^\infty$. By the functorial construction of the ind-category, the functors $Res, Coind$ and Ind extend with the same properties.

We are now in the following setting: Two abelian categories

$$\mathcal{C} = Rep(GL(m|n))^\infty, \mathcal{D} = Rep(P(m|n)^+)^\infty$$

which are related by two exact functors $U = Res$ and $F = Coind$

$$Hom_{\mathcal{C}}(FX, Y) \cong Hom_{\mathcal{D}}(X, UY).$$

A standard construction in model category allows to transfer a model structure on a model category \mathcal{D} to a model structure on a category \mathcal{C} provided there is a Quillen adjunction between them: An adjunction (F, U) such that F maps cofibrations to cofibrations and U maps fibrations to fibrations. In the case the model structure on \mathcal{D} is cofibrantly generated, these conditions simplify, but in our specific situation we will have to sacrifice the finite dimensionality and pass to the ind categories in order to satisfy the requirements.

Hence the stable model structure on $Rep(P(m|n)^+)^\infty$ defines a model structure on $Rep(GL(m|n))^\infty$. Every model category comes with two distinguished classes of objects, the *fibrant* objects, namely the objects X such that $0 \rightarrow X$ (where 0 is the initial object) is in \mathcal{L} , and the *cofibrant* objects X where $X \rightarrow *$ (where $*$ is the terminal object) is in \mathcal{R} . In our model category every object is fibrant, so we also consider the trivially fibrant objects X where $X \rightarrow *$ is in $\mathcal{R} \cap \mathcal{W}$. The cofibrant objects define the full subcategory \mathcal{C}_+ and the trivially fibrant objects the full subcategory \mathcal{C}_- . Then \mathcal{C}_- is the ind-category of the anti Kac objects \mathcal{T}_- and \mathcal{C}_+ is the ind-category of the Kac objects \mathcal{T}_+ . These two categories satisfy

$$Ext^1(\mathcal{C}_+, \mathcal{C}_-) = 0$$

and form a cotorsion pair on $\mathcal{T}_{m|n}^\infty$.

Every model category \mathcal{C} can be localized by the weak equivalences \mathcal{W} . The localization is called the *homotopy category* $Ho\mathcal{C}$ of \mathcal{C} . By definition of our model structure on $\mathcal{T}_{m|n}^\infty$, a morphism f is in \mathcal{W} if and only if $Res(f)$ is a weak equivalence for the stable model structure on $Rep(P(m|n)^+)^\infty$. It follows that the kernel of the functor $\mathcal{T}_{m|n}^\infty \rightarrow Ho\mathcal{T}_{m|n}^\infty$ to the homotopy category consists of the trivially fibrant objects \mathcal{C}_- , the inductive limits of objects in $\mathcal{T}_{m|n}$ with an anti Kac filtration. If we take instead the parabolic subgroup $P(m|n)^-$ of lower triangular block matrices for the definition of our model structure, the roles of $\mathcal{C}_+, \mathcal{T}_+, \mathcal{C}_-$ and \mathcal{T}_- switch.

One of the most important features of a model category is that every object X has a *cofibrant replacement* $QX \rightarrow X$ with $QX \in \mathcal{C}_+$ and kernel in \mathcal{C}_- . While these cofibrant replacements may seem to be a bit abstract on first

sight, note that in the $GL(m|n)$ -case a cofibrant replacement, say of $L(\lambda)$, is given by an exact sequence

$$0 \rightarrow A \rightarrow QL(\lambda) \rightarrow L(\lambda) \rightarrow 0$$

with $QL(\lambda) \in \mathcal{T}_+^\infty$ and $A \in \mathcal{T}_-^\infty$. The minimal cofibrant replacements (called the *minimal model* of $L(\lambda)$), if they exist, have a rather special form. We show in lemma 10.6 that atypical irreducible objects have a minimal model. To capture this we define a canonical degree filtration 14.1 (analog to the weight filtration in algebraic geometry) for each object with a Kac filtration. By lemma 14.1 each object $M \in \mathcal{T}_+$ has a filtration by submodules $F_i(M) \in \mathcal{T}_+$ such that

$$\dots \subseteq F_{i-1}(M) \subseteq F_i(M) \subseteq F_{i+1}(M) \subseteq \dots$$

and

$$F_i(M)/F_{i-1}(M) = \bigoplus_{\lambda} V(\lambda)$$

holds for certain Kac modules $V(\lambda) \in \mathcal{T}_+$ of degree $\deg(\lambda) = i$. This degree filtration extends to \mathcal{C}_+ and in particular to the minimal model of $L(\lambda)$. The minimal model lies in the exact subcategory \mathcal{C}_+^{pol} : objects M with a degree filtration F such that $F_k(M) = 0$ for some $k \in \mathbb{N}$ and $\dim gr_i^F(M) < C \cdot P(i)$ for all i where $C = C(M)$ is a constant and $P = P(M)$ a polynomial. To a Kac module $V(\lambda)$ we assign the power series

$$q^{-\deg(\lambda)}[V(\lambda)] \in K_0(\mathcal{T})[[q^{-1}]]$$

and extend this to sequential inductive limits of Kac-modules of polynomial growth via the degree filtration. Likewise we can associate a power series to any object in \mathcal{C}_- of polynomial growth. To link this to irreducible representations, we assign to $L(\lambda)$ the power series $q^{-\deg(\lambda)}[L(\lambda)]$. This extends to the exact subcategory $\mathcal{C}^{pol} \subset \mathcal{C}$ of inductive limits of polynomial growth of finite dimensional modules. Under the ring homomorphism $K_0(\mathcal{C}_+^{pol})$ to $K_0(\mathcal{C}^{pol})$ given by

$$q^{-\deg(\lambda)}[V(\lambda)] \mapsto \sum_L q^{-\deg(L)}[L]$$

where L runs over the irreducible constituents of $V(\lambda)$, the minimal model

$$0 \rightarrow A \rightarrow \Omega L(\lambda) \rightarrow L(\lambda) \rightarrow 0$$

gives via identifications in the power series ring the formula

$$[L(\lambda)] = [\Omega L(\lambda)] - [A].$$

Since the class of a Kac object V is the same as the one of the anti Kac object V^* for the twisted dual $(\)^*$ on $\mathcal{T}_{m|n}$, this can be seen as analogous to the resolutions of $L(\lambda)$ by Kac objects used for example by Serganova [Se96].

A particular important feature of the $GL(m|n)$ -case is the existence of a monoidal structure on $Ho\mathcal{C}$ such that $\mathcal{C} \rightarrow Ho\mathcal{C}$ is a tensor functor. We discuss this in more detail in section 1.6.

1.2. Induced model structures coming from Frobenius pairs. We axiomatize the setting of the $GL(m|n)$ -case now in the hope that the abstract mechanism to operate with Kac modules of $\mathcal{T}_{m|n}$ might be useful in other contexts. The representation categories are replaced by two abelian Frobenius categories \mathcal{C}, \mathcal{D}

$$\begin{aligned} Rep(GL(m|n))^\infty &\rightsquigarrow \mathcal{C} \\ Rep(P(m|n)^+)^\infty &\rightsquigarrow \mathcal{D} \end{aligned}$$

and the functors Res and $Coind$ by two functors U and F

$$\begin{aligned} Res &\rightsquigarrow U \\ Coind &\rightsquigarrow F. \end{aligned}$$

We assume now that we are given the following data - called a *Frobenius pair* $(\mathcal{C}, \mathcal{D})$ - as in section 4.1:

- (1) two abelian Frobenius categories \mathcal{C}, \mathcal{D} such that \mathcal{D} satisfies the additional conditions **FC.1** - **FC.4** (described in section 3), and
- (2) an adjoint pair of functors U, F between them satisfying

$$Hom_{\mathcal{C}}(FX, Y) \cong Hom_{\mathcal{D}}(X, UY)$$

such that U and F are exact and U is faithful.

Under these conditions the cofibrantly generated stable model structure on \mathcal{D} induces a model structure on \mathcal{C} . More precisely the model structure $(\mathcal{L}, \mathcal{R}, \mathcal{W})$ on \mathcal{C} satisfies $f \in \mathcal{W}$ if and only if $U(f) \in \mathcal{W}_D$ where the latter is defined via stable equivalence.

If we denote by \mathcal{C}_+ the cofibrant objects and by \mathcal{C}_- the trivially fibrant objects in this model structure, this defines a cotorsion pair in the sense of [BR07] [Ho02] on \mathcal{C} , and so in particular

$$Ext^1(\mathcal{C}_+, \mathcal{C}_-) = 0.$$

We stress that we do not obtain just a single model structure on \mathcal{C} in this way. Indeed any such pair $(\mathcal{C}, \mathcal{D})$ will give rise to a different cotorsion pair, a different model structure and a different homotopy category.

Theorem. (Theorem 5.1) *The homotopy category $Ho\mathcal{C}$ of the model category \mathcal{C} is equivalent as a triangulated category to the stable category $\mathcal{C}_+/P_{\mathcal{C}}$.*

1.3. Comodule categories. We apply this construction to the case where \mathcal{C} is the category of (graded) comodules of a supercommutative Hopf algebra A over a field k (with $\text{char}(k) \neq 2$). Then \mathcal{C} is the ind-category of the category \mathcal{T} of finite-dimensional comodules, and it is a Frobenius category if \mathcal{T} is one. If $A \rightarrow B$ is a quotient of supercommutative Hopf algebras, then their comodule categories \mathcal{C} and \mathcal{D} are related by induction and restriction functors. These need not satisfy the strong conditions imposed above. We call a pair (A, B) or the corresponding pair of affine supergroup schemes (H, G) where $H \subset G$, a Frobenius pair if the comodule categories $\mathcal{C} = \text{Comod}(A)$ and $\mathcal{D} = \text{Comod}(B)$ are Frobenius categories and $G_0 \subset H \subset G$ where G_0 is the underlying even algebraic group of G . In particular for any Frobenius pair the functor $\text{Res} : \text{Rep}(G)^\infty \rightarrow \text{Rep}(H)^\infty$ has a left and right adjoint which are isomorphic. In this situation we obtain a model structure on \mathcal{C} as in section 4. We point out that the same constructions also define a model structure (and hence a cotorsion pair) on the ind-category of $\text{Rep}(G)$ if $H \subset G$ are finite groups (considered as algebraic groups) over a field of characteristic $p \neq 2$ such that $|H|$ is not prime to p . We have not explored this further.

1.4. Monoidal model structures. So far we have not used any monoidal properties of our categories. Since the comodule category \mathcal{C} is a tensor category, we want of course that $\text{Ho}\mathcal{C}$ is again a tensor category such that the localization functor $\mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ is a tensor functor. This requires that the model structure on \mathcal{C} is compatible with the usual tensor product on \mathcal{C} and \mathcal{C} carries a monoidal model structure.

Theorem. (Theorem 7.1) \mathcal{C} is a monoidal model category and the functor $\gamma : \mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ is a tensor functor.

We may pass from a supercommutative Hopf algebra to the associated affine supergroup scheme G . Then the category \mathcal{T} of finite dimensional comodules is equivalent as a tensor category to the finite-dimensional algebraic representations of G . If G is an algebraic supergroup and has reductive even part (e.g. G is a basic supergroup such as $GL(m|n)$ or $OSp(m|2n)$), then the algebraic representations \mathcal{T} are a Frobenius category; and so our construction yields a cotorsion pair and a model structure on $\mathcal{C} \simeq \text{Rep}(G)^\infty$ for any embedded subgroup $G_0 \subset H \subset G$ with reductive even part H_0 . The kernel of $\gamma : \mathcal{C} \rightarrow \text{Ho}\mathcal{C}$ are then simply the representations which restrict to a projective representation on the subgroup. While the construction of the model structure works in this generality, one needs to choose H carefully to get an interesting theory alike to the $GL(m|n)$ -case.

1.5. Categories with weights. Now let \mathcal{T} be the category of finite dimensional representations of a (connected) algebraic supergroup G with reductive G_0 over an algebraically closed field k of characteristic $\text{char}(k) = 0$ and let \mathcal{C} be its ind-category. There is a notion of weights with an ordering \leq defined between weights. Then we can define $\mathcal{T}^{\leq w}$ and $\mathcal{C}^{\leq w}$ to be the full

subcategories of objects whose simple subquotients $L(\lambda)$ all satisfy $\lambda \leq w$. We axiomatize this situation and assume from section 9 onwards that our categories \mathcal{C}, \mathcal{D} are *categories with weights*, a weaker notion than that of a highest weight category. We then show

Theorem. (Theorem 9.3) *Every object in $\mathcal{C}^{\leq w}$ has a cofibrant replacement Z where Z is a direct sum of an injective object and an object in $\mathcal{C}^{\leq w}$.*

This theorem is important because it gives of some control on the morphism spaces in $Ho\mathcal{C}$ since

$$Hom_{Ho\mathcal{C}}(X, Y) \simeq Hom_{\mathcal{C}}(QX, Y) / \sim$$

by theorem 5.1. As an application we deduce the following important vanishing theorem:

Theorem. (Theorem 11.2) *If $(\mathcal{C}, \mathcal{D})$ are categories with weights*

$$Hom_{Ho\mathcal{C}}(QL(\mu), L(\lambda)) = 0 \quad \text{hence} \quad [L(\mu), L(\lambda)] = 0$$

for any irreducible objects $L(\lambda), L(\mu)$ for which $\mu < \lambda$.

Under some additional natural conditions A.1 - A.4 formulated in section 11.2 we deduce from this

Corollary. (Theorem 11.5) *If assumptions A.1 - A.4 hold, $[L(\lambda), L(\lambda)] = k \cdot id_{L(\lambda)}$.*

The most important special for us occurs if $\mathcal{C} = Rep(G)^\infty$ where G is an algebraic supergroup with reductive even part G_0 , e.g.

$$G = GL(m|n), \quad OSp(m|2n), \quad P(n), \quad Q(n)$$

or one of the exceptional simple supergroups. Any subgroup H with $G_0 \subset H \subset G$ defines a model structure on $Rep(G)^\infty$ (e.g. the upper triangular block matrices in G or a maximal parabolic containing G_0), but it is not even clear in the $OSp(m|2n)$ -case what the appropriate analogue of $P(m|n)^+ \subset GL(m|n)$ should be. Put $\mathcal{T} = Rep_k(G)$ and $\mathcal{T}_H = Rep_k(H)$ (or a related tensor category $Rep_k(\mu, G)$ etc.). Attached to the pair (H, G) we consider the ind categories \mathcal{C} of \mathcal{T} and \mathcal{D} of \mathcal{T}_H . We consider the following chain of functors

$$\gamma : \mathcal{C} \rightarrow Ho\mathcal{C} = \mathcal{C}_+ / \sim_{stable}$$

Let $\mathcal{H} = Ho\mathcal{T}$ be the full triangulated tensor subcategory of $Ho\mathcal{C}$ generated by the image of \mathcal{T} under γ . Then there is the functor

$$\gamma : \mathcal{T} \rightarrow \mathcal{H} = Ho\mathcal{T}$$

which in general is neither surjective nor injective on the set of morphisms.

Conjecture. (Conjecture 11.9) *If assumptions A.1 - A.4 hold and $\mathcal{C} = Rep(G)^\infty$ for an algebraic supergroup with reductive even part,, $\dim[X, Y] < \infty$ for any $X, Y \in \mathcal{T}$.*

This conjecture is a theorem in the $GL(m|n)$ -case.

1.6. The $GL(m|n)$ -case. In part II of our paper we study the aforementioned case of the Frobenius pair $(P(m|n)^+, GL(m|n))$. Our main technical result theorem 13.10 is that for any object X in \mathcal{T} there exists a cofibrant replacement $q_X : QX \rightarrow X$ with particularly nice properties.

Theorem. (Theorem 13.10) *For any object X in \mathcal{T} there exists a cofibrant replacement $q_X : QX \rightarrow X$ in \mathcal{C} with the following property: For any Y in \mathcal{T} there exists a subobject $K' \in QX$ of finite codimension contained in $\text{Kern}(q_X)$ such that $K' \in \mathcal{C}_-$ and such that $\text{Hom}_{\mathcal{C}}(K', Y) = 0$.*

Of course this is proven by giving an explicit construction of a cofibrant replacement $q : \Omega \rightarrow \mathbf{1}$ for the trivial representation in lemma 13.7 and then forming $QX = \Omega \otimes X$. An immediate consequence is the finite dimensionality of $[X, Y]$ for any $X, Y \in \mathcal{T}$.

We already explained in the beginning of the introduction that these cofibrant replacements should be seen as an abstract version of resolutions of objects by Kac objects. Similarly the dimension of the Hom space $[L(\lambda), L(\mu)]$ has a more concrete interpretation in the $GL(m|n)$ -case. Note that

$$[L(\lambda), L(\mu)] = \text{Hom}_{\mathcal{C}_+}(QL(\lambda), L(\mu)) = \text{Hom}_{\mathcal{C}}(QL(\lambda), L(\mu))$$

if $QL(\lambda)$ is clean by corollary 10.7. Using that $QL(\lambda) \in \text{Ind}(\mathcal{T}_+)$ and that

$$\text{Hom}_{\mathcal{C}}(V(\lambda), V(\mu)^*) = \delta_{\lambda\mu} k$$

by [Ge98, Proposition 3.6.2], one proves

$$\dim \text{Hom}_{\mathcal{C}}(QL(\lambda), V(\mu)^*) = [QL(\lambda) : V(\mu)],$$

the latter being the multiplicity of $V(\mu)$ in $QL(\lambda)$. Since every morphism of $QL(\lambda) \rightarrow L(\mu)$ extends via $L(\mu) \hookrightarrow V(\mu)^*$ to a morphism to $V(\mu)^*$, we obtain

$$\dim[L(\lambda), L(\mu)] \leq [QL(\lambda) : V(\mu)].$$

We do not know any direct representation theoretic interpretation of $\dim[L(\lambda), L(\mu)]$.

1.7. A second interpretation of $\text{Ho}\mathcal{T}$. The construction of well-behaved cofibrant replacements of objects in $\mathcal{T} = \mathcal{T}_{m|n}$ also enables us to give a different interpretation of $\text{Ho}\mathcal{T}$:

Theorem. (Theorem 13.13) *$\text{Ho}\mathcal{T}$ is equivalent as a tensor category to the Verdier quotient of the stable category $\overline{\mathcal{T}}$ by the thick tensor ideal \mathcal{T}_- of anti Kac modules.*

This theorem should not be read as the statement that we should view $\text{Ho}\mathcal{T}$ simply as that Verdier quotient. Instead both interpretations have their advantages. While theorem 13.13 gives a more concrete description of $\text{Ho}\mathcal{T}$, the important cofibrant replacements (i.e. the infinite resolutions by Kac objects) live naturally in the model theoretic interpretation.

The category $Ho\mathcal{T}$ is a k -linear rigid symmetric monoidal category with $End_{Ho\mathcal{T}}(\mathbf{1}) = k$, and the tensor ideal of negligible morphisms \mathcal{N} (the largest proper tensor ideal of $Ho\mathcal{T}$) is defined (see section 16).

Theorem. (Theorem 16.4) *For $GL(m|n)$ the quotient $Ho\mathcal{T}/\mathcal{N}$ is the semisimple representation category of an affine supergroup scheme.*

Parts of our motivation to study $Ho\mathcal{T}$ comes from our search of understanding the complicated monoidal structure of $\mathcal{T}_{m|n}$. Recent results in this area include the classification of thick ideals of $\mathcal{T}_{m|n}$ in [BKN17], a semisimplicity theorem about the Duflo-Serganova functor [HW14], a structural computation of tensor products up to superdimension 0 [He15] [HW15] and explicit tensor product decompositions for special classes of representations [He17]. One of the interesting aspects of the homotopy category is that it is the natural habitat of the Duflo-Serganova cohomology functor $DS : \mathcal{T}_{m|n} \rightarrow \mathcal{T}_{m-1|n-1}$ and its extension $DS : \mathcal{C}_{m|n} \rightarrow \mathcal{C}_{m-1|n-1}$ to the ind completion. Indeed the DS functor factorizes over the homotopy category and induces a tensor functor (see section 15.3)

$$\begin{aligned} DS : Ho\mathcal{C}_{m|n} &\rightarrow Ho\mathcal{C}_{m-1|n-1} \\ DS : Ho\mathcal{T}_{m|n} &\rightarrow Ho\mathcal{T}_{m-1|n-1} \end{aligned}$$

which might allow us to study the homotopy categories or their semisimple quotients inductively. Despite all these results the overall understanding of the tensor category \mathcal{T} is rather poor. Passing to $Ho\mathcal{T}$ simplifies the monoidal structure at the prize of complicating the category, as the following special case shows.

1.8. The $GL(m|1)$ -case. In the final sections we look at the $GL(m|1)$ -case. We compute morphism spaces between irreducible objects and use this to determine the indecomposable objects in $Ho\mathcal{T}_{m|1}$. In the homotopy category $Ho\mathcal{T}_{m|1}$ every indecomposable representation of non-vanishing superdimension becomes isomorphic to an irreducible representation. Hence the irreducible representations in $Ho\mathcal{T}/\mathcal{N}$ are parametrized by the atypical weights. Therefore $Ho\mathcal{T}/\mathcal{N}$ agrees with the quotient $\mathcal{I}_{m|1}/\mathcal{N}$ where $\mathcal{I}_{m|1}$ is the full tensor subcategory of $Rep(GL(m|1))$ of direct summands in iterated tensor products of irreducible representations. The latter has been determined in [He15] and we obtain

Proposition. (Proposition 18.2) *$Ho\mathcal{T}/\mathcal{N}$ is monoidal equivalent to the super representations of $GL(m|1) \times GL(1)$.*

The entire quotient $\mathcal{T}_{m|1}/\mathcal{N}$ is tensor equivalent to the super representations of $(GL(m-1) \times GL(1) \times GL(1))$. Hence passing to $Ho\mathcal{T}$ simplifies the complete quotient $\mathcal{T}_{m|1}/\mathcal{N}$, but retains all the information about the tensor products of irreducible representations. This might generalize to the $GL(m|n)$ -case. In fact let $\mathcal{I}_{m|n}$ denote the full tensor subcategory of $\mathcal{T}_{m|n}$ of direct summands in iterated tensor products of irreducible representations

and consider the full triangulated subcategory $Ho\mathcal{I}_{m|n}$ in $Ho\mathcal{T}_{m|n}$ generated by the image of $\mathcal{I}_{m|n}$. Then it seems plausible that

$$\mathcal{I}_{m|n}/\mathcal{N} \simeq Ho\mathcal{I}_{m|n}/\mathcal{N}.$$

In particular each indecomposable object of non-vanishing superdimension in $\mathcal{I}_{m|n}$ would stay indecomposable in $Ho\mathcal{C}$.

Part 1. Induced model structures on categories of comodules

2. BACKGROUND ON MODEL CATEGORIES

2.1. Model structures. For categories we assume that the morphisms between two objects form a set. A category is small if its objects define a set. A category \mathcal{C} has all small limits and colimits, if limits and colimits exist for all functors from small categories to \mathcal{C} . A category \mathcal{C} with small limits and colimits is a model category in the sense of [Ho99] if \mathcal{C} has a model structure. A model structure consists of classes $\mathcal{W}, \mathcal{L}, \mathcal{R}$ of morphisms (weak equivalences, cofibrations, fibrations) such that if two of three morphisms $f, g, f \circ g$ are in \mathcal{W} also the third is in \mathcal{W} .

We require the following three axioms [Ho99, Definition 1.1.3]: the retract axiom, the lifting axiom and the factorization axiom. The *lifting axiom* postulates the existence of a lifting h for commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow i & \nearrow h & \downarrow p \\ B & \xrightarrow{g} & D \end{array}$$

where $i \in \mathcal{W} \cap \mathcal{L}$ and $p \in \mathcal{R}$, or where $i \in \mathcal{L}$ and $p \in \mathcal{W} \cap \mathcal{R}$ (trivial fibrations). The *factorization axiom* states that every morphism f can be (functorially) written in the form $f = \psi \circ \varphi$ for certain $\varphi \in \mathcal{W} \cap \mathcal{L}$ and $\psi \in \mathcal{R}$, and also for certain $\varphi \in \mathcal{L}$ and $\psi \in \mathcal{W} \cap \mathcal{R}$. The *retract axiom* states that \mathcal{L}, \mathcal{R} or \mathcal{W} are stable under retracts.

2.2. Morphisms, fibrations and cofibrations. We recall the following definitions [Ho99, Definition 2.1.7]:

Definition 2.1. Let I be a set of morphisms in \mathcal{C} .

- (1) A morphism $p : C \rightarrow D$ is called I -injective, if it has the right lifting property (see diagram above) with respect to all morphisms $i : A \rightarrow B$ in I . The class of I -injective maps is denoted $Iinj$.
- (2) A morphism $i : A \rightarrow B$ is called I -projective, if i has the left lifting property with respect to all morphisms $p : C \rightarrow D$ in I . The class of I -projective maps is denoted $Iproj$.

- (3) A morphism is an I -cofibration if it has the left lifting property with respect to all I -injective maps. The class of I -cofibrations is the class $(Inj)proj$ and is denoted $Icof$.
- (4) A morphism is an I -fibration if it has the left lifting property with respect to all I -projective maps. The class of I -fibrations is the class $(Iproj)inj$ and is denoted $Ifib$.

The trivial fibrations $\mathcal{R} \cap \mathcal{W}$ are the \mathcal{L} -injective morphisms and $\mathcal{L} \cap \mathcal{W}$ are the \mathcal{R} -projective morphisms ([Ho99, lemma 1.1.10]). Since $f = p \circ i$ with $p \in \mathcal{R} \cap \mathcal{W}$ and $i \in \mathcal{L}$ implies $i \in \mathcal{W}$ for $f \in \mathcal{W}$, hence \mathcal{L} and \mathcal{R} determine

$$\mathcal{W} = (\mathcal{R} \cap \mathcal{W}) \circ (\mathcal{L} \cap \mathcal{W}) .$$

\mathcal{L} and \mathcal{R} are closed under compositions, (trivial) cofibrations are stable under pushout and (trivial) fibrations are stable under pullback [Ho99, Corollary 1.1.11]. \mathcal{C} has initial objects 0 and terminal objects $*$.

Definition 2.2. Let \mathcal{C}_+ resp. \mathcal{C}_- denote the full subcategory of objects X in \mathcal{C} for which $0 \rightarrow X$ is in \mathcal{L} resp. $X \rightarrow *$ is in $\mathcal{R} \cap \mathcal{W}$. Objects in \mathcal{C}_+ are called cofibrant and objects in \mathcal{C}_- trivially fibrant.

2.3. Cofibrant generation. Suppose \mathcal{C} admits arbitrary small limits and colimits. An object $X \in \mathcal{C}$ is called small, if it is κ -small for some cardinal κ : for all κ -filtered ordinals λ and all λ -sequences Y_i of morphisms in I the canonical morphism $colim_{i < \lambda} Hom(X, Y_i) \rightarrow Hom(X, colim_{i < \lambda} Y_i)$ is an isomorphism ([Ho99, p.29]). There is a similar notion for smallness with respect to a subcollection of the morphisms of \mathcal{C} . A model category is cofibrantly generated, if there exist sets of morphisms J and I , such that the domains of the morphisms in I (resp. J) are small with respect to I -cellular (resp. J -cellular) maps and if

- The fibrations \mathcal{R} are the J -injective morphisms $Jinj$
- The trivial fibrations $\mathcal{W} \cap \mathcal{R}$ are the I -injective morphisms $Iinj$.

Then $\mathcal{L} = (Inj)proj = Icof$ and $\mathcal{L} \cap \mathcal{W} = (Jinj)proj = Jcof$. Hence \mathcal{L} and $\mathcal{L} \cap \mathcal{W}$ are uniquely determined by I and J , such that $I \subset \mathcal{L}$ and $J \subset \mathcal{L} \cap \mathcal{W}$.

2.4. Quillen adjoint functors. Quillen adjoint functors are adjoint functors $F : \mathcal{D} \rightarrow \mathcal{C}$ and $U : \mathcal{C} \rightarrow \mathcal{D}$

$$Hom_{\mathcal{C}}(FX, Y) = Hom_{\mathcal{D}}(X, UY)$$

between model categories \mathcal{D} and \mathcal{C} such that one of the following three equivalent conditions holds (see [DS95, p.43])

- F maps (trivial) cofibrations to (trivial) cofibrations
- U maps (trivial) fibrations to (trivial) fibrations.
- F maps cofibrations to cofibrations and U maps fibrations to fibrations.

If \mathcal{D} is cofibrantly generated by J and I this holds if $FI \subset \mathcal{L}_{\mathcal{C}}$ and $FJ \subset \mathcal{L}_{\mathcal{C}} \cap \mathcal{W}_{\mathcal{C}}$ (see [Ho99, p.14 and p.36, Lemma 2.1.20]).

2.5. Monoidal model structure. A symmetric monoidal category (\mathcal{C}, \otimes) (see [Ho99, p.101 ff]) will be called closed monoidal, if internal $\mathcal{H}om$'s exist with functorial isomorphisms

$$\mathit{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathit{Hom}_{\mathcal{C}}(X, \mathcal{H}om(Y, Z))$$

and the properties of [Ho99, Definition 4.1.13]. A model structure on a closed symmetric monoidal category \mathcal{C} is called a symmetric monoidal model category if it satisfies the following two conditions [Ho99, Definition 4.2.6]:

- (1) The tensor functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is a Quillen bifunctor [Ho99, Definition 4.2.1].
- (2) If the unit $\mathbf{1}_{\mathcal{C}}$ is not cofibrant, factor $0 \rightarrow \mathbf{1}$ into a cofibration and a trivial fibration $q : Q\mathbf{1} \rightarrow \mathbf{1}$. Then we require that $q \otimes id : Q\mathbf{1} \otimes X \rightarrow \mathbf{1} \otimes X$ and $id \otimes q : X \otimes Q\mathbf{1} \rightarrow X \otimes \mathbf{1}$ are in \mathcal{W} for all cofibrant X .

