SYMPLECTIC STRUCTURE ON A MODULI SPACE OF SHEAVES ON THE CUBIC FOURFOLD

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Abstract. A 10-dimensional symplectic moduli space of torsion sheaves on the cubic 4-fold is constructed. It parametrizes the stable rank 2 vector bundles on the hypeplane sections of the cubic 4-fold which are obtained by Serre's construction from normal elliptic quintics. The natural projection to the dual projective 5-space parametrizing the hyperplane sections is a Lagrangian fibration. The symplectic structure is closely related (and conjecturally, is equal) to the quasi-symplectic one, induced by the Yoneda pairing on the moduli space.

Introduction

Mukai constructed in [Muk] an example of a symplectic moduli space of torsion sheaves on a K3 surface S, whose supports run over a complete linear system on S. This moduli space is identic to the relative compactified Jacobian of the linear system, and the fibers, that is the Jacobians of the individual curves, are Lagrangian subvarieties with respect to the symplectic structure.

We searched for an analogous situation with the relative intermediate Jacobian of a family of higher dimensional varieties in place of the usual Jacobian of a family of curves. Thus, the corresponding torsion sheaves should be with supports of dimension at least 3, and their restrictions to the supports should be of rank > 1. We found a 10-dimensional component of the moduli space of stable sheaves whose supports are hyperplane sections of a 4-dimensional cubic in $\mathbb{P}^5$ with desired properties. This is a new example of a moduli space of sheaves possessing a symplectic structure; the known ones parametrize sheaves on manifolds with holomorphic 2-forms (see [Muk] for K3 or abelien surfaces, [K] for hyperkaehler varieties, [Tyu] for surfaces with $p_g > 0$), and the symplecitc structure comes essentially from the 2-form on the base. In our example, the base is very far from having holomorphic 2-forms: it is a Fano variety.

The fiber of our variety, parametrizing sheaves with a fixed support (a cubic 3-fold), was constructed in our previous paper [Ma-Ti]. We studied there the Abel–Jacobi map of the family of elliptic quintics lying on a general cubic threefold X. We proved that it factors through a 5-dimensional moduli component M_X of stable rank 2 vector bundles $\mathcal{E}$ on X with Chern numbers $c_1 = 0, c_2 = 2$, whose general point represents a vector bundle obtained by Serre's construction from an elliptic quintic $C \subset X$. The elliptic quintics mapped to a point of the moduli space vary in a 5-dimensional projective space inside the Hilbert scheme Hilb_X^{5n}, and the map Ψ from the moduli space to the intermediate Jacobian $J(X)$ is étale on the open set representing (smooth) elliptic quintics which are not contained in a hyperplane. This map is uniquely determined by the choice of a reference quintic in X. In fact, the elliptic quintics represent $c_2(\mathcal{E}(1))$, so one can interprete Ψ as c_2 with values in algebraic cycles modulo rational equivalence, composed with the Abel–Jacobi map on cycles.

In the present paper, we relativize this construction over the family of nonsingular hyperplane sections of a cubic fourfold Y. This gives a 10-dimensional moduli component $\mathfrak{M}_Y$ of torsion sheaves on Y with supports on the hypeplane sections $X = H \cap Y$, whose restrictions to X are

1991 *Mathematics Subject Classification.* 14J60, 14J45.
Partially supported by the grant INTAS-OPEN-97-2072.

rank two vector bundles from M_X. We show that the Yoneda pairing induces a quasi-symplectic structure on it, that is a holomorphic nondegenerate 2-form which is not necessarily closed, and that the natural projection $p : \mathfrak{M} \longrightarrow \check{\mathbb{P}}^5$ is a Lagrangian fibration, where $\check{\mathbb{P}}^5$ is the base of the family of hyperplanes in $\mathbb{P}^5$. This means that the fibers M_X, parametrizing sheaves with fixed support X, are Lagrangian submanifolds of $\mathfrak{M}$.

We study also the relation of the Yoneda quasi-symplectic structure to the symplectic one constructed by Donagi–Markman [D-M, Example 8.22] on $\mathbf{J}_Y$. It satisfies a similar property: the fibers $J(Y \cap H)$ are Lagrangian. In general, there is no map between the two 10-dimensional varieties, so we cannot hope that one quasi-symplectic structure is induced by the other. Nevertheless, the map from $\mathfrak{M}_Y$ to $\mathbf{J}_Y$ exists locally over $\check{\mathbb{P}}^5$ in the classical (or étale) topology, and we prove that these local maps respect the symplectic structures up to the addition of a 2-form which is a local section of $p^*\Omega^2_{\check{\mathbb{P}}^5}$.

According to [Il-Ma], the degree of the above map Ψ is 1, so M_X can be identified with an open subset of $J^2(X)$, and $\mathfrak{M}_Y$ with that of some torsor under the relative intermediate Jacobian $\mathbf{J}_Y$ of the family of nonsingular hyperplane sections of Y.

Unfortunately, we do not see any reasonable approach to the proof of the closedness of the Yoneda quasi-symplectic structure. Remark, that Tyurin and Mukai did not prove the closedness of their 2-forms. Finally it turned out that their 2-forms are closed [O'G], [Bot], [Hu-Le], but the method for proving this in the cited papers reduces essentially the question to the closedness of the representatives for the Atiyah class of a coherent sheaf. Our quasi-symplectic structure does not involve components of the Atiyah class, so this method does not work. Nevertheless, we prove the existence of a symplectic structure on $\mathfrak{M}_Y$ which is in the same relation to that of Donagi–Markman: the above local maps between $\mathfrak{M}_Y$ to $\mathbf{J}_Y$ respect the symplectic structures up to the addition of a 2-form which is a local section of $p^*\Omega^2_{\check{\mathbb{P}}^5}$. We define it in fact on the torsor, containing $\mathfrak{M}_Y$.

Now, we will briefly describe the contents of the paper by sections.

In Section 1, we obtain the preparatory information on the cohomology and the Ext groups of the sheaves from $\mathfrak{M}_Y$. The calculations use the local-to-global spectral sequence for Ext's and the projective geometry of elliptic quintics.

In Section 2, we prove that the Yoneda pairing $\mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \xrightarrow{\text{Yoneda}} \mathrm{Ext}^2(\mathcal{E},\mathcal{E}) \simeq \mathbb{C}$ for $\mathcal{E} \in \mathfrak{M}_Y$ is skew-symmetric and non-degenerate. We show that the complex lines $\mathrm{Ext}^2(\mathcal{E},\mathcal{E})$ for different $\mathcal{E}$'s fit together into the trivial line bundle over $\mathfrak{M}_Y$. Hence the Yoneda coupling defines a genuine 2-form $\Lambda_{\mathfrak{M}_Y}$ on $\mathfrak{M}_Y$, which is the wanted quasi-symplectic structure.

