Max Planck Institut für Mathematik Bonn

Mathematische Arbeitstagung 2005 Talk: Black Holes and Arithmetic

Notes based on the talk of Gregory W. Moore (Rutgers University)

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Contents

1 Setting the Scene: Black Holes in String Compactification and the Attractor Mechanism

I am going to talk about some connections between black holes and certain things that come up in number theory. I specially consider the type II string theory on a Calabi-Yau 3-fold X.

$$\text{Spacetime} \ = \ \text{product} \ = \mathbb{R}^4 \times X. \tag{1}$$

When X is small (in Planck units), physics at ordinary energies is described by a generalization of Einstein's general relativity. Technically "$d = 4, N = 2$ supergravity "

$$\text{Action} \ = \frac{1}{16\pi G_N} \int d^4 R(g) + \underbrace{\cdots\cdots\cdots\cdots}_{\text{many other terms}} \tag{2}$$

These other terms include, typically, lots of abelian gauge fields: generalizations of Maxwell's theory of a connection on a principal $U(1)$ bundle.

The underlying gauge fields are actually described by differential K-theory - but that is an other story - The important thing for today is that particles in the theory - or states in the Hilbert space of the quantum theory carry electric and magnetic charge. The electric *and* magnetic charge lattice is

$$K^1(X) \tag{II B}$$

$$K^0(X) \tag{II A}$$

for general Calabi-Yau we think of charge vectors as

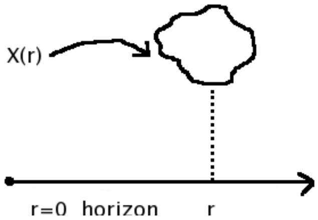
$$\gamma \in H^3(X;\mathbb{Z}) \quad \text{for} \quad \text{IIB} \tag{3}$$

Now,

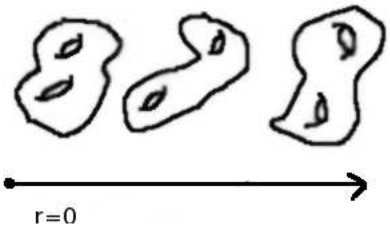
1. since we have a generalization of GR we can have black hole solutions,
2. since we have $U(1)$ gauge fields we can have *charged black hole* solutions.
3. Moreover since $X \times \mathbb{R}^4$ has covariantly constant spinors we can have *supersymmetric spherically symmetric* charged black hole solutions.

The spacetime for these solutions turns out to take the following form:
Spherical Symmetry $\Rightarrow$

$$ds^2 = -e^{+2U(r)}dt^2 + e^{-2U(r)}dt^2_{s^2} \tag{4}$$



The Calabi-Yau metric varies as a function of r.

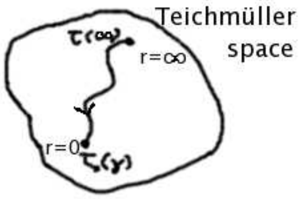


Turns out: In the IIB description Kähler class = constant
Complex structure evolves by *gradient flow* $\dot{\tau} = -\nabla U$.

To define the gradient flow we use $g_{\text{Weil-Peterson}}$ and

$$\begin{aligned} U &= \log |Z(\gamma, \tau)|^2 \\ |Z(\gamma,\tau)|^2 &= \frac{\left| \int \gamma \wedge \Omega \right|^2}{\int \Omega \wedge \bar{\Omega}} \end{aligned}$$

where $\Omega = \Omega^{3,0}$ = nowhere zero holomorphic 3-form that trivializes the canonical bundle. Moreover, the flow always reaches an attractive fixed point at the horizon of the black hole which is entirely determined by γ



Fixed point of the flow : $\tau_*(\gamma)$. $\delta\Omega$ has no overlap with γ. By Griffiths transversality that means

$$\boxed{\gamma \in H^{3,0}(X, \mathbb{C}) + H^{0,3}(X, \mathbb{C}) \qquad \text{attractor equation}}$$

The integral charge γ determines an isolated point in complex structure moduli space by this condition on the Hodge structure.

2 Some Examples of Attractors

The attractor equation can be explicitly analyzed for some simple Calabi-Yau manifolds and the answer turns out to be nice.

a.) $X = T^6$ 3-dimensional complex torus.

Find $\tau_*(\gamma)$: X_* is isogenous to a product of 3 elliptic curves $E_{\tau_0} \times E_{\tau_0} \times E_{\tau_0}$. Moreover $E_{\tau_0} = \mathbb{C}/\mathbb{Z} + \tau_0\mathbb{Z}$ has complex multiplication.