3. THE STABLE MODULE CATEGORY OF A FROBENIUS CATEGORY

An abelian (or more generally an exact) category \mathcal{D} is a *Frobenius category* if it has enough projectives and enough injectives, and if the subcategories $\mathcal{P}_{\mathcal{D}}$ of projective objects and the subcategory $\mathcal{I}_{\mathcal{D}}$ of injective objects coincide $\mathcal{P}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}}$.

Attached to an exact category \mathcal{D} with enough injective objects is its stable category $\overline{\mathcal{D}}$ [Ha] which is a suspended category. It has the same objects as \mathcal{D} . A morphism of $\overline{\mathcal{D}}$ is an equivalence class \overline{f} of a morphism $f : X \rightarrow Y$ in \mathcal{D} modulo the subgroup of morphisms factoring through an injective module. Objects $X, Y \in \mathcal{D}$ are called stably equivalent if they become isomorphic in $\overline{\mathcal{D}}$. The suspension $SX = X[1]$ of X is defined via an exact sequence (X, IX, SX) , where $i : X \rightarrow IX$ is an injective resolution and $SX = IX/i(X)$. Any morphism $f : X \rightarrow Y$ lifts to a morphism $If : IX \rightarrow IY$, hence defines a suspension morphism $SX \rightarrow SY$ whose equivalence class Sf is well defined in $\overline{\mathcal{D}}$, i.e. independent of the resolutions and the choice of the lift If . Associated to an exact sequence (X, Y, Z, j, p) in \mathcal{D} is a standard triangle $(X, Y, Z, i, p, \partial)$ in $\overline{\mathcal{D}}$, where $\partial : Z \rightarrow SX$ is the well defined class of the right vertical arrow

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y & \xrightarrow{p} & Z \\ \parallel & & \vdots & & \vdots \\ X & \xrightarrow{i} & IX & \xrightarrow{\quad} & SX \end{array}$$

in $\overline{\mathcal{D}}$. A triangle (A, B, C, a, b, c) in $\overline{\mathcal{D}}$ is called distinguished, if it is isomorphic to a standard triangle in $\overline{\mathcal{D}}$. Thus $\overline{\mathcal{D}}$ becomes a suspended category. If the exact category \mathcal{D} is a Frobenius category, $\overline{\mathcal{D}}$ is a triangulated category (see [Ch11], [Ha] or [Ke96, p.9]). This is shown by using the loop functor defined by a projective resolution in a similar way. The following lemma is well-known.

Lemma 3.1. *If $\overline{\mathcal{D}}$ is the stable category of a Frobenius category \mathcal{D} , then for $n \geq 1$*

$$\text{Ext}_{\mathcal{D}}^n(A, B) \cong \text{Hom}_{\overline{\mathcal{D}}}(A, B[n]) .$$

If \mathcal{D} has arbitrary coproducts, the stable category $\overline{\mathcal{D}}$ has arbitrary coproducts as well since the coproduct of objects X_i , $i \in I$, in $\overline{\mathcal{D}}$ is represented by the coproduct $\bigoplus_{i \in I} X_i$ in \mathcal{D} (both categories have the same objects). Furthermore coproducts of projective objects are projective. From [Ne01, Proposition 1.6.8 and Lemma 3.2.10] we easily conclude

Lemma 3.2. *Let \mathcal{D} be a Frobenius category with arbitrary coproducts. Let \mathcal{H} be a quotient category of the triangulated stable category $\overline{\mathcal{D}}$ of \mathcal{D} divided by a thick triangulated subcategory. Then \mathcal{H} is pseudo-abelian and has arbitrary coproducts, and the functor $\mathcal{D} \rightarrow \mathcal{H}$ commutes with arbitrary coproducts.*

A particular example of a Frobenius category is the category of left R -modules of a Frobenius ring as in [Ho99, 2.2]. In this setting the stable category of R -mod carries a cofibrantly generated model structure [Ho99, Theorem 2.2.12]. A general Frobenius category \mathcal{D} carries a model structure for which the associated homotopy category is the stable category [Li17]. Under mild additional conditions $\overline{\mathcal{D}}$ is the homotopy category of a cofibrantly generated model structure on \mathcal{D} .

Assume the properties **FC.1-FC.4**.

FC.1 \mathcal{D} has small limits and colimits

FC.2 Any object of \mathcal{D} is small with respect to \mathcal{D} in the sense of [Ho99, p.29]

FC.3 There exists a set $P_{\mathcal{D}}$ of projective objects in \mathcal{D} being generators of \mathcal{D} in the sense that any nontrivial object X of \mathcal{D} admits a nontrivial morphism $P \rightarrow X$ for some $P \in P_{\mathcal{D}}$. We call $P_{\mathcal{D}}$ an admissible set of projectives and denote by J the set of monomorphisms $0 \rightarrow P$ for $P \in P_{\mathcal{D}}$.

FC.4 There exists a set I of monomorphisms $i : A \rightarrow B$ in \mathcal{D} containing J such that $X \in \mathcal{D}$ is in $\mathcal{I}_{\mathcal{D}}$ if and only if $i^* : \text{Hom}_{\mathcal{D}}(B, X) \rightarrow \text{Hom}_{\mathcal{D}}(A, X)$ is surjective for all $i \in I$. We call I an admissible set of monomorphisms.

We ignore whether these conditions can be relaxed. The proof of the following lemma is analog to the proofs of [Ho99] and [Li17] and will be skipped.

Lemma 3.3. *A Frobenius category \mathcal{D} with the properties FC1-4 carries a cofibrantly generated model structure called the stable model structure. The fibrations are the epimorphisms, the trivial fibrations are the split epimorphisms with kernel in $\mathcal{I}_{\mathcal{D}}$. The cofibrations are the monomorphisms, and the trivial cofibrations are the split monomorphisms with cokernel in $\mathcal{P}_{\mathcal{D}}$. Every object is fibrant and cofibrant. The associated homotopy category $Ho\mathcal{D}$ is the stable category $\overline{\mathcal{D}}$ of \mathcal{D} . The morphisms in \mathcal{W} consist of the morphisms for*

which $\text{Ext}^i(X, f)$ and $\text{Ext}^i(f, X)$ are isomorphisms for all objects $X \in \mathcal{D}$ and all $i \geq 1$. The model structure is cofibrantly generated by I and J as above.

4. INDUCED MODEL STRUCTURE

4.1. Induced model structures. The following construction of D. M. Kan can be used to lift model structures from a cofibrantly generated model category to another [Hi03, Theorem 11.3.1, Theorem 11.3.2]. Let \mathcal{D} be a cofibrantly generated model category generated by $J_{\mathcal{D}}$ and $I_{\mathcal{D}}$ (we may assume $I_{\mathcal{D}}$ contains $J_{\mathcal{D}}$). Let \mathcal{C} be a category with small limits and colimits and adjoint functors

$$F : \mathcal{D} \rightarrow \mathcal{C} \quad , \quad U : \mathcal{C} \rightarrow \mathcal{D}$$

such that $\text{Hom}(FX, Y) = \text{Hom}(X, UY)$. Suppose U is faithful. Put $J = F(J_{\mathcal{D}})$ and $I = F(I_{\mathcal{D}})$ and suppose

$$U(\text{Jcof}) \subset \mathcal{W}_{\mathcal{D}}$$

(and certain smallness conditions in \mathcal{C} automatically fulfilled in our later cases). Then there exists a model structure $(\mathcal{L}, \mathcal{R}, \mathcal{W})$ on \mathcal{C} generated by J and I such that: $f \in \mathcal{W}$ if and only if $U(f) \in \mathcal{W}_{\mathcal{D}}$. The functors (F, U) define a Quillen adjunction.

The abelian case. We will only apply this in the case where \mathcal{C} and \mathcal{D} are abelian categories. So let us always assume this in the following. Recall that for an adjoint pair of functors $F : \mathcal{D} \rightarrow \mathcal{C}$ and $U : \mathcal{C} \rightarrow \mathcal{D}$ between abelian categories F preserves colimits and U preserves limits (e.g. [GM96, p.137]). Hence, if F is left exact (U is right exact), then F (resp. U) is exact and U preserves injectives (resp. F preserves projectives). The adjunction morphisms $ad : FUX \rightarrow X$ are epimorphisms if U is faithful $U(Z) = 0 \Leftrightarrow Z = 0$ ([GM96, p.61 formula II.16]). Similarly the adjunction morphisms $Y \rightarrow UFY$ are monomorphisms if F is faithful $F(Z) = 0 \Leftrightarrow Z = 0$.

Proposition 4.1. *Assume that \mathcal{C} and \mathcal{D} are abelian Frobenius categories, and that \mathcal{D} satisfies the finiteness assumptions **FC.1** - **FC.4**. Assume also that F and U are exact. Consider the stable model structure on \mathcal{D} cofibrantly generated by the set $J_{\mathcal{D}} = \{0 \rightarrow X, X \in P_{\mathcal{D}}\}$ and by a set $I_{\mathcal{D}} \supset J_{\mathcal{D}}$ of admissible monomorphisms, where $\mathcal{W}_{\mathcal{D}}$ is defined by stable equivalence.*

- Then

$$\text{Jcof} \subset \mathcal{W} \cap \{\text{split mono}\}.$$

- There exists a model structure $(\mathcal{L}, \mathcal{R}, \mathcal{W})$ on \mathcal{C} generated by J and I (as defined above) such that: $f \in \mathcal{W}$ if and only if $U(f) \in \mathcal{W}_{\mathcal{D}}$. The functors (F, U) define a Quillen adjunction.

Definition 4.2. We call a pair of abelian Frobenius categories \mathcal{C} and \mathcal{D} with an adjoint pair of functors U, F satisfying the conditions of proposition 4.1 a Frobenius pair.

We only have to prove part (1) of proposition 4.1. This claim immediately follows from the next two lemmas, since $U(\mathcal{P}_{\mathcal{C}}) \subset \mathcal{P}_{\mathcal{D}}$.

Lemma 4.3. *The morphisms $p : C \rightarrow D$ in $\mathcal{R} = \text{Jinj}$ are the epimorphisms in \mathcal{C} . Every object of \mathcal{C} is fibrant.*

Proof. Let Q be the kokernel of p . Notice $Q = 0$ if and only if $UQ = 0$, since U is faithful. If UQ were not zero, choose a nontrivial morphism $u : P \rightarrow UQ$ for $P \in \mathcal{P}_{\mathcal{D}}$ by **FC.3**, lift it to a morphism $P \rightarrow UD$ (projectivity of P) and consider the adjoint morphism $g : FP \rightarrow D$. Since $0 \rightarrow FP$ is in J , for $p \in \mathcal{R}$ the morphism g lifts by definition to a morphism $h : FP \rightarrow C$. This implies that the adjoint morphism $FP \rightarrow Q$ attached to $u : P \rightarrow UQ$ is zero, since it factorizes over the zero morphism $C \rightarrow D \rightarrow Q$. Hence u itself is zero. Contradiction. \square

The proof of the next lemma is similar to [Ho99, Lemma 2.2.11] and will be skipped.

Lemma 4.4. *The morphisms $i : A \rightarrow B$ in $\mathcal{Rproj} = \text{Jcof}$ are the split monomorphisms in \mathcal{C} with projective kokernel.*

4.2. Explicit description of the induced model structure. In fact we now explicitly describe these model structures via [Ho99, 2.1.19]. Again let us first ignore some important finiteness conditions that have to be imposed. The reason is that these finiteness conditions are satisfied for categories of comodules of supercommutative Hopf algebras A (where we apply this). We usually call the functor U the restriction functor.

Step I. Define \mathcal{W} as before by $f \in \mathcal{W}$ if and only if $U(f) \in \mathcal{W}_{\mathcal{D}}$. Then the two out of three property holds and \mathcal{W} is closed under retracts

Step II. Define $J = F(\mathcal{J}_{\mathcal{D}})$ and $I = F(\mathcal{I}_{\mathcal{D}}) \supset J$ as in the last section. Put $\mathcal{R} = \text{Jinj}$ and $\mathcal{L} = \text{Icof}$. Notice $I \subset \text{Icof} = \mathcal{L}$. For $(\mathcal{L}, \mathcal{R}, \mathcal{W})$ to be a model structure it suffices by the smallness property stated above that

- (1) $\text{Jcof} \subset \mathcal{W}$ (obvious by lemma 4.4),
- (2) $\text{Iinj} \subset \mathcal{W}$, and then also $\subset \mathcal{W} \cap \mathcal{R}$ since $\mathcal{R} = \text{Jinj} \supset \text{Iinj}$
- (3) $(\text{Jinj}) \cap \mathcal{W} =: \mathcal{R} \cap \mathcal{W} \subset \text{Iinj}$

The last property might be replaced by $(\text{Icof}) \cap \mathcal{W} =: \mathcal{L} \cap \mathcal{W} \subset \text{Jcof}$, also denoted property 3'. Since all four conditions necessarily hold in a model category, this shows that conditions 1,2,3 imply 3'. Since $J \subset I$ in our situation, we also have $\text{Jcof} \subset \text{Icof} = \mathcal{L}$ and therefore by property 1

$$(\text{Icof}) \cap \mathcal{W} = \mathcal{L} \cap \mathcal{W} = \text{Jcof}$$

once we have shown the properties 2 and 3.

Property 2. $\text{Iinj} \subset \mathcal{W}$ follows from the next

Lemma 4.5. *Iinj consists of the epimorphisms $p \in \mathcal{R}$ with kernel K so that UK is injective.*

Proof. Recall that I was the class of all morphisms $F(i)$ for admissible monomorphisms $i : A \rightarrow B$. Consider an I -injective morphism $p : X \rightarrow Y$

$$\begin{array}{ccc} F(A) & \longrightarrow & X \\ F(i) \downarrow & \nearrow h & \downarrow p \\ F(B) & \longrightarrow & Y \end{array}$$

Recall $J \subset I$, hence $Iinj \subset Jinj$, and therefore p must be an epimorphism by lemma 4.3. Put $K = \text{kern}(p)$ and $Q = K\text{okern}(i)$. We now pass to the restriction diagram using adjunction

$$\begin{array}{ccc} & UK & \\ & \downarrow & \\ A & \longrightarrow & UX \\ i \downarrow & \nearrow & \downarrow U(p) \\ B & \longrightarrow & UY \\ \downarrow & & \\ Q & & \end{array}$$

Hence $U(p)$ has to be $I_{\mathcal{D}}$ -injective with respect to the class $I_{\mathcal{D}}$ of all admissible monomorphisms in \mathcal{D} . We have shown that this is equivalent to $U(p)$ is a split epimorphism with injective kernel UK . Hence K is in \mathcal{C}_- and $p : X \rightarrow Y$ is in \mathcal{W} . \square

Property 3 is now an immediate consequence of lemma 4.3 and 4.5.

Corollary 4.6. *An object K is in \mathcal{C}_- if and only if UK is injective (if and only if $K \rightarrow 0$ is in $\mathcal{R} \cap \mathcal{W} = Iinj$).*

Obviously $\mathcal{I}_{\mathcal{C}} \subset \mathcal{C}_-$ and \mathcal{C}_- is closed under retracts.

Corollary 4.7. *The morphisms in $\mathcal{R} \cap \mathcal{W} = Iinj$ are the epimorphisms p with kernel in $K \in \mathcal{C}_-$.*

Property 3' now holds automatically, and using lemma 4.4 gives

Corollary 4.8. *The morphisms in $\mathcal{L} \cap \mathcal{W}$ are the split monomorphisms with projective cokernel.*

We now describe the class \mathcal{L} . We start with a technical lemma. In an abelian category \mathcal{C} we consider diagrams

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow i & \nearrow h & \downarrow p \\ B & \xrightarrow{g} & D \end{array}$$

with epimorphism p (with kernel K) and monomorphism i (with kokernel Q) and look for liftings h making the diagram commutative.

Lemma 4.9. [BR07, Chapter VIII Lemma 3.1] *Under the assumption $Ext^1(Q, K) = 0$ the diagram above has the lifting property in the situation above (i monomorphism and p epimorphism).*

Define \mathcal{L}_- to be the class of morphisms

$$\mathcal{L}_- = \{\mathcal{C}_- \rightarrow 0\}proj .$$

Then $\mathcal{L} = Icof \subset \mathcal{L}_-$ is a consequence of lemma 4.5. Since F and U are exact, F preserves monomorphisms and projectives, and U preserves injectives and monomorphisms (since U was supposed to be faithful). If \mathcal{C} has enough injectives, this implies that any morphism in \mathcal{L}_- is a monomorphism. Indeed it suffices to observe that any $A \in \mathcal{C}$ can be embedded into an injective object L of \mathcal{C} . Since $\mathcal{I}_{\mathcal{C}} \subset \mathcal{C}_-$ the lifting property then forces any $i \in \mathcal{L}_-$ to be a monomorphism.

Lemma 4.10. \mathcal{L} is the class of monomorphisms in \mathcal{C} whose kokernel is in \mathcal{C}_+ .

Proof. We know $\mathcal{L} \subset \mathcal{L}_-$ and any morphism $i : A \rightarrow B$ in \mathcal{L}_- is a monomorphism, since \mathcal{C} has enough injectives. Let Q denote the kokernel of i . Since the class of cofibrations is always closed under pushouts ([Ho99, cor.1.1.11]), $i \in \mathcal{L}$ implies that $0 \rightarrow B/i(A) = Q$ is in \mathcal{L} , i.e. cofibrant

$$\begin{array}{ccc} A & \longrightarrow & 0 \\ \downarrow i \in \mathcal{L} & & \downarrow \\ B & \longrightarrow & B/i(A) \end{array}$$

Hence $Q \in \mathcal{C}_+$ is a necessary condition (any extension E of Q by $K \in \mathcal{C}_-$ gives rise to a morphisms $p : E \rightarrow Q$ in $\mathcal{R} \cap \mathcal{W}$). For the converse we have to show the lifting property for epimorphisms p with kernel in \mathcal{C}_- . The claim follows from lemma 4.9 since $Q \in \mathcal{C}_+$ and $K \in \mathcal{C}_-$ □

Corollary 4.11. $\mathcal{C}_- \cap \mathcal{C}_+ = \mathcal{I}_{\mathcal{C}}$.

Proof. From the description of \mathcal{L} from above and the description of \mathcal{W} we conclude: $i \in \mathcal{L} \cap \mathcal{W}$ if and only if i is monomorphisms with kokernel $Q \in \mathcal{C}_+ \cap \mathcal{C}_-$. On the other hand we know that $\mathcal{L} \cap \mathcal{W}$ are the split monomorphisms with injective kokernel. Considering this for the monomorphisms $i : 0 \rightarrow Q$, the corollary follows. □

Corollary 4.12. (1) *The fibrations \mathcal{R} are the epimorphisms.*
 (2) *The trivial fibrations $\mathcal{R} \cap \mathcal{W}$ are the epimorphisms with kernel in \mathcal{C}_- .*
 (3) *The cofibrations \mathcal{L} are the monomorphisms with kokernel in \mathcal{C}_+ .*
 (4) *The trivial cofibrations $\mathcal{L} \cap \mathcal{W}$ are the split monomorphisms with kokernel in $\mathcal{C}_+^{triv} = \mathcal{P}_{\mathcal{C}}$.*

4.3. Cotorsion pairs. Model categories whose morphisms satisfy certain good properties as in corollary 4.12 give rise to cotorsion pairs as described in [BR07] [Ho02]. Recall from [BR07, Chapter V Definition 3.1] the following definition: A cotorsion pair in an abelian category \mathcal{C} is a pair of full subcategories $(\mathcal{X}, \mathcal{Y})$ closed under isomorphisms and direct summands satisfying

- (1) $Ext^1(\mathcal{X}, \mathcal{Y}) = 0$.
- (2) For any object $C \in \mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow Y_C \longrightarrow X_C \longrightarrow C \longrightarrow 0$$

with $Y_C \in \mathcal{Y}$ and $X_C \in \mathcal{X}$.

- (3) For any object $C \in \mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow C \longrightarrow Y^C \longrightarrow X^C \longrightarrow 0$$

with $Y^C \in \mathcal{Y}$ and $X^C \in \mathcal{X}$.

Furthermore [BR07, Chapter I Definition 2.1] a torsion pair in \mathcal{C}' (\mathcal{C}' triangulated with suspension Σ) is a pair of strict full subcategories $(\mathcal{X}, \mathcal{Y})$ satisfying

- (1) $Hom(\mathcal{X}, \mathcal{Y}) = 0$.
- (2) $\Sigma\mathcal{X} \subset \mathcal{Y}$ and $\Sigma^{-1}\mathcal{Y} \subset \mathcal{X}$.
- (3) For any $C \in \mathcal{C}'$ there exists a triangle

$$X_C \longrightarrow C \longrightarrow Y^C \longrightarrow \Sigma(X_C)$$

in \mathcal{C} such that $X_C \in \mathcal{X}$ and $Y^C \in \mathcal{Y}$.

Theorem 4.13. *S.1 $\mathcal{C}_+^{triv} = Proj$.*

S.2 The pairs $(\mathcal{C}_+, \mathcal{C}_-)$ and $(\mathcal{C}_+^{triv}, \mathcal{C})$ are cotorsion pairs in \mathcal{C} .

S.3 $(\mathcal{C}_+, \mathcal{C}_-)$ and $(\mathcal{C}_+^{triv} = Proj, \mathcal{C})$ define torsion pairs in the triangulated category $\bar{\mathcal{C}}$. In particular any map between a cofibrant object X and a trivially fibrant object Y factors through a projective object.

S.4 \mathcal{C}_- and \mathcal{C}_+ are closed under extensions and cokernels of monics and kernels of epis.

S.5 \mathcal{C}_+ is a Frobenius category.

Proof. S.1: By [BR07, Chapter VIII Lemma 2.2] $\mathcal{C}_+ \cap \mathcal{C}_- = \mathcal{C}_-^{triv}$, therefore $\mathcal{C}_+^{triv} = Proj$.

S.2 is [BR07, Chapter VIII Proposition 3.4].

S.3 follows from [BR07, Chapter VIII Corollary 3.7] and [BR07, Chapter I Proposition 2.6] (see also [BR07, Chapter VIII Proposition 2.1]).

S.4 follows from [BR07, Chapter V Lemma 2.4] and [BR07, Chapter I Corollary 2.9].

S.5 is [BR07, Chapter VI Theorem 2.1].

□

Remark 4.14. The torsion pair is hereditary (see [BR07, Chapter I Definition 2.5]) and the categories \mathcal{C}_- and \mathcal{C}_+ are resolving and coresolving.

4.4. Permanence properties of cofibrant objects. We discuss some easy consequences for \mathcal{C}_+ and \mathcal{C}_- .

Lemma 4.15. *For a monomorphism in \mathcal{C}_+*

$$i : N \hookrightarrow N'$$

any morphism

$$\varphi : N \rightarrow M \quad , \quad M \in \mathcal{C}_-$$

can be extended to a morphism $\varphi' : N' \rightarrow M$ such that $\varphi' \circ i = \varphi$.

Proof. We may replace $\varphi : N \rightarrow M$ by $\varphi : N \rightarrow I$ for some injective object by theorem 4.13. Then the assertion becomes obvious. \square

Corollary 4.16. *\mathcal{C}_+ is closed under sequential direct limits with monomorphic transition maps. Similarly \mathcal{C}_- is closed under sequential projective limits with surjective transition maps.*

Proof. Suppose $X = \text{colim}_i X_i$ with $X_i \in \mathcal{C}_+$. Consider an extension E with class in $\text{Ext}^1(X, M)$ and $M \in \mathcal{C}_-$. Its pullback to each X_i splits. Let $\lambda_i : X_i \rightarrow E$ be a splitting morphism. It is unique up to a morphism $X_i \rightarrow M$. Since $\varphi : \lambda_{i+1}|_{X_i} - \lambda_i : X_i \rightarrow M$ can be extended to X_{i+1} by the last lemma, one can choose λ_{i+1} in such a way that $\varphi = 0$. Proceeding inductively gives a compatible system of splittings which define a splitting $\lambda : X = \text{colim}_i X_i \rightarrow E$. Hence $\text{Ext}^1(X, M) = 0$. The second assertion can be proven similarly. \square

Corollary 4.17. *For $C \in \mathcal{C}_+$ we have $\text{Ext}^i(C, M) = 0$ for all $M \in \mathcal{C}_-$ and all $i \geq 1$.*

Proof. Choose a monomorphism $M \hookrightarrow I$ with $I \in \mathcal{I}_{\mathcal{C}} \subset \mathcal{C}_-$. Then $I/M \in \mathcal{C}_-$, hence $\text{Ext}^2(C, M) \cong \text{Ext}^1(C, I/M) = 0$. Then proceed by induction on i to complete the proof. \square

5. THE HOMOTOPY CATEGORY $\text{Ho}\mathcal{C}$

We now discuss properties of the homotopy category of \mathcal{C} and give a more explicit description as the stable category of the category of cofibrant objects. We first recall some background about cofibrant replacements and the homotopy category.

5.1. Cofibrant replacements. Every object $X \in \mathcal{C}$ has a cofibrant replacement $q : QX \rightarrow X$ where QX is cofibrant and q is a trivial fibration. We denote by Q the cofibrant replacement functor $\mathcal{C} \rightarrow \mathcal{C}_+$, $X \mapsto QX$. Note that the morphism q defines an extension

$$0 \rightarrow K \rightarrow QX \rightarrow X \rightarrow 0$$

with kernel $K \in \mathcal{C}_-$, i.e. $U(K) \in \mathcal{I}_{\mathcal{D}}$.

5.2. The homotopy category. The homotopy category of \mathcal{C} is the localization $Ho\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$ by the weak equivalences. The functor $\gamma : \mathcal{C} \rightarrow Ho\mathcal{C}$ is then the identity on objects and takes morphisms in \mathcal{W} to isomorphisms. It is universal with this property in the sense of [Ho99, Lemma 1.2.2]. For $X, Y \in \mathcal{C}$ there are natural isomorphisms

$$Hom_{Ho\mathcal{C}}(\gamma X, \gamma Y) \simeq Hom_{\mathcal{C}}(QX, RY) / \sim$$

where \sim denotes the homotopy equivalence [Ho99, Theorem 1.2.10]. We often abbreviate $Hom_{Ho\mathcal{C}}(X, Y)$ or $Hom_{\mathcal{C}}(\gamma X, \gamma Y)$ as $[X, Y]$.

The category $\mathcal{E} = \mathcal{C}_+$ is an exact subcategory of the abelian category \mathcal{C} in the sense of [Ke96] (section 4 and 5) (and similarly is \mathcal{C}_-). In particular \mathcal{E} is closed under extensions. \mathcal{E} contains $\mathcal{I}_{\mathcal{C}} = \mathcal{P}_{\mathcal{C}}$. Since $\mathcal{P}_{\mathcal{C}} \subset \mathcal{P}_{\mathcal{E}}$ and $\mathcal{I}_{\mathcal{C}} \subset \mathcal{I}_{\mathcal{E}}$, the category \mathcal{E} has enough injectives and projectives.

Theorem 5.1. *H.1 \mathcal{C}_+ is a Frobenius category. Hence $\bar{\mathcal{C}}_+ = \mathcal{E}/\mathcal{I}_{\mathcal{E}}$ is a triangulated category.*

H.2 $Ho\mathcal{C}$ is a triangulated category and quasi-equivalent to the stable category of \mathcal{C}_+

$$Ho\mathcal{C} = \bar{\mathcal{C}}_+ ,$$

and the functor $\mathcal{C} \rightarrow Ho\mathcal{C}$ is k -linear. In particular

$$[X, Y] = Hom(QX, Y) / \sim_{stable} .$$

H.3 The functors F, U preserve distinguished triangles and the shift functor.

Proof. (1) **H.1** follows from [BR07, Chapter VI Theorem 2.1].
 (2) **H.2** follows from [BR07, Chapter VIII Theorem 4.2, Corollary 4.5].
 (3) **H.3** follows from **H.1** and **H.2**.

□

6. COMODULES AND SUPERCOMMUTATIVE HOPF ALGEBRAS

The main example of a Frobenius pair in this paper comes from representation theory. Let $\pi : A \rightarrow B$ be a surjective homomorphism of supercommutative Hopf algebras. It corresponds to an embedding $H \subset G$ of supergroups. This embeddings induces a restriction functor $Res : Rep(G) \rightarrow Rep(H)$ which is an exact faithful tensor functor. We discuss its left and right adjoint (called coinduction and induction). Under suitable conditions on H and G this will give rise to a Frobenius pair.

6.1. k -linear tensor categories with finiteness conditions. For a k -linear abelian category \mathcal{T} consider the following finiteness conditions

Condition (F). Every object has finite length and all morphism spaces $Hom(X, Y)$ have finite k -dimension.

Condition (G). The category \mathcal{T} is monoidal and there exists an object X such that any object is a subquotient of some tensor power of X .

Modules. If conditions (F) and (G) hold, then the abelian category admits a projective generator ([Del90, Proposition 2.14]). Hence by Morita's theorem it is equivalent to the category of right modules of some k -algebra of finite rank ([Del90, Corollary 2.17]). In particular there exist enough injectives and projectives. By assumption (F) the category is nice in the sense of [Ge98, p.371]: Indecomposable objects in \mathcal{T} have unique simple quotients (and by duality unique simple submodules) and have local endomorphism rings. Any object satisfies the Krull-Schmidt theorem, i.e. is isomorphic to a finite direct sum of indecomposable objects. The isomorphism classes in \mathcal{T} of these indecomposable objects (up to permutation of the summands) do not depend on the decomposition. Up to isomorphism objects have unique projective covers resp. injective hulls. For any object X there are only finitely many indecomposable objects Y with $Hom(Y, X) \neq 0$.