In Section 3, we describe the symplectic structure $\Lambda_{\mathbf{J}_Y}$ of Donagi–Markman on $\mathbf{J}_Y$. It is determined by the one on the Fano 4-fold Z of Y, parametrizing lines $l \subset Y$. The existence and the uniqueness of the latter was proved by Beauville–Donagi [B-D], but they did not give any construction of it. Section 4 provides an explicit construction of the symplectic structure of Beauville–Donagi on Z. The idea is to mimic the proof of the Tangent Bundle Theorem for the Fano surface of a cubic 3-fold. Essentially, this proof is the calculation of the differential of the Abel–Jacobi map of lines via the Clemens– Griffiths–Welters diagram (see Sect. 2 in [We]) for the triple $l \subset X \subset \mathbb{P}^4$. We adapt it to the triple $l \subset Y \subset \mathbb{P}^5$, and it gives (after an obvious relativization) a tensor field $\wedge_Z : \ \mathcal{T}Z \ \tilde{\rightarrow} \ \Omega_Z$. The Bochner principle sais that the global holomorphic tensor fields on Z are generated by the Beauville–Donagi form, so $\wedge_Z$ has to be proportional to it.

In Section 5, we compare the quasi-symplectic structures on $\mathfrak{M}_Y$ and on $\mathbf{J}_Y$. These two manifolds are fibered over the same base $U \subset \check{\mathbb{P}}^5$ parametrizing the nonsingular hyperplane sections of Y, have Lagrangian fibers, and the fibers are related by the étale map of [Ma-Ti]. So we can identify pointwise the tangent spaces of two respective fibers, as well as their normal spaces, which are identified with the tangent space to U. The first thing to notice is that the two quasi-symplectic structures define the same pairing between the tangent and the normal space of the fiber. This gives a pointwise identification of two quasi-symplectic structures. Next, we consider a local (in the classical topology) fiberwise map $\Psi_{\mathcal{U}} : \mathfrak{M}_Y \to \mathbf{J}_Y$, which is the Abel–Jacobi map determined by a family of reference elliptic quintics in the hyperplane sections of Y over a small disk $\mathcal{U} \subset U$. We show that $\Lambda_{\mathfrak{M}_Y}, \Lambda_{\mathbf{J}_Y}$ differ by a section of $p^*\Omega^2_{\mathcal{U}}$. In conclusion, we show that the Yoneda quasi-symplectic structure can be replaced by a symplectic structure with the same relation to $\Lambda_{\mathbf{J}_Y}$ and discuss the extensions to the bigger open set $\overline{U}$ parametrizing hypeplane sections with at most one nondegenerate double point as singularity.

Acknowledgements. The first author acknowledges with pleasure the hospitality of the ICTP at Trieste and of the MPIM at Bonn, where he made a part of the work on the paper.

1. Computing Ext's

Let $Y \in \mathbb{P}^5$ be a nonsingular cubic fourfold, and $X = Y \cap \mathbb{P}^4$ its general hyperplane section. Let $\mathcal{E}$ be the vector bundle obtained from a projectively normal elliptic quintic curve $C \subset X \subset \mathbb{P}^4$ by Serre's construction:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow 0 \ , \tag{1}$$

where $\mathcal{I}_C = \mathcal{I}_{C,X}$ is the ideal sheaf of C in X. Since the class of C modulo algebraic equivalence is $5l$, where l is the class of a line, the sequence (1) implies that $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = 2l$. Moreover, $\det \mathcal{E}$ is trivial, and hence $\mathcal{E}$ is self-dual as soon as it is a vector bundle (that is, $\mathcal{E}^* \simeq \mathcal{E}$). See [Ma-Ti, Sect. 2] for further details on this construction.

Here, we will consider $\mathcal{E}$ as a torsion sheaf on Y, supported on X.

Lemma 1.1. *We have*

$$\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathcal{E} \otimes \mathcal{E} \text{ if } q = 0, \\ \mathcal{E} \otimes \mathcal{E}(1) \text{ if } q = 1, \\ 0 \ \text{ otherwise.} \end{cases}$$

Proof. Choose a local trivialization of $\mathcal{E}$:

$$\mathcal{E}|_U \simeq (\mathcal{O}_X \oplus \mathcal{O}_X)|_U.$$

This reduces the problem to the computaion of $\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$. In using the resolution

$$0 \longrightarrow \mathcal{O}_Y(-1) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

we arrive at the conclusion that $\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$ for $q = 0$, $\mathcal{N}_{X/Y} \simeq \mathcal{O}_X(1)$ for $q = 1$, and 0 for $q \geq 2$. Taking into accout the functorial behavior of the last isomorphism with respect to the choice of bases in $\mathcal{O}_X$, we obtain the wanted formulas with $\mathcal{E}^* \otimes \mathcal{E}$ instead of $\mathcal{E} \otimes \mathcal{E}$, which is the same by self-duality. □

Lemma 1.2. *The following assertions are true:*

(i) $h^p(\mathcal{E} \otimes \mathcal{E}) = \dim \ \mathrm{Ext}^p_X(\mathcal{E} \otimes \mathcal{E}) = \begin{cases} 1 \text{ if } q = 0, \\ 5 \text{ if } q = 1, \\ 0 \ \text{ otherwise.} \end{cases}$

(ii) $h^0(\mathcal{E}(1)) = 6$ *and* $h^i(\mathcal{E}(1)) = 0$ *for* $i > 0$.

(iii) For $k \geq 1$, $h^0(\mathcal{I}_C(k)) = \binom{4+k}{4} - \binom{1+k}{4} - 5k$, *and* $h^i(\mathcal{I}_C(k)) = 0$ *if* $i > 0$.

(iv) $h^i(\mathcal{E}(j)) = 0$ *for all* $i \in \mathbb{Z}, j = 0, -1, -2$.

Proof. (i), (ii) follow from [Ma-Ti, Lemma 2.1, c); Lemma 2.7, b)]. (iii) is verified for $k \geq 2$ in the proof of Lemma 2.1, loc. cit. and is obviously extended to $k = 1$ by considering the exact triple

$$0 \longrightarrow \mathcal{I}_C(k) \longrightarrow \mathcal{O}_X(k) \longrightarrow \mathcal{O}_C(k) \longrightarrow 0 \tag{2}$$

with $k = 1$. C is not contained in a hyperplane by projective normality, so $h^0(\mathcal{I}_C(1)) = 0$. As $h^i(\mathcal{O}_X(1)) = h^i(\mathcal{O}_C(1))$ equals 5 for $i = 0$ and is zero for all $i > 0$, we obtain the wanted assertion.

(iv) for $j = 1$ is nothing but [Ma-Ti, Lemma 2.7, a)]. By Serre duality (on X), it remains to verify (iv) only for $j = 0$. This follows from (iii) with $k = 1$, the exact sequence (1), twisted by $\mathcal{O}_X(-1)$, and from $H^i(\mathcal{O}_X(-1)) = 0$ for $i > 0$. □

Corollary 1.3. *The local-to-global spectral sequence*

$$E_2^{pq} = H^p(Y, \mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{E})) \Longrightarrow \mathrm{Ext}^{p+q}_Y(\mathcal{E}, \mathcal{E})$$

degenerates in its second term: $E_2^{pq} = E_\infty^{pq}$. *Hence* $\mathrm{Ext}^0_Y(\mathcal{E}, \mathcal{E}) = \mathbb{C}$, *that is,* $\mathcal{E}$ *is simple,* $\mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{E})$ *fits into the exact sequence*

$$0 \longrightarrow H^1(\mathcal{E} \otimes \mathcal{E}) \longrightarrow \mathrm{Ext}^1_Y(\mathcal{E}, \mathcal{E}) \longrightarrow H^0(\mathcal{E} \otimes \mathcal{E}(1)) \longrightarrow 0,$$

and $\mathrm{Ext}^n_Y(\mathcal{E}, \mathcal{E}) = H^{n-1}(\mathcal{E} \otimes \mathcal{E}(1))$ *for* $n \geq 2$.