In fact $\tau_0 = i\sqrt{I_4(\gamma)}$ where $I_4(\gamma)$ =quartic polynomial in γ, related to $E_{7,7}$,

$$\underbrace{H^{\mathrm{odd}}(T^6;\mathbb{Z})}_{\substack{R\\ \text{spinor of } \mathrm{S0}(6,6)}} \oplus \underbrace{\underbrace{\mathbb{I}^{6,6}}_{NS}}_{\text{vector of } \mathrm{S0}(6,6)} = \substack{\text{56 dim. sympl. lattice}\\ E_7 \subset \mathrm{Sp}(56) \text{ defined by } I_4}$$

b.) Let

$$\begin{aligned} X &= S \times T^2 \qquad \text{with} \qquad S = \text{ K3 surface} \\ H^3(X) &\cong H^2(S) \oplus H^2(S) \\ \gamma &= p_{\mathrm{mag}} \oplus q_{\mathrm{el.}} \\ \Omega^{3,0} &= \Omega^{2,0} \wedge dz \end{aligned}$$

$\mathbb{I}'^{9,3} = H^2(S;\mathbb{Z}) \supset NS(S)$ is orthogonal to $\langle p, \rho\rangle = T_{p,q}$, rank$p = 20$

$$H^{2,0} \oplus H^{0,2} = T_{p,q} \otimes \mathbb{C} \tag{5}$$

"Singular K3 surface" discussed by P.-S. + Shaf., Deligne, Shioda-Inose. We call them "attractive K3 surfaces".

Classified by Shioda-Inose:

1-1 correspondence with even integral symmetric bilinear forms

$$\begin{pmatrix} p^2 & p\cdot q \\ p\cdot q & q^2 \end{pmatrix}$$

Let

$$\tau_{p,q} := \frac{p\cdot q + i\sqrt{-D_{p,q}}}{p^2} \quad , \quad D_{p,q} = (p\cdot q)^2 - p^2q^2$$

Morally, the S-I surface is Kummer$\left(E_{\tau_{p,q}} \times E_{\tau'_{p,q}}\right)$ with $\tau' = \frac{-pq+i\sqrt{-D}}{2}$.

$$\text{Attractive CY } = S_{p,q} \times E_{\tau_{p,q}}$$

Again elliptic curves with complex multiplication are selected by the attractor mechanism.

What is about more general Calabi-Yau manifolds?

1. Rank 2 attractor $H^{3,0} + H^{0,3} = T \otimes \mathbb{C}$ is rationally generated. (In particular all rigid CY's)

   Discussion with M.Nori (1999): he told us he could prove these are arithmetic. Nontrivial new examples? Hard to find.

2. For E_τ both τ and $j(\tau)$ are arithmetic. But $\tau \to j(\tau)$ is just the mirror map.

Conjecture: Not only ist the attractor variety arithmetic but also the flat coordinates X^I, F_I are arithmetic. i.e. the mirror map enjoys arithmetic properties analogous to those of the j-function.

3 Black Hole Degeneracies and BPS States

So far we discussed how we associate to a charge $\gamma \in H^3(X;\mathbb{Z})$ a distinguished complex structure $\to$ distinguished CY $\to$ distinguished supergravity solution.

But string theory is a quantum mechanical theory.

Associated with strings on $\mathbb{R}^4 \times X$, there should be a Hilbert space of states and this Hilbert space is *graded* by electric and magnetic charge

$$\mathcal{H} = \bigoplus_{\gamma \in K^1(X)} \mathcal{H}_\gamma$$

(Aside: Actually, the Hilbert space is really a representation of a Heisenberg extension of $K(X)$, but if $\operatorname{Tors} K(X) = 0$ we can ignore this.)

Within each superselection sector $\mathcal{H}_\gamma$. There is a distinguished subspace the space of BPS states, $\mathcal{H}_\gamma^{\mathrm{BPS}}$. It is a finite dimensional vectorspace.

In each sector there is a lower bound on the spectrum of the Hamiltonian

$$E \geq |Z(\gamma)|$$

Definition: $\mathcal{H}_\gamma^{\mathrm{BPS}}$ = states which saturate the bound.

What is the connection to the black holes?

For large charges γ the black hole is a semiclassical description of a quantum state.

(The lightwaves in this room are a solution to Maxwell's equations, but really it is only a semiclassical description of a quantum state of photons.)