Comodules. By [Si02, Theorem 2.13] a k -linear abelian category \mathcal{T} is k -linear equivalent to the category of k -finite dimensional comodules over some k -coalgebra A if and only if the finiteness conditions (F) hold. By [B77, p.141-146] in the category \mathcal{C} of all comodules of a fixed k -coalgebra A an injective comodule L in \mathcal{C}' is a direct sum $L = \bigoplus_{\mu} J_{\mu}$ of indecomposable injective subcomodules J_{μ} , and each J_{μ} contains a unique simple subcomodule X_{μ} . All simple A -comodules are finite dimensional, and the comodules J_{μ} are injective hulls in \mathcal{T} and hence are k -finite dimensional. Hence injective objects $L \in \mathcal{C}$ are direct sums of k -finite dimensional injective objects. In particular \mathcal{C} has enough injectives.

6.2. Supercommutative Hopf algebras. For a supercommutative Hopf algebra A [Ma13] [We09] over an arbitrary field k let now

- \mathcal{T} the category of k -finite dimensional graded A -comodules, and
- $\mathcal{C} = \mathcal{T}^{\infty}$ be the abelian category of all graded A -comodules.

These k -linear tensor categories \mathcal{T} always satisfy the finiteness condition (F). Condition (G) holds if A is finitely generated over k and $char(k) \neq 2$. Condition (G) follows from the fact that every faithful representation of an algebraic supergroup is a tensor generator as in the algebraic group case [DM82, Proposition 2.20] [Wa79, Section 3.5], and its proof is virtually the same as in the classical case [Del82, Proposition 3.1] using [Wes09, Proposition 9.3.1] that every finite-dimensional representation V of G is a submodule of $k[G]^{\dim V}$ (See [CH17, Section 7]). We list some general important properties of these categories

- (1) \mathcal{C} has all small limits and colimits ([Ho99, Corollary 2.5.6]).
- (2) All objects in \mathcal{C} are small ([Ho99, cor 2.5.7]).
- (3) Any $X \in \mathcal{C}$ is an inductive union $X = \text{colim}_i X_i$ of $X_i \in \mathcal{T}$ ([Ho99, Lemma 2.5.1]). It is easy to see that $\mathcal{C} = \mathcal{T}^{\infty}$ is the ind-category of \mathcal{T} in the sense of [Del02].
- (4) \mathcal{C} is a closed symmetric monoidal category ([Ho99, p. 63] and [Ho99, 2.5.1]).

- (5) The abelian category \mathcal{C} only depends on the coalgebra structure of A .
- (6) An A -module $L \in \mathcal{C}$ is injective if and only if L has the lifting property for all monomorphisms $i : A \rightarrow B$ in \mathcal{T} ([Ho99, Proposition 2.5.8]). The isomorphism classes of monomorphisms in \mathcal{T} are a set ([Ho99, Definition 2.5.12]).
- (7) $\mathcal{I}_{\mathcal{C}}$ is a tensor ideal ([Ho99, Proposition 2.5.8]).
- (8) A is injective ([Ho99, Proposition 2.5.8]).
- (9) The isomorphism classes of indecomposable injective modules are a set of generators. Each of them is a direct summand of A .
- (10) Coproducts of injectives are injective ([Ho99, Proposition 2.5.8]). \mathcal{C} has enough injectives.
- (11) Any injective comodule is a direct sum of finite dimensional indecomposable injective comodules.
- (12) Every comodule M has a nontrivial socle $\text{soc}(M)$, the sum of the simple subcomodules of M . The socle can be described by a wedge construction, and defines a left exact functor $\text{soc} : \mathcal{C} \rightarrow \mathcal{C}$ [B77, p.142].
- (13) Indecomposable injective comodules have irreducible socle.
- (14) Properties **FC.1-FC.4** hold for the category \mathcal{C} .

Let \mathcal{C}' be a full subcategory of \mathcal{C} containing \mathcal{T} closed under direct summands. Then an injective module $L \in \mathcal{I}_{\mathcal{C}'}$ is injective in the category of all A -comodules.

Remark 6.1. We point out that all direct limits (colimits) in the comodule category are sequential direct limits. In particular direct limits are exact. We will use this frequently without further justification.

6.3. Frobenius categories. From now on we also make the crucial

Assumption. \mathcal{T} is a Frobenius category.

Example 6.2. The category of finite dimensional representations $\text{Rep}(G, \epsilon)$ of an algebraic supergroup in the sense of Deligne [Del02] over a field of characteristic 0 is equivalent as a tensor category to a category of graded A -comodules of a supercommutative Hopf algebra A finitely generated over k . If the even subgroup G_0 is reductive over k , $\text{Rep}(G, \epsilon)$ is a Frobenius category.

Lemma 6.3. $\mathcal{P}_{\mathcal{T}} \subset \mathcal{P}_{\mathcal{C}}$. Projective objectives in $\mathcal{C} = \mathcal{T}^{\infty}$ are injective $\mathcal{P}_{\mathcal{C}} \subset \mathcal{I}_{\mathcal{C}}$, and there are enough projectives in \mathcal{C} . P is projective in \mathcal{C} if and only if P is a retract of a direct sum of projectives in \mathcal{T} .

Proof. 1) Projectives P in \mathcal{T} are projective in \mathcal{C} . Simply check the lifting property for $g : P \rightarrow Y$ with respect to an epimorphism $p : X \rightarrow Y$ in \mathcal{C} . For this replace Y by $g(P)$ and X by $p^{-1}(g(P))$. Since $g(P) \in \mathcal{T}$, there exists a subobject $X' \in \mathcal{T}$ of X such that $p : X' \rightarrow Y'$ is an epimorphism. So the lifting property in \mathcal{C} reduces to the lifting property in \mathcal{T} .

2) Given $X = \text{colim}_i X_i$ in \mathcal{C} (for objects $X_i \in \mathcal{T}$) with injective transition maps $f_i : X_i \rightarrow X_{i+1}$ choose $P_i \in \mathcal{P}_{\mathcal{T}}$ together with surjections $\pi_i : P_i \rightarrow X_i$

$$\begin{array}{ccccc}
 \tilde{P}_{i+1} = P_{i+1} \oplus P_i & \longrightarrow & P_{i+1} & \xrightarrow{\pi_{i+1}} & X_{i+1} \\
 & \searrow^{g_i \oplus id} & \uparrow^{g_i} & \nearrow & \uparrow^{f_i} \\
 & & P_i & \xrightarrow{\pi_i} & X_i
 \end{array}$$

The transition morphisms f_i can be lifted to morphisms $g_i : P_i \rightarrow P_{i+1}$. [$f_i \circ \pi_i : P_i \rightarrow X_{i+1}$ can be lifted to a morphism $g_i : P_i \rightarrow P_{i+1}$ such that $\pi_{i+1} \circ g_i = f_i \circ \pi_i$, since P_i is projective.] Now replace P_{i+1} by $\tilde{P}_{i+1} = P_{i+1} \oplus P_i$ and π_{i+1} by $\tilde{\pi}_{i+1} = \pi_{i+1} \circ \text{proj}_1$ and g_i by $\tilde{g}_i = g_i \oplus id_{P_i}$. Repeating this inductively all \tilde{g}_i become monomorphisms between finite dimensional projective modules. Its limit $P = \text{colim}_i \tilde{P}_i$ is isomorphic to a direct sum of finite dimensional projective modules. Choose a well ordering. The image of \tilde{P}_i in \tilde{P}_{i+1} is isomorphic to \tilde{P}_i , hence injective. Therefore it has a complement P'_{i+1} , again finite dimensional projective and injective. If $P_0 = 0$, then $P = \bigoplus P'_i$; and we have constructed a surjection $P \rightarrow X$. Direct sums of projective objects are projective, hence P is projective and \mathcal{C} has enough projectives.

3) Now all P'_i are projective and in \mathcal{T} , hence also injective. A direct sum of injective objects in the category of A -comodules is injective. This proves that P is also injective. Now suppose X was a projective object. Then X is a summand of P , since P surjects onto X by our construction. Hence X is a summand of an injective object, hence itself injective. \square

Since any $I \in \mathcal{I}_{\mathcal{C}}$ is a direct sum of finite dimensional injective objects (and each of them is in $\mathcal{I}_{\mathcal{T}} = \mathcal{P}_{\mathcal{T}}$), any injective object is a direct sum of projective objects. A direct sum of projectives is projective. Hence $\mathcal{I}_{\mathcal{T}} \subset \mathcal{P}_{\mathcal{T}}$. Together with the previous lemma this gives

Corollary 6.4. *If \mathcal{T} is a Frobenius category, then \mathcal{C} is a Frobenius category.*

6.4. The stable model structure. The assumption for the existence of the stable model structure are satisfied (see lemma 3.3): Let J be the set of isomorphism classes of indecomposable injective A -comodules (these are all finite dimensional) (or better the morphisms $0 \rightarrow \text{Inj}$). Let I be the set of isomorphism classes of monomorphisms $i : A \rightarrow B$ between finite dimensional A -comodules. Then I and J define the cofibrantly generated stable model structure. We will later show (in a slightly more general context) that this stable model structure is a monoidal model structure.

6.5. Induced model structure. We now specialize the construction of the induced model structure of subsection 4.1 to the comodule category $\mathcal{C} = \mathcal{T}^{\infty}$. We assume in this section that the comodule categories are Frobenius categories. Let $\pi : A \rightarrow B$ be a surjective homomorphism of supercommutative

Hopf algebras. It corresponds to an embedding $H \subset G$ of supergroups. This embeddings induces a restriction functor $Res : Rep(G) \rightarrow Rep(H)$ which is an exact faithful tensor functor. We discuss its left and right adjoint (called coinduction and induction) and their extensions to the ind-category.

6.6. Frobenius extensions and adjunctions. We review some facts about Frobenius extensions following [BF93]. Let $S \subset R$ be a subring and α an automorphism of S . If M is an S -module, let ${}_{\alpha}M$ denote the S -module with action $s * m = \alpha(s)m$. Then $Hom_S(R, {}_{\alpha}S)$ is the set of additive maps $f : R \rightarrow S$ such that $f(rs) = \alpha(s)f(r)$ for all $s \in S, r \in R$. It is an (R, S) -bimodule via $(r \cdot f \cdot s) = f(xs)$.

Definition 6.5. R is an α -Frobenius extension of S if

- (1) R is a finitely generated S -module and
- (2) there exists an isomorphism $\varphi : R \rightarrow Hom_S(R, {}_{\alpha}S)$ of (R, S) -bimodules.

The Frobenius extensions is said to be free if R is a free S -module.

Now let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over a field k of characteristic $\neq 2$ and let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ be a subalgebra with $\mathfrak{g}_0 \subset \mathfrak{h}$ (and hence $\mathfrak{g}_0 = \mathfrak{h}_0$).

The map $f : \mathfrak{h}_0 \rightarrow \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ defined by $f(a)(y+\mathfrak{h}) = [a, y] + \mathfrak{h}$ for $a \in \mathfrak{h}, y \in \mathfrak{g}$ is a homomorphism of Lie superalgebras. We also define the linear functional

$$\lambda(a) = -str(f(a))$$

which vanishes on $[\mathfrak{h}, \mathfrak{h}] + \mathfrak{h}_1$. There is a unique automorphism α of $\mathcal{U}(\mathfrak{h})$ such that

$$\alpha(a) = \begin{cases} a + \lambda(a)1 & \text{for } a \in \mathfrak{h}_0 \\ (-1)^n a & \text{for } a \in \mathfrak{h}_1 \end{cases}$$

where n is the codimension of \mathfrak{h} in \mathfrak{g} .

Theorem 6.6. ([BF93, Theorem 2.2]) *Let $\mathfrak{h} \subset \mathfrak{g}$ be a sub Lie superalgebra of codimension n that contains \mathfrak{g}_0 . Then the extension $\mathcal{U}(\mathfrak{g}) : \mathcal{U}(\mathfrak{h})$ is a free α -Frobenius extension.*

Theorem 6.7. ([NT60, Page 96]) *For any α -Frobenius extension there is a natural equivalence*

$$R \otimes_S V \cong Hom_S(R, {}_{\alpha}V).$$

In particular the left and right adjoint of $Res : Rep(\mathfrak{g}) \rightarrow Rep(\mathfrak{h})$ are isomorphic and therefore exact. For the special case of induction and coinduction in the $\mathfrak{gl}(m|n)$ -case see [Ge98, Proposition 2.1.1] where also the automorphism α is described explicitly.

If G is an affine supergroup and \mathfrak{g} its Lie superalgebra, we have a fully faithful tensor functor $Rep(G) \rightarrow Rep(\mathfrak{g})$ (note that $Rep(G) = Rep(G_0, \mathfrak{g})$, the \mathfrak{g} -representations such that the restriction to \mathfrak{g}_0 is an algebraic representation of G_0). Consider an embedding $G_0 \subset H \subset G$ of affine supergroups. Then

$\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})$ is a Frobenius extension by theorem 6.6. For $V \in \text{Rep}(H) \subset \text{Rep}(\mathfrak{h})$, its induction and coinduction is again algebraic, i.e. in $\text{Rep}(G)$.

Corollary 6.8. *Let H be an affine supergroup such that $G_0 \subset H \subset G$. Then Res has a left and right adjoint and $\text{Coind} \cong \text{Ind}$. Both adjoints are exact.*

6.7. Limit to the ind-category and the induced model structure.

We recall from [KS06, Proposition 6.1.9] that the passage from a category \mathcal{C} to $\text{Ind}(\mathcal{C})$ is functorial in the sense that there is a unique extension of a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ to the ind-category

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \text{Ind}(\mathcal{C}) & \xrightarrow{IF} & \text{Ind}(\mathcal{C}'). \end{array}$$

On objects $X = (X_i)_i$ in $\text{Ind}(\mathcal{C})$ the extension IF of F is defined by

$$IF(X) = \varinjlim_i F(X_i).$$

Recall that the morphism spaces in $\text{Ind}(\mathcal{C})$ are given by

$$\text{Hom}_{\text{Ind}(\mathcal{C})}(X, Y) = \varprojlim_i \varinjlim_j \text{Hom}_{\mathcal{C}}(X_i, Y_j).$$

Then the map

$$IF : \text{Hom}_{\text{Ind}(\mathcal{C})}(X, Y) \rightarrow \text{Hom}_{\text{Ind}(\mathcal{C}')} (IF(X), IF(Y))$$

is given by

$$\varprojlim_i \varinjlim_j \text{Hom}_{\mathcal{C}}(X_i, Y_j) \rightarrow \varprojlim_i \varinjlim_j \text{Hom}_{\mathcal{C}'}(FX_i, FY_j).$$

Therefore the functors Res and $\text{Ind} \cong \text{Coind}$ extend to the ind-category of $\text{Rep}(G)$ with the same adjunction properties. Since \varinjlim is an exact functor, the extended functors are exact as well. Using the equivalence between $\text{Rep}(G)$ and $\text{Comod}(A)$ for $A = k[G]$ we get the same adjunction properties for the Hopf algebra quotient $A \rightarrow B$ corresponding to $H \subset G$. We use the notation

$$\begin{aligned} U &= I\text{Res} : \text{Rep}(G)^\infty \rightarrow \text{Rep}(H)^\infty, \\ F &= I\text{Coind} : \text{Rep}(H)^\infty \rightarrow \text{Rep}(G)^\infty \end{aligned}$$

and likewise for the corresponding functors on the comodule categories.

Corollary 6.9. *Let H be an affine supergroup such that $G_0 \subset H \subset G$ and the associated Hopf algebra quotient $A \rightarrow B$. For $\mathcal{C} = \text{Comod}(A)$ and $\mathcal{D} = \text{Comod}(B)$ we have*

$$\text{Hom}_{\mathcal{C}}(FX, Y) = \text{Hom}_{\mathcal{D}}(X, UY)$$

where U and F are exact and U is faithful.

Now let $H \subset G$ be the pair of affine supergroups with Hopf algebra quotient $A \rightarrow B$.

Definition 6.10. We call (H, G) as well as the pair of corresponding supercommutative Hopf algebras (A, B) a *Frobenius pair* if $G_0 \subset H \subset G$ and the categories $\mathcal{C} = \text{Comod}(A)$ and $\mathcal{D} = \text{Comod}(B)$ are Frobenius categories.

In particular for any Frobenius pair the functor $\text{Res} : \text{Rep}(G)^\infty \rightarrow \text{Rep}(H)^\infty$ has a left and right adjoint which are isomorphic. By definition any Frobenius pair (H, G) defines a pair of comodule categories $(\mathcal{C}, \mathcal{D})$ which is a Frobenius pair in the sense of section 4. In particular the homotopy category of (H, G) is defined.

Example 6.11. Our main example of a Frobenius pair (H, G) in part 2 is the following: Let $H = P$ be the parabolic subgroup of upper triangular block matrices in the general linear supergroup $G = GL(m|n)$.

Remark 6.12. If $H \subset G$ are finite groups over a field of characteristic $p > 0$ such that $|H|$ is not prime to p , we may consider H and G as finite algebraic groups. The finite dimensional representations of these algebraic groups coincide with the modules of the group algebras $k[H]$ and $k[G]$. Both categories are Frobenius categories, and induction is exact and is isomorphic to the coinduction functor [J03, Chapter 3, Chapter 8]. The formal extension to the ind-category shows that for each subgroup $H \subset G$ there exists an associated model structure on the ind-category of $\text{Rep}(G)$, an associated cotorsion pair and an associated homotopy quotient. This model structure is however uninteresting unless $p \nmid |G : H|$. We have not verified that this model structure is closed monoidal.

Definition 6.13. The induced model structure. Let A be a supercommutative Hopf algebra over a field k with a quotient $A \rightarrow B$ such that (A, B) is a Frobenius pair. Let \mathcal{C} be the category of all graded comodules over A endowed with the induced model category structure via the quotient $A \rightarrow B$ where we use the stable model structure on $\text{Comod}(B)$. We call this model structure on \mathcal{C} the *induced model structure attached to (A, B)* .

6.8. Setup and further conventions. In the following sections 8 - 12.2 the reader is invited to assume that we are in the setting of definition 6.13, i.e. $\mathcal{C} = \mathcal{T}^\infty$ is the category of comodules of a supercommutative Hopf algebra. However many results hold if we just assume that \mathcal{T} is a k -linear abelian Frobenius category satisfying conditions (F) and (G). Then \mathcal{T} is equivalent to the category of finite dimensional comodules of some coalgebra A (see section 6.1) and we denote by \mathcal{C} its ind completion. We assume then that \mathcal{C} carries a model structure induced by the stable model structure from a Frobenius pair $(\mathcal{C}, \mathcal{D})$ in the sense of definition 4.2. An exception are the results of section 7: We have not verified that the model structure on \mathcal{C} is a monoidal model structure in the more general situation.

For a k -linear abelian Frobenius category \mathcal{T} satisfying conditions (F) and (G) we put

$$\mathcal{T}_- = \mathcal{C}_- \cap \mathcal{T}, \quad \mathcal{T}_+ = \mathcal{C}_+ \cap \mathcal{T}.$$

We denote the full triangulated subcategory generated by the image of \mathcal{T} in $\text{Ho}\mathcal{C}$ by $\text{Ho}\mathcal{T}$.

7. THE MONOIDAL MODEL STRUCTURE

7.1. Monoidal model structures. We assume in this section that we are in the setting of definition 6.13.

Theorem 7.1. *The induced model structure 6.13 is a monoidal model structure.*

In order to prove this theorem we have to verify that the model structure on \mathcal{C} satisfies the pushout-product axiom and the unit axiom [Ho99, Definition 4.2.6] as in section 2.5. The unit axiom - a condition on the cofibrant replacement $q : Q\mathbf{1} \rightarrow \mathbf{1}$ of the unit object - is easily verified: Indeed $f \otimes id_Z \in \mathcal{W}$ if and only if $U(f \otimes id_Z) = U(f) \otimes id_{U(Z)} \in \mathcal{W}_{\mathcal{D}}$. The latter is obvious, since stable equivalence is preserved by tensor products with id_K (for any $K \in \mathcal{D}$). So it remains to show that the pushout-product axiom [Ho99, Definition 4.2.6.1] holds. For this it is very convenient ([Ho99, Cor.4.2.5]) that the model structure is cofibrantly generated by J and I , where J is the set of morphisms $0 \rightarrow Z$ for $Z \in F(\mathcal{I}_{\mathcal{D}})$ and $I \subset \mathcal{L}_+$ defined in section 4.1.

We start with a technical lemma.

Lemma 7.2. *We have natural isomorphisms $\varphi : F(A \otimes UB) \cong F(A) \otimes B$ in \mathcal{C} .*

Proof. We construct φ by adjunction. For this it suffices to construct a homomorphism in $\text{Hom}_{\mathcal{D}}(A \otimes UB, UFA \otimes UB)$, namely $ad(A \rightarrow UFA) \otimes id_{UB}$. It is enough to show that $U(\varphi)$ is an isomorphism. Alternatively use twice adjunction

$$\text{Hom}(F(A \otimes B), C) = \text{Hom}(A, \mathcal{H}om(UB, UC))$$

and similarly

$$\text{Hom}(F(A) \otimes B, C) = \text{Hom}(A, U\mathcal{H}om(B, C)) ,$$

so it suffices to give an isomorphism

$$U\mathcal{H}om(B, C) \cong \mathcal{H}om(UB, UC) .$$

The proof is now basically an unravelling of the definition of the comodule structure of $\mathcal{H}om(B, C)$ as in [Ho99, p. 63]. First notice that $B^* \hookrightarrow A^*$ becomes a subalgebra where A^*, B^* are the duals of A, B [Ho99, 2.5.1]. Comodules V of A define tame A^* -modules V^* , and this defines an equivalence of categories (see [Ho99, Proposition 2.5.5]). The functor U corresponds to the restricting the tame A^* -module to the corresponding tame B^* -module.

The internal Hom-module is constructed in loc. cit. as a tame A^* -module $\text{Hom}_k(M, N)$ attached to tame A^* -modules M, N . From the definition of the A^* -action it can be verified that the defining action commutes with the restriction to the subalgebra B^* (the restriction functor U). The formula involves a basis b_i of B with dual basis b_i^* of B^* and

- $\chi^* : B^* \rightarrow B^*$ (dual of the antipode $\chi : B \rightarrow B$)
- $\Delta^* : B^* \rightarrow \text{Hom}_k(B, B^*)$ (dual of comultiplication)
- $f : M \rightarrow N$ k -linear
- $u \in B^*$, so that $b_i^u := \chi * (\delta^*(u)(b_i)) \in B^*$

with $uf \in \text{Hom}_k(M, N)$ defined by

$$(uf)(x) = \sum_i b_i^*(f(b_i^u x))$$

for $x \in M$. The similar formula for the action of A^* reduces to this formula defining the action of B^* , if there is a k -basis a_i of A , such that the dual basis a_i^* contains the dual basis b_i^* as a subset. This is possible provided there exists a splitting $A = B \otimes_k V$ for a finite dimensional k -algebra V (see [We09]) such that the quotient map $A \rightarrow B$ is induced from an algebra map $V \rightarrow k$. This is the situation we are considering. Indeed by [We09, Page 16] A can be written as the tensor product of two supercommutative Hopf algebras

$$A = A_0 \otimes \Lambda^\bullet(\theta_1, \dots, \theta_s)$$

where $A_0 = k[G_0] = A/J$ and the $\{\theta_i\}$ are an A_0 -basis of J/J^2 . □

We now prove theorem 7.1

Proof. We use [Ho99, Corollary 4.2.5]. For the definition of $f \square g$ (the pushout-product) we refer to [Ho99, Definition 4.2.1].

1) We first verify $f \square g \in \mathcal{L} \cap \mathcal{W}$ for $f \in I$ and $g \in J$ or vice versa: For $g : 0 \rightarrow Z$ and $f : X \rightarrow Y$ we have

$$f \square g = f \otimes id_Z : X \otimes Z \rightarrow Y \otimes Z .$$

Now $Z \in \mathcal{P}_C = \mathcal{I}_C$, since F maps projectives to projectives. Hence $X \otimes Z$ and $Y \otimes Z$ are in \mathcal{I}_D . Since any $f \in I$ is a monomorphism, also the morphism $f \otimes id_Z$ is a monomorphism. This shows $f \otimes id_Z$ is a split monomorphisms with projective cokernel. Hence it is in $\mathcal{L} \cap \mathcal{W}$.

2) Next we show $f \square g \in \mathcal{L}$ for $f, g \in I$. For this we use the previous lemma 7.2. For $f = X \rightarrow Y \in I$ and $f' = F(i) : F(A) \rightarrow F(B)$ in I we find that $f \square f'$

$$X \otimes F(B) \bigoplus_{X \otimes F(A)} Y \otimes F(A) \rightarrow Y \otimes F(B)$$

can be identified with the morphism

$$F(UX \otimes B) \bigoplus_{F(UX \otimes A)} F(UY \otimes A) \rightarrow F(UY \otimes B)$$

using lemma 7.2. But this last morphism is $F(U(f)\square i)$ for the pushout product $U(f)\square i$ of $U(f) : UX \rightarrow UY$ and $i : A \rightarrow B$. Since the pushout square $u\square v$ of two monomorphisms u, v is a monomorphism, $U(f)\square i$ is a monomorphism. Hence we see that $F(U(f)\square i) \in I \subset \mathcal{L}$. \square

Corollary 7.3. $C \in \mathcal{C}_-$ implies $\mathcal{H}om(B, C) \in \mathcal{C}_-$.

Proof. Since $UC \in \mathcal{I}_{\mathcal{D}}$ also $\mathcal{H}om(UB, UC) \in \mathcal{I}_{\mathcal{D}}$ using that $\mathcal{H}om$ is right adjoint to the tensor functor. Hence $U\mathcal{H}om(B, C) \in \mathcal{I}_{\mathcal{D}}$ by the isomorphism established in the proof of lemma 7.2. \square

Corollary 7.4. $B \in \mathcal{C}_-$ implies $\mathcal{H}om(B, C) \in \mathcal{C}_-$. In particular, $\mathcal{T}_- = \mathcal{C}_- \cap \mathcal{T}$ is closed under duality.

Proof. $UB \in \mathcal{I}_{\mathcal{D}} = \mathcal{P}_{\mathcal{D}}$ implies also $\mathcal{H}om(UB, UC) \in \mathcal{I}_{\mathcal{D}}$ by adjunction, since $\mathcal{P}_{\mathcal{D}}$ is a tensor ideal. Hence $U\mathcal{H}om(B, C) \in \mathcal{I}_{\mathcal{D}}$. \square

7.2. Monoidal model categories and homotopy quotients. For a symmetric monoidal model category the homotopy functor

$$\gamma : \mathcal{C} \rightarrow Ho\mathcal{C}$$

is a tensor functor [Ho99, Page 116, Theorem 4.3.2]. If \mathcal{C} is a pointed symmetric monoidal model category, then $Ho\mathcal{C}$ is a closed monoidal pretriangulated category ([Ho99, Page 174, Theorem 6.6.3]).

We recall some basic facts about rigid categories. If X^\vee is a left-dual object to an object X in a monoidal category in the sense of [EGNO15, Definition 2.10.1], there exist morphisms $ev_X : X^\vee \otimes X \rightarrow \mathbf{1}$ (the evaluation) and $coev_X : \mathbf{1} \rightarrow X \otimes X^\vee$ (the coevaluation) and similarly for right dual objects. A left or right-dual object is unique up to isomorphism. If X, Y admit left or right duals, the dual morphism $f^\vee : Y^\vee \rightarrow X^\vee$ is defined. An object in a monoidal category is called rigid if it has left and right duals. A monoidal category \mathcal{C} is called rigid if every object is rigid. For $\mathcal{C} = Comod(A)$ the notions of left and right dual coincide.

Lemma 7.5. *The homotopy category $Ho\mathcal{T}$ is rigid.*

Proof. If X is rigid and F a tensor functor, then it is easy to see that $F(X^\vee)$ is dual to $F(X)$. Suppose X is rigid. If $X = X_1 \oplus X_2$ we obtain $id_X = e + (1 - e)$ for $e^2 = e$ where e is the idempotent projecting to X_1 . Then $(e^\vee)^2 = e^\vee$ where $e^\vee : X_1^\vee \rightarrow X^\vee$. Furthermore

$$id_{X^\vee} = (id_X)^\vee = e^\vee + (id_{X^\vee} - e^\vee).$$

Since X is rigid, we have natural adjunction morphisms [EGNO15, Proposition 2.10.8]

$$Hom(A \otimes X, B) \cong Hom(A, X^\vee \otimes B).$$

We then obtain the commutative diagram

$$\begin{array}{ccc} \text{Hom}(A \otimes X, B) & \xrightarrow{\cong} & \text{Hom}(A, X^\vee \otimes B) \\ \uparrow & & \uparrow \\ \text{Hom}(A \otimes X_1, B) & \xrightarrow{\cong} & \text{Hom}(A, X_1^\vee \otimes B) \end{array}$$

where the left vertical map is induced by e and the right vertical map by e^\vee . This implies that $eX = X_1$ and $(1 - e)X = X_2$ are rigid. \square

Lemma 7.6. *For objects Y in \mathcal{T} and $X, Z \in \mathcal{C}$ one has*

$$[X \otimes Y, Z] = [X, Y^\vee \otimes Z] .$$

7.3. Some monoidal properties of \mathcal{C}_\pm . The injective A -comodules $\mathcal{I}_\mathcal{C}$ define a tensor ideal: $A \in \mathcal{I}_\mathcal{C}$ and $B \in \mathcal{C}$ implies $A \otimes B \in \mathcal{I}_\mathcal{C}$. Therefore the tensor product functor $- \otimes Y$ is an exact functor and preserves injectives. By the Frobenius property it also preserves projectives.

Lemma 7.7.

$$Z \in \mathcal{I}_\mathcal{C} \implies \text{Hom}(Y, Z) \in \mathcal{I}_\mathcal{C}$$

and

$$\text{Hom}(Y, Z) \in \mathcal{P}_\mathcal{C}$$

for $Y \in \mathcal{P}_\mathcal{C}$

Proof. Since the functors $\text{Hom}_\mathcal{C}(X \otimes Y, Z)$ and $\text{Hom}_\mathcal{C}(X, \text{Hom}(Y, Z))$ are right exact in X , using projective resolutions of X in \mathcal{C} , we obtain

$$\text{Ext}_\mathcal{C}^i(X \otimes Y, Z) \cong \text{Ext}_\mathcal{C}^i(X, \text{Hom}(Y, Z)) .$$

Now $Z \in \mathcal{I}_\mathcal{C}$ is equivalent to $\text{Ext}^1(\mathcal{C}, Z) = 0$. By the last formula this implies

$$Z \in \mathcal{I}_\mathcal{C} \implies \text{Hom}(Y, Z) \in \mathcal{I}_\mathcal{C} .$$

An analogous argument gives the second statement. \square

Lemma 7.8. $\text{Hom}(\mathcal{C}_+, \mathcal{C}_-) \subset \mathcal{I}_\mathcal{C}$.