Proof. This follows immediately from Lemma 1.1 and (i) of Lemma 1.2. □

Lemma 1.4. *The following assertions hold:*

(i) $H^i(\mathcal{E} \otimes \mathcal{E}) = H^i(\mathcal{E} \otimes \mathcal{I}_C(1))$ *and* $H^i(\mathcal{E} \otimes \mathcal{E}(1)) = H^i(\mathcal{E} \otimes \mathcal{I}_C(2))$ *for all* $i \in \mathbb{Z}$.

(ii) $H^i(\mathcal{E} \otimes \mathcal{E}(1)) = 0$ *for* $i = 2, 3$, *and* $\chi(\mathcal{E} \otimes \mathcal{E}(1)) = h^0(\mathcal{E} \otimes \mathcal{E}(1)) - h^1(\mathcal{E} \otimes \mathcal{E}(1)) = 4$.

Proof. The first isomorphism is proved in [Ma-Ti]. For the second one, look at the exact sequences

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{E}(1) \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(2) \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathcal{I}_C \otimes \mathcal{E}(2) \longrightarrow \mathcal{E}(2) \longrightarrow \mathcal{E}(2) \otimes \mathcal{O}_C \longrightarrow 0, \tag{4}$$

obtained by tensoring (1),(2) with $\mathcal{E}$.

The assertion (i) follows from (3) and Lemma 1.2, (iv).

The assertions (ii) and (iv) of Lemma 1.2 give sufficiently many values of $\chi(\mathcal{E}(k))$, which is a cubic polynomial in k by Riemann–Roch–Hirzebruch, that one can deduce that $\chi(\mathcal{E}(k)) = k(k+1)(k+2)$. Thus, $\chi(\mathcal{E}(2)) = 24$. We have also deg $\mathcal{E}(2) \otimes \mathcal{O}_C = 20$, hence $\chi(\mathcal{E}(2) \otimes \mathcal{O}_C) = 20$, and this allows to determine the remaining Euler numbers from (3),(4), thus proving $\chi(\mathcal{E} \otimes \mathcal{E}(1)) = 4$.

Now, use again (4) to show that $H^i(\mathcal{E} \otimes \mathcal{I}_C(2)) = 0$ for $i = 2, 3$. First, show that $H^1(\mathcal{E}(2) \otimes \mathcal{O}_C) = 0$. By (1), $\mathcal{E}(1)|_C \simeq \mathcal{N}_{C/X}$. Hence $H^1(\mathcal{E}(2) \otimes \mathcal{O}_C) = H^1(\mathcal{N}_{C/X}(1)) = H^0(\mathcal{N}^*_{C/X}(-1))^* = H^0(\mathcal{N}_{C/X}(-3))^*$ (we used the fact that $\mathcal{N}_{C/X}(-1)$ is self-dual). Consider the natural inclusion $\mathcal{N}_{C/X}(-2) \subset \mathcal{N}_{C/\mathbb{P}^4}(-2)$ and apply [Hu, Proposition V.2.1] : $H^1(\mathcal{N}^*_{C/\mathbb{P}^4}(2) \otimes \mathcal{M}) = 0$ for any invertible $\mathcal{M}$ of degree 0. By Serre duality, $H^0(\mathcal{N}_{C/\mathbb{P}^4}(-2)) = 0$. Hence $H^0(\mathcal{N}_{C/X}(-2)) = 0$, and all the more $H^0(\mathcal{N}_{C/X}(-3)) = 0$. Hence $H^1(\mathcal{N}_{C/X}(1)) = 0$.

Next, $H^i(\mathcal{E}(2)) = 0$ for $i > 0$ by the same argument as the one applied in the proof of (ii) of Lemma 1.2. Now (4) gives the wanted assertion. □

Lemma 1.5. *We have,* $h^0(\mathcal{I}^2_C(3)) = 0$, $h^1(\mathcal{E}(2) \otimes \mathcal{I}_C) = 1$ *and hence* $h^0(\mathcal{E}(2) \otimes \mathcal{I}_C) = 5$.

Proof. As in the proof of Lemma VI.3.1 in [Hu], we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow H^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \otimes \mathcal{O}_C \xrightarrow{\alpha} \mathcal{N}^*_{C/\mathbb{P}^4}(2) \longrightarrow 0, \tag{5}$$

where $\mathcal{F} = \ker \alpha$ is a rank two vector bundle with $\det \mathcal{F} = \mathcal{O}_C(-1)$ and $h^0(\mathcal{F}) = 0$. It is either the direct sum of two line bundles $\mathcal{F} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\deg M_i \geq -5$, or fits into the non-split exact sequence

$$0 \longrightarrow \mathcal{O}_C(-3o) \otimes \mathcal{M} \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_C(-2o) \otimes \mathcal{M} \longrightarrow 0,$$

where $\mathcal{M}$ is a theta characteristic on C, and o a point of C such that $\mathcal{O}_C(1) \simeq \mathcal{O}_C(5o)$. This implies that $h^1(\mathcal{F}(1)) = 0$. By (5), the map

$$H^0(\mathcal{I}_{C/\mathbb{P}^4}(2)) \otimes H^0(\mathcal{O}_C(1)) \longrightarrow H^0(\mathcal{N}^*_{C/\mathbb{P}^4}(3))$$

is surjective. This map factors obviously through $H^0(\mathcal{I}_{C/\mathbb{P}^4}(3))$, hence the map

$$H^0(\mathcal{I}_{C/\mathbb{P}^4}(3)) \longrightarrow H^0(\mathcal{N}^*_{C/\mathbb{P}^4}(3))$$

is also surjectve. By Riemann-Roch, $h^0(\mathcal{F}(1)) = 5$, and by (5), $h^0(\mathcal{N}^*_{C/\mathbb{P}^4}(3)) = 25 - 5 = 20$, and $h^1(\mathcal{N}^*_{C/\mathbb{P}^4}(3)) = 0$. By [Ma-Ti, Lemma 4.2], $h^0(\mathcal{I}_{C/\mathbb{P}^4}(3)) = 20$, hence, the exact sequence

$$0 \longrightarrow \mathcal{I}^2_{C/\mathbb{P}^4}(3) \longrightarrow \mathcal{I}_{C/\mathbb{P}^4}(3) \longrightarrow \mathcal{N}^*_{C/\mathbb{P}^4}(3) \longrightarrow 0$$

yields $h^0(\mathcal{I}^2_{C/\mathbb{P}^4}(3)) = 0$. By projective normality of C, $h^1(\mathcal{I}_{C/\mathbb{P}^4}(3)) = 0$, hence $h^1(\mathcal{I}^2_{C/\mathbb{P}^4}(3)) = 0$.

Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{C/\mathbb{P}^4} & \longrightarrow & \mathcal{I}^2_{C/\mathbb{P}^4}(3) & \longrightarrow & \mathcal{I}^2_{C/X}(3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4} & \longrightarrow & \mathcal{I}_{C/\mathbb{P}^4}(3) & \longrightarrow & \mathcal{I}_{C/X}(3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{N}^*_{C/\mathbb{P}^4}(3) & \longrightarrow & \mathcal{N}^*_{C/X}(3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6}$$

The left column gives $h^1(\mathcal{I}_{C/\mathbb{P}^4}) = 0$, and the first row $h^0(\mathcal{I}^2_{C/X}(3)) = h^1(\mathcal{I}_{C/\mathbb{P}^4}) = 0$. The lower row gives $h^0(\mathcal{N}^*_{C/X}(3)) = 20$, $h^1(\mathcal{N}^*_{C/X}(3)) = 0$. The middle row gives $h^0(\mathcal{I}_{C/X}(3)) = 19$, $h^1(\mathcal{I}_{C/X}(3)) = 0$, which implies also $h^1(\mathcal{I}^2_{C/X}(3)) = 1$.

Now, look at another commutative diagram in which all the triples are exact:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(1) \arrow[r] \arrow[d, equal] & \mathcal{E}(2)\otimes\mathcal{I}_C \arrow[r] \arrow[d] & \mathcal{I}_C^2(3) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(1) \arrow[r, "\lambda"] & \mathcal{E}(2) \arrow[r, "\mu"] \arrow[d] & \mathcal{I}_C(3) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_{C/X}(1) \arrow[r, equal] \arrow[d] & \mathcal{N}^*_{C/X}(3) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{7}$$

¿From the middle row, we see that μ is surjective on global sections. We can think of elements of $H^0(\mathcal{E}(2)\otimes\mathcal{I}_C)$ as of sections from $H^0(\mathcal{E}(2))$ vanishing on C. Thus, a section $s\in H^0(\mathcal{E}(2))$ is in $H^0(\mathcal{E}(2)\otimes\mathcal{I}_C)$ if and only if $\mu(s)\in H^0(\mathcal{I}^2_{C/X}(3))$. Hence $H^0(\mathcal{E}(2)\otimes\mathcal{I}_C)=\mu^{-1}(H^0(\mathcal{I}^2_{C/X}(3)))$ and

$$h^0(\mathcal{E}(2)\otimes\mathcal{I}_C)=h^0(\mathcal{I}^2_{C/X}(3)))+\dim\ker\mu=5,$$

and $h^1(\mathcal{E}(2)\otimes\mathcal{I}_C)=1$ by Lemma 1.4, (ii). □

Corollary 1.6. $\dim \mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E})=10,\ \dim\mathrm{Ext}^2_Y(\mathcal{E},\mathcal{E})=1,\ \mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})=0$ *for* $i>2$.

Proof. Corollary 1.3 and the last lemma imply the result. □

2. Quasi-symplectic structure on $\mathfrak{M}_Y$

Let Y be a nonsingular cubic fourfold. The notation X will be used for any of the nonsingular hyperplane sections of Y. Denote by $\mathcal{M}$ the union of components of the moduli space of simple sheaves on Y containing the classes $[\mathcal{E}]$ of the sheaves $\mathcal{E}$ defined by (1). According to [Ma-Ti], the family of elliptic normal quintics in a general cubic threefold is irreducible, so $\mathcal{M}$ will be irreducible in the case if Y is general. Let $\mathfrak{M}\subset\mathcal{M}$ be the subset of classes of sheaves $\mathcal{E}$ defined by (1) for all projectively normal elliptic quintics C lying in all nonsingular hyperplane sections of Y.

Proposition 2.1. *(i) $\mathfrak{M}$ is an open subset of $\mathcal{M}$ contained in the smooth locus of the moduli space of simple sheaves.*

(ii) For all i,p,q, the groups $\mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})$, $H^p(Y,\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}))$ for different $\mathcal{E}$ fit into vector bundles $\mathcal{T}^i$, resp. $\mathcal{T}^{p,q}$ on $\mathfrak{M}$. That is, there exist such vector bundles $\mathcal{T}^i$, $\mathcal{T}^{p,q}$ on $\mathfrak{M}$ that for every $\mathcal{E}$ with $e=[\mathcal{E}]\in\mathfrak{M}$, there are canonical isomorphisms $\mathcal{T}^i\otimes\mathbb{C}(e)=\mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})$, $\mathcal{T}^{p,q}\otimes\mathbb{C}(e)=H^p(Y,\mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}))$. There is a natural including $\mathcal{T}^{i,0}\subset\mathcal{T}^i$.

$\mathcal{T}^1$ is canonically isomorphic to the tangent bundle of $\mathfrak{M}$, and $\mathrm{rk}\,\mathcal{T}^1=\dim\mathfrak{M}=10$, $\mathrm{rk}\,\mathcal{T}^2=1$.

(iii) The Yoneda coupling

$$\mathrm{Ext}^i_Y(\mathcal{E},\mathcal{E})\times\mathrm{Ext}^j_Y(\mathcal{E},\mathcal{E})\longrightarrow\mathrm{Ext}^{i+j}_Y(\mathcal{E},\mathcal{E})$$

defines a skew-symmetric bilinear form

$$\Lambda:\mathcal{T}^1\times\mathcal{T}^1\longrightarrow\mathcal{T}^2, \tag{8}$$

and $\mathcal{T}^{1,0}$ is isotropic with respect to Λ.

Proof. (i) For the smoothness, remark that $\dim \mathfrak{M} \leq 10 = \dim \mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E})$, and there is an equality if and only if $\mathcal{M}$ is smooth at $[\mathcal{E}]$. But it is indeed an equality, because $\mathfrak{M}$ contains explicitly a 10-dimensional family of pairwise non-isomorphic sheaves: we have 5 parameters specifying the support X of the sheaf, and the moduli space M_X with fixed support X is smooth of dimension 5 by [Ma-Ti]. The openness of $\mathfrak{M}$ in the moduli space of sheaves on Y follows exactly as in loc. cit., Corollary 5.5 in using an appropriate relativization of Serre's construction.

The existence of the sheaves $\mathcal{T}^i$ in (ii) and of the bilinear form in (iii) follow from Proposition 2.2 of [Muk] and from Theorem 1.21, (4) of [Sim]. The existence of the $\mathcal{T}^{p,q}$ is obtained by an obvious modification of the arguments of [Muk]. The skew symmetry follows from the smoothness: the Yoneda square $z \circ z$ of an element $z \in \mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E})$ is exactly the first obstruction map $\mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E}) \longrightarrow \mathrm{Ext}^2_Y(\mathcal{E},\mathcal{E})$, and it is zero by smoothness of $\mathcal{M}$ at $\mathcal{E}$.