For the BPS black holes $\mathcal{H}_\gamma^{\mathrm{BPS}}$ is a space of *groundstates*, so

$$S_{\mathrm{BPS}} = \log \dim \mathcal{H}_\gamma^{\mathrm{BPS}} \tag{6}$$

is a kind of entropy. Now, 30 years ago Beckstein and Hawking discovered that the laws of black hole mechanics are formally equivalent to the laws of thermodynamics and in this equivalence:

$$S_{\mathrm{BH}} = \frac{\text{Area Horizon}}{4} \tag{7}$$

This was just a formal equivalence, so it was a great step forward when Strominger and Vafa pointed out that for certain supersymmetric 5D black holes (K3$\times S^1$)

$$S_{\mathrm{BPS}} \sim S_{\mathrm{BH}}$$

asymptotically for large charges.

This is why physicists are interested in determining S_{BPS} as a function of γ. From the classical supergravity point of view one evaluates

$$\frac{A}{4} = \pi \left| Z(\gamma, \tau_*(\gamma)) \right|^2 \tag{8}$$

In particular, for K3$\times T^2$

$$\frac{A}{4} = \pi \sqrt{-D_{p,q}} = \pi \sqrt{p^2 q^2 - (p \cdot q)^2} \tag{9}$$

So we have some definite prediction for asymptotics of $\log \dim \mathcal{H}_\gamma^{\mathrm{BPS}}$

- Test this?
- What are the exact degeneracies like?
  - governed by some smooth functions $f : H^{\mathrm{odd}}(X; \mathbb{R}) \to \mathbb{R}$
  - or are they line coefficients of cusp forms for the modular group where $\frac{a_n}{n^{(n-1)/2}}$ can only be described as a probality distribution?

To say more we need some concrete description of $\mathcal{H}_\gamma^{\mathrm{BPS}}$. There are different descriptions of the same space related by "string dualities".

$$\text{BPS states} \leftrightarrow \text{wrapped branes}$$

Where roughly speaking - a brane is a kind of (differential) K-homology cycle submanifold, vector bundle with connections,...

Very roughly: We have three descriptions

(a.) $\gamma \in H^3(X, \mathbb{Z}) \quad \mathcal{M}_{\mathrm{SLAG}}(\gamma)$:

D3 branes wrap SLAG cycles in homology class γ.

$$\mathcal{H}_\gamma^{\mathrm{BPS}} = H^*(\mathcal{M}_{\mathrm{SLAG}}(\gamma)) \tag{10}$$

(b.) Mirror picture $\tilde{\gamma} \in H^{\text{ev}}(\tilde{X}, \mathbb{Z})$ $\tilde{\gamma}$ determines Chern classes of coherent sheaves (element of odd derived category) $\tilde{\gamma} = \text{ch}\mathcal{E}\sqrt{\text{Td}}$

$$\mathcal{H}^{\text{BPS}}_{\gamma} = H^*(\mathcal{M}_{\text{sheaves}}(\tilde{\gamma})) \tag{11}$$

(c.) When X is K3 fibered there is a third description using the heterotic string

$$\text{IIA}/K3 \times T^2 \cong \text{Het}/T^6$$

More generally

$$\begin{array}{ccccc} \text{IIA}/K3 \to & X & \cong & \text{Het}/K3' \times T^2 \\ & \downarrow & & \\ & \mathbb{P}^1 & & \end{array} \tag{12}$$

under this isomorphism.

If $\tilde{\gamma}$ corresponds to classes of sheaves supported on a fiber. Then

$$\mathcal{H}^{\text{BPS}}_{\tilde{\gamma}} = \begin{array}{c} \text{subspace of a Fock space} \\ \text{explicit, rigorous} \\ \text{construction using vertex} \\ \text{operator algebra techniques} \end{array}$$

Detailes would take too long. Important point is that the description is very explicit and it is "easy" to count dimension in this case.

Example:

$$\begin{aligned} H^{\text{ev}}(K3 \times T^2) &= H^*(K3) \oplus H^*(K3) \\ \tilde{\gamma} &= p_{\text{mag}} \oplus q_{\text{el}} \end{aligned}$$

$p = 0$: supported on fiber.

$$T\text{duality} \to \dim \mathcal{H}^{\text{BPS}}_{(0,q_{\text{el}})} = d\left(\frac{1}{2}q_{\text{el}}^2\right)$$

Where $\sum d(N)q^N = \frac{1}{\eta^{24}}$ counts levels in Fock spaces of 24 oscillators and is directly related to Göttsche's

$$\sum q^N \chi\left(\text{Hilb}^N K3\right) = \frac{q}{\eta^{24}}$$

For more general K3 fibrations the explicit vertex operator description gives $\frac{1}{\eta^{24}} \to$ modular forms of negative weight. Formula has the slope

$$\sum_{g \in \Gamma} \frac{1}{\eta} \left(\prod_j \frac{\theta(\epsilon_j(g))}{\theta(\epsilon'_j(g))} \right)$$

very explicit.