Proof. For $X \in \mathcal{C}$ and fixed $X_\pm \in \mathcal{C}_\pm$ we have $\text{Ext}^1(X, \text{Hom}(X_-, X_+)) = \text{Ext}_\mathcal{C}^1(X \otimes X_-, X_+) \subset \text{Ext}^1(\mathcal{C}_+, \mathcal{C}_-) = 0$. \square

Lemma 7.9. \mathcal{C}_\pm are tensor ideals

$$X \in \mathcal{C}_\pm, Y \in \mathcal{C} \implies X \otimes Y \in \mathcal{C}_\pm .$$

Proof. For $X \in \mathcal{C}_-$ this follows from $U(X \otimes Y) = UX \otimes UY \in \mathcal{I}_\mathcal{D} \otimes D \subset \mathcal{I}_\mathcal{D}$. The statement about \mathcal{C}_+ follows from $X \in \mathcal{C}_- \Rightarrow \text{Hom}(Z, X) \in \mathcal{C}_-$ and $\text{Ext}_\mathcal{C}^1(Y \otimes Z, \mathcal{C}_-) = \text{Ext}^1(Y, \text{Hom}(Z, \mathcal{C}_-)) = 0$. \square

Lemma 7.10. *Suppose a full subcategory $\mathcal{C}' \subset \mathcal{C}$ is a tensor ideal, closed under retracts such that \mathcal{C}' is not quasi-equivalent to \mathcal{C} . Then for rigid objects $X \in \mathcal{C}'$ the dimension $\dim(X)$ vanishes.*

Proof. $\dim(X) \neq 0$ implies that the maps $coev$ and $\chi(X)^{-1}ev$ defines a split summand $\mathbf{1} \subset X \otimes X^\vee$. If $X \in \mathcal{C}'$, then also $X \otimes X^\vee \in \mathcal{C}'$, hence also any direct summand is in \mathcal{C}' . But $\mathbf{1} \in \mathcal{C}'$ would imply $X = \mathbf{1} \otimes X \in \mathcal{C}'$ for all $X \in \mathcal{C}$. This proves the claim. \square

Lemma 7.11. *Morphisms from \mathcal{C}_- to \mathcal{T}_+ are stably equivalent to zero if $\mathcal{T}_+^\vee = \mathcal{T}_+$.*

Proof. For $X_+ \in \mathcal{T}_+$ suppose $X_+^\vee = \mathcal{H}om(X_+, \mathbf{1}) \in \mathcal{T}_+$. For $X_+ \in \mathcal{C}_+$ and

$$\varphi : X_- \rightarrow X_+$$

the associated morphism $\Phi = ev \circ (\varphi \otimes X_+^\vee)$

$$\Phi : X_- \otimes X_+^\vee \rightarrow \mathbf{1}$$

factorizes uniquely over the projective hull $p : P(\mathbf{1}) \rightarrow \mathbf{1}$, since $X_- \otimes X_+^\vee \subset \mathcal{I}_{\mathcal{C}} = \mathcal{P}_{\mathcal{C}}$. Notice that here we use that $\mathbf{1}$ (the class of the trivial comodule) is simple. Hence φ , which is the composite $X_- \rightarrow X_+ \otimes X_+^\vee \otimes X_+ \rightarrow P(\mathbf{1}) \otimes X_+ \rightarrow X_+$, factorizes over the projective object $P(\mathbf{1}) \otimes X_+$. \square

8. CLEAN DECOMPOSITIONS

Suppose \mathcal{T} is an abelian k -linear Frobenius category satisfying (F) and (G) as in section 6.8.

Definition 8.1. An object X of \mathcal{C} is *clean* if it does not contain an injective subobject.

The next lemma follows easily from a Krull-Schmidt decomposition.

Lemma 8.2. *Suppose N, M are clean objects. Then $M \oplus N$ is clean.*

Lemma 8.3. *Any object $M \in \mathcal{C}$ decomposes into a direct sum $M = I \oplus N$, where I is injective and N is clean, i.e. does not contain an injective subobject. For two such decompositions $M = I \oplus N = I' \oplus N'$ the morphisms β, γ are isomorphisms*

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & & & & I' & & \\
 & & \nearrow \beta & & \uparrow & & \\
 0 & \longrightarrow & I & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & & & \uparrow \sim & & \nearrow \sim \\
 & & & & N' & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
 \end{array}$$

Finally, $M = I \oplus N$ and $M' = I' \oplus N'$ in \mathcal{C} are stably equivalent if and only if their clean components N and N' are isomorphic in \mathcal{C} .

Proof. Existence. Any $M \in \mathcal{T}$ decomposes by Krull-Schmidt into a finite direct sum of indecomposable objects and the claim follows easily. In general $M = \operatorname{colim} M_i$ for $M_i \in \mathcal{T}$ with monomorphic transition morphisms. Since injectives are projective, the argument above shows that clean decompositions $M_i = I_i \oplus N_i$ can be chosen such that the transition morphism map the injective subobjects I_i into the chosen injective subobject using transfinite induction. Hence $I = \operatorname{colim} I_i$ is a subobject of M with quotient $M/I \cong N = \operatorname{colim} N_i$. Since I is injective, we get $M \cong I \oplus N$ and N is clean. If N contains a nontrivial injective object, then in particular an indecomposable injective object. Therefore it suffices that N does not contain an injective object I from \mathcal{T} . But this is obvious, since such an object I must be contained in some N_i . This is impossible, since all N_i are clean.

Suppose $I \oplus N = M = I' \oplus N'$ are two decompositions. Let the projections from M to N' and I' denote α and β . Any injective object decomposes as $I = \bigoplus_{\nu} I_{\nu}$ for indecomposable injectives $I_{\nu} \in \mathcal{T}$. The socle of each I_{ν} is a simple object $L_{\nu} \in \mathcal{T}$. Hence if $\alpha(\operatorname{soc}(I_{\nu})) \neq 0$, this implies that $\alpha : I_{\nu} \rightarrow N'$ is injective. But this is impossible, since N' is clean, and implies $\alpha(\operatorname{soc}(I_{\nu})) = 0$. Since this holds for all summands I_{ν} , it implies $\alpha(\operatorname{soc}(I)) = 0$. Therefore β must be injective on $\operatorname{soc}(I)$, and hence $\beta : I \rightarrow I'$ is injective, since otherwise $0 \neq \operatorname{soc}(\operatorname{Kern}(\beta)) \subset \operatorname{soc}(I)$ would be annihilated by β . On the other hand $R' = \operatorname{Kern}(\beta)$ which forces $R' \cap \beta(I) = 0$. By the snake lemma we get an exact kokernel sequence

$$0 \rightarrow R' \rightarrow M/I \rightarrow I'/\beta(I) \rightarrow 0.$$

Since $M/I = R$ and since $I'/\beta(I)$ is injective, this implies $R \cong R' \oplus (I'/\beta(I))$. Accordingly R' is clean and we obtain $R \cong R'$ and $I'/\beta(I) = 0$. Therefore $\beta : I \cong I'$ must be an isomorphism. The last assertion is obvious. \square

9. CONSTRUCTION OF COFIBRANT REPLACEMENTS

Suppose \mathcal{T} is an abelian k -linear Frobenius category satisfying (F) and (G) as in section 6.8 and $\mathcal{C} = \mathcal{T}^{\infty}$. Recall that \mathcal{C} is equipped with an induced model structure coming from a Frobenius pair. In particular we have two abelian model categories \mathcal{C} and \mathcal{D} with a Quillen adjunction given by F and U . Let \mathcal{C} be cofibrantly generated by $J = F(\mathcal{F}_{\mathcal{D}})$ and $I = F(\mathcal{I}_{\mathcal{D}})$.

Definition 9.1. Let $\mathcal{C}^{\leq w}$ and $\mathcal{D}^{\leq w}$ be full subcategories of \mathcal{C} and \mathcal{D} which are closed under limits and colimits, hence in particular under pushouts. Assume the following two conditions:

- (1) The restriction functor $U : \mathcal{C} \rightarrow \mathcal{D}$ satisfies $U : \mathcal{C}^{\leq w} \rightarrow \mathcal{D}^{\leq w}$.
- (2) The induction functor $F : \mathcal{D} \rightarrow \mathcal{C}$ satisfies $F : \mathcal{D}^{\leq w} \rightarrow \mathcal{C}^{\leq w}$.

Then we say that \mathcal{C} and \mathcal{D} are *categories with weights*.

Remark 9.2. Consider the category \mathcal{T} of finite dimensional representations of a (connected) algebraic supergroup G with reductive G_0 over an algebraically closed field k of characteristic $\text{char}(k) = 0$. Let \mathcal{C} be its ind-category. There is a notion of weights with an ordering \leq defined between weights. Then we can define $\mathcal{T}^{\leq w}$ and $\mathcal{C}^{\leq w}$ to be the full subcategories of objects whose simple subquotients $L(\lambda)$ all satisfy $\lambda \leq w$. This ordering need not be the usual weight ordering, but could be quite arbitrary.

Weight truncation. Given $M \in \mathcal{C}$ there exist objects $M_{\leq w}$ and $M^{\leq w}$ in $\mathcal{T}^{\leq w}$, such that $M_{\leq w}$ is the maximal subobject of M in $\mathcal{T}^{\leq w}$ and $M^{\leq w}$ is the maximal quotient object in $\mathcal{T}^{\leq w}$. There exists an obvious morphism $\text{can} : M_{\leq w} \rightarrow M^{\leq w}$ functorial in M . These constructions are functorial in the following sense: For a morphism $f : M \rightarrow N$ the composed morphism $M \rightarrow N \rightarrow N^{\leq w}$ uniquely factorizes over $M \rightarrow M^{\leq w}$. The induced morphism $M^{\leq w} \rightarrow N^{\leq w}$ will again be denoted $f^{\leq w}$ by abuse of notation

$$\begin{array}{ccccc} M_{\leq w} & \hookrightarrow & M & \twoheadrightarrow & M^{\leq w} \\ \downarrow f & & \downarrow f & & \downarrow f^{\leq w} \\ N_{\leq w} & \hookrightarrow & N & \twoheadrightarrow & N^{\leq w} \end{array}$$

If i is a monomorphism, $i^{\leq w}$ need not be a monomorphism.

Theorem 9.3. *Any morphism p in $\mathcal{C}^{\leq w}$ can be factorized into a morphism in \mathcal{L} and a morphism in $\mathcal{R} \cap \mathcal{W}$, such that $\varphi : X \rightarrow Z$ where Z is a direct sum of an injective object and an object in $\mathcal{C}^{\leq w}$. Every object in $\mathcal{C}^{\leq w}$ has a cofibrant replacement Z of this form.*

Proof. Step 1). Recall that \mathcal{C} is cofibrantly generated by $J = F(J_{\mathcal{D}})$ and $I = F(I_{\mathcal{D}})$. For a morphism $p : X \rightarrow Y$ between objects in $\mathcal{C}^{\leq w}$ we will construct a factorization $p = \psi \circ \varphi$ with $\varphi \in \mathcal{L}$ and $\psi \in \mathcal{R} \cap \mathcal{W}$ for $\varphi : X \rightarrow Z$ such that

$$Z \in \mathcal{C}^{\leq w}.$$

Fix $p : G^0 = X \rightarrow Y$. To each $i \in I$ corresponds a morphism

$$i : A_i \hookrightarrow B_i$$

in \mathcal{D} . Let $\nu_i : A_i \hookrightarrow I(A_i)$ denote the injective hull. Let $I^{\leq w} \subset I$ be the subset of all monomorphisms $i : A_i \rightarrow B_i$ in \mathcal{D} , where $A_i \in \mathcal{D}^{\leq w}$. For i consider the set $S(i)$ of all morphisms f, g in \mathcal{C} , such that (f, g) makes the following diagram commutative

$$\begin{array}{ccc} F(A_i) & \xrightarrow{f} & G^0 \oplus A_i \\ \downarrow F(\nu_i \oplus i^{\leq w}) & & \downarrow p, 0 \\ F(I(A_i) \oplus B_i^{\leq w}) & \xrightarrow{g} & Y \end{array}$$

for $F(i^{\leq w}) : F(A_i^{\leq w}) \rightarrow F(B_i^{\leq w})$ and $p : G^0 = X \rightarrow Y$. Every $f = (f_X, f_{A_i})$ has two components, where f_{A_i} can be chosen unconditionally, for example $f_{A_i} = id_{A_i}$ or $f_{A_i} = 0$.

Step 2). Put

$$L = \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f,g) \in S(i)} F(A_i)$$

and

$$I = \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f,g) \in S(i)} F(I(A_i)).$$

We extend the map $p : X \rightarrow Y$ to a map $X \oplus L \rightarrow Y$ which is zero on L . It factorizes over the the pushout $G^1 \in \mathcal{C}^{\leq w}$ as indicated in the next diagram

$$\begin{array}{ccc} \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f,g) \in S(i)} F(A_i) & \xhookrightarrow{\bigoplus(f)} & G^0 \oplus L \\ \downarrow \bigoplus_{I^{\leq w}, S(i)} F(\nu_i \oplus i^{\leq w}) & & \downarrow i_1 \\ \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f,g) \in S(i)} F(I(A_i) \oplus B_i^{\leq w}) & \xrightarrow{\bigoplus g} & G^1 \\ & & \downarrow p_1 \\ & & Y \end{array}$$

Since $F(\nu_i)$ is a monomorphism, the left vertical map (abbreviated $F(M) \rightarrow F(N)$ in the following) is in \mathcal{L} , and therefore also its pushout $i_1 \in \mathcal{L}$. From the injectivity of i_1 we get an exact sequence

$$0 \rightarrow (G^0 \oplus L) \rightarrow G^1 \rightarrow F(N)/F(M) \rightarrow 0 .$$

Hence $G^1/(X \oplus L)$, and then also G^1/X , is in \mathcal{C}_+ .

On the other hand

$$0 \rightarrow I \rightarrow G^1 \rightarrow ((X \oplus L) \oplus (\bigoplus B_i^{\leq w}))/F(M) \rightarrow 0$$

since the upper horizontal map is injective. The right hand side R is in $\mathcal{C}^{\leq w}$ by our assumptions. Thus by a clean decomposition of R we get

$$G^1 \cong N^1 \oplus I^1 \quad , \quad (N^1 \in \mathcal{C}^{\leq w} \text{ clean} , I^1 \in \mathcal{I}_{\mathcal{C}}) .$$

Step 3). Now iterate the construction by replacing $p : G^0 = X \rightarrow Y$ by $p_1 : G^1 \rightarrow Y$, and so on. This gives a chain of diagrams with $i_k \in \mathcal{L}$ and $G^k \in \mathcal{C}^{\leq w}$

$$\begin{array}{ccc} G^k & \xhookrightarrow{i_k} & G^{k+1} \\ \downarrow p_k & & \downarrow p_{k+1} \\ Y & \xlongequal{\quad} & Y \end{array}$$

Put $G^\infty = \text{colim}_k G_k$. Then there exists a canonical factorization of p

$$X = G^0 \hookrightarrow \dots \hookrightarrow G^k \hookrightarrow G^{k+1} \hookrightarrow \dots \hookrightarrow G^\infty \rightarrow Y .$$

The morphism $X \rightarrow G^\infty$ is in \mathcal{L} : It is a monomorphism. Its cokernel is a sequential colimit of objects in \mathcal{C}_+ with injective transition maps. Hence its cokernel is in \mathcal{C}_+ .

Step 4). We claim $G^\infty \cong N \oplus I$ with clean $N \in \mathcal{C}^{\leq w}$ and injective I . To see this we write the transition monomorphism $G^k \rightarrow G^{k+1}$ as a matrix. Then the matrix entry

$$\gamma : I^k \rightarrow N^{k+1}$$

is not a monomorphism. Indeed, since I^k is a direct sum of irreducible injective objects with irreducible socle and since N^{k+1} is clean, $\gamma(\text{soc}(I^k)) = 0$. Thus the matrix entry

$$\alpha : I^k \rightarrow I^{k+1}$$

must satisfy $\alpha : \text{soc}(I^k) \hookrightarrow I^{k+1}$. This forces

$$\alpha : I^k \hookrightarrow I^{k+1} .$$

Hence we may write $I^{k+1} = \alpha(I^k) \oplus I'$. Define

$$\tilde{\gamma} : I^{k+1} \rightarrow \alpha(I^k) \cong I^k \rightarrow N^{k+1}$$

and the isomorphism

$$\rho = \begin{pmatrix} id_{I^{k+1}} & -\tilde{\gamma} \\ 0 & id_{N^{k+1}} \end{pmatrix} .$$

Then $\rho \circ i_k$ maps I^k into I^{k+1} . Replacing i_{k+1} by $i_{k+1} \circ \rho^{-1}$, we may therefore assume

$$i_k : I^k \hookrightarrow I^{k+1} .$$

In the limit we obtain an exact sequence

$$0 \rightarrow \text{colim}_k I^k \rightarrow \text{colim}_k G^k \rightarrow \text{colim}_k N_k \rightarrow 0 .$$

Since $I = \text{colim}_k I^k$ is injective, and since $N = \text{colim}_k N^k$ is clean and in $\mathcal{C}^{\leq w}$, our claim follows.

Step 5). We want to show that $p_\infty : G^\infty \rightarrow Y$ is in $\mathcal{R} \cap \mathcal{W}$. For this consider a possible diagram for some $j \in I$ (i.e. a monomorphism $j : A_j \rightarrow B_j$ in \mathcal{D})

Diagram 1:

$$\begin{array}{ccc} F(A_j) & \xrightarrow{f} & G^\infty \\ \downarrow F(j) & \nearrow ? & \downarrow p_\infty \\ F(B_j) & \xrightarrow{g} & Y \end{array}$$

We have to show that there exists a lifting h in order to prove $p_\infty \in \mathcal{R} \cap \mathcal{W}$. Since each $F(A_j)$ is finite dimensional, the morphism $f : F(A_j) \rightarrow G^\infty$

factorizes in the form $f = p_k \circ f'$ over some G^k . This gives the left square of the next diagram

$$\begin{array}{ccccc}
 F(A_j) & \xrightarrow{f'} & G^k & \xrightarrow{i_k} & G^{k+1} \\
 \downarrow F(j) & & \downarrow p_k & \nearrow \text{dotted} & \downarrow p_{k+1} \\
 F(B_j) & \xrightarrow{g} & Y & \xlongequal{\quad} & Y
 \end{array}$$

To find the desired lift to G^∞ , it would suffice to find a lifting $h : F(B_j) \rightarrow G^{k+1}$ of the outer diagram. To discuss this we switch sides to obtain the new diagram

$$\begin{array}{ccccc}
 F(A_j) & \xrightarrow{f'} & G^k & \xrightarrow{p_k} & Y \\
 \downarrow F(j) & & \downarrow i_k & & \parallel \\
 F(B_j) & \xrightarrow{\text{dotted } h} & G^{k+1} & \xrightarrow{p_{k+1}} & Y \\
 & \searrow \text{curved } g & & & \uparrow
 \end{array}$$

To construct such an h , making this diagram commutative, amounts to construct a morphism

$$h : F(B_j) \rightarrow G^{k+1} = (G^k \oplus L^k \oplus F(N^k)) / F(M^k)$$

Step 6) (*Weight factorization*). Any morphism $f : F(A) \rightarrow X \in \mathcal{C}^{\leq w}$ corresponds to a morphism $A \rightarrow UX$. Since $UX \in \mathcal{D}^{\leq w}$, this morphism uniquely factorizes over the quotient $A \rightarrow A^{\leq w}$. Hence by adjunction $f : F(A) \rightarrow X$ canonically factorizes in the form

$$f : F(A) \dashrightarrow F(A^{\leq w}) \longrightarrow X ,$$

and similar for B .

Step 7). By the weight factorization property, and since G^∞ and Y are in $\mathcal{C}^{\leq w}$, diagram 1 factorizes over the following

Diagram 2:

$$\begin{array}{ccc}
 F(A_j^{\leq w}) & \xrightarrow{f} & G^\infty \\
 \downarrow F(j^{\leq w}) & \nearrow \text{dotted } h' & \downarrow p_\infty \\
 F(B_j^{\leq w}) & \xrightarrow{g} & Y
 \end{array}$$

Hence a lift h' in diagram 2 provides us with a lift in our original diagram 1. This reduction step allows us to assume

$$A_j = A_j^{\leq w}$$

and j by $j^{\leq w} : A_j = A_j^{\leq w} \rightarrow B_j^{\leq w}$ without restriction of generality. Since this new $j^{\leq w}$ need not be a monomorphism any longer, we now replace diagram 2 by the more convenient

Diagram 3: $h = (f, h')$

$$\begin{array}{ccc}
 F(A_j^{\leq w}) & \xrightarrow{f} & G^\infty \\
 \downarrow F(\nu_j \oplus j^{\leq w}) & \nearrow h & \downarrow p_\infty \\
 F(I(A_j^{\leq w}) \oplus B_j^{\leq w}) & \xrightarrow{g \circ pr_2} & Y
 \end{array}$$

Claim. Any lift h in diagram 3 gives a lift h' of diagram 2.

Proof of the claim. The lift $h = (h_1, h_2)$ is given by a pair of morphisms $h_1 : F(I(A_j)) \rightarrow G^\infty$ and $h_2 : F(B_j^{\leq w}) \rightarrow G^\infty$ such that

- $p_\infty \circ f = g \circ F(j^{\leq w})$
- $p_\infty \circ h_2 = g$
- $p_\infty \circ h_1 = 0$
- $h_2 \circ F(j^{\leq w}) = f - h_1 \circ F(\nu_j)$.

The desired lift h' should satisfy

- $p_\infty \circ f = g \circ F(j^{\leq w})$
- $p_\infty \circ h' = g$
- $h' \circ F(j^{\leq w}) = f$.

For this we put $h' = h_2 + \lambda$, where $\lambda : F(B_j^{\leq w}) \rightarrow G^\infty$ is chosen such that

$$p_\infty \circ \lambda = 0$$

$$\lambda \circ F(j^{\leq w}) = h_1 \circ F(\nu_j) .$$

To construct λ by adjunction, we construct the corresponding $\Lambda : B_j^{\leq w} \rightarrow UG^\infty$ via $\lambda = \text{adj} \circ F(\Lambda)$. For the definition of Λ consider the next diagram

$$\begin{array}{ccccc}
 A_j^{\leq w} & \xrightarrow{\nu_j} & I(A_j^{\leq w}) & \xrightarrow{H_1} & UG^\infty \\
 \downarrow j & \nearrow \exists u & \downarrow & & \parallel \\
 B_j & \longrightarrow & I(A_j^{\leq w})^{\leq w} & \xrightarrow{\exists v} & UG^\infty \\
 \downarrow & & & & \parallel \\
 B_j^{\leq w} & \xrightarrow{\Lambda} & & & UG^\infty
 \end{array}$$

(A curved arrow labeled $j^{\leq w}$ points from B_j to $B_j^{\leq w}$)

where u exists, since $I(A_j)$ is injective. The existence of the morphisms v, Λ follows from weight factorization, if G^∞ is in $\mathcal{C}^{\leq w}$ and then $UG^\infty \in \mathcal{D}^{\leq w}$. Since we only know that $G^\infty = N^\infty \oplus I^\infty$ with $N^\infty \in \mathcal{C}^{\leq w}$, we still have to show that we may assume $G^\infty = N^\infty$ without restriction of generality. For this we should temporarily return to the beginning of step 5) as we do in the next step.

Step 8) (Replacing G^∞ by $N = N^\infty$). Consider the lifting condition for a diagram

$$\begin{array}{ccc}
 F(A_j) & \xrightarrow{(f_N, f_I)} & N \oplus I \\
 \downarrow j & \nearrow (h_N, h_I) & \downarrow p_N + p_I \\
 F(B_j) & \xrightarrow{g} & Y
 \end{array}$$

Since I is injective and j is a monomorphism, we can choose once and for all $h_I : B_j \rightarrow I$. The lifting condition then is equivalent to

$$h_N \circ F(j) = f_N \quad , \quad p_N \circ h_N = g - p_I \circ h_I .$$

This allows us to replace our lifting problem for G^∞ to a lifting property with G^∞ replaced by N^∞ by using the new diagram

$$\begin{array}{ccc}
 F(A_j) & \xrightarrow{f_N} & N \\
 \downarrow j & \nearrow h_N & \downarrow p_N \\
 F(B_j) & \xrightarrow{g - p_I \circ h_I} & Y
 \end{array}$$

instead of the old one.

Step 9). Now we combine the results from step 5) -step 8). We can assume $A_j = A_j^{\leq w}$ and we can replace $j : A_j \rightarrow B_j$ by the redundant morphism

$(\nu_j, j^{\leq w}) : A_j \rightarrow I(A_j) \oplus B_j^{\leq w}$. This defines an element in $j \in I^{\leq w}$. So it suffices now finally to find a lift for very special morphisms in $I^{\leq w}$ as in the next

Diagram 4:

$$\begin{array}{ccc}
 F(A_j) & \xrightarrow{f} & G^\infty \\
 \downarrow F(\nu_j \oplus j^{\leq w}) & \nearrow ? & \downarrow p_\infty \\
 F(I(A_j) \oplus B_j^{\leq w}) & \xrightarrow{g} & Y
 \end{array}
 , \quad j \in I^{\leq w} .$$

instead of diagram 1. As explained in step 5) the morphism f factorizes over some $f' : F(A_j) \rightarrow G^k$, and it suffices to find the desired lift $h : F(I(A_j) \oplus B_j^{\leq w}) \rightarrow G^{k+1}$ on the next level $k+1$ of the construction of G^∞ .

The data from diagram 4 give a triple (j, f, g) defining one of the summands that appear in the construction of the pushout G^{k+1} . Notice $j \in I^{\leq w}$ and choose $f = (f', 0) : F(A_j) \rightarrow G^k \oplus L^k$. This being said, we map $F(I(A_j) \oplus B_j^{\leq w})$ into the summand corresponding to the index (j, f, g) by the identity map, and we map it into all other summands by zero. Thus we obtain a commutative diagram

$$\begin{array}{ccccc}
 & & & & f \\
 & & & & \curvearrowright \\
 F(A_j) & \xrightarrow{(j, f, g)} & \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f, g) \in S(i)} F(A_i) & \xrightarrow{\bigoplus f} & G^k \oplus L^k \\
 \downarrow F(j) & & \downarrow \bigoplus_{I^{\leq w}, S(i)} F(\nu_i \oplus i^{\leq w}) & & \downarrow i_k \\
 F(I(A_j) \oplus B_j^{\leq w}) & \xrightarrow{(j, f', g)} & \bigoplus_{i \in I^{\leq w}} \bigoplus_{(f, g) \in S(i)} F(I(A_i) \oplus B_i^{\leq w}) & \xrightarrow{\bigoplus g} & G^{k+1} \\
 & & & & \downarrow p_k \\
 & & & & Y
 \end{array}$$

h

Then $i_k \circ f = h \circ F(j)$ and $p_{k+1} \circ h = g$. Composing h with the morphism $G^{k+1} \rightarrow G^\infty$ defines the lift we were looking for. \square

10. MINIMAL MODELS

Suppose \mathcal{T} is an abelian k -linear Frobenius category satisfying (F) and (G) as in section 6.8.

10.1. Minimal cofibrant replacements. Let \mathcal{C} be an abelian model category in which every object is fibrant.

Definition 10.1. Let \mathcal{C} be an abelian model category in which every object is fibrant. A cofibrant replacement $q : QX \rightarrow X$ of $X \in \mathcal{C}$ is called *minimal* (or *minimal model*) if an element $\varphi \in \text{End}_{\mathcal{C}}(QX)$ is a unit, i.e. in $\text{Aut}_{\mathcal{C}}(QX)$, if and only if φ is in \mathcal{W} .

Lemma 10.2. *Suppose $q : QX \rightarrow X$ is minimal. Then any cofibrant replacement $q' : Q'X \rightarrow X$ is of the following form: there exists a projective object P and an isomorphism $\tau : Q'X \cong P \oplus QX$ such that $q' = q \circ \text{pr}_{QX} \circ \tau$.*

Proof. Suppose $q : QX \rightarrow X$ is minimal. Suppose $q' : Q'X \rightarrow X$ is another cofibrant replacement. Then by the lifting property there exist morphisms $\alpha \in \mathcal{W}$ and $\beta \in \mathcal{W}$ (not uniquely) such that $q = q' \circ \alpha$ and $q' = q \circ \beta$. Hence $q = q \circ (\beta \circ \alpha)$. Notice that $\varphi = \beta \circ \alpha \in \text{End}_{\mathcal{C}}(QX)$ is in \mathcal{W} . If $q : QX \rightarrow X$ is minimal, the φ is an isomorphism of QX . Then

$$\alpha : QX \rightarrow Q'X \quad , \quad s \circ \alpha = \text{id}_{QX}$$

is a retract of $Q'X$ for $s = \varphi^{-1} \circ \beta$. Hence α is a monomorphism in \mathcal{W} , and QX is a direct summand of $Q'X$

$$Q'X = QX \oplus P$$

where P is in \mathcal{C}_- . But P is in \mathcal{C}_+ , hence in $\mathcal{P}_{\mathcal{C}}$. Then $P = \text{Kern}(s)$ is also in the kernel of $q' : Q'X \rightarrow X$, since $q' = q \circ \beta = q \circ \varphi \circ s$. Hence

$$\text{Kern}(q') = \text{Kern}(q) \oplus P \quad , \quad P \in \mathcal{P}_{\mathcal{C}} .$$

□

Remark 10.3. Suppose $QX = A \oplus P$ for some projective $P \neq 0$. Then $QX \rightarrow A \rightarrow QX$ is in $\text{End}_{\mathcal{C}}(QX) \cap \mathcal{W}$, but not in $\text{Aut}_{\mathcal{C}}(QX)$. Hence QX is not minimal. Hence minimal models are unique up to isomorphism.