By general properties of spectral sequences of composite functors with a ring structure, the Yoneda coupling induces on the bi-graded object E_2^{pq} a natural ring structure

$$H^p(Y, \mathcal{E}xt^q_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E})) \times H^{p'}(Y, \mathcal{E}xt^{q'}_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E})) \longrightarrow H^{p+p'}(Y, \mathcal{E}xt^{q+q'}_{\mathcal{O}_Y}(\mathcal{E},\mathcal{E}))$$

The isotropy of $H^1(\mathcal{E}\otimes\mathcal{E})$ follows from the fact that $H^2(\mathcal{E}\otimes\mathcal{E}) = 0$ (Lemma 1.2, (i)). ☐

In order to get a quasi-symplectic structure, we have to verify that Λ is non-degenerate, and that $\mathcal{T}^2$ is a trivial line bundle.

By the isotropy of $H^1(\mathcal{E}\otimes\mathcal{E})$, the non-degeneracy of the coupling $\mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E}) \times \mathrm{Ext}^1_Y(\mathcal{E},\mathcal{E}) \longrightarrow \mathrm{Ext}^2_Y(\mathcal{E},\mathcal{E})$ is equivalent to that of

$$H^1(\mathcal{E}\otimes\mathcal{E}) \times H^0(\mathcal{E}\otimes\mathcal{E}(1)) \longrightarrow H^1(\mathcal{E}\otimes\mathcal{E}(1)) \tag{9}$$

Now, in using the isomorphism $\wedge^2\mathcal{E} \simeq \mathcal{O}_X$, we can identify $H^0(\mathcal{E}\otimes\mathcal{E}(1))$ with $H^0(\wedge^2\mathcal{E}(1)) = H^0(\mathcal{O}_X(1))$ via the natural map $\mathcal{E}\otimes\mathcal{E}\longrightarrow\wedge^2\mathcal{E}$. The fact that the resulting map is an isomorphism follows from the factorisation

$$H^0(\mathcal{E}\otimes\mathcal{E}(1)) = H^0(\mathcal{I}_C\otimes\mathcal{E}(2)) = H^0(\mathcal{O}_X(1)),$$

coming from (3) and from the upper row of (7).

Lemma 2.2. *The coupling (9) is identified via the above isomorphism with the coupling of plain multiplication of elements of $H^1(\mathcal{E}\otimes\mathcal{E})$ by linear forms:*

$$H^1(\mathcal{E}\otimes\mathcal{E}) \times H^0(\mathcal{O}_X(1)) \longrightarrow H^1(\mathcal{E}\otimes\mathcal{E}(1)). \tag{10}$$

Proof. To see that these are indeed the same, we will express both of them in a local basis of $\mathcal{E}$. The latter is just the multiplication of coefficients:

$$(\sum a_{ij} e_i \otimes e_j\ ,\ l\ ) \ \mapsto\ \sum l a_{ij} e_i \otimes e_j\ .$$

For the former, notice that $\mathcal{E}\otimes\mathcal{E}$ contains the trivial subbundle $\wedge^2\mathcal{E}$ as a direct summand. By the above considerations, $H^0(\mathcal{E}\otimes\mathcal{E}(1)) = H^0(\wedge^2\mathcal{E}(1))$. Hence every section from $H^0(\mathcal{E}\otimes\mathcal{E}(1))$ can be written locally as $l(e_2\otimes e_1 - e_1\otimes e_2)$. The self-duality of $\mathcal{E}$ being given by a trivialization of $\wedge^2\mathcal{E}$, we can assume that $e_2\wedge e_1 = e_2\otimes e_1 - e_1\otimes e_2$ is the trivializing section, and then the dual basis of $(e_j) = (e_1, e_2)$ is $((-1)^{\bar{j}} e_{\bar{j}}) = (e_2, -e_1)$, where $\bar{j} = 3 - j$. Hence the contraction of middle terms

$$(a_1\otimes a_2, b_1\otimes b_2) \mapsto \frac{a_2\wedge b_1}{e_2\wedge e_1} a_1\otimes b_2$$

giving the Yoneda coupling is the identity $e_i\otimes e_j \mapsto e_i\otimes e_j$ on the basic sections. This implies that the Yoneda coupling is given by multiplication of coefficients, $(a_{ij}, l) \mapsto l a_{ij}$. ☐

Proposition 2.3. *The coupling (10) is non-degenerate.*

Proof. We are to verify, that for any linear form $l \in H^0(\mathcal{O}_X(1))$, the map

$$\cdot l : H^1(\mathcal{E} \otimes \mathcal{E}) \longrightarrow H^1(\mathcal{E} \otimes \mathcal{E}(1)) (\simeq \mathbb{C})$$

is surjective. By Lemma 1.4, the last map is identified with

$$\cdot l : H^1(\mathcal{I}_C \otimes \mathcal{E}(1)) \longrightarrow H^1(\mathcal{I}_C \otimes \mathcal{E}(2)).$$

Using (6), as it is and twisted by $\mathcal{O}(-1)$, and Lemma 1.4, (i), we can continue the sequence of identifications:

$$\begin{tikzcd} H^1(\mathcal{I}_C \otimes \mathcal{E}(1)) \arrow[r, "\sim"] \arrow[d, "\cdot l"] & H^1(\mathcal{I}^2_{C/X}(2)) \arrow[r, "\sim"] \arrow[d, "\cdot l"] & H^2(\mathcal{I}_{C/\mathbb{P}^4}(-1)) \arrow[d, "\cdot l"] & H^1(\mathcal{O}_C(-1)) \arrow[l, "\sim"'] \arrow[d, "\cdot l"] \\ H^1(\mathcal{I}_C \otimes \mathcal{E}(2)) \arrow[r, "\sim"] & H^1(\mathcal{I}^2_{C/X}(3)) \arrow[r, "\sim"] & H^2(\mathcal{I}_{C/\mathbb{P}^4}) & H^1(\mathcal{O}_C) \arrow[l, "\sim"'] \end{tikzcd} \tag{11}$$

Only the isomorphisms involving $\mathcal{I}^2_{C/X}$ are not obvious. Those involving $H^1(\mathcal{I}^2_{C/X}(3))$ are established in the proof of Lemma 1.5. We will verify now the isomorphisms involving $H^1(\mathcal{I}^2_{C/X}(2))$. To this end, write down the diagram (6) twisted by $\mathcal{O}(-1)$:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{I}_{C/\mathbb{P}^4}(-1) \arrow[r] \arrow[d] & \mathcal{I}^2_{C/\mathbb{P}^4}(2) \arrow[r] \arrow[d] & \mathcal{I}^2_{C/X}(2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^4}(-1) \arrow[r] \arrow[d] & \mathcal{I}_{C/\mathbb{P}^4}(2) \arrow[r] \arrow[d] & \mathcal{I}_{C/X}(2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_C(-1) \arrow[r] \arrow[d] & \mathcal{N}^*_{C/\mathbb{P}^4}(2) \arrow[r] \arrow[d] & \mathcal{N}^*_{C/X}(2) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{12}$$

The left column yields for the vector $(h^i(\mathcal{I}_{C/\mathbb{P}^4}(-1)))$ the values $(0, 0, 5, 0, \dots)$. Similarly the left column of (6) gives $(h^i(\mathcal{I}_{C/\mathbb{P}^4}(2))) = (5, 0, 0, \dots)$. Now, the middle row of (12) gives $(h^i(\mathcal{I}_{C/X}(2))) = (5, 0, 0, \dots)$. Further, $\mathcal{N}^*_{C/X}(2) \simeq \mathcal{N}_{C/X}$, and by [Ma-Ti, Lemma 5.1, e)], $(h^i(\mathcal{N}^*_{C/X}(2))) = (10, 0, 0, \dots)$. We proved in the previous section that $h^0(\mathcal{I}^2_{C/X}(3)) = 0$, and so much the more $h^0(\mathcal{I}^2_{C/X}(2)) = 0$. Hence, from the last column of (12) we can deduce that $(h^i(\mathcal{I}^2_{C/X}(2))) = (0, 5, 0, \dots)$.