What about more general charges? Nothing exact is known with any confidence for $\dim \mathcal{H}_\gamma^{\rm BPS}$ in 4D. About 8 or 9 years ago DVV conjectured on exact formula for $K3 \times T^2$:

$$H^{\rm ev}(K3 \times T^2) = \underset{p}{H^*(K3)} \oplus \underset{q}{H^*(K3)}$$

Now T duality says:

$$\dim \mathcal{H}_\gamma^{\rm BPS} = D\left(\frac{1}{2}p^2, \frac{1}{2}q^2, p\cdot q\right)$$

Then DVV conjectured

$$\sum p^N q^M y^L \, D(N,M,L) \stackrel{?}{=} \frac{1}{\Phi} = \frac{1}{pq\prod_{(n,m,l)>0}(1-p^n q^m y^l)^{c(nm,l)}} \qquad (n,m,l) \in \mathbb{R}^{2,1}$$

$$\mathcal{E}\ell\ell(q,y;K3) = \sum c(n,l) q^n y^l = \frac{1}{\Delta_s^2} \tag{13}$$

It has always been an intriguing formula - Gritsenko and Nikulin showed that this is Δ_s^2 and Δ_s = Denom. product for a generalized Kac-Moody algebra. On the other hand, the argument for it was not convincing even by the standards of the physicists! So nobody believed it. But recently D. Shih, A. Strominger, X. Yin gave a completely independent argument by relating 4D to better understood 5D degeneracies and using the rigorous formula

$$\sum p^N \mathcal{E}\ell\ell(q,y;\mathrm{Sym}^N X) = \frac{1}{\prod_{\substack{n>0 \\ m\geq 0}}(1-p^n q^m y^l)^{c(nm,l)}} \tag{14}$$

So maybe we have to take it more seriously.

Puzzle: It is true that

$$D(N,M,L) \sim e^{\pi\sqrt{4NM-L^2}} = e^{\pi\sqrt{p^2q^2-(p\cdot q)^2}}$$

but it also predicts non zero degeneracies for states with $4NM - L^2 <$ 0 - which ought not to be there (physically: negative area) In fact - by the Rademacher formula. These negative discriminant degeneries completly determine the others.

$$\frac{1}{\Phi} = \sum p^N \underbrace{\phi_N(q,y)}_{\text{jacobi form of negative wt}}$$

$$\phi_N(q,y) = \sum \underbrace{f_{N,\mu}}_{\text{apply Radermacher to this}} \Theta_\mu$$

4 The OSV Conjecture

But what can we say about exact formulae for $\dim \mathcal{H}_\gamma^{\mathrm{BPS}}$ for other Calabi-Yau's X?

There was no good idea until last when Ooguri-Strominger-Vafa following up on work of DeWit, Cardoso, Mohaupt, made a striking conjecture. To motivate it let's rewrite the a.e.'s. Choose a symplectic basis α^I, β_I for $H_3(X, \mathbb{Z})$

$$X^I = \int_{\alpha^I} \Omega \qquad F_I = \int_{\beta^I} \Omega \tag{15}$$

The a.e.'s are then

$$\begin{array}{lrcl} (\alpha) & \mathrm{Re} X^I & = & p^I \\ (\beta) & \mathrm{Re} F_I & = & q_I \end{array}$$

Solve (α):

$$X^I = p^I + i\phi^I \qquad , \phi^I \text{ real.}$$

Now there exists a prepontential

$$\mathcal{F}^{(0)}\left(X^I\right) \quad \text{with} \quad F_I = i\frac{\partial \mathcal{F}^{(0)}}{\partial X^I} \tag{16}$$

so the second equation can be written as

$$\frac{\partial \mathcal{F}^{(0)}}{\partial \phi_I} = -q_I \tag{17}$$

where we define

$$\mathcal{F}^{(0)}\left(p^I, \phi^I\right) := 2\mathrm{Re} F^{(0)}\left(p^I + i\phi^I\right) \tag{18}$$

Moreover the classical formula for the entropy turns out to be equivalent to:

$$S(p^I, q_I) = \mathcal{F}^{(0)} - \phi^I q_I \tag{19}$$

We recognize the entropy is a Legendre transform of the prepotential! Cardoso, DeWit, Mohaupt asked what happens to these formulae when we take into account certain corrections to the Einstein equations which are predicted by string theory.

Their answer: (They didn't say it this way)

Let us work near a point of maximal unipotent monodromy, so it exists distinguished period $X^0 \neq 0$, X^A-remaining periods.