Remark 10.4. Not every indecomposable object has a minimal model, see example 17.8 for an example in the $GL(m|1)$ -case.

Lemma 10.5. *A cofibrant replacement $q : QX \rightarrow X$ is minimal if and only if QX is clean (i.e. without injective subobject). A minimal model $q : QX \rightarrow X$ is unique up to isomorphism.*

Proof. If QX is minimal, then X is clean by remark 10.3. If QX is clean, let $\gamma : QX \rightarrow QX$ to be an endomorphism, and assume $\gamma \in \mathcal{W}$. It can be factorized $\gamma = u \circ v$, where $u \in \mathcal{R} \cap \mathcal{W}$ and $v \in \mathcal{L} \cap \mathcal{W}$. Hence $v : QX \rightarrow M$ is a monomorphism with injective cokernel I' , and $u : M \rightarrow QX$ is an epimorphism with kernel $I \in \mathcal{C}_-$. Then $M \in \mathcal{C}_+$, since QX and I' are in \mathcal{C}_+ ,

This implies $I \in \mathcal{C}_- \cap \mathcal{C}_+$, hence I is injective too.

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \uparrow & & & \\
 & & & I' & & & \\
 & \beta & \nearrow & \uparrow & & & \\
 0 & \longrightarrow & I & \longrightarrow & M & \xrightarrow{u} & QX \longrightarrow 0 \\
 & & & & \uparrow & \nearrow \gamma & \\
 & & & & QX & & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
 \end{array}$$

Since QX is clean, the morphisms β and γ are isomorphisms. This proves $\gamma \in \text{Aut}_{\mathcal{C}}(QX)$. \square

Consider an arbitrary cofibrant replacement

$$0 \rightarrow K \rightarrow QX \rightarrow X \rightarrow 0$$

with $QX \in \mathcal{C}_+$ and $K \in \mathcal{C}_-$. We decompose $QX = I \oplus N$ into an injective I and a clean summand N . Notice $N, I \in \mathcal{C}_+$. Then there are the following exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Q_1 & \longrightarrow & X & \longrightarrow & Q_2 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & N & \longrightarrow & N \oplus I & \longrightarrow & I \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & K \cap N & \longrightarrow & K & \longrightarrow & \text{im}(K) \longrightarrow 0
 \end{array}$$

Lemma 10.6. *Every simple object $X \in \mathcal{T}$ not in \mathcal{C}_- has a minimal model in \mathcal{C} . If \mathcal{C} and \mathcal{D} are categories with weights and if the simple object X has weight $\leq w$, then the minimal model QX of X is in the subcategory $\mathcal{C}^{\leq w}$.*

Proof. For the first assertion assume that X is a simple object in \mathcal{C} in the diagram above. Then the top horizontal exact sequence implies that either $Q_1 = 0, Q_2 \cong X$ or $Q_1 \cong X, Q_2 = 0$:

a) In the case $Q_1 = 0$ we have $N = K \cap N$, hence $K = (K \cap N) \oplus (K \cap I)$. Thus $K \cap N \in \mathcal{C}_-$ is a summand of $K \in \mathcal{C}_-$. Hence $K \cap N$ and $\text{im}(K)$ are both in \mathcal{C}_- . Secondly q factorizes $N \oplus I \rightarrow (N \oplus I)/K \cong I/\text{im}(K) \cong X$. Since I and $\text{im}(K)$ are in \mathcal{C}_- , this would imply

$$X \in \mathcal{C}_- .$$

Then $QX \in \mathcal{C}_-$, hence $QX \in \mathcal{C}_- \cap \mathcal{C}_+ = \mathcal{I}_{\mathcal{C}}$. Thus

$$X \cong 0 \quad , \quad \text{in } Ho\mathcal{C} .$$

b) In the other case $Q_2 = 0$ we have $Im(K) = I$, hence $Im(K)$ is projective. The projection $K \rightarrow im(K)$ therefore splits by a morphism $s : im(K) \rightarrow K$. Then $K \cong s(im(K)) \oplus (K \cap N) \in \mathcal{C}_-$ again implies $(K \cap N) \in \mathcal{C}_-$. On the other hand there is the exact sequence

$$0 \rightarrow (K \cap N) \rightarrow N \rightarrow N/(N \cap K) \cong X \rightarrow 0 .$$

Since $N \in \mathcal{C}_+$ and $(K \cap N) \in \mathcal{C}_-$, this implies that $N \rightarrow X$ is a cofibrant replacement. Since N is clean, this proves the first statement. On the other hand there exist a cofibrant replacement $q : QX \rightarrow X$ with $QX \cong N \oplus I$, where I is injective and N is of weight $\leq w$ as shown for arbitrary objects in $\mathcal{C}^{\leq w}$ in section 9. Since we know that X admits a minimal model, then $q(I) = 0$ and $(QX = N, q|_N)$ defines a minimal model, where $QX \in \mathcal{C}^{\leq w}$. \square

10.2. More on morphisms in $Ho\mathcal{T}$. By theorem H.2 $[X, Y] = Hom(QX, Y) / \sim_{stable}$. Two maps $f : QX \rightarrow Y$, $g : QX \rightarrow Y$ are equivalent if and only if their difference factors over a projective module P

$$\begin{array}{ccc} f - g : QX & \longrightarrow & Y \\ & \searrow & \uparrow \\ & & P \end{array}$$

Suppose $Y \in \mathcal{T}$. It suffices to consider any projective cover $\pi_Y : P(Y) \rightarrow Y$ instead of arbitrary P . Indeed, assume $QX \rightarrow Y$ factorizes over $\pi : P \rightarrow Y$ as above. Then π factorizes over the surjection $\pi_Y : P(Y) \rightarrow Y$ (since P is projective). If $\lambda : QX \rightarrow P(Y)$ is surjective, then $P(Y)$ splits and becomes a summand of QX . Suppose QX is clean and Y is simple, then the image of λ is contained in the radical of $P(Y)$ and hence $f = g$.

Corollary 10.7. *If QX is clean and $Y \in \mathcal{T}$ is simple, then*

$$[X, Y] = Hom_{\mathcal{C}}(QX, Y) .$$

In particular

$$[V, Y] = Hom_{\mathcal{C}}(V, Y)$$

for clean $V \in \mathcal{C}_+$.

Corollary 10.8. *Suppose X admits a minimal model and suppose Y is simple. Then a morphism $f : X \rightarrow Y$ becomes zero in $Ho\mathcal{C}$ if and only if f is zero in \mathcal{C} .*

Remark 10.9. For categories with weights this applies if X and Y are simple objects in \mathcal{C} .

Proof. The category $Ho\mathcal{C}$ is obtained as the localization of the stable category $\overline{\mathcal{C}}$ by $\mathcal{R} \cap \mathcal{W}$. Hence f becomes zero if and only if there exists $s \in \mathcal{R} \cap \mathcal{W}$ such that $s \circ f = 0$ in the stable category. Indeed we may even assume that $s = q$ is the cofibrant replacement $q : QX \rightarrow X$. In other words $q \circ f = 0$ in $Hom_{\overline{\mathcal{C}}}(QX, Y)$. Now suppose that X admits a minimal model, i.e. suppose

that QX is clean, and suppose that Y is simple. Then this amounts to $q \circ f = 0$ in $\text{Hom}_{\mathcal{C}}(QX, Y)$ by corollary 10.7. But $QX \rightarrow X$ is surjective. Hence $q \circ f = 0$ implies $f = 0$ in \mathcal{C} . \square

11. CATEGORIES WITH WEIGHTS AND VANISHING THEOREMS

11.1. Vanishing theorem. The ind-category \mathcal{C} decomposes into blocks \mathcal{C}^Λ . Consider a poset structure on the set Λ of isomorphism classes λ of simple objects in a block. For $\lambda \in \Lambda$ define full subcategories $\mathcal{C}^{\leq \lambda} \subset \mathcal{C}^\Lambda$ and similarly for \mathcal{T} to consist of all objects with simple subquotients $\leq \lambda$. We define $\mathcal{C}^{< \lambda} \subset \mathcal{C}^\Lambda$ and similarly for \mathcal{T} to consist of all objects with simple subquotients $\leq \lambda$ but $\neq \lambda$. Then by definition

$$\text{Hom}_{\mathcal{C}}(\mathcal{C}^{< \lambda}, L(\mu)) = 0$$

for all $\lambda \leq \mu$. The subcategories $\mathcal{C}^{< \lambda}$ and $\mathcal{C}^{\leq \lambda}$ are closed under limits and colimits. Suppose that the blocks $\Lambda_{\mathcal{C}}$ of \mathcal{C} can be identified with the blocks $\Lambda_{\mathcal{D}}$ of \mathcal{D} via the functors F and U . Then it makes sense to assume $F : \mathcal{D}^{\leq \lambda} \rightarrow \mathcal{C}^{\leq \lambda}$ and likewise for U . We continue with our assumption from section 9 that \mathcal{C} and \mathcal{D} define *categories with weights* as in definition 9.1.

Example 11.1. These assumptions above and the Frobenius assumptions for \mathcal{C} and \mathcal{D} hold, if \mathcal{C} is the ind-category of representations of an algebraic supergroup H over an algebraically closed field of characteristic zero whose underlying even subgroup H_0 is a reductive k -group, and if \mathcal{D} is chosen to be the ind-category of representations of an algebraic supergroup H' with $H_0 \subset H' \subset H$, where F and U respectively are the usual induction and restriction functors.

Theorem 11.2. *Under these assumptions*

$$\text{Hom}_{\text{Ho}\mathcal{C}}(QL(\mu), L(\lambda)) = 0 \quad \text{hence} \quad [L(\mu), L(\lambda)] = 0$$

for any irreducible objects $L(\lambda), L(\mu)$ for which $\mu < \lambda$.

Proof. For $X \in \mathcal{C}^{\leq \mu}$ we choose a cofibrant replacement $QX = N \oplus I$, whose clean component N is in $\mathcal{C}^{\leq \mu}$ as shown in section 9. Then $\text{Hom}_{\text{Ho}\mathcal{C}}(QL(\mu), L(\lambda))$ is

$$\text{Hom}_{\bar{\mathcal{C}}}(QL(\mu), L(\lambda)) = \text{Hom}_{\bar{\mathcal{C}}}(I, L(\lambda)) \oplus \text{Hom}_{\bar{\mathcal{C}}}(N, L(\lambda)) = 0,$$

since $I = 0$ in $\bar{\mathcal{C}}$ and since $\text{Hom}_{\bar{\mathcal{C}}}(N, L(\lambda))$ is a quotient of $\text{Hom}_{\mathcal{C}}(N, L(\lambda))$, which is zero by weight reasons. \square

11.2. Finiteness theorems. We now discuss some finiteness theorems or conjectures under certain additional conditions. These theorems hold in our main example $(P(m|n)^+, GL(m|n))$. It is plausible that they all hold for categories with weights.

For a Frobenius pair $(\mathcal{C}, \mathcal{D})$ consider the following chain of functors

$$\gamma : \mathcal{C} \rightarrow \text{Ho}\mathcal{C} = \mathcal{C}_+ / \sim_{\text{stable}}.$$

Let $\mathcal{H} = Ho\mathcal{T}$ be the full triangulated tensor subcategory of $Ho\mathcal{C}$ generated by the image of \mathcal{T} under γ . Then there is the functor

$$\gamma : \mathcal{T} \rightarrow \mathcal{H} = Ho\mathcal{T}$$

which in general is neither surjective nor injective on the set of morphisms. We assume now the following additional assumptions on our categories with weights:

Assumptions I. Let \mathcal{C} and \mathcal{D} be Frobenius categories with an interval finite poset Λ of weights, such that

- A.1** Every simple object X has a weight λ in this poset such that $X \in \mathcal{T}^{\leq \lambda}$ and $Hom_{\mathcal{C}}(X, \mathcal{C}^{\leq \lambda'}) = 0$ for all $\lambda' < \lambda$
- A.2** $\mathcal{C}^{\leq \lambda}$ is closed under extensions.
- A.3** For every simple object $X = L(\lambda)$ of weight λ there exists a monomorphism $L(\lambda) \hookrightarrow K_-(\lambda)$ in \mathcal{C} with $K_-(\lambda) \in \mathcal{T}_-$ and cokernel in $\mathcal{C}^{< \lambda}$. Here $\mathcal{C}^{< \lambda}$ denotes the full subcategory generated by the objects in $\mathcal{C}^{\leq \lambda'}$ for $\lambda' < \lambda$. Let $Ho\mathcal{C}^{\leq \lambda}$ denote the full image of $\mathcal{C}^{\leq \lambda}$.
- A.4** For every simple object $X = L(\lambda)$ of weight λ there exists an epimorphism $K_+(\lambda) \rightarrow L(\lambda)$ in \mathcal{C} with $K_+(\lambda) \in \mathcal{T}_+$ and kernel in $\mathcal{C}^{< \lambda}$.

Example 11.3. For the Frobenius pair $(P(m|n)^+, GL(m|n))$ put $K_-(\lambda) = V(\lambda)^*$ (anti Kac module) and $K_+(\lambda) = V(\lambda)$ (Kac module).

Theorem 11.4. *If assumptions A.1 - A.4 hold, then the shift functor induces a functor*

$$[1] : Ho\mathcal{T}^{\leq \lambda} \rightarrow Ho\mathcal{T}^{< \lambda} .$$

Proof. Obviously we obtain $L(\lambda)[1] \cong \text{cokern}(L(\lambda) \rightarrow K_+(\lambda)) \in \mathcal{T}^{< \lambda'}$ for simple objects $X = L(\lambda)$ of weight λ' in $\mathcal{T}^{\leq \lambda}$. Since $\lambda' \leq \lambda$ this implies the claim. \square

Theorem 11.5. *If assumptions A.1 - A.4 hold and $X \in \mathcal{T}$ is simple,*

$$[X, X] \cong k \cdot id_X .$$

Proof. We use the exact sequence

$$0 \longrightarrow K \longrightarrow K_+(\lambda) \longrightarrow L(\lambda) \longrightarrow 0$$

for $K \in \mathcal{C}^{< \lambda}$. Since $\gamma : \mathcal{C} \rightarrow Ho\mathcal{C}$ is exact, it induces a distinguished triangle. We apply $Hom_{Ho\mathcal{C}}(-, L(\lambda))$ and obtain

$$[K[1], L(\lambda)] \longrightarrow [L(\lambda), L(\lambda)] \longrightarrow [K_+(\lambda), L(\lambda)] \longrightarrow [K, L(\lambda)] .$$

Since $K, K[1] \in \mathcal{C}^{< \lambda}$ we obtain

$$Hom_{Ho\mathcal{C}}(L(\lambda), L(\lambda)) \simeq Hom_{Ho\mathcal{C}}(K_-(\lambda), L(\lambda)) .$$

Since we can suppose that $K_+(\lambda)$ is clean, the latter equals $Hom_{\mathcal{C}}(K_+(\lambda), L(\lambda))$ which is one-dimensional. \square

11.3. Irreducible objects in $Ho\mathcal{T}$. We now add another assumption (the typicality axiom):

T.1 For simple $X = L(\lambda)$ let $P(\lambda) \rightarrow L(\lambda)$ denote the projective hull. Then there exists a chain of surjections $P(\lambda) \rightarrow V(\lambda) \rightarrow L(\lambda)$ with $V(\lambda) \in \mathcal{T}_+$ such that $V(\lambda) \cong P(\lambda)$ implies $L(\lambda) \cong V(\lambda)$, hence $L(\lambda) \cong P(\lambda)$ is projective.

We remark that a simple object $X = L(\lambda)$ becomes zero in $Ho\mathcal{T}$ if and only if $X \in \mathcal{T}_-$.

Theorem 11.6. *If assumption T.1 holds, a simple object X becomes zero in $Ho\mathcal{T}$ if and only if X is projective in \mathcal{T} .*

Proof. If X is projective, then X becomes zero in $\bar{\mathcal{C}}$ and hence zero in $Ho\mathcal{T}$. Conversely, if id_X becomes zero in $Ho\mathcal{T}$, then the epimorphism

$$p : V = V(\lambda) \rightarrow L(\lambda) = X$$

factorizes over a projective module. This implies $p = 0$ in \mathcal{C} , and hence a contradiction if V is clean. If V is not clean, then the typicality axiom implies $P(\lambda) \cong V(\lambda)$, and $V(\lambda)$ is projective. The typicality axiom then furthermore implies that $L(\lambda)$ is projective in \mathcal{C} . \square

11.4. Representations of supergroups. Consider an algebraic supergroup G and a subgroup H such that (H, G) is a Frobenius pair. We assume that the reduced groups G_0 and H_0 are reductive, e.g. G is a basic classical supergroup such as $GL(m|n)$, $OSp(m|2n)$, $P(n)$ or $Q(n)$. Put $\mathcal{T} = Rep_k(G)$ and $\mathcal{T}_H = Rep_k(H)$ (or a related tensor category $Rep_k(\mu, G)$ etc.). Attached to the pair (H, G) we consider the ind-categories \mathcal{C} of \mathcal{T} and \mathcal{D} of \mathcal{T}_H , and the associated tensor functors

$$\begin{aligned} \gamma : \mathcal{C} &\rightarrow Ho\mathcal{C} \\ \gamma : \mathcal{T} &\rightarrow \mathcal{H} = Ho\mathcal{T} . \end{aligned}$$

Recall that $Ho\mathcal{T}$ is rigid. Hence

Corollary 11.7. *The homotopy category $Ho\mathcal{T}$ is a k -linear symmetric rigid monoidal category satisfying $End(\mathbf{1}) = k$.*

Corollary 11.8. *For X in \mathcal{T} and $Y \in \mathcal{T}_+$ we have $[X, Y] \cong Hom_{\bar{\mathcal{T}}}(X, Y)$. If $Y = V(\lambda)$ and X is simple this is equal to $Hom_{\mathcal{T}}(X, Y)$.*

Proof. Use $[X, Y] \cong [Y^\vee, X^\vee] = Hom_{\bar{\mathcal{T}}}(Y^\vee, X^\vee) \cong Hom_{\bar{\mathcal{T}}}(X, Y)$, since Y^\vee is cofibrant. If $Y = V(\lambda)$, then $Y^\vee \in \mathcal{C}_+$ is clean. If X^\vee is simple, hence $[Y^\vee, X^\vee] = Hom_{\mathcal{T}}(Y^\vee, X^\vee) = Hom_{\mathcal{T}}(X, Y)$. \square

Conjecture 11.9. *If assumptions A.1 - A.4 hold, for all objects $X, Y \in \mathcal{T}$*

$$\dim_k([X, Y]) < \infty .$$

Remark 11.10. It is enough to prove the conjecture for $[L, \mathbf{1}]$ for irreducible L whose weight is neither bigger or smaller than the weight of $\mathbf{1}$. Using $[X, Y] \simeq [X \otimes Y^\vee, \mathbf{1}]$ we can assume that Y is trivial. We claim that it

then suffices to show that $\dim_k[X, \mathbf{1}]$ is finite-dimensional for irreducible $X \simeq L(\lambda)$. We induct on the length of X and suppose $\dim_k[X, \mathbf{1}] < \infty$ for $l(X) < n$ for $n \geq 2$. Let $l(X) = n$. Then embedding of the socle gives a distinguished triangle

$$\mathrm{soc}(X) \longrightarrow X \longrightarrow X' \longrightarrow \mathrm{soc}(X)[1]$$

with $l(\mathrm{soc}(X)) < n$ and $l(X') < n$. If we apply the functor $[\mathbf{1}]$ to this triangle, then the finite-dimensionality of $[\mathrm{soc}(X), \mathbf{1}]$ and $[X', \mathbf{1}]$ forces $\dim_k[X, \mathbf{1}] < \infty$. Now consider $[L(\lambda), \mathbf{1}]$. For irreducible objects $[L(\lambda), L(\mu)] = 0$ if $\mu > \lambda$ by theorem 11.2 and $[L(\lambda), L(\lambda)] = k \cdot \mathrm{id}$. Hence we assume now $\lambda > w(\mathbf{1})$. Then there are finitely many weights w' satisfying $w(\mathbf{1}) \leq w' < \lambda$. Assume now by induction that the statement holds for all $[L(w'), \mathbf{1}]$ with $w(\mathbf{1}) \leq w' < \lambda$. Then the morphism $K_+(\lambda) \rightarrow L(\lambda)$ gives a distinguished triangle in $\mathrm{Ho}\mathcal{C}$

$$K \longrightarrow K_+(\lambda) \longrightarrow L(\lambda) \longrightarrow K[1].$$

We apply the functor $[\mathbf{1}]$. Now K and $K[1]$ are in $\mathcal{C}^{<\lambda}$ by assumption and theorem 11.4, and therefore $[K[1], \mathbf{1}]$ is finite dimensional. But $[K_+(\lambda), \mathbf{1}]$ is also finite dimensional since $K_+\lambda$ is cofibrant and clean.

Lemma 11.11. *The conjecture holds for $(P(m|n)^+, GL(m|n))$.*

Proof. Follows immediately from the explicit construction of cofibrant replacements in theorem 13.10. \square

Since $\mathrm{Ho}\mathcal{T}$ is rigid, the monoidal ideal \mathcal{N} of negligible morphisms is defined (see section 16.4 for more details).

Conjecture 11.12. *Assume that G is basic classical and H satisfies $G_0 \subset H \subset G$. The quotient $\mathrm{Ho}\mathcal{T}/\mathcal{N}$ is the semisimple representation category of an affine supergroup scheme.*

We prove this in the $G = GL(m|n)$ and $H = P(m|n)^+$ -case in theorem 16.4.

Remark 11.13. An indecomposable object $X \in \mathrm{Ho}\mathcal{T}$ is in the kernel of $\mathrm{Ho}\mathcal{T} \rightarrow \mathrm{Ho}\mathcal{T}/\mathcal{N}$ if and only if $\mathrm{sdim}(X) = 0$ [He15]. Suppose $\mathcal{C}_\pm \neq \mathcal{C}$ to exclude trivial cases. Then $X \in \mathcal{T}_\pm = \mathcal{T} \cap \mathcal{C}_\pm$ implies $\mathrm{sdim}(X) = 0$ by lemma 7.10.

Example 11.14. If $\mathcal{T} = \mathcal{T}_{m|n}$ and \mathcal{T}_\pm are the tensor ideals of Kac and anti Kac modules in $\mathcal{T}_{m|n}$ respectively, then it is well-known [He15] that the superdimension is zero for every object in \mathcal{T}_\pm .

12. ISOGENIES

12.1. Isogenies I. Assume \mathcal{T} is a k -linear abelian Frobenius category satisfying properties (F) and (G) as in section 6.8. For $\mathcal{E} = \mathcal{C}_+$ consider the full triangulated subcategory \mathcal{F} of objects stably equivalent to objects in

\mathcal{T}_+ . Its image $\overline{\mathcal{F}}$ in $\overline{\mathcal{E}}$ is quasi-equivalent to the image of \mathcal{T}_+ in $\overline{\mathcal{E}}$. Moreover, every object in \mathcal{E} isomorphic in $\overline{\mathcal{E}}$ to an object in \mathcal{T}_+ is in \mathcal{F} by definition.

Lemma 12.1. $\overline{\mathcal{F}}$ is a thick triangulated subcategory of $\overline{\mathcal{E}}$.

Remark 12.2. Similarly the full image of \mathcal{T}_- defines a thick triangulated subcategory of the stable category $\overline{\mathcal{T}}$ of \mathcal{T} .

Proof. $\overline{\mathcal{F}}$ is closed under the suspension and loop functor. Thickness: We have to show $\overline{\mathcal{F}}$ is closed under direct summands and $A, B \in \overline{\mathcal{F}}$ implies $C \in \overline{\mathcal{F}}$ for distinguished triangles $(A, B, C, \alpha, \beta, \gamma)$ in $\overline{\mathcal{E}}$. An equivalent characterization is: For distinguished triangles $(A, B, C, \alpha, \beta, \gamma)$ in $\overline{\mathcal{E}}$ such that $C \in \overline{\mathcal{F}}$ and such that $\alpha : A \rightarrow B$ factorizes over an object in $\overline{\mathcal{F}}$ it follows that A, B are also in $\overline{\mathcal{F}}$.

Suppose $U \oplus V = C \in \overline{\mathcal{F}}$. Then $C = C_{clean} \oplus I$ for injective I and finite dimensional $C_{clean} \in \mathcal{E}$. Using the clean decompositions $U = U_{clean} \oplus I_U$ and $V = V_{clean} \oplus I_V$ we can assume U, V to be clean. We already have shown that U, V clean implies $U \oplus V$ clean. Hence $U \oplus V = C_{clean}$ is finite dimensional. Hence U and V are finite dimensional.

Suppose we are given a distinguished triangle $(A', B', C', \alpha, \beta, \gamma)$ in $\overline{\mathcal{E}}$ such that A', B' are in $\overline{\mathcal{F}}$. We have to show $C' \in \overline{\mathcal{F}}$. Obviously we may replace the triangle by an isomorphic standard triangle. So let us assume $(A, B, C, \alpha, \beta, \gamma)$ is a standard triangle in $\overline{\mathcal{E}}$, hence there is an exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

in \mathcal{E} defined by morphisms $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ in \mathcal{E} such that $A, B \in \mathcal{F}$. We have to show $C \in \mathcal{F}$. Using a clean decomposition $A_{clean} \oplus I$ of A , we may write $B = I \oplus B'$ such that B'_{clean} is in \mathcal{T}_+ . Hence replacing A by A_{clean} and B by B' we may assume that $A \in \mathcal{T}_+$, and $B = B_{clean} \oplus J$ for injective J and $B_{clean} \in \mathcal{T}_+$. Hence there exists an exact sequence in \mathcal{C}

$$0 \rightarrow B_{clean}/(A \cap B_{clean}) \rightarrow C \rightarrow J/im(A) \rightarrow 0.$$

Since $im(A)$ is finite dimensional and $J = \bigoplus_{\nu \in X} I_\nu$, we can assume that $im(A)$ is contained in a finite sum $\bigoplus_{\nu \in X_0} I_\nu$ given by suitable finite subset X_0 of the index set X . Hence $J/im(A) \cong J' \oplus J''$ isomorphic to the direct sum of the injective comodule $J' = \bigoplus_{\nu \notin X_0} I_\nu$ and the finite dimensional comodule $J'' = (\bigoplus_{\nu \in X_0} I_\nu)/im(A)$. The summand J' is projective, hence splits in the exact sequence above; hence $C \cong J'' \oplus C'$, where C' is a finite dimensional extension of the finite dimensional comodules $B_{clean}/(A \cap B_{clean})$ and J'' . Hence C_{clean} is a finite dimensional. Since C is in \mathcal{E} this implies $C \in \mathcal{F}$. \square

12.2. Isogenies II. We now suppose additionally that we are in the situation of definition 6.10 to ensure that $Ho\mathcal{T}$ is a monoidal category. Let Σ denote the class of morphisms s in $\overline{\mathcal{E}}$, whose cone is in the subcategory $\overline{\mathcal{F}}$. We call morphisms in Σ isogenies. As shown in [Ve77, p.279ff], the class of

morphisms Σ admits a calculus of right and left fractions, since $\overline{\mathcal{F}}$ is a thick subcategory. This defines a triangulated localization functor

$$\overline{\mathcal{E}} \rightarrow \overline{\mathcal{E}}[\Sigma^{-1}].$$

Let \mathcal{H} denote the full subcategory of objects in $\overline{\mathcal{E}}$ which are isomorphic to cofibrant replacements of objects which are stably isomorphic to objects in \mathcal{T} . This is a full triangulated subcategory of $\overline{\mathcal{E}}$, i.e stable under suspension and loop functor, and closed under extension meaning that for a distinguished triangle (A, B, C) the condition $A, C \in \mathcal{H}$ implies $B \in \mathcal{H}$.

Then

$$\overline{\mathcal{F}} \subset \mathcal{H},$$

and \mathcal{H} is a symmetric monoidal rigid subcategory of $\overline{\mathcal{E}}$ such that $\overline{\mathcal{F}}$ is a tensor ideal in \mathcal{H} . Indeed for $X = I \otimes N$ and $q : QY \rightarrow Y$ with $N \in \mathcal{T}_+$ and $Y \in \mathcal{T}$ we have $X \otimes QY \rightarrow X \otimes Y$ is a cofibrant replacement of $X \otimes Y = I \otimes Y \oplus N \otimes Y$, which is stably equivalent to $N \otimes Y \in \mathcal{T}_+$ and hence is in $\overline{\mathcal{F}}$. This implies that the localization functor

$$\mathcal{H} \rightarrow \mathcal{H}[\Sigma^{-1}]$$

is a triangulated tensor functor.

Part 2. The homotopy category associated to $GL(m|n)$

We now study one particular Frobenius pair: We consider the embedding of the parabolic subgroup of upper triangular block matrices in $GL(m|n)$. We construct an explicit cofibrant replacement for any $X \in \mathcal{T}$ and deduce from this an another description of $Ho\mathcal{T}$. We show that the semisimplification $Ho\mathcal{T}_{m|n}/\mathcal{N}$ is the semisimple representation category of a supergroup scheme. In the $GL(m|1)$ -case we determine this semisimple quotient. For more background on $\mathcal{T}_{m|n} = Rep(GL(m|n))$ we refer to [HW14].