We see already that $h^2(\mathcal{I}_{C/\mathbb{P}^4}(-1)) = h^1(\mathcal{I}^2_{C/X}(2)) = 5$. Now, let us verify that the connecting homomorphism $H^1 \longrightarrow H^2$ of the first row is indeed an isomorphism. Look at the second column of (12). By [Hu, Proposition V.2.1], $(h^i(\mathcal{N}^*_{C/\mathbb{P}^4}(2))) = (5, 0, 0, \dots)$. We saw in the previous section that $h^0(\mathcal{I}^2_{C/\mathbb{P}^4}(3)) = 0$, and so much the more $h^0(\mathcal{I}^2_{C/\mathbb{P}^4}(2)) = 0$. Hence $(h^i(\mathcal{I}^2_{C/\mathbb{P}^4}(2))) = (0, 0, 0, \dots)$. Hence all the connecting homomorphisms of the first row are isomorphisms.

Now, coming back to (11), we see that the vertical arrow on the right hand side is obviously surjective, and we are done. $\square$

Theorem 2.4. *The line bundle $\mathcal{T}^2$ on $\mathfrak{M}$ is trivial, hence the skew-symmetric pairing Λ defined in (8) is a well defined 2-form on $\mathfrak{M}$:*

$$\Lambda : \mathcal{T}\mathfrak{M} \xrightarrow{\sim} \Omega^1_{\mathfrak{M}}, \quad \Lambda \in H^0(\Omega^2_{\mathfrak{M}}). \tag{13}$$

The fibers of the natural map $p : \mathfrak{M} \to \check{\mathbb{P}}^5$, $\mathcal{E} \to < \mathrm{Supp}(\mathcal{E}) >$, are Lagrangian subvarieties with respect to Λ.

Proof. By Proposition 2.1, for any point $e = [\mathcal{E}] \in \mathfrak{M}$, one has a canonical isomorphism $\mathcal{T}^2 \otimes \mathbf{C}(e) \simeq \mathrm{Ext}^2_Y(\mathcal{E}, \mathcal{E})$. It is enough to show that there is a canonical isomorphism

$$\alpha : \ \mathrm{Ext}^2_Y(\mathcal{E}, \mathcal{E}) \overset{can}{\simeq} \mathbf{C}. \tag{14}$$

To this end, fix an equation $f_Y = 0$ of the cubic Y in $\mathbb{P}^5$ and take any section $h \in H^0(\mathcal{O}_Y(1))$ vanishing on $X = \mathrm{Supp}(\mathcal{E})$; this section h defines an isomorphism $\phi_h : \mathcal{O}_X(1) \xrightarrow{\sim} \mathcal{N}_{X/Y}$ such that the substitution $(h \mapsto \lambda h)$ induces $\phi_h \mapsto \lambda^{-1}\phi_h$.

Next, by Lemma 1.1 there is a canonical isomorphism

$$\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{E}) \overset{can}{\xrightarrow{\sim}} \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{N}_{X/Y}. \tag{15}$$

Now take a general section $0 \neq s \in H^0(\mathcal{E}(1))$ such that $C = (s)_0$ is a smooth projectively normal elliptic quintic. By computations of Section 1,

$$h^i(\det \mathcal{E} \otimes \mathcal{I}^2_{C,H}(3)) = h^i(\mathcal{I}^2_{C,H}(3)) = 0, \quad i = 1, 2, \tag{16}$$

where $H =< C >=< X >$ is the span of C, or of X, in $\mathbb{P}^5$. The section s defines an exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{s} \mathcal{E}(1) \xrightarrow{\cdot \wedge s} \mathcal{I}_{C,X}(2) \otimes \det \mathcal{E} \to 0. \tag{17}$$

Twisting it by $\mathcal{E}^\vee$ and using the canonical isomorphism $\mathcal{E}^\vee \otimes \det \mathcal{E} \overset{can}{\xrightarrow{\sim}} \mathcal{E}$, we obtain the exact triple $0 \to \mathcal{E}^\vee \to \mathcal{E}^\vee \otimes \mathcal{E}(1) \xrightarrow{\varepsilon_s} \mathcal{E} \otimes \mathcal{I}_{C/X}(2) \to 0$. Thus we have a composed isomorphism

$$\gamma_1 : \mathrm{Ext}^2_Y(\mathcal{E}, \mathcal{E}) \underset{Cor.1.3}{\overset{can}{\simeq}} H^1(\mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{E})) \overset{can}{\simeq} H^1(\mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{N}_{X/Y}) \underset{\sim}{\xrightarrow{\phi_h^{-1}}}$$
$$H^1(\mathcal{E}^\vee \otimes \mathcal{E}(1)) \underset{\sim}{\xrightarrow{H^1(\varepsilon_s)}} H^1(\mathcal{E} \otimes \mathcal{I}_{C,X}(2)) \tag{18}$$

which is proportional, by the above, to h and to s [1]. Next, similar to (17) we have a morphism $\mathcal{E} \otimes \mathcal{I}_{C,X}(2) \xrightarrow{\cdot \wedge s} \mathcal{I}^2_{C,X}(3) \otimes \det \mathcal{E}$ inducing a map $H^1(\mathcal{E} \otimes \mathcal{I}_{C,X}(2)) \to H^1(\mathcal{I}^2_{C,X}(3) \otimes \det \mathcal{E})$ which gives, in composition with γ_1, the isomorphism

$$\gamma_2 : \mathrm{Ext}^2_Y(\mathcal{E}, \mathcal{E}) \xrightarrow{\sim} H^1(\mathcal{I}^2_{C,X}(3) \otimes \det \mathcal{E}), \quad \gamma_2 \sim (h, s^2). \tag{19}$$

The exact triples $0 \to \mathcal{I}_{C,H} \xrightarrow{\cdot f_Y} \mathcal{I}^2_{C,H}(3) \to \mathcal{I}^2_{C,X}(3) \to 0$ and $0 \to \mathcal{I}_{C,H} \to \mathcal{O}_H \to \mathcal{O}_C \to 0$ give together with (16) and Serre duality on C a chain of canonical isomorphisms:

$$H^1(\mathcal{I}^2_{C,X}(3) \otimes \det \mathcal{E}) \simeq H^2(\mathcal{I}_{C,H} \otimes \det \mathcal{E}) \simeq H^1(\det \mathcal{E} \otimes \mathcal{O}_C) \simeq H^0((\det \mathcal{E})^\vee \otimes \omega_C)^\vee. \tag{20}$$