It is easier to say it in the mirror picture $\tilde{X}$: so we work at "large radius".

Introduction of the Gromov-Witten potential

$$F_{\mathrm{top}}(\lambda, t^A) \quad \sim \quad i\lambda^{-2} C_{\mathrm{ABC}} t^A t^B t^C + i c_{2,A} t^A + \sum_{\substack{n \geq 0 \\ \beta \in H_2(\tilde{X}, \mathbb{Z})}} N_{h,\beta} e^{2\pi i t \cdot \beta} \lambda^{2h-2}$$

$$\begin{aligned} t^A = X^A / X^0 &= \text{complexified Kähler class} \\ N_{h,\beta} &= \text{Gromov-Witten invariant} \end{aligned}$$

Definition: $\mathcal{F}\left(p^I, \phi^I\right) := 2\mathrm{Re} F_{\mathrm{top}} \left(\lambda = \frac{4\pi}{p^0 + i\phi^0}, t^A = \frac{p^A + i\phi^A}{p^0 + i\phi^0} \right)$

the we just replace $\mathcal{F}^{(0)} \to \mathcal{F}$. ! Now recall $\dim \mathcal{H}^{\mathrm{BPS}}_\gamma = e^{S_{\mathrm{BPS}}}$

OSV conjectured

$$\begin{aligned} \dim \mathcal{H}^{\mathrm{BPS}}(p,q) &\stackrel{?}{=} \int d\phi \left| \Psi_{\mathrm{top}} \left(p^I + i\phi^I \right) \right|^2 e^{-q_I \phi^I} \\ &= \int d\phi e^{\mathcal{F} - q_I \phi^I} \\ \Psi_{\mathrm{top}}\left(X^I\right) &= e^{F_{\mathrm{top}}(X^I)} \end{aligned}$$

Formula must be put in heavy quotation marks

- $\dim \mathcal{H}^{\mathrm{BPS}}(p,q) \rightsquigarrow$ index-like sum "helicity supertrace"
- F_{top} only exists as an asymptotic expansion for $\lambda \to 0$
- Contour? measure?

Still, it is an intriguing formula.

It would relate sheaves on Calabi-Yau manifolds to Gromov-Witten theory. Rather like the conjectural relation between Donaldson-Thomas and Gromov-Witten invariants (but not the same). For certain limits of charges you can evaluate the integral reliably in saddle point approximation.

Moreover - for K3 fibered CY and certain special charges we can use the heterotic dual to compute reliably the black hole degeneracies.

$\Rightarrow$ We can test the OSV conjecture in these cases.

This was done by A. Dabholkar, F. Denef, G. Moore, B. Pioline

Example: IIA/K3 $\times T^2 = $ Het$/T^6$

$$F_{\mathrm{top}} = -i\lambda^{-2} C_{ab} t^a t^b t^1 - 24 \log \Delta(t^1)$$

1. Evaluate integral on RHS: $16 \hat{I}_{13}\left(4\pi\sqrt{N-1}\right)\left(1 + \mathcal{O}\left(e^{-4\pi\sqrt{N}}\right)\right)$

   $\hat{I}_\nu(z) = 2\pi \left(\frac{z}{2\pi}\right)^{-\nu} I_\nu(z)$

2. Evaluate LHS: $p_{24}(N)$: Rademacher formula

$$
\begin{aligned}
p_{24}(N) &= \sum_{c=1}^{\infty} K_{c,N} \hat{I}_{13}\left(\frac{4\pi}{c}\sqrt{N-1}\right) \qquad \text{exact!} \\
&= 16\left(\hat{I}_{13}(4\pi\sqrt{N-1}) + 2^{-14} e^{i\pi N} \hat{I}_{13}(2\pi\sqrt{N-1}) + \ldots\right)
\end{aligned}
$$

Thus the agreement goes well beyond the exspected $\exp(4\pi\sqrt{N})$.

We get all orders in the $\frac{1}{N}$ expansion! But nonperturbatively in N its wrong.

We checked in a large collection of examples of K3-fibred CY's and found broad agreement of this type.

There were some cases where the Bessel function comes out wrong - but those cases are murky - the OSV conjecture is ill-defined because the charges are such that the $F_{\rm top}$ is evaluated for a singular CY. These examples have the defect that they are "small black holes" (horizon area is classically zero, ir is only nonzero due to quantum corrections.)

There is no test at present for "large black holes".

5 Summary/Conclusion

1. Attractor mechanism has some intriguing connection to arithmetic. Physical significance of arithmetic properties?

2. New life for old DVV formula - but puzzles.

3. OSV conjecture needs significant improvement, but has a lot of truth in it.