13. COFIBRANT REPLACEMENTS AND AN EXPLICIT DESCRIPTION OF $Ho\mathcal{T}$

13.1. Representations of $GL(m|n)$. Let k be an algebraically closed field of characteristic zero. We adopt the notations of [HW14]. With $GL(m|n)$ we denote the general linear supergroup and by $\mathfrak{g} = \mathfrak{gl}(m|n)$ its Lie superalgebra. We assume without loss of generality $m \geq n$. A representation ρ of $GL(m|n)$ is a representation of \mathfrak{g} such that its restriction to \mathfrak{g}_0 comes from an algebraic representation of $G_0 = GL(m) \times GL(n)$. We denote by $\mathcal{T} = \mathcal{T}_{m|n}$ the category of all finite dimensional representations with parity preserving morphisms. The irreducible representations in $\mathcal{T}_{m|n}$ are parametrized by their highest weight with respect to the Borel subalgebra of upper triangular matrices. A weight $\lambda = (\lambda_1, \dots, \lambda_m \mid \lambda_{m+1}, \dots, \lambda_{m+n})$ of an irreducible representation in \mathcal{R}_n satisfies $\lambda_1 \geq \dots \lambda_m, \lambda_{m+1} \geq \dots \lambda_{m+n}$ with integer entries. The Berezin determinant of the supergroup G defines a one dimensional representation Ber . Its weight is given by $\lambda_i = 1$ and $\lambda_{m+i} = -1$

for $i = 1, \dots, n$. For each weight λ we also have the parity shifted irreducible representation $\Pi L(\lambda)$. Both \vee and $*$ (the twisted dual) define contravariant functors on $\mathcal{T}_{m|n}$. We denote by \mathcal{T}_+ the tensor ideal of modules with a filtration by Kac modules in \mathcal{T} and by \mathcal{T}_- the tensor ideal of modules with a filtration by anti Kac modules. We abbreviate $\mathcal{T}_{n|n} = \mathcal{T}_n$.

13.2. Two Frobenius pairs for $GL(m|n)$. We write P^+ for the maximal parabolic subgroup of upper triangular block matrices and P^- for the maximal parabolic of lower triangular block matrices. By [Ge98, Lemma 3.3.1]

$$V(\lambda)^* = \text{Coind}_{P^-}^G L_{P^-}(\lambda) = \text{Ind}_{P^-}^G L_{P^-}(\lambda - 2\rho_1).$$

Lemma 13.1. ([Ge98, Proposition 3.6.2])

- (1) For $M \in \mathcal{T}_{m|n}$ the following are equivalent:
 - M has a filtration by Kac modules.
 - $\text{Ext}^1(M, V^*(\mu)) = 0$ for all $\mu \in X^+$.
 - $\text{Res}_{P^-} M$ is projective in $\text{Rep}(P^-)$.
- (2) For $M \in \mathcal{T}_{m|n}$ the following are equivalent:
 - M has a filtration by anti Kac modules.
 - $\text{Ext}^1(M, V(\mu)) = 0$ for all $\mu \in X^+$.
 - $\text{Res}_{P^+} M$ is projective in $\text{Rep}(P^-)$.

Corollary 13.2. For the Frobenius pair (P^+, G) the subcategory $\mathcal{C}_- \cap \mathcal{T}_{m|n}$ equals the tensor ideal of modules with an anti Kac filtration and $\mathcal{C}_+ \cap \mathcal{T}_{m|n}$ equals the tensor ideals of modules with a Kac filtration.

Remark 13.3. Note that our notation for modules with Kac resp. anti Kac filtrations shows that we always consider the case $H = P^+$. If we would exchange P^+ with P^- , this would switch the roles of \mathcal{T}_+ and \mathcal{T}_- and our notation in section 13.1 for modules with a Kac or anti Kac filtration would be inconsistent.

13.3. Axiomatic description of the highest weight structure. We fix until the end of the article \mathcal{D} such that \mathcal{T}_- is the category of representations with anti Kac flags. In other words: \mathcal{D} is the ind-category of $\text{Rep}(P(m|n)^+)$ where $P(m|n)^+$ is the parabolic subgroup of upper triangular block matrices in $GL(m|n)$.

As an abelian category \mathcal{T} splits into blocks \mathcal{T}_Λ , each of which is a highest weight category with duality [CPS88]. The standard modules in this highest weight structure are the Kac modules $V(\lambda)$. We now axiomatize the situation of the $\mathcal{T}_{m|n}$ -case and consider an abelian category $\mathcal{T} = \text{Rep}(G)$ for some supergroup G satisfying the following sets of assumptions.

First list of assumptions. As an abelian category \mathcal{T} splits into blocks \mathcal{T}_Λ , each of which is a highest weight category with duality [CPS88] in the following way: Each block Λ has the structure of an interval finite poset such that the elements λ correspond to representatives $L(\lambda)$ of isomorphism

classes of simple objects in the block \mathcal{T}_Λ . Each $L(\lambda)$ has a projective cover $P(\lambda) \in \mathcal{T}_\Lambda$. Furthermore for $\lambda \in \Lambda$ there exist objects $V(\lambda) \in \mathcal{T}_+ \cap \mathcal{T}_\Lambda$ such that

- (1) There exists an epimorphism $P(\lambda) \rightarrow V(\lambda)$, and the kernel has a finite filtration whose successive quotients are of the form $V(\nu)$ for certain $\nu \in \Lambda$ such that $\nu > \lambda$,
- (2) There exists an epimorphism $V(\lambda) \rightarrow L(\lambda)$ such that $L(\lambda)$ is the cosocle of $V(\lambda)$, so that the kernel (radical) has a finite filtration whose successive quotients are of the form $L(\mu)$ for certain $\mu \in \Lambda$ such that $\mu < \lambda$.

Second list of assumptions. Let $V = V(\mathbf{1})$ in \mathcal{T}_+ be the standard module corresponding to the trivial object and $P = P(\mathbf{1})$ the projective hull of $\mathbf{1}$. These objects are defined for a highest weight category. We abbreviate $V(\mathcal{L}) = V \otimes \mathcal{L}$. We now assume that the following additional assumptions hold:

- (1) There exists an antiinvolutive \otimes -functor $*$ inducing an equivalence of the tensor categories $*$: $\mathcal{T} \rightarrow \mathcal{T}^{op}$, which permutes the subcategories \mathcal{T}_- and \mathcal{T}_+ so that $Y^* \cong Y$, if Y is simple or if Y is an indecomposable projective object.
- (2) There exists an invertible simple object \mathcal{L} in \mathcal{T} , such that
- (3) there exists an injection $i : V(\mathcal{L}) \rightarrow P$,
- (4) and there exists a surjection $\pi : P \rightarrow V(\mathcal{L})^*$.
- (5) V (hence $V(\mathcal{L})$) is rigid with a Loewy filtration of length r with r pairwise non-isomorphic simple constituents L_i .
- (6) The kernel of $\pi \circ i : V(\mathcal{L}) \rightarrow V(\mathcal{L})^*$ is the radical of $V(\mathcal{L})$, i.e. $V(\mathcal{L})$ divided by the kernel is isomorphic to \mathcal{L} .
- (7) The Jordan-Hölder constituent \mathcal{L} is the highest weight representation in P and has multiplicity one.

Property 5 is of auxiliary nature. It will not be used in the following except that it allows to verify the other properties in the case where $\mathcal{T} = \mathcal{T}_{m|n}$.

Lemma 13.4. *Under the assumption 1. above the subcategories \mathcal{T}_- and \mathcal{T}_+ are stable under the Tannaka duality functor ${}^\vee$.*

Proof. Assumption 1) implies that as a tensor functor $*$ commutes with the Tannaka duality ${}^\vee$. Since \mathcal{T}_- is preserved by ${}^\vee$, therefore 1) implies that also \mathcal{T}_+ is preserved by ${}^\vee$. Indeed, for $X \in \mathcal{T}_+$ we get $X^* \in \mathcal{T}_-$ and hence $(X^\vee)^* \cong (X^*)^\vee \in \mathcal{T}_-$. Therefore $X^\vee = (X^\vee)^{**} \in \mathcal{T}_+$. \square

Lemma 13.5. *Axioms 1.-7. are satisfied in the case $GL(m|n)$.*

Proof. For $\mathcal{T}_{m|n}$ these conditions hold for $\mathcal{L} = \text{Ber}^n$. Then \mathcal{L} is the dual of the socle of the Kac module V of the trivial representation

$$0 \longrightarrow I \xrightarrow{a} V \xrightarrow{b} \mathbf{1} \longrightarrow 0,$$

hence \mathcal{L}^{-1} is the cosocle of V^* , and $\mathbf{1}$ is the cosocle

$$0 \longrightarrow J \longrightarrow V^*(\mathcal{L}) \longrightarrow \mathbf{1} \longrightarrow 0 ,$$

of $V^*(\mathcal{L}) = V^* \otimes \mathcal{L}$. The rigidity assertion 5) has been shown in [BS12],[BS11],[BS10]. Notice also $V^\vee \cong \mathcal{L} \otimes V$. In particular by property 5) the constituents L_i of the Loewy filtration of V satisfy

$$L_i^\vee \cong L_{r-i} \otimes \mathcal{L} .$$

Property 7) follows from the fact that \mathcal{L} is the highest weight constituent of P and also follows from loc. cit. The Loewy length is $r = n$ by [SZ12, Theorem 3.2]. Property 3) and 4) and also 7) follow for $m + n$ from

$$\mathcal{L} \otimes V \otimes V^* = P \oplus Q ,$$

where Q is a projective object of atypicality $< n$ (see lemma 14.10). If 3), 4), 5) hold for $\mathcal{T}_{n|n}$, they hold for $\mathcal{T}_{m|n}$ by using the block equivalence to the principal block in $\mathcal{T}_{n|n}$. The inclusion $\mathbf{1} \rightarrow V^*$ induces the embedding $i : \mathcal{L} \otimes V \rightarrow P \subset P \oplus Q$; similarly the projection $V \rightarrow \mathbf{1}$ induces the surjection $\pi : P \oplus Q \rightarrow \mathcal{L} \otimes V^*$, since π is necessarily trivial on Q . \square

So let us now take all these properties for granted. For simplicity we could assume $\mathcal{T} = \mathcal{T}_{m|n}$ (for $m \geq n$) in order to ensure that these conditions hold. Then we obtain

Lemma 13.6. *The restriction of $V^* \otimes \mathcal{L}$ under $U : \mathcal{C} \rightarrow \mathcal{D}$ is projective and the restriction of V to \mathcal{D} decomposes in the following way*

$$U(V) \cong U(I) \oplus \mathbf{1} .$$

Proof. Since $V \in \mathcal{T}_+$ the first property 1) implies $V^* \in \mathcal{T}_-$, hence $V^* \otimes \mathcal{L} \in \mathcal{T}_-$. Concerning the second assertion this implies that $U(V^* \otimes \mathcal{L})$ is projective and that $U(\pi) : U(P) \rightarrow U(V(\mathcal{L})^*) = U(V^* \otimes \mathcal{L})$ splits

$$U(P) \cong U(\text{Kern}(\pi)) \oplus U(V(\mathcal{L})^*) .$$

Now $D = U(i(V(\mathcal{L}))) \cap U(V(\mathcal{L})^*) \neq 0$, since by the properties 6) and 7) the Jordan-Hölder constituent $U(\mathcal{L})$ of $U(P)$ is obtained from $U(i(V(\mathcal{L}))) \subset U(P)$ but not obtained from $U(\text{Kern}(\pi))$. To proof our second assertion it would suffice to show $D \cong U(\mathcal{L})$. Indeed, since $\text{Kern}(\pi)$ contains the radical of $i(V(\mathcal{L}))$ by property 6), the splitting of $U(\pi)$ then induces a splitting of $U(i(V(\mathcal{L})))$

$$U(i(V(\mathcal{L}))) \cong U(\text{Kern}(\pi) \cap i(V(\mathcal{L}))) \oplus U(\mathcal{L}) .$$

Tensoring with \mathcal{L}^{-1} gives the required isomorphism $U(V) \cong U(I) \oplus \mathbf{1}$. \square

13.4. Construction of cofibrant replacements. Recall that $V = V(\mathbf{1})$ and $I = V(\mathbf{1})/\mathbf{1}$. Now consider in the category \mathcal{C} the objects $P = \bigoplus_{i=0}^{\infty} P_i$, $Q = \bigoplus_{i=0}^{\infty} Q_i$ and $R = \bigoplus_{i=0}^{\infty} R_i$ for

$$\begin{aligned} R_i &= (I \otimes I^*)^{\otimes(i+1)} \\ Q_i &= (I \otimes V^*) \otimes (I \otimes I^*)^{\otimes i} \bigoplus V \otimes (I \otimes I^*)^{\otimes i} \\ P_i &= I \otimes (I \otimes I^*)^{\otimes i}. \end{aligned}$$

We define morphisms $\alpha_i : P_i \rightarrow Q_i$ by $\alpha \otimes id_{(I \otimes I^*)^{\otimes i}}$ for

$$\alpha : I \hookrightarrow (I \otimes V^*) \bigoplus V ,$$

where α is the diagonal map obtained from the two morphisms $id_I \otimes b^* : I = I \otimes \mathbf{1} \hookrightarrow I \otimes V^*$ and $a : I \hookrightarrow V$. Similarly define morphisms $\beta_i : Q_i \rightarrow R_i$ by $\beta \otimes id_{(I \otimes I^*)^{\otimes i}}$ for the epimorphism

$$\beta : (I \otimes V^*) \bigoplus V \longrightarrow I \otimes I^* ,$$

where β is of projection onto $(I \otimes V^*)$ followed by the epimorphism $id_I \otimes a^* : I \otimes V^* = I \otimes I^*$. Finally define for $i \geq 1$ morphisms $\gamma_i : Q_i \rightarrow R_{i-1}$ by $\gamma \otimes id_{(I \otimes I^*)^{\otimes i}}$ for the epimorphism

$$\gamma : (I \otimes V^*) \bigoplus V \longrightarrow V \rightarrow \mathbf{1} ,$$

where γ is of projection onto V followed by the epimorphism $b : V \rightarrow \mathbf{1}$. Put $\gamma_0 = 0$. The maps $(\alpha_i)_{i \geq 0}$ and $(\gamma_i)_{i \geq 1} - (\beta_i)_{i \geq 0}$ define a complex in \mathcal{C}

$$0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0 .$$

Let $\Omega = \text{Kern}(\beta)/\text{Im}(\alpha)$ be its cohomology. The composition of epimorphisms $\Omega \rightarrow \Omega_0 \rightarrow V \rightarrow \mathbf{1}$ defines an epimorphism $q : \Omega \rightarrow \mathbf{1}$.

Lemma 13.7. *Ω is cofibrant in \mathcal{C} . There exists an epimorphism*

$$q : \Omega \rightarrow \mathbf{1}$$

with kernel in \mathcal{C}_- . Hence Ω is a cofibrant replacement of $\mathbf{1}$.

Proof. The inclusion $V \rightarrow Q_0$ on the second summand of Q_0 induces a complex map, which defines a monomorphism on cohomology $V \hookrightarrow \Omega$ with quotient $\Omega/V \cong \Omega \otimes (I \otimes I^*)$. Similarly $V \otimes (I \otimes I^*) \hookrightarrow \Omega \otimes (I \otimes I^*)$ has quotient isomorphic to $\Omega \otimes (I \otimes I^*)^{\otimes 2}$. Iterating this gives short exact sequences

$$0 \rightarrow \Omega_i \rightarrow \Omega \rightarrow \Omega \otimes (I \otimes I^*)^{\otimes i} \rightarrow 0 .$$

The kernels Ω_i define an increasing sequence of sub-comodules of Ω

$$V = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \dots$$

such that $\Omega = \text{colim } \Omega_i$. Since

$$\Omega_{i+1}/\Omega_i \cong V \otimes (I \otimes I^*)^{\otimes i}$$

is in \mathcal{T}_+ , all the comodules Ω_i are in \mathcal{T}_+ . Hence $\Omega \in \mathcal{C}_+$. This shows that Ω is cofibrant. The kernel K of $q : \Omega \rightarrow \mathbf{1}$ is the cohomology of the complex

$$P \rightarrow Q' \rightarrow R ,$$

for $Q' = \text{Kern}(Q \rightarrow \mathbf{1})$. Since U is an exact functor and commutes with direct sums, we can compute $U(R)$ from the complex $U(P) \rightarrow U(Q') \rightarrow U(R)$. Since $U(V) \cong U(I) \oplus \mathbf{1}$ splits, $U(Q)$ simplifies

$$U(Q_i) \cong U(V^*) \otimes U(I \otimes I^*)^{\otimes i} \bigoplus (U(I) \oplus \mathbf{1}) \otimes U(I \otimes I^*)^{\otimes i}$$

Thus we obtain

$$U(Q')/U(P) \cong \bigoplus_{i=0}^{\infty} U(V^*) \otimes U(I \otimes I^*)^{\otimes i} \oplus \bigoplus_{i=1}^{\infty} U(I \otimes I^*)^{\otimes i}$$

so that the kernel of $U(Q')/U(P) \rightarrow U(R)$ becomes

$$U(R) \cong \bigoplus_{i=0}^{\infty} U(V^*) \otimes U(I \otimes I^*)^{\otimes i} .$$

Since $U(V^*)$ is projective by the last lemma, $U(V^*) \otimes U(I \otimes I^*)^{\otimes i}$ is projective as well. Hence $U(R)$ is a direct sum of projectives objects in \mathcal{D} , hence projective in \mathcal{D} . Thus $R \in \mathcal{C}_-$. \square

Example 13.8. In the $GL(1|1)$ -case $V(\mathbf{1})$ has the composition factors $\mathbf{1}$ and Ber^{-1} . Therefore $I \otimes I^* \cong Ber^{-2}$. Accordingly Ω_{i+1}/Ω_i equals

$$V \otimes (I \otimes I^*)^{\otimes i} \cong V \otimes (Ber^{-2})^{\otimes i} \cong V(Ber^{-2i}).$$

We can find embeddings $K \otimes (I \otimes I^*) \hookrightarrow K$ with cokernel isomorphic to $V^* \otimes I$. Indeed $P_{i+1} = P_i \otimes (I \otimes I^*)$, $Q_{i+1} = Q_i \otimes (I \otimes I^*)$ and $R_{i+1} = R_i \otimes (I \otimes I^*)$, and similarly for the complex maps. This defines a short exact sequences of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & P \otimes (I \otimes I^*) & \longrightarrow & Q' \otimes (I \otimes I^*) & \longrightarrow & R \otimes (I \otimes I^*) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P & \longrightarrow & Q' & \longrightarrow & R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus (I \otimes V^*) \oplus R_0 & \longrightarrow & R_0 \longrightarrow 0 \end{array}$$

whose cohomology sequence gives the short exact sequence in \mathcal{C}

$$0 \rightarrow K \otimes (I \otimes I^*) \rightarrow K \rightarrow I \otimes V^* \rightarrow 0 .$$

By i -fold iteration this gives short exact sequences

$$\begin{aligned} 0 &\rightarrow K^{(i)} \rightarrow K \rightarrow K_i \rightarrow 0 \\ K^{(i)} &\cong K \otimes (I \otimes I^*)^{\otimes i} \end{aligned}$$

Hence the objects $K^{(i)}$ and K_i are in \mathcal{C}_- .

Hence K has a descending chain of subcomodules $K^{(i)}$ in \mathcal{C}

$$K = K^{(0)} \supset K^{(1)} \supset K^{(2)} \supset K^{(3)} \dots$$

whose successive quotients K_i are in \mathcal{T}_- . By the construction the weights of all irreducible constituents of K and of $I \otimes I^*$ are < 0 . Hence we get from simple weight reasons

Theorem 13.9. *For any Y in \mathcal{T} there exists an integer n such that $\text{Hom}_{\mathcal{C}}(K^{(n)}, Y) = 0$.*

Theorem 13.10. *For any object X in \mathcal{T} there exists a cofibrant replacement $q_X : QX \rightarrow X$ in \mathcal{C} with the following property. For any Y in \mathcal{T} there exists a subobject $K' \in QX$ of finite codimension contained in $\text{Kern}(q_X)$ such that $K' \in \mathcal{C}_-$ and such that $\text{Hom}_{\mathcal{C}}(K', Y) = 0$.*

Proof. Consider $QX = \Omega \otimes X$ for $q_X = q \otimes X$ and $K' = K^{(n)} \otimes X$ for n large enough, such that $\text{Hom}_{\mathcal{C}}(K^{(n)}, X^\vee \otimes Y) = 0$. \square

Since $[X, Y] = \text{Hom}_{\text{HoC}}(QX, Y) / \sim$, the two theorems imply immediately the following important corollary.

Corollary 13.11. *For $X, Y \in \mathcal{T}$ we have $\dim[X, Y] < \infty$.*

Remark 13.12. For a way to see the cofibrant replacement as a Kac resolution see section 1.1. An estimate for the dimension of $[L(\lambda), L(\mu)]$ can be found in section 1.6. We do not know a direct representation theoretic meaning of this dimension.

13.5. A second interpretation of HoT . The full image category of \mathcal{T}_- in $\overline{\mathcal{T}}$ is a triangulated subcategory. It is thick, since $X \cong A \oplus B$ for $X \in \mathcal{T}_-$ implies $P \oplus X \cong P' \oplus A \oplus B$ in \mathcal{T} , hence $X' \cong A' \oplus B'$ for the clean components X', A', B' of X, A, B . Let hoT be the quotient category of the triangulated stable category $\overline{\mathcal{T}}$ by the thick subcategory $\overline{\mathcal{T}}_-$. There is a natural tensor functor

$$\text{hoT} \rightarrow \text{HoT} .$$

Fix objects X and Y in \mathcal{T} . Then morphisms $X \rightarrow Y$ in hoT are (certain equivalence classes of diagrams) of the form (see [Ne01])

$$\begin{array}{ccc} & Z & \\ s \swarrow & & \searrow f \\ X & & Y \end{array}$$

for morphisms $s : Z \rightarrow X$ and $f : Z \rightarrow X$ in $\overline{\mathcal{T}}$ with $Z \in \mathcal{T}$ (hence s and f are classes of morphisms in \mathcal{T} that are still denoted s and f by abuse of notation) such that the cone of s is in \mathcal{T}_- . Since $\overline{\mathcal{T}}_-$ maps to zero under the functor $\gamma : \overline{\mathcal{T}} \rightarrow \text{HoT}$, defined as a full subcategory of HoC , the morphism $s : Z \rightarrow X$ in $\overline{\mathcal{T}}$ becomes an isomorphism $\gamma(s)$ in HoT . In the manner

diagrams are composed and the equivalence classes are defined, it is easy to see that we obtain an induced functor

$$\gamma : ho\mathcal{T} \rightarrow Ho\mathcal{T} ,$$

which maps the equivalence class of the diagram $X \leftarrow Z \rightarrow Y$ to $\gamma(f) \circ \gamma(s)^{-1}$. Let us show

Theorem 13.13. *The functor γ induces a k -linear equivalence of tensor categories*

$$\overline{\mathcal{T}}/\overline{\mathcal{T}}_- =: ho\mathcal{T} \cong Ho\mathcal{T}$$

between the quotient of the stable category by the thick ideal of anti Kac modules and the homotopy category $Ho\mathcal{T}$.

Proof. We have to show that the induced map

$$\gamma : Hom_{ho\mathcal{T}}(X, Y) \rightarrow Hom_{Ho\mathcal{T}}(X, Y)$$

is an isomorphism.

$X \leftarrow Z \rightarrow Y$ for $f : Z \rightarrow Y$ is equivalent to zero if and only if there exists a morphism s' in $\overline{\mathcal{T}}$ with cone in $\overline{\mathcal{T}}_-$ such that $f \circ s' = 0$ in $\overline{\mathcal{T}}$ (see [Ne01]). On the other hand recall $Hom_{Ho\mathcal{T}}(X, Y) = Hom_{\overline{\mathcal{C}}}(QX, Y)$. Since $\gamma(s) : Z \rightarrow X$ is an isomorphism, there exists a commutative diagram in $Ho\mathcal{T}$

$$\begin{array}{ccc} QX & & \\ \downarrow q_X & \searrow q & \\ X & \xleftarrow{s} & Z & \xrightarrow{f} & Y \end{array}$$

such that $\gamma(s)^* : [X, Y] \cong [Z, Y]$ and $[Z, Y] = Hom_{\overline{\mathcal{C}}}(QX, Z)$, since QX is cofibrant and Z is fibrant. Hence $\gamma(f) \circ \gamma(s)^{-1}$ is equivalent to zero in $Ho\mathcal{T}$ if and only if $f \circ q = 0$. This is where the last theorem comes in. Since $Z \in \mathcal{T}$ it implies that q is trivial on a subobject K' of QX such that $Z' = QX/K' \in \mathcal{T}$. Hence $s' : Z' \rightarrow Z$ is well defined in \mathcal{T} , such that $f \circ s' = 0$. But there also exists a distinguished triangle

$$C_q \rightarrow C_{q_X} \rightarrow C_s \rightarrow C_q[1] .$$

Since $C_{q_X} = K[1] \in \mathcal{C}_-$ and $C_s \in \mathcal{C}_-$ this implies $C_q \in \mathcal{C}_-$. Hence $C_q \in \mathcal{T}_-$. Therefore $K' \in \mathcal{C}_-$ and

$$K' \rightarrow C_q \rightarrow C_{s'} \rightarrow K'[1]$$

implies $C_{s'} \in \mathcal{C}_-$. But then already $C_{s'} \in \mathcal{T}_-$. Therefore $f \circ s' = 0$ implies that the class of $X \leftarrow Z \rightarrow Y$ is the zero morphism $X \rightarrow Y$ in $ho\mathcal{T}$. This shows that γ is faithful. The fullness of γ is shown similarly. Any morphism in $[X, Y]$ is represented by a morphism $q : QX \rightarrow Y$ similarly as in the

diagram above. Since $q_X \oplus q$ is trivial on some $K' \subset \text{Kern}(q_X)$ with finite quotient $Z' = QX/K'$ we obtain a diagram in \mathcal{T}

$$\begin{array}{ccc} & Z' & \\ s' \swarrow & & \searrow f' \\ X & & Y \end{array}$$

(s' is induced by q_X and f is induced by q) such that $\gamma(s')^{-1} \circ \gamma(f) \in [X, Y]$ represents the morphism we started from. \square

Accordingly we will identify the categories $ho\mathcal{T}$ and $Ho\mathcal{T}$ in the following. Note however that it is important for us to have both interpretations of $Ho\mathcal{T}$. While the interpretation of $Ho\mathcal{T}$ as a Verdier quotient looks more down to earth, the cofibrant replacements are only visible when we use the model structure on $Ind(\mathcal{T})$.

13.6. Remarks on the Balmer spectrum. Balmer [Ba05] defined for a tensor triangulated category the notion of its spectrum by equipping the set of all prime ideals (proper thick tensor ideals such that $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$) with a Zariski topology. The category $Ho\mathcal{T}$ is a triangulated tensor category in the sense of Balmer [Ba05]. Hence its spectrum $Spc(Ho\mathcal{T})$ is defined. By [BKN17] the spectrum of the stable category $\overline{\mathcal{T}}$ is homeomorphic

$$Spc(\overline{\mathcal{T}}) \simeq Proj(N - Spec(S^\bullet(\mathfrak{f}_1^*)))$$

where $N = Norm_{G_0(\mathfrak{f}_1)}$ and the detecting subalgebra \mathfrak{f} . Formation of the spectrum is a contravariant functor, and if $F : \mathcal{K} \rightarrow \mathcal{L}$ is an essentially surjective tensor triangulated functor, the induced map $Spc(F) : Spc(\mathcal{L}) \rightarrow Spc(\mathcal{K})$ of locally ringed spaces is injective. More specifically, let $q : \mathcal{K} \rightarrow \mathcal{L} = \mathcal{K}/\mathcal{J}$ be the localization functor where \mathcal{J} is a thick tensor ideal. Then the associated map $Spc(q) : Spc(\mathcal{L}) \rightarrow Spc(\mathcal{K})$ induces a homeomorphism between $Spc(\mathcal{L})$ and the subspace

$$\{\mathcal{P} \in Spc(\mathcal{K}) \mid \mathcal{J} \subset \mathcal{P}\} \subset Spc(\mathcal{K})$$

of those thick prime ideals containing \mathcal{J} . In our case this applies to $Ho\mathcal{T} \cong \overline{\mathcal{T}}/\overline{\mathcal{T}}_-$, but doesn't give a concrete description of $Spc(Ho\mathcal{T})$ in this way. Note that by [BKN17] the thick tensor ideals of \mathcal{T} are in bijection with specialization closed (union of closed sets) subsets of $N - Proj(S^\bullet(\mathfrak{f}_1^*))$ by assigning to a thick tensor ideal the union of the support varieties of its elements. However the support varieties of anti Kac modules (or modules with a filtration by anti Kac modules) don't seem to have a known description.

14. THE DEGREE FILTRATION AND COFIBRANT REPLACEMENTS

14.1. Degree filtration of Kac objects. We show that every Kac object has a canonical degree filtration. By using the cofibrant replacement of an

arbitrary X we can also endow X with such a filtration in the ind-category. This filtration could be seen as an analogue of Deligne's weight filtration.

To $\lambda = (\lambda_1, \dots, \lambda_m \mid \lambda_{m+1}, \dots, \lambda_{m+n})$ we associate the bidegree

$$(d, d') = \left(\sum_{i=0}^m \lambda_i, \sum_{i=1}^n \lambda_{m+i} \right).$$

By the description of the blocks [BS12] $d - d'$ only depends on the block of $L(\lambda)$. If we fix the block, we can therefore think of d as the relevant degree and we define therefore

$$\text{deg}(\lambda) = \sum_i^n \lambda_i.$$

Recall that \mathcal{T}_+ denotes the tensor ideal of modules with a filtration by Kac modules in $\mathcal{T}_{m|n}$ and \mathcal{T}_- the tensor ideal of modules with a filtration by anti Kac modules in $\mathcal{T}_{m|n}$.

Lemma 14.1. *Each $M \in \mathcal{T}_+$ has a canonical degree filtration, i.e. a filtration by submodules $F_i(M) \in \mathcal{T}_+$ such that*

$$\dots \subseteq F_{i-1}(M) \subseteq F_i(M) \subseteq F_{i+1}(M) \subseteq \dots$$

and

$$F_i(M)/F_{i-1}(M) = \bigoplus_{\lambda} V(\lambda)$$

holds for certain Kac modules $V(\lambda) \in \mathcal{T}_+$ of degree $\text{deg}(\lambda) = i$. This filtration is inherited to retracts N of M so that $F_i(N) = N \cap F_i(M)$. The filtration is functorial with respect to morphisms.