Next, $\phi_h : \mathcal{O}_X(1) \xrightarrow{\sim} \mathcal{N}_{X/Y}$, $\phi_h \sim h$, composed with $\psi : \mathcal{N}_{C/X} \xrightarrow{\sim} \mathcal{E}(1)|_C$, $\psi \sim s$, and with the canonical isomorphisms $\omega_C \simeq \omega_Y \otimes \mathcal{N}_{X/Y} \otimes \det \mathcal{N}_{C/X}$, $\omega_Y \simeq \mathcal{O}_Y(-3)$, gives the

[1] We denote this as follows: $\gamma_1 \sim (h, s)$.

following isomorphism: $\chi : \omega_C \xrightarrow{\sim} \det\mathcal{E}|_C, \quad \chi \sim (h, s^2)$. Composing it with (20), we obtain the isomorphism

$$\theta : \ H^1(\mathcal{I}^2_{C,X}(3)\otimes\det\mathcal{E}) \overset{can}{\simeq} H^0((\det\mathcal{E})^\vee\otimes\omega_C)^\vee \xrightarrow{\sim} H^0((\det\mathcal{E}|_C)^\vee\otimes\det\mathcal{E}|_C) \overset{can}{\simeq} \mathbf{C},$$
$$\theta\sim(h^{-1}, s^{-2}). \quad (21)$$

Now, composing (19) and (21), we get the desired isomorphism α in (14) with $\alpha \sim (h^0, s^0)$, that is α does not depend on the choice of scalar multiples of h and s. Thus, to show that α is a canonical isomorphism, it is enough to check that α does not depend on the choice of the point $z = \mathbf{C}s \in \mathbb{P}(H^0(\mathcal{E}(1))$. The last assertion is clear, since, by construction, α is defined for *any* point $z \in \mathbb{P}(H^0(\mathcal{E}(1))$ [2], thus giving a morphism of trivial bundles $\alpha : \mathrm{Ext}^2_Y(\mathcal{E},\mathcal{E})\otimes\mathcal{O}_{\mathbb{P}(H^0(\mathcal{E}(1))}\longrightarrow\mathbf{C}\otimes\mathcal{O}_{\mathbb{P}(H^0(\mathcal{E}(1))}$. We have $\alpha\in H^0(\mathcal{O}_{\mathbb{P}(H^0(\mathcal{E}(1))})$, hence α is constant as a function on the projective 5-space $\mathbb{P}(H^0(\mathcal{E}(1))$.

The tangent distribution of the fibers of p is the vector bundle $\mathcal{T}^{1,0}$, and its isotropy was proved in Proposition 2.1. This implies that the fibers of p are Lagrangian.

□

3. Symplectic structure on $\mathbf{J}_Y$

In this section, Y will denote a nonsingular cubic fourfold in $\mathbb{P}^5$, $h : \mathbf{J} = \mathbf{J}_Y\longrightarrow U$ the relative intermediate Jacobian of the family $\mathbf{X}\longrightarrow U \subset \check{\mathbb{P}}^5$ of nonsingular hyperplane sections over an open subset U in the dual projective space $\check{\mathbb{P}}^5$. X will denote any of the smooth hyperplane sections $H\cap Y$ for $H\in U$. $\mathbf{X}$ is the scheme of zeros of a section of $\mathcal{O}_\Pi(3,0)$ in the incidence divisor $\Pi = \{(x,H)\in\mathbb{P}^5_U \mid x\in H\}$, where $\mathbb{P}^5_U = \mathbb{P}^5\times U \subset \mathbb{P}^5\times\check{\mathbb{P}}^5$. We can fix this section, as well as the cubic form defining Y in $\mathbb{P}^5$, and consider all the isomorphisms depending only on these data as canonical ones.

According to Donagi–Markman [D-M, 8.22], $\mathbf{J}$ has a natural (up to a constant factor) symplectic structure $\alpha_{\mathbf{J}}\in H^0(\mathbf{J},\Omega^2_{\mathbf{J}})$ for which h is a Lagrangian fibration, that is the fibers J_X of h, the intermediate Jacobians of the hyperplane sections $X = H\cap Y$, are Lagrangian submanifolds of $\mathbf{J}$. The idea is to recover the symplectic structure from the isomorphisms $\mathcal{N}_{L/\mathbf{J}}\simeq\Omega^1_L$ that it defines for Lagrangian submanifolds L. For any symplectic structure α on $\mathbf{J}$ for which h is a Lagrangian fibration, the above isomorphisms for Lagrangian submanifolds $L = J_X$ glue together to give an isomorphism of vector bundles $\sigma(\alpha) : \mathcal{T}_U\longrightarrow h_*\Omega^1_{\mathbf{J}/U}$.

Conversely, let $\sigma\in\mathrm{Hom}(\mathcal{T}_U, h_*\Omega^1_{\mathbf{J}/U}) = H^0(U,\Omega^1_U\otimes h_*\Omega^1_{\mathbf{J}/U})$ be any isomorphism. The natural exact triple

$$0\longrightarrow h^*\Omega^1_U\longrightarrow\Omega^1_{\mathbf{J}}\longrightarrow\Omega^1_{\mathbf{J}/U}\longrightarrow 0 \quad (22)$$

implies another one, obtained by taking $\wedge^2$:

$$0\longrightarrow h^*\Omega^2_U\longrightarrow\mathcal{F}\longrightarrow h^*\Omega^1_U\otimes\Omega^1_{\mathbf{J}/U}\longrightarrow 0, \quad (23)$$

where $\mathcal{F}\subset\Omega^2_{\mathbf{J}}$ is the vector subbundle of sections for which $\mathcal{T}_{\mathbf{J}/U}$ is an isotropic distribution. Let $\delta : H^0(\mathbf{J}, h^*\Omega^1_U\otimes\Omega^1_{\mathbf{J}/U})\longrightarrow H^1(\mathbf{J}, h^*\Omega^2_U)$ be the connecting homomorphism defined by (23). The following assertion is obvious:

---

[2] In fact, for any $z = \mathbf{C}s\in\mathbb{P}(H^0(\mathcal{E}(1))$, it follows from the stability and from the local freeness of $\mathcal{E}$, that the scheme $C = (s)_0$ of zeros of s is a locally complete intersection of pure codimension 2 in X, i.e. a curve, so that the map $\alpha : \ \mathrm{Ext}^2_Y(\mathcal{E},\mathcal{E})\to\mathbf{C}$ is well defined.