Proof. Every M in \mathcal{T}_+ admits a filtration by objects in \mathcal{T}_+ whose graded pieces are Kac modules. To show the existence as in our claim it suffices to show that $\text{Ext}^1(V(\rho_1), V(\rho_2)) = 0$ holds for $\text{deg}(\rho_1) \leq \text{deg}(\rho_2)$. Since all Jordan-Hoelder constituents $L(\tau)$ of $V(\rho_1)$ have degree $\text{deg}(\tau) \leq \text{deg}(\rho_1)$, it suffices to show $\text{Ext}^1(V(\rho_1), L(\tau)) = 0$ for $\text{deg}(\rho_1) \leq \text{deg}(\tau)$. The dimension of $\text{Ext}^1(V(\rho), L(\tau))$ can be expressed as the coefficient $p_{\rho, \tau}^{(1)}$ of the linear term of the Kazhdan-Lusztig polynomial $p_{\rho, \tau}$ [BS10]. By [MS11, Lemma 6.10] and [BS10, Lemma 5.2] $p_{\rho, \tau}^{(1)} \neq 0$ if and only if τ is obtained from ρ by interchanging the labels at the ends of one of the cups in the cup diagram of ρ . Since this operation increases the degree of ρ , we must have $\text{deg}(\tau) > \text{deg}(\rho)$ to get a nonvanishing $p_{\rho, \tau}^{(1)}$. Hence the coefficient must be zero for $\text{deg}(\rho) \geq \text{deg}(\tau)$. The uniqueness is proved by induction on the length of such filtrations. The minimal nontrivial filtration submodule N is uniquely characterized by the maximal degree highest weight vectors in M . Then consider M/N and proceed by induction. Concerning retracts it suffices that \mathcal{T}_+ is closed under retracts, and hence so is \mathcal{T}_+ . \square

Lemma 14.2. *Every object $M \in \mathcal{C}_+$ is isomorphic to an inductive limit of finite dimensional Kac objects. In particular the degree filtration extends to \mathcal{C}_+ .*

Proof. The given object $M \in \mathcal{C}_+$ is stably isomorphic to $\Omega \otimes M$, hence we may replace M by $\Omega \otimes M$. Hence we may suppose

$$M \cong \bigcup_i \Omega \otimes M_i \cong \bigcup_{i,j} \Omega_j \otimes M_i$$

for $\Omega = \bigcup_j \Omega_j$. Since the Ω_j are finite dimensional Kac objects $\Omega_j, \Omega_j \otimes M_i$ is a Kac object as well. \square

Given an object $M \in \mathcal{C}_-$, we can dualize it via $()^*$ (the extension of the twisted dual to the ind completion) to obtain an object in \mathcal{C}_+ with its canonical degree filtration. Dualizing the filtration steps, equips $M \in \mathcal{C}_-$ with a dual degree filtration.

Corollary 14.3. *Every object $M \in \mathcal{C}_-$ is a sequential projective limit of finite dimensional anti Kac objects. It carries a descending degree filtration whose graded pieces are direct sums of finite dimensional anti Kac modules.*

Example 14.4. (*Degree filtration of projective objects*) The degree filtration of a maximal atypical projective cover $P(\tau)$ is as follows using the known filtration of $P(\tau)$ by Kac modules as in [BS11, Theorem 5.1]. Let $L(\rho)$ denote the constituent of highest weight in $P(\tau)$. Then there are 2^n weights μ_1, \dots, μ_{2^n} whose weight diagrams are obtained from the labeled cup diagram of τ by interchanging the labels at the ends of the n cups in all possible ways. Enumerate these 2^n distinct weights as μ_1, \dots, μ_{2^n} so that $\mu_i > \mu_j$ in the Bruhat order implies $i < j$. Then $\rho = \mu_1$ and $\mu_{2^n} = \tau$. The projective cover has then a filtration by submodules $M(i)$

$$\{0\} = M(0) \subset M(1) \subset \dots \subset M(2^n) = P(\tau)$$

such that

$$M(i)/M(i-1) \cong V(\mu_i)$$

for each $i = 1, \dots, 2^n$. Note that the enumeration in the Bruhat order also implies that $\mu_i > \mu_j$ implies $i < j$ in the degree ordering. The quotient $F_i(P(\tau))/F_{i-1}(P(\tau)) = \bigoplus_\lambda V(\lambda)$ is the direct sum of the $V(\mu_j)$ with $\deg(\mu_j) = i$. If $\deg(\rho) = k$, then $M(2^n)/M(2^n - 1) = V(\tau)$ and $M(1)/M(0) = M(1) = V(\rho)$.

Example 14.5. (*Degree filtration of $V(\nu) \otimes L(\mu)$*) (see [Se96, Corollary 5.2] for a variant) For maximal atypical $L(\lambda) \in \mathcal{T}_n$ we write $L_0(\lambda)$ for the irreducible $Gl(n) \times Gl(n)$ -module with highest weight $(\lambda_1, \dots, \lambda_n) \times (\lambda_{n+1}, \dots, \lambda_{2n})$. We denote the restriction of $L(\lambda)$ to $G_0 = Gl(n) \times Gl(n)$ by $Res_{G_0}(L(\lambda)) = L_{G_0}(\lambda)$. The restriction decomposes into a direct sum of irreducible representations, and the representation $L_0(\lambda)$ is the irreducible representation in this decomposition of largest degree.

For maximal atypical $L(\nu)$, $L(\mu)$ we determine the canonical degree filtration of the maximal atypical summand V of $V(\nu) \otimes L(\mu)$. The restriction $L_{G_0}(\nu) = \text{Res}_{G_0}(L(\nu))$ of $L(\nu)$ to the subgroup G_0 decomposes into a direct sum of irreducible representations and we denote the direct sum of the irreducible G_0 representations with weight of degree i by $L_{G_0}(\nu)^i$. Then $i \leq \text{deg}(\nu)$. Since $\bigoplus_{j \geq i} L_{G_0}(\nu)^j$ is stable under the super parabolic $P \subset G$ and since Ind_P^G is an exact functor we obtain from Frobenius reciprocity

$$\tilde{V} := L(\nu) \otimes V(\mu) = L(\nu) \otimes \text{Ind}_P^G L_0(\mu) \simeq \text{Ind}_P^G (L_{G_0}(\nu) \otimes L_0(\mu)).$$

Then the degree filtration of \tilde{V} has the form

$$F_i(\tilde{V}) = \text{Ind}_P^G \left(\bigoplus_{j \leq i} L_{G_0}(\nu)^{j - \text{deg}(\mu)} \otimes L_0(\mu) \right).$$

The associated graded modules are the Kac objects

$$\text{gr}^i(\tilde{V}) = \bigoplus V(L_{G_0}(\nu)^{i - \text{deg}(\mu)} \otimes L_0(\mu))$$

in \mathcal{C}^+ . The projection V of \tilde{V} onto the principal block Γ has the same structure except that only those irreducible G_0 -representations in $L_{G_0}(\nu)^{i - \text{deg}(\mu)}$ contribute which give Kac modules in Γ .

14.2. Polynomial growth and power series. We now define two subcategories

$$\mathcal{T}_+ \subset \mathcal{C}_+^{\text{pol}} \subset \mathcal{C}_+^{\text{fin}} \subset \mathcal{C}_+.$$

Definition 14.6. Let $\mathcal{C}_+^{\text{pol}}$, $\mathcal{C}_+^{\text{fin}}$ be the full subcategories of \mathcal{C}_+ with the following objects:

- $\mathcal{C}_+^{\text{fin}}$: objects M with a degree filtration F such that $F_k(M) = 0$ for some $k \in \mathbb{N}$ and $\dim \text{gr}_i^F(M) < \infty$ for all i .
- $\mathcal{C}_+^{\text{pol}}$: objects M with a degree filtration F such that $F_k(M) = 0$ for some $k \in \mathbb{N}$ and $\dim \text{gr}_i^F(M) < C \cdot P(i)$ for all i where $C = C(M)$ is a constant and $P = P(M)$ a polynomial.

Lemma 14.7. *The subcategories $\mathcal{C}_+^{\text{pol}}$, $\mathcal{C}_+^{\text{fin}}$ are exact subcategories in \mathcal{C}_+ and closed under tensor products. Their images in the homotopy category are triangulated monoidal categories.*

Both statements are obvious (closure under tensor product follows from the classical behaviour of weights in tensor products over $GL(m) \times GL(n)$). In particular the Grothendieck group K_0 of $\mathcal{C}_+^{\text{pol}}$, $\mathcal{C}_+^{\text{fin}}$ is defined.

We consider formal power series of the form

$$\sum_{i < k} [M_i] q^i$$

for $[M_i] \in K_0(\mathcal{T})$. Then we have a homomorphism

$$K_0(\mathcal{C}_+^{\text{fin}}) \rightarrow K_0(\mathcal{T})[[q^{-1}]], \quad [M] \mapsto [\text{gr}_i^F(M)] q^i.$$

Any $M \in \mathcal{T}$ is isomorphic in $Ho\mathcal{T}$ to $M \otimes \Omega$ and lies in the image of \mathcal{C}_+^{fin} . Therefore we can assign to M a formal power series as above. If M is irreducible, or more generally, if M has a minimal model, then we obtain a canonical power series in $K_0(\mathcal{T})[[q^{-1}]]$ associated to M . We remark that the minimal model lies in \mathcal{C}_+^{pol} . Indeed this is already true for $M \otimes \Omega$ by the description of Ω_{i+1}/Ω_i in section 13. By lemma 10.2 the minimal model for M is obtained from $M \otimes \Omega$ by projecting to the clean component. As for Kac objects we can define analogs of \mathcal{C}_+^{fin} and \mathcal{C}_+^{pol} for anti Kac objects using corollary 14.3 and then define an associated power series

Example 14.8. We compute the minimal model $\Omega(L(a))$ of an irreducible representation $L(a) = L(a| - a)$ in the principal block of $GL(1|1)$ in section 17.3. The Kac modules in $\Omega(L(a))$ are $V(a), V(a-2), V(a-4), \dots$ and the anti Kac modules in A are $V(a-1)^*, V(a-3)^*, V(a-5)^*, \dots$. Therefore the power series associated to $\Omega(L(a))$ is

$$\begin{aligned} & [V(a)]q^a + [V(a-2)]q^{a-2} + [V(a-4)]q^{a-4} + \dots \\ & = ([L(a)] + [L(a-1)])q^a + ([L(a-2)] + [L(a-3)])q^{a-2} + \dots \end{aligned}$$

The power series associated to A is similar, but involves only odd powers of q .

A variant. We would like to read the exact sequence associated to a cofibrant replacement as an equality between power series. As example 14.8 shows, the power series of $\Omega(L(a))$ and A might not have any cancellations. We identify $\mathcal{C}_+^{pol}, \mathcal{C}_+^{fin}$ and $\mathcal{C}_-^{pol}, \mathcal{C}_-^{fin}$ with their image in the ring of formal power series. We also use $q^{-deg(\lambda)}$ to obtain a formal power series with finite principal part. Then we have three different non-unital rings of formal power series:

- The \mathcal{C}_+^{pol} version: Here we give $V(\lambda)$ degree $deg(\lambda)$ and assign to $V(\lambda)$ the power series $q^{-deg(\lambda)}[V(\lambda)]$. This construction extends to sequential inductive limits of Kac-modules of polynomial growth.
- The \mathcal{C}_-^{pol} version: Here we give $V(\lambda)^*$ degree $deg(\lambda)$ and assign to $V(\lambda)^*$ the power series $q^{-deg(\lambda)}[V(\lambda)^*]$. This extends to sequential projective limits of anti Kac-modules of polynomial growth.
- The \mathcal{C}^{pol} version: Here we give $L(\lambda)$ degree $deg(\lambda)$ and assign to $L(\lambda)$ the power series $q^{-d(\lambda)}[L(\lambda)]$. This extends to inductive limits of polynomial growth of finite dimensional modules.

There is a natural ring isomorphism between $K_0(\mathcal{C}_+^{pol})$ and the $K_0(\mathcal{C}_-^{pol})$ induced by $()^*$. There is a natural ring homomorphism from $K_0(\mathcal{C}_+^{pol})$ to $K_0(\mathcal{C}^{pol})$ given by

$$q^{-deg(\lambda)}[V(\lambda)] \mapsto \sum_L q^{-deg(L)}[L]$$

where L runs over the irreducible constituents of $V(\lambda)$. Now $0 \rightarrow A \rightarrow \Omega(M) \rightarrow M \rightarrow 0$ for finite dimensional M and its cofibrant replacement of

polynomial growth $\Omega(M)$ with $A \in \mathcal{C}_-$ gives via identifications in the power series ring \mathcal{C}^{pol} the formula

$$[M] = [\Omega(M)] - [A].$$

14.3. The tensor product $V(\mathbf{1}) \otimes V(\mathbf{1})^*$. We give some estimates on the weights appearing in the cofibrant replacement Ω of $\mathbf{1}$. Recall that Ω has an increasing sequence of sub-comodules of Ω

$$V = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \dots$$

such that

$$\Omega_{i+1}/\Omega_i \cong V \otimes (I \otimes I^*)^{\otimes i}.$$

We analyse now the filtration step Ω_{i+1}/Ω_i . For simplicity we specialize in this section to the $m = n$ (and assume $m = n \geq 2$) since many calculations in a maximal atypical block can be reduced to calculations in the principal block of $\mathcal{T}_{n|n}$. In order to understand the tensor product $I \otimes I^*$ better, we first analyze the $V \otimes V^*$ tensor product. We recall from [HW14, Proposition 27.4]:

Proposition 14.9. [BSch17, Theorem B.17] *The space of matrices $M_{nn}(k)$ is a $GL(n, k) \times GL(n, k)$ -module in a natural way by left and right multiplication, hence also the Grassmann algebra $\Lambda := \Lambda^\bullet(M_n(k))$. As a representation of $GL(n, k) \times GL(n, k)$ we have*

$$\Lambda^\bullet(M_{nn}(k)) \cong \bigoplus_{\rho} \rho^\vee \boxtimes \rho^*$$

where $\rho = \rho_\lambda$ runs over all partitions in

$$P(n, n) = \{\lambda \in \mathbf{Z}^n \mid n \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}.$$

We also note that there exist 2^n symmetric Young diagrams with $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\lambda_1 \leq n$.

We also recall from [HW14] that $V(\mathbf{1})$ has a decreasing filtration (the radical filtration) of $GL(n|n)$ -subrepresentations with $n + 1$ irreducible graded pieces L_i such that $L_0 = k$ is the maximal irreducible quotient representation. The highest weights of the L_i can be computed from [BS11, Theorem 5.2] to be the duals

$$\lambda_{(i)}^\vee = (0, \dots, 0, -i, \dots, -i) \quad , \quad \text{for } i = 0, \dots, n$$

of the basic selftransposed weights λ_i in $P(n, n)$. In particular the cosocle of V^* consists of Ber^{-n} .

Lemma 14.10. *The tensor product $V \otimes V^*$ decomposes as*

$$V \otimes V^* \cong P(Ber^{-n}) \oplus Q$$

where Q is of atypicality less than n .

Proof. First note that

$$V \otimes V^* \in \mathcal{C}_+ \otimes \mathcal{C}_- \subset P_{\mathcal{C}}$$

is projective in \mathcal{T} since $\mathcal{C}_+ \cap \mathcal{C}_- = Proj$ and \mathcal{C}_{\pm} are tensor ideals. From $V = F(\mathbf{1})$ and $F(X)^* \cong Ind_{P_-}^G(X^* \otimes Ber^{-n})$ [Ge98, Proposition 2.1.1] we obtain as a P -module

$$W^* = F(\mathbf{1})^* = \Lambda^\bullet(G/P_-) \otimes Ber^{-n} \cong \Lambda^\bullet(P_+) \otimes Ber^{-n} = Ind_{G_0}^{P_+}(Ber^{-n}).$$

In other words

$$F(U(V^*)) \cong F(Ind_{G_0}^{P_+}(Ber^{-n})) \cong F_0(Ber^{-n})$$

where F_0 denotes induction from the group G_0 . Hence we get

$$V \otimes V^* \cong F(\mathbf{1}) \otimes V^* \cong F(U(V^*)) = F_0(Ber^{-n}).$$

Any filtration of $U(V^*)$ by irreducible P -modules, for instance the one by the Grassmann degree, has as irreducible graded pieces the $\binom{2n}{n}$ modules $\rho_\alpha \boxtimes \rho_{\alpha^*}$ of the quotient group G_0 of P for $\alpha \in P(n, n)$ (considered as representations of P). It induces a filtration of

$$V \otimes V^* = F(U(V^*))$$

by the $\binom{2n}{n}$ Kac-modules corresponding to the representations $\rho_\alpha \boxtimes \rho_{\alpha^*}$ of G for $\alpha \in P(n, n)$. Hence $V \otimes V^*$ inherits a Kac filtration whose maximal atypical graded pieces correspond to the 2^n self transposed $\alpha = \alpha^*$ weights. Since the degree of atypicality is a block invariant we can decompose $V \otimes V^*$ in the form

$$V \otimes V^* = P \oplus Q$$

where $P \in \mathcal{T}^n$ has atypicality n and $Q \in \mathcal{T}^{<n}$ is in the direct sum of blocks of atypicality $< n$.

Therefore P is projective with a filtration containing as graded pieces the 2^n maximal atypical Kac-modules defined by the 2^n highest weights $\alpha = \alpha^* \in P(n, n)$ (each with multiplicity one). Obviously $P(Ber^{-n})$ must be a summand of P , since Ber^{-n} is in the cosocle of P . The claim now follows since $P(Ber^{-n})$ contains 2^n Kac-constituents. \square

Since $V \otimes V^* \in \mathcal{C}_+$ and $V \otimes V^* \in \mathcal{C}_-$, by the vanishing of $Ext(\mathcal{C}_+, \mathcal{C}_-)$ -groups and

$$\dim_k Hom_{\mathcal{T}}(V(\lambda), V(\mu)^*) = \delta_{\lambda\mu}$$

we obtain from the fact that P has a filtration by 2^n Kac modules (and similarly a filtration by 2^n anti Kac modules)

$$\dim_k End_{\mathcal{T}}(P) = \dim_k Hom_{\mathcal{T}^n}(V \otimes V^*, V \otimes V^*) = 2^n.$$

For $V(\lambda) \in \mathcal{T}^n$ notice $\dim_k \text{Hom}_{\mathcal{T}^n}(V \otimes V^*, V(\lambda)^*) = 0$ or 1 depending on whether λ is one of the self transposed weights $\alpha \in P(n, n)$. Hence for maximal atypical λ we get

$$\dim_k \text{Hom}_{\mathcal{T}}(P, V(\lambda)^*) = 1 \text{ or } 0 ,$$

depending on whether λ is one of the self transposed weights $\alpha \in P(n, n)$ or not. On the other hand for $V(\lambda) \in \mathcal{T}^n$ we have

$$\begin{aligned} \text{Hom}_{\mathcal{T}^n}(P, V(\lambda)^*) &= \text{Hom}_{\mathcal{T}^n}(V \otimes V^*, V(\lambda)^*) = \text{Hom}_{\mathcal{T}}(V \otimes V^*, V(\lambda)^*) \\ &= \text{Hom}_{\mathcal{T}}(F_0(\text{Ber}^{-n}), V(\lambda)^*) = \text{Hom}_G(\text{Ber}^{-n}, V(\lambda)^*) \end{aligned}$$

by Frobenius reciprocity. We have deduced the following result:

Corollary 14.11. *For maximal atypical irreducible weights λ the following assertions are equivalent*

- (1) $\lambda = \lambda^*$, and hence $\lambda \in P(n, n)$ (and there are 2^n such λ).
- (2) $V(\lambda)$ is a Kac-constituent of the projective module $P = P(\text{Ber}^{-n})$ in the category \mathcal{T} .
- (3) The restriction of $V(\lambda)^*$ to G_0 contains the representation Ber^{-n} .
- (4) The restriction of $V(\lambda)^*$ to G_0 contains the representation Ber^{-n} with multiplicity one.
- (5) The restriction of $V(\lambda)$ to G_0 contains the representation Ber^{-n} .
- (6) $V(\lambda)$ contains Ber^{-n} as a constituent in the category \mathcal{T} .

For the equivalence of 3. and 4. recall that $V(\lambda)$ and $V(\lambda)^*$ have the same simple constituents. For the last implication use that property 2) is equivalent to the last property by the BGG formula $[P(\text{Ber}^{-n}) : V(\lambda)] = [V(\lambda) : \text{Ber}^{-n}]$.

14.4. **Estimates for $I \otimes I^*$.** The i -th filtration step of Ω is given by

$$\Omega_{i+1}/\Omega_i \cong V \otimes (I \otimes I^*)^{\otimes i}.$$

Since $I \subset V(\mathbf{1})$ we obtain $V(\mathbf{1}) \rightarrow I^*$

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ 0 & \longrightarrow & I \otimes I^* & \longrightarrow & V \otimes I^* & \longrightarrow & I^* \longrightarrow 0 \\ & & & & \uparrow & & \\ & & & & V \otimes V^* & & \\ & & & & \uparrow & & \\ & & & & V \otimes \mathbf{1} \cong V & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where $V \otimes V^* \cong P(\text{Ber}^{-n})$ up to contributions of lower atypicality by lemma 14.10. The filtration of the module $P(\text{Ber}^{-n})$ via Kac modules $V(\mathbf{1}), \dots, V(\text{Ber}^n)$ has been described in section 14.3. The Kac filtration of

$V \otimes I^*$ misses exactly the Kac module $V(\mathbf{1})$ in $P(\text{Ber}^{-n})$; and the module $I \otimes I^*$ lacks exactly a copy of I^* in comparison to $V \otimes I^*$.

The highest weight in a Kac module is always in the top. The maximal atypical composition factors of $I \otimes I^*$ are those of $P(\text{Ber}^{-n})$ with those of one $V(\mathbf{1})$ and I missing. The constituents in I are just the constituents in the tops of the Kac modules in $P(\text{Ber}^{-n})$. Therefore the largest weights in I come from the Kac module $V(0, \dots, 0, -1)$: Its top $[0, \dots, 0, -1]$ is not in $I \otimes I^*$, but the constituents $[0, \dots, 0, -2]$ and $[0, \dots, 0, -1, -1]$ (see [BS11, Theorem 5.2] for the description of the Loewy layers) in the next radical layer are. The smallest weight in $P(\text{Ber}^{-n})$ and in $I \otimes I^*$ is the socle Ber^{-2n} of $P(\text{Ber}^{-n})$ of degree $-2n^2$.

Corollary 14.12. *Let $L(\lambda)$ be a composition factor of $I \otimes I^*$. Then*

$$-2n^2 \leq \text{deg}(\lambda) \leq -2.$$

In particular $V \otimes (I \otimes I^)$ is a Kac object with weights between $-3n^2$ and -2 .*

Example 14.13. We assumed in this section $m = n \geq 2$. The $GL(1|1)$ -case was already treated in example 13.8. In this case $\Omega_{i+1}/\Omega_i \cong V(\text{Ber}^{-2i})$.

15. RESTRICTION AND DS -COHOMOLOGY

15.1. Restriction I. For $m = m_1 + m_2$ and $n = n_1 + n_2$ consider the super subgroups

$$GL(m_1|n_1) \times GL(m_2|n_2) \hookrightarrow GL(m|n)$$

of the supergroup $GL(m|n)$ defined in terms of matrices by

$$\begin{pmatrix} A' & 0 & B' & 0 \\ 0 & A'' & 0 & B'' \\ C' & 0 & D' & 0 \\ 0 & C'' & 0 & D'' \end{pmatrix}$$

This defines a restriction functor res

$$\mathcal{T}_{m|n} \rightarrow \mathcal{T}_{m_1|n_1} \times \mathcal{T}_{m_2|n_2} .$$

This is an exact tensor functor. We easily see

$$res(\text{Ber}) \cong \text{Ber} \boxtimes \text{Ber} .$$

Similarly we obtain a functor for the corresponding ind-categories

$$\mathcal{C}_{m|n} \rightarrow \mathcal{C}_{m_1|n_1} \times \mathcal{C}_{m_2|n_2} ,$$

denoted $res : \mathcal{C} \rightarrow \mathcal{C}' \times \mathcal{C}''$ for simplicity.

These restriction functors are exact tensor functors. Since the restrictions of projective comodules are projective comodules, this induces a monoidal triangulated functor between the triangulated tensor categories defined by the stable categories

$$\bar{\mathcal{C}} \rightarrow \bar{\mathcal{C}}' \times \bar{\mathcal{C}}'' .$$

Note that

$$P = \begin{pmatrix} A' & * & B' & * \\ * & A'' & * & B'' \\ 0 & 0 & D' & * \\ 0 & 0 & * & D'' \end{pmatrix} \subset GL(m|n)$$

contains

$$P' \times P'' = \begin{pmatrix} A' & 0 & B' & 0 \\ 0 & A'' & 0 & B'' \\ 0 & 0 & D' & 0 \\ 0 & 0 & 0 & D'' \end{pmatrix} \subset GL(m_1|n_1) \times GL(m_2|n_2)$$

Therefore $U(X) \in \mathcal{I}_{\mathcal{D}}$ for $X \in \mathcal{C}$ implies $(U' \times U'')(res(X)) = res(U(X)) \subset res(\mathcal{I}_{\mathcal{D}}) \subset \mathcal{I}_{\mathcal{D}'} \times \mathcal{I}_{\mathcal{D}''}$. In other words

$$res : \mathcal{C}_- \rightarrow \mathcal{C}'_- \times \mathcal{C}''_- .$$

Thus we get an induced functor between the triangulated quotient categories

$$\overline{\mathcal{E}} \rightarrow \overline{\mathcal{E}'} \times \overline{\mathcal{E}''} .$$

15.2. Restriction II. Similarly we may embed $GL(m-k|n-k)$ as an outer block matrix in $Gl(m|n)$

$$\varphi_{n,m} : GL(m-k|n-k) \hookrightarrow GL(m|n) .$$

Analogous to the preceding discussion we obtain induced functors

$$res : Ho\mathcal{C}_{m|n} \rightarrow Ho\mathcal{C}_{m-k|n-k}$$

and similarly

$$res : Ho\mathcal{T}_{m|n} \rightarrow Ho\mathcal{T}_{m-k|n-k} .$$

15.3. The functor DS . We recall from the article [HW14] that we have a tensor functor $DS : \mathcal{T}_{m|n} \rightarrow \mathcal{T}_{m-1|n-1}$ attached to the choice of an odd element $x \in \mathfrak{g}_1$ satisfying $[x, x] = 0$. Since $[x, x] = 0$ we get

$$2 \cdot \rho(x)^2 = [\rho(x), \rho(x)] = \rho([x, x]) = 0$$

for any algebraic representation (V, ρ) of $GL(m|n)$ in $\mathcal{C}_{m|n}^\infty$. We fix now

$$x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \text{ for } y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The cohomological tensor functor DS is defined as

$$DS = DS_{n,n-1} : \mathcal{C}_{m|n} \rightarrow \mathcal{C}_{m-1|n-1}$$

via $DS_{n,n-1}(V, \rho) = V_x := Kern(\rho(x))/Im(\rho(x))$.

Lemma 15.1. *The functor DS factorizes over the homotopy category and induces tensor functors*

$$\begin{aligned} DS : Ho\mathcal{C}_{m|n} &\rightarrow Ho\mathcal{C}_{m-1|n-1} \\ DS : Ho\mathcal{T}_{m|n} &\rightarrow Ho\mathcal{T}_{m-1|n-1}. \end{aligned}$$

Proof. It was proven in [HW14, Theorem 4.1] that the kernel of $DS : \mathcal{T}_{m|n} \rightarrow \mathcal{T}_{m-1|n-1}$ equals \mathcal{T}_- . A module X is in \mathcal{C}_- if and only if its restriction to $P(m|n)^+$ is projective and therefore injective. Since any injective module is a direct sum of injective finite dimensional modules, we obtain

$$DS(X) = DS\left(\bigoplus X_i\right) = \bigoplus DS(X_i) = 0$$

where used that $x \in P(m|n)^+$. So $\ker(DS) = \mathcal{C}_-$. A cofibrant replacement of $X \in \mathcal{C}$ defines an exact sequence

$$0 \longrightarrow K^- \longrightarrow QX \xrightarrow{q} X \longrightarrow 0$$

with $K^- \in \mathcal{C}_-$. We apply DS to this sequence. Since $DS(\mathcal{C}_-) = 0$ and the functor DS is weakly exact in the sense of [HW14, Lemma 2.1], this implies that $DS(q) : DS(QX) \rightarrow DS(X)$ is an isomorphism. Let $f \in [X, Y]$ be an arbitrary morphism and recall that $[X, Y] = Hom_{\mathcal{C}}(QX, Y)/\sim$. Therefore we obtain from $f : QX \rightarrow Y$ the commutative diagram

$$\begin{array}{ccc} DS(QX) & \xrightarrow{DS(q)} & DS(X) \\ \downarrow & \swarrow & \\ DS(Y) & & \end{array}$$

which shows that $DS : \mathcal{C}_{m|n} \rightarrow \mathcal{C}_{m-1|n-1}$ factorizes over $Ho\mathcal{C}_{m|n}$. We obtain all in all a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{m|n} & \longrightarrow & Ho\mathcal{C}_{m|n} \\ \downarrow DS & \swarrow & \downarrow DS \\ \mathcal{C}_{m-1|n-1} & \longrightarrow & Ho\mathcal{C}_{m-1|n-1} \end{array}$$

which defines the induced functor $DS : Ho\mathcal{C}_{m|n} \rightarrow Ho\mathcal{C}_{m-1|n-1}$. This functor can be also simply restricted to the $Ho\mathcal{T}$ -case. \square

Remark 15.2. This result extends to the more general functors $DS_{m-k|n-k} : \mathcal{T}_{m|n} \rightarrow \mathcal{T}_{m-k|n-k}$ considered in [HW14].

Remark 15.3. If we choose the other Frobenius pair (i.e. exchange P^+ with P^-) to define the homotopy category, we get a similar result for the DS functor associated to the element $\sigma(x) \in \mathfrak{gl}(m|n)_1$.

16. SEMISIMPLE QUOTIENTS

16.1. Supertannakian categories. A k -linear tensor category \mathcal{T} over a field k (in the sense of [Del02]) is a small abelian k -linear symmetric closed monoidal rigid category with $End_{\mathcal{T}}(1) \cong k$. If \mathcal{T} admits a super fibre functor over an extension field of k , it is a supertannakian category and the finiteness condition (F) holds.

For a symmetric k -linear tensor category the symmetric group S_m acts on $X^{\otimes m}$ for any $X \in \mathcal{T}$. If k is of characteristic zero, the irreducible representations σ_α of the group S_m define the Schur functors $S = S^\alpha : X \rightarrow S^\alpha(X) = Hom_{S_m}(\sigma_\alpha, X^{\otimes m})$ where α is a partition of m . Special cases are the symmetric or alternating m -th powers of X . By Deligne [Del02] a k -linear tensor category \mathcal{T} over an algebraically closed field k of characteristic 0 is supertannakian if and only if every object is annihilated by some Schur functor (Schur finiteness). Any supertannakian category over an algebraically closed field of characteristic 0 is tensor equivalent to the representation category $Rep(G, \epsilon)$ of an affine supergroup scheme over k .

16.2. Tensor generators. A tensor generator in the sense of [Del02, 0.1] is an object $Y \in \mathcal{T}$ such that any other object in \mathcal{T} is obtained by iterated application of the operations \oplus, \otimes, \vee and subquotients. A supertannakian category has a tensor generator if and only if G is of finite type, i.e. an algebraic supergroup. We also say that it is an algebraic tensor category. By the usual comodules-representations correspondence, an algebraic tensor category over an algebraically closed field of characteristic zero is equivalent to the tensor category of finite dimensional graded A -comodules of a supercommutative Hopf algebra A finitely generated over k .

If $\mathcal{T} \rightarrow \mathcal{T}'$ is a k -linear tensor functor, \mathcal{T} is a k -linear tensor category and \mathcal{T}' is k -linear symmetric (closed) monoidal category, then the full image subcategory is a full k -linear symmetric (closed) monoidal rigid subcategory of \mathcal{T}' . Schur finiteness will be inherited from \mathcal{T} , whereas properties (F) and (G) might not be inherited to the full image subcategory of \mathcal{T} in \mathcal{T}' .

16.3. Ideals. An ideal \mathcal{J} of a k -category \mathcal{H} is a collection of k -subvectorspaces $\mathcal{J}(X, Y) \subset Hom(X, Y)$ for all $X, Y \in \mathcal{H}$ such that $f\mathcal{J}(X, Y)g \in \mathcal{J}(X', Y')$ holds for all $g \in Hom(X', X)$ and $f \in Hom(Y, Y')$. This defines the k -linear quotient category \mathcal{H}/\mathcal{J} with morphisms $Hom(X, Y)/\mathcal{J}(X, Y)$ and the same objects as in \mathcal{H} . If \mathcal{H} is k -linear, so is \mathcal{H}/\mathcal{J} .

Example 16.1. The radical $rad(X, Y)$ is the ideal, which is defined by: $f \in rad(X, Y)$ if and only if $id_X - gf$ is invertible for all $g \in Hom(Y, X)$.

16.4. Negligible morphisms. For a symmetric monoidal k -category \mathcal{H} an ideal \mathcal{J} is called a *monoidal ideal*, if it is stable under tensor products with id_Z for all objects $Z \in \mathcal{H}$ [AK02, section 6]. In this case \mathcal{H}/\mathcal{J} inherits

a symmetric monoidal structure and the quotient functor $\mathcal{H} \rightarrow H/J$ is a tensor k -functor, and rigid objects in \mathcal{H} map to rigid objects in \mathcal{H}/J .

Example 16.2. If \mathcal{H} is a symmetric monoidal rigid k -linear category with $\text{End}(1) = k$, then the monoidal ideal $\mathcal{N}(X, Y)$ is defined by the morphisms $f \in \text{Hom}(X, Y)$ such that $\text{tr}(g \circ f) = 0$ holds for all $g \in \text{Hom}(Y, X)$. This is the ideal of negligible morphisms $\mathcal{N} = \mathcal{N}_{\mathcal{H}}$.

Lemma 16.3. ([AK02, 7.1.4])

- (1) The ideal \mathcal{N} is the largest monoidal ideal of \mathcal{H} distinct from \mathcal{H} .
- (2) If \mathcal{I} is a monoidal ideal such that \mathcal{H}/\mathcal{I} is semisimple, then $\mathcal{I} = \mathcal{N}$.

16.5. Semisimplicity. By [AK02, 2.1.2] a small k -linear category \mathcal{H} is semisimple if and only if

- The radical ideal vanishes $\text{rad}(\mathcal{H}) = 0$,
- $\text{Hom}(X, X)$ is a semi-simple Artin ring for all objects $X \in \mathcal{H}$.

In this situation, if \mathcal{H} is semisimple pseudo-abelian and k -linear, then \mathcal{H} is an abelian category [AK02].

16.6. The quotient by negligible morphisms. In the $GL(m|n)$ -case the vanishing and finiteness theorems of section 11 hold. Then $\mathcal{H} = \text{Ho}\mathcal{T}$ is a k -linear rigid symmetric monoidal category, and we have shown $\text{End}_{\mathcal{H}}(1) = k$. Hence the ideal of negligible morphisms \mathcal{N} is defined.

Theorem 16.4. For $GL(m|n)$ over an algebraically closed field k of characteristic 0 the following holds for $\text{Ho}\mathcal{T}$:

- (1) We have the relation $\mathcal{N} \supset \mathcal{R}$ and \mathcal{H}/\mathcal{N} is semisimple.
- (2) The quotient $\text{Ho}\mathcal{T}/\mathcal{N}$ is the semisimple representation category of an affine supergroup scheme.

For the proof we use the following criterion due to André and Kahn [AK05, Théorème 1]:

Proposition 16.5. Let \mathcal{A} be k -linear symmetric monoidal category, rigid, with $\text{End}_{\mathcal{A}}(1) = k$ and $\text{char}(k) = 0$. Suppose there is an extension L/k and a k -linear tensor functor $H : \mathcal{A} \rightarrow \mathcal{V}$ into an abelian L -linear symmetric monoidal rigid category, in which the Hom -spaces are finite-dimensional and the trace of a nilpotent endomorphism vanishes. Then $\mathcal{R} \subset \mathcal{N}$ and therefore \mathcal{A}/\mathcal{N} is semisimple.

Proof. We prove in section 18.2 by direct computations that $\text{Ho}\mathcal{T}_{m|1}/\mathcal{N}$ is a supertannakian category for the Frobenius pair $(GL(m|1), P(m|1)^+)$ where $P(m|1)^+$ denotes the upper parabolic in $GL(m|1)$. For $G = GL(m|n)$ we then obtain an induced restriction functor

$$\text{res} : \text{Ho}\mathcal{T}_{m|n} \rightarrow \text{Ho}\mathcal{T}_{m-n+1|1}$$

as in section 15.2. The functor

$$\text{Ho}\mathcal{T}_{m|n} \rightarrow \text{Ho}\mathcal{T}_{m-n+1|1} \rightarrow \text{Ho}\mathcal{T}_{m-n+1|1}/\mathcal{N}$$

satisfies the criterion of proposition 16.5. Therefore $\mathcal{R} \subset \mathcal{N}$ and $Ho\mathcal{T}/\mathcal{N}$ is semisimple. This implies that $Ho\mathcal{T}/\mathcal{N}$ is abelian. Since Schur finiteness is inherited via tensor functors and every object in \mathcal{T} is Schur finite, these quotients are supertannakian categories and we can apply Deligne’s theorem. \square

17. THE CASE $GL(m|1)$: MORPHISMS AND COFIBRANT REPLACEMENTS

Let $G = GL(m|1)$. As before we choose $A \rightarrow B$ to correspond to the inclusion $P = G_0 \oplus G_{+1} \hookrightarrow G$. For the corresponding model structure on the category $Ind(\mathcal{T}_{m|1})$ we obtain the homotopy category $Ho\mathcal{C}$. We now compute the morphisms in $Ho\mathcal{C}$ between the simple objects of \mathcal{T} .

17.1. Representations of $GL(m|1)$. The Kac modules in the category \mathcal{T} are either irreducible projective (λ typical) or have length two (λ atypical) with two atypical composition factors. The category \mathcal{T} decomposes into blocks \mathcal{T}^Λ . The *Ext*-quiver of an atypical block has been described in [Ge98]. The irreducible representations in a given block can be parametrized by the integers, and we denote representatives of the simple objects of such a block \mathcal{T}^Λ by $L(i)$ for $i \in \mathbb{Z}$ for some arbitrarily chosen simple object $L(0) = L(\lambda)$ of this block. The Kac module $V = V(0)$ is an extension with simple cosocle $L(0)$ and simple socle $L(-1)$.

By the classification of the indecomposable objects in \mathcal{T}^Λ the non-projective indecomposable modules correspond to intervals on the numberline. More precisely for every interval $[a, b]$ we have two indecomposable modules with composition factors $L(a), \dots, L(b)$. The indecomposable module with socle $L(a), L(a + 2), \dots$ and cosocle $L(a + 1), L(a + 3), \dots$ is denoted $R[a, \dots, b]$. Its twisted dual is $B[a, \dots, b] = R[a, \dots, b]^*$ with cosocle $L(a), L(a + 2), \dots$ and socle $L(a + 1), L(a + 3), \dots$. The Kac- and anti Kac-modules are then given by

$$V(a) = R[a, a + 1], \quad V(a)^* = B[a, a + 1].$$

If $R[a, \dots, b]$ has even length, it has a filtration by the Kac modules $V(a + 1), \dots, V(b)$ and $B[a, \dots, b]$ has a filtration by the anti Kac-modules $V(a)^*, \dots, V(b - 1)^*$.

Remark 17.1. This notation differs from the one used in [He15]. There we use the notation $I^+[a, b]$ for the unique indecomposable module with $L(b) \in top(I^+[a, b])$ and $I^-[a, b] = I^+[a, b]^*$ for its twisted dual.

17.2. Morphisms.

Lemma 17.2. *We have $[L(i), L(j)] = 0$ unless $i \geq j$ and $i \equiv j$ modulo 2, where $[L(i), L(j)] \cong k$.*

Proof. We apply the functor $Hom_{Ho\mathcal{C}}(-, L(u))$ to the exact sequence

$$0 \rightarrow L(-1) \rightarrow V \rightarrow L(0) \rightarrow 0$$

defined by the Kac object V . Since $V \in \mathcal{C}_+$ is cofibrant, we obtain

$$[V, L(u)] = k$$

for $u = 0$ and 0 for $u \neq 0$ since $\text{Hom}_{\mathcal{C}}(V, L(u)) = k$ for $u = 0$ and zero otherwise. Furthermore for $u = 0$ this morphism can not be factorized over a projective object, since K is clean and $L(u)$ is simple. Now put $u = 0$ (for simplicity). The long exact homotopy sequence then implies

$$[L(-1)[i], L(0)] \cong [L(0)[i-1], L(0)]$$

for all $i \leq -1$ and all $i \geq 2$. Furthermore for $i = 0, 1$ we have an exact sequence

$$0 \rightarrow [L(0)[-1], L(0)] \rightarrow [L(-1), L(0)] \rightarrow k \rightarrow [L(0), L(0)] \rightarrow [L(-1)[1], L(0)] \rightarrow 0$$

We already know

$$L(n+1) = L(n)[-1],$$

since $L(n) \cong V(n+1)^*/L(n+1)$ for the anti Kac module $V(n+1)^*$, which becomes zero in $\text{Ho}\mathcal{T}$. Hence

$$[L(-1-i), L(0)] \cong [L(1-i), L(0)]$$

for all $i \leq -1$ and all $i \geq 2$. Furthermore for $i = 0, 1$ we have an exact sequence

$$0 \rightarrow [L(1), L(0)] \rightarrow [L(-1), L(0)] \rightarrow k \rightarrow [L(0), L(0)] \rightarrow [L(-2), L(0)] \rightarrow 0$$

Since $[L(-2), L(0)] = [L(-1), L(0)] = 0$ by theorem 11.2, this implies

$$[L(i), L(0)] = 0$$

for all odd i and all even $i \leq -2$, and

$$[L(i), L(0)] \cong k$$

for all even $i \geq 0$. □

Hence the triangulated category \mathcal{H} decomposes into blocks \mathcal{H}^Λ , and each block decomposes into two subblocks

$$\mathcal{H}^\Lambda = \mathcal{H}_{ev}^\Lambda \oplus \mathcal{H}_{odd}^\Lambda$$

such that $\mathcal{H}_{odd}^\Lambda = \mathcal{H}_{ev}^\Lambda[1]$. The images of the simple objects in the block \mathcal{T}^Λ are identified with the integers. Those in \mathcal{H}_{ev}^Λ are identified with the even integers, and the morphisms in \mathcal{H}_{ev}^Λ arise $\text{Hom}(2j, 2i) = k \cdot f_{ij}$ for a nonzero morphism f_{ij} if $j \geq i$ and $\text{Hom}(2j, 2i) = 0$ otherwise, such that

$$f_{ij} \circ f_{jk} = f_{ik} \quad , \quad i \leq j \leq k .$$

Lemma 17.3. *$L(u)$ and $L(v)$ for $u \neq v$ atypical are isomorphic in $\text{Ho}\mathcal{C}$ if and only if $u = v$.*

Proof. Assume $v \neq u$ and assume $L(u) \simeq L(v)$ in $Ho\mathcal{C}$. Then we may assume that the weight of u is smaller than the weight of v without restriction of generality. Any such isomorphism is represented by a homotopy class of a morphism f in $Hom(QL(u), L(v))$ ([Ho99, Theorem 1.2.10ii])

$$f : QL(u) \rightarrow L(v) .$$

Since all weights in $\Omega = QL(u)$ are $\leq u$ and hence $< v$, this implies $f = 0$. Now f becomes an isomorphism in $Ho\mathcal{C}$ if and only if f is a weak equivalence ([Ho99, Theorem 1.2.10iv]). If $f = 0$ is a weak equivalence, we can factor $f = \psi \circ \varphi$ into a split monomorphism $\varphi : QL(u) \rightarrow Z$ with projective kernel and a surjective morphism $\psi : Z \rightarrow L(v)$ with kernel in \mathcal{C}_- . But then $\psi = 0$, since $f = 0$. This implies $L(v) = 0$. Contradiction. \square

- Lemma 17.4.** (1) *If the length of $B = B[a, \dots, b]$ is even, B becomes isomorphic to zero in $Ho\mathcal{C}$.*
 (2) *If the length of $B = B[a, \dots, b]$ is odd, $B \simeq L(b)$ in $Ho\mathcal{C}$.*
 (3) *If the length of $R = R[a, \dots, b]$ is even, B is indecomposable in $Ho\mathcal{C}$.*
 (4) *If the length of $R = R[a, \dots, b]$ is odd, $R \simeq L(a)$ in $Ho\mathcal{C}$.*

Proof. If the length of B is even, it is in \mathcal{T}_- . If the length is odd, the quotient morphism $B[a, \dots, b] \rightarrow L(b)$ has kernel in \mathcal{T}_- . If the length of R is even, it is in \mathcal{T}_+ . Since R is cofibrant $[R, R] = Hom_{\overline{\mathcal{C}}}(R, R)$. Since the latter is one-dimensional, R is indecomposable. If the length of R is odd, then the morphism $L(a) \rightarrow R[a, \dots, b]$ has cokernel in \mathcal{T}_- and therefore $L(a) \simeq R$ in $Ho\mathcal{C}$. \square

Any object in \mathcal{C} is a direct sum of indecomposable modules. Those in \mathcal{C}_- become isomorphic to zero in $Ho\mathcal{C}$. Those in \mathcal{C}_+ stay indecomposable unless they are projective. All the remaining ones become isomorphic in $Ho\mathcal{C}$ to the image of some simple module $L(u)$.

17.3. Cofibrant replacements. In this section we explicitly determine the minimal models of the simple objects.

Cofibrant replacements. Projective simple objects X in \mathcal{C} are cofibrant. Atypical simple objects are not cofibrant. Objects in \mathcal{C}_+ are cofibrant. For $X \in \mathcal{C}_-$ a projective resolution $q : P \rightarrow X \rightarrow 0$ defines a cofibrant replacement $QX \cong P$ of X . We now construct an explicit cofibrant replacement $q : QX \rightarrow X$ for atypical simple modules $X = L(u)$ as a sequential inductive limit

$$\Omega = \underset{i}{\text{co lim}} \Omega_i$$

of subobjects

$$\Omega_i = R[u - 1 - 2i, \dots, u]$$

with the obvious inclusion morphisms $\Omega_i \hookrightarrow \Omega_{i+1}$ (see [Ge98] for further details). $\Omega_{i+1}/\Omega_i \cong R[u - 1 - 2i, u - 2i]$ is in $F(\mathcal{D}) \subset \mathcal{C}_+$. This shows $\Omega_i \in \mathcal{C}_+$, since \mathcal{C}_+ is closed under extensions. Ω is a union of the Ω_i . Since \mathcal{C}_+ is closed under monomorphic sequential colimits we obtain

Lemma 17.5. Ω is cofibrant.

$$\begin{array}{ccccccc}
 & & L(u-4) & & L(u-2) & & L(u) \\
 & \swarrow & / & \backslash & / & \backslash & / \\
 & & L(u-5) & & L(u-3) & & L(u-1) \\
 & \swarrow & & \swarrow & & \swarrow & \\
 & & & & & &
 \end{array}$$

There exists an exact sequence

$$0 \rightarrow R \rightarrow \Omega \rightarrow L(u) \rightarrow 0,$$

where R is isomorphic to the cohomology of the complex

$$\bigoplus_{i=1}^{\infty} L(u-2i-1) \rightarrow \bigoplus_{i=1}^{\infty} R[u-2i-1, u-2i] \oplus R[u-2i, u-2i+1]^* \rightarrow \bigoplus_{i=1}^{\infty} L(u-2i).$$

Notice that the simple module $L(u-2i)$ is a quotient of $R[u-2i-1, u-2i]$ and $R[u-2i, u-2i+1]^*$. Similarly the simple module $L(u-2i-1)$ is a submodule of $R[u-2i-1, u-2i]$ and $R[u-2i, u-2i-1]^*$.

Lemma 17.6. The restriction $UR[u-1, u] = UL(u-1) \oplus UL(u)$ splits in \mathcal{D} .

Proof. The projective $P = P[u-2, u-1, u-1, u]$ contains $R[u-1, u]$ via the standard embedding. P has a filtration by two anti Kac modules V and V' , which under the restriction functor U become indecomposable projectives in \mathcal{D} . Therefore the anti Kac filtration splits in \mathcal{D} , hence

$$UR[u-1, u] = (UR[u-1, u] \cap UV) \oplus (UR[u-1, u] \cap UV').$$

On the other hand the anti Kac filtration on P cuts out the standard filtration on $R[u-1, u] \subset P$ with the graded pieces $L(u-1)$ and $L(u)$. \square

Lemma 17.7. R is in \mathcal{C}_- . Hence Ω is a cofibrant replacement $Q'X$ of the simple module $X = L(u)$.

Proof. The exact sequences

$$0 \rightarrow L(u-1) \rightarrow R[u-1, u] \rightarrow L(u) \rightarrow 0$$

split in \mathcal{D} after applying the restriction functor U by lemma 17.6. This implies

$$UR \cong \bigoplus_{i=1}^{\infty} UR[u-2i, u-2i+1]^*.$$

All $UR[u-2, u-1]^*$ are injective, hence UR is injective in \mathcal{D} . \square

Example 17.8. Consider the indecomposable module $X = R[a, a+1, a+2]$ and the cofibrant object $Q'X = \Omega(L(a)) \oplus P[a, a+1, a+1, a+2]$. There is a morphism $q' : Q'X \rightarrow X$ with kernel $K \cong \text{kern}(q : \Omega(L(a+2)) \rightarrow L(a+2)) \in \mathcal{C}_-$, hence $Q'X$ is a cofibrant replacement of X . This shows

that both arrows, the natural inclusion and q' , are in \mathcal{W} , and therefore the composite morphism

$$\Omega(L(a)) \hookrightarrow Q'X \rightarrow X$$

is in \mathcal{W} . This again implies

$$\Omega(L(a)) \cong X$$

in HoC . Since $q'(P[a, a+1, a+1, a+2]) \neq 0$, there does not exist a minimal model for X by the last lemma.

Example 17.9. In general $\text{End}_{\mathcal{C}}(X) \rightarrow [X, X]$ is not surjective. Put $X = L(0) \oplus L(-2)$, then $\text{End}_{\mathcal{C}}(X) = k^2$, but $[X, X]$ also contains a nilpotent radical generated by the morphism $f_{-2,0}$.

18. THE CASE $GL(m|1)$: SEMISIMPLE QUOTIENTS

18.1. Semisimplicity of HoT/\mathcal{N} . Let us write $\text{HoT}^{ss} = \text{HoT}/\mathcal{N}$. The indecomposable objects in \mathcal{T}_+ become isomorphic to zero in HoT . Those in \mathcal{T}_- become zero in HoT^{ss} . All the remaining ones become isomorphic in HoT , hence in HoT^{ss} , to the image of some simple module $L(u)$ by the previous section. Hence HoT^{ss} is a semisimple category.

Lemma 18.1. *HoT^{ss} is a semisimple k -linear rigid closed monoidal tensor category with $\text{End}(\mathbf{1}) = k$. Its simple objects are parameterized by the atypical weights. It is of the form $\text{Rep}(G', \epsilon)$ for some supergroup G .*

18.2. The quotient $\text{Rep}(GL(m|1))/\mathcal{N}$ and consequences. We want to describe $\text{Rep}(G', \epsilon)$ explicitly. Recall from [Del02, Example 0.4 (ii)] that if G is an affine supergroup scheme and μ_2 acts on G by the parity automorphism, $\text{Rep}(\mu_2 \times G, \epsilon = (-1, e))$ is the category of super representations of G .

We recall now from [He17] results about the tensor product decomposition of simple $GL(m|1)$ -modules. Any irreducible module $L(\lambda)$ can be written uniquely in the form $L(\tilde{\lambda}) \otimes \text{Ber}^{s_\lambda}$ where $L(\tilde{\lambda})$ is a direct summand in a space of mixed tensors $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ for some r, s , and s_λ is an explicit shift factor that can be read off from the cup diagram of λ . The irreducible mixed tensors generate a tensor category isomorphic to $\text{Rep}(GL(m-1))$ in the quotient category $\text{Rep}(GL(m|1))/\mathcal{N}$ and $L(\tilde{\lambda})$ corresponds to an irreducible $GL(m-1)$ -representation $L(\text{wt}(\tilde{\lambda}))$. Therefore the Tannaka group generated by the irreducible $GL(m|1)$ -modules in $\text{Rep}(GL(m|1))/\mathcal{N}$ is $GL(m|1) \times GL(1)$, so that the Tannaka category is equivalent to the super representations of $GL(m-1) \times GL(1)$, i.e.

$$\begin{aligned} (\text{Rep}(GL(m|1))/\mathcal{N})^{irr} &\simeq \text{Rep}(\mathbb{Z}/2\mathbb{Z} \times (GL(m-1) \times GL(1)), \epsilon = (-1, e)), \\ L(\tilde{\lambda}) \otimes \text{Ber}^{s_\lambda} &\mapsto L(\text{wt}(\tilde{\lambda})) \times \text{det}^{s_\lambda}. \end{aligned}$$

Proposition 18.2. *The quotient $\text{HoT}^{ss} \cong \text{Rep}(G', \epsilon)$ is equivalent to*

$$\text{HoT}^{ss} \simeq \text{Rep}(\mathbb{Z}/2\mathbb{Z} \times (GL(m-1) \times GL(1)), \epsilon = (-1, e)).$$

Proof. The irreducible representations in $Ho\mathcal{T}^{ss}$ and $(Rep(GL(m|1))/\mathcal{N})^{irr}$ are both parametrized by the atypical weights. They obey the same tensor product decomposition and the categories are semisimple. Therefore one can write down an isomorphism between the Grothendieck semirings of these two categories which lifts to an isomorphism of groups. For further details we refer to the identical proof in [He15, Theorem 5.12]. \square

18.3. Isogenies: Semisimplicity. Recall from the last section that the triangulated category \mathcal{H} decomposes into blocks \mathcal{H}^Λ where the images of the simple objects in the block \mathcal{T}^Λ are identified with the integers.

Lemma 18.3. *All f_{ik} from section 17.2 are isogenies, i.e. contained in Σ .*

Proof. For the indecomposable modules $B[2i, \dots, 2j]$ the quotient morphism $s : B[2i, \dots, 2j] \rightarrow L(2j)$ is a weak equivalence in \mathcal{W} . On the other hand, the quotient morphism $f : B[2i, \dots, 2j] \rightarrow L(2i)$ in \mathcal{T} has kernel in \mathcal{T}_+ , hence induces an isogeny in \mathcal{H} . The composition $f \circ s^{-1} : L(2j) \rightarrow L(2i)$ is well defined in \mathcal{H} and is the morphism f_{ij} mentioned above up to a constant. \square

All objects in \mathcal{T}_\pm become isomorphic to zero in $\mathcal{H}[\Sigma^{-1}]$. By lemma 18.3 the image of a block in $\mathcal{H}[\Sigma^{-1}]$ has up to isomorphism two indecomposable elements (note $Hom_{\mathcal{H}[\Sigma^{-1}]}(X, X) = k \cdot id \cong k$ for simple atypical objects X), namely one representative in $\mathcal{H}_{ev}^\Lambda[\Sigma^{-1}]$ and one in $\mathcal{H}_{odd}^\Lambda[\Sigma^{-1}]$.

Lemma 18.4. *$\mathcal{H}[\Sigma^{-1}]$ is a semisimple abelian category.*

Proof. The pair

$$(\mathcal{H}_{ev}[\Sigma^{-1}], \mathcal{H}_{odd}[\Sigma^{-1}])$$

for

$$\mathcal{H}_{ev}[\Sigma^{-1}] = \bigoplus_{\Lambda} \mathcal{H}_{ev}^\Lambda[\Sigma^{-1}], \quad \mathcal{H}_{odd}[\Sigma^{-1}] = \bigoplus_{\Lambda} \mathcal{H}_{odd}^\Lambda[\Sigma^{-1}]$$

is a torsion pair on $\mathcal{H}[\Sigma^{-1}]$ in the sense of [AN12] (more precisely $\mathcal{H}_{odd}[\Sigma^{-1}]$ is a cluster tilting subcategory [KR07]). The heart of a torsion pair is an abelian category. Here the heart equals the quotient $\mathcal{H}[\Sigma^{-1}]/\mathcal{H}_{odd}[\Sigma^{-1}] \simeq \mathcal{H}_{ev}[\Sigma^{-1}]$. The latter is therefore abelian and hence also semisimple. \square

As for the quotient by the negligible morphisms this implies

Corollary 18.5. *$\mathcal{H}[\Sigma^{-1}] \cong Rep(\mu, \tilde{G})$ for of a reductive algebraic super groupscheme \tilde{G} over k .*

18.4. Isogenies: The reductive group \tilde{G} . The representation

$$\Pi = Ber^{-1} \otimes \Lambda^{m-1}(V) = L(0, 0, \dots, 0, -1|1)$$

is the socle of the Kac module $V(\mathbf{1})$. Therefore it sits in the exact sequence

$$0 \longrightarrow \mathbf{1} \longrightarrow V(\mathbf{1})^* \longrightarrow \Pi \longrightarrow 0 .$$

Since $V(\mathbf{1})^*$ is zero in $Ho\mathcal{T}$ this implies the isomorphism

$$\Pi \simeq \mathbf{1}[1]$$

in $\mathcal{H}o\mathcal{T}$. The Kac module $V(Ber)$ has constituents Ber and $\Lambda^{m-1}(V)$. Since $V(Ber)$ and $V(Ber)^*$ are both trivial in $\mathcal{H}[\Sigma^{-1}]$, we obtain the isomorphism $Ber \simeq \Lambda^{m-1}(V)[1]$ in $\mathcal{H}[\Sigma^{-1}]$. Together with $Ber \simeq \Lambda^{m-1}(V)[1]$ in $\mathcal{H}o\mathcal{T}$ this implies $\Pi^2 \simeq \mathbf{1}$ in $\mathcal{H}[\Sigma^{-1}]$ for $\Pi \simeq \mathbf{1}[1]$.

Since $\mathcal{H} \rightarrow \mathcal{H}[\Sigma^{-1}]$ is a full tensor functor into a semisimple tensor category, it factorizes by [He15]

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{H}[\Sigma^{-1}]. \\ & \searrow & \nearrow \exists \phi \\ & \mathcal{H}/\mathcal{N} & \end{array}$$

The irreducible representations in $\mathcal{H}[\Sigma^{-1}]$ are now parametrized by the irreducible representations of $GL(m-1) \times \mathbb{Z}/2\mathbb{Z}$: Indeed the atypical blocks are in bijection with the irreducible representations of $GL(m-1)$ (every block contains exactly one irreducible mixed tensor [He17, Lemma 8.1]) and every block gives two irreducible objects in $\mathcal{H}[\Sigma^{-1}]$, represented by the mixed tensor $L(\tilde{\lambda})$ in the given block and $L(\tilde{\lambda}) \otimes \Pi$.

Corollary 18.6. $\mathcal{H}[\Sigma^{-1}]$ is tensor equivalent to the super representations of $GL(m-1) \times \mathbb{Z}/2\mathbb{Z}$.

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