Lemma 3.1. *$h^*\sigma$ lifts to a section $\alpha \in H^0(\mathbf{J}, \mathcal{F})$ if and only if $\delta(h^*\sigma) = 0$. α is determined up to the addition of a 2-form $h^*\beta$, where $\beta \in H^0(U, \Omega^2_U)$. The lift becomes unique, if one imposes the condition that the zero section $O \subset B$ be a Lagrangian subvariety (that is, $\alpha|_O \equiv 0$).*

We will formulate the result of Donagi–Markman in a form, convenient to our purposes. Let Z be the "Fano scheme" of Y, that is the Hilbert scheme parametrizing all the lines in Y. It is a smooth projective irreducible 4-fold. Beauville–Donagi [B-D] proved that Z is an irreducible symplectic variety, deformation equivalent to the Fujiki–Beauville 4-fold $S^{[2]}$ for a K3 surface S; let α_Z be its symplectic structure, unique up to proportionality. Let $\pi : \mathbf{F} \longrightarrow U$ be the relative Fano surface of $\mathbf{X}$ over U, that is the family of surfaces $F(X)$ parametrizing the lines in the hyperplane sections $X = H \cap Y$ with $H \in U$. By Clemens–Griffiths [Cl-Gr], $\mathbf{J}/U$ is canonically isomorphic to the relative Albanese variety $\mathrm{Alb}(\mathbf{F}/U)$ and to $\mathrm{Pic}^0(\mathbf{F}/U)$. By Voisin [V], the Fano surfaces $F(X)$ are Lagrangian in Z and form an open set in the Hilbert scheme of Z. By Theorem 8.1 of [D-M], the relative Picard variety $P = \mathrm{Pic}(\mathbf{L}/B)$ of the universal family of Lagrangian subvarieties of a smooth projective symplectic variety V over an open set $B \subset \mathrm{Hilb}(V)$ has also a symplectic structure for which P/B is a Lagrangian fibration. The application of this result to the family $\mathbf{F}/U$ yields a symplectic 10-fold $\mathrm{Pic}^0(\mathbf{F}/U) = \mathrm{Alb}(\mathbf{F}/U) = \mathbf{J}$.

Theorem of Donagi–Markman. For any $H \in U$, let $X = Y \cap H$, $F = F(X)$, $\sigma_H : \mathcal{N}_{F/Z} \longrightarrow \Omega^1_F$ the isomorphism, induced by α_Z, and $H^0(\sigma_H) : H^0(\mathcal{N}_{F/Z}) \longrightarrow H^0(\Omega^1_F)$ the induced map on global sections. With natural identificatons $H^0(F, \mathcal{N}_{F/Z}) = T_{[F]}\mathrm{Hilb}(Z) = T_{[F]}U$ and $H^0(F, \Omega^1_F) = \Omega^1_{\mathrm{Alb}(F)}|_0 = \Omega^1_{J(X)}|_0$, let $\sigma : \mathcal{T}_U \longrightarrow h_*\Omega^1_{\mathbf{J}/U}$ be the isomorphism obtained by the relativization of the construction of $H^0(\sigma_H)$ over U. Then σ satisfies the hypotheses of Lemma 3.1, and hence lifts to a unique symplectic structure $\alpha_{\mathbf{J}}$ on $\mathbf{J}$ for which the fibers and the zero section of h are Lagrangian submanifolds.

Our goal now is to identify σ as the canonical isomorphism, constructed in Lemma 3.2 below.

Let $\mathrm{pr}_1, \mathrm{pr}_2$ be the natural projections onto the two factors of $\mathbb{P}^5 \times U$, and $\mathcal{O}(i,j) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^5}(i) \otimes \mathrm{pr}_2^* \mathcal{O}_{\check{\mathbb{P}}^5}(j)$.

Lemma 3.2. *There exist canonical isomorphisms*

$$h_*\Omega^1_{\mathbf{J}/U} = \pi_*\Omega^1_{\mathbf{F}/U} = R^1 \mathrm{pr}_{2*} \Omega^2_{\mathbf{X}/U} = \mathrm{pr}_{2*} \mathcal{O}_{\mathbf{X}}(1,1) = \mathcal{T}_U.$$

Proof. The first two isomorphisms follow from the definitions of the intermediate Jacobian and of the Albanese variety, as well as the identification $\mathbf{J} = \mathrm{Alb}(\mathbf{F}/U)$ mentioned above. The third one is a relative version of the tangent bundle formula for the intermediate Jacobian of a cubic 3-fold. To prove it, write down the relative conormal bundle sequence of $\mathbf{X} \subset \Pi$ over U, taken to the third exteriour power and twisted by $\mathcal{N}_{\mathbf{X}/\Pi}$:

$$0 \longrightarrow \Omega^2_{\mathbf{X}/U} \longrightarrow \Omega^3_{\Pi/U} \otimes \mathcal{N}_{\mathbf{X}/\Pi} \longrightarrow \Omega^3_{\mathbf{X}/U} \otimes \mathcal{N}_{\mathbf{X}/\Pi} \longrightarrow 0.$$

Applying $R^i \mathrm{pr}_{2*}$ and using the Bott formulae, we obtain the canonical isomorphism

$$\mathrm{pr}_{2*}(\Omega^3_{\mathbf{X}/U} \otimes \mathcal{N}_{\mathbf{X}/\Pi}) \xrightarrow{\sim} R^1 \mathrm{pr}_{2*} \Omega^2_{\mathbf{X}/U}. \tag{24}$$

We have $\mathcal{N}_{\mathbf{X}/\Pi} = \mathcal{O}_{\mathbf{X}}(3,0) = \mathcal{O}_\Pi(3,0)|_{\mathbf{X}}$, $\Omega^3_{\mathbf{X}/U} = \omega_{\Pi/U} \otimes \mathcal{N}_{\mathbf{X}/\Pi}$ and $\omega_{\Pi/U} = \mathcal{O}_\Pi(-5,1)$, so both sheaves in (24) are canonically isomorphic to $\mathrm{pr}_{2*} \mathcal{O}_{\mathbf{X}}(1,1)$.

It is also well known that the incidence divisor Π is identified with the projectivized cotangent bundle of U: $\Pi = \mathbb{P}(\mathcal{T}_U^\vee)$, hence there are canonical isomorphisms $\mathcal{T}_U^\vee = \mathrm{pr}_{2*} \mathcal{O}_{\Pi/U}(1) = \mathrm{pr}_{2*} \mathcal{O}_\Pi(1,1) = \mathrm{pr}_{2*} \mathcal{O}_{\mathbf{X}}(1,1)$.

∎

The Clemens–Griffiths Tangent Bundle Theorem for the Fano surface $F(X)$ of a cubic 3-fold X states, that there is an isomorphism $\mathcal{T}_{F(X)} \simeq \mathcal{S}_{G(1,\mathbb{P}^4)}|_{F(X)}$, where $G(1,\mathbb{P}^n)$ denotes the Grassmannian of lines in $\mathbb{P}^n$ and $\mathcal{S}_{G(1,\mathbb{P}^n)}$ the universal rank 2 vector bundle on it. We are going to relativize this result, in order to compute the sheaf $\Omega^1_{\mathbf{F}/U}$.

Let $G = G(1,\mathbb{P}^5)$, $\mathcal{S}_G$ the universal vector bundle, $\mathbb{P}^5 \xleftarrow{q_G} \Gamma_G \xrightarrow{p_G} G$ the universal family of lines together with its natural projections, $Y \xleftarrow{q} \Gamma \xrightarrow{p} Z$ its restriction to the lines in Y, and $\mathbf{X} \xleftarrow{\boldsymbol{q}} \mathbf{\Gamma} \xrightarrow{\boldsymbol{p}} \mathbf{F}$ the relativization over U. Let p_i be the projection to the three factors of $\mathbb{P}^5 \times \check{\mathbb{P}}^5 \times G$, and $\mathcal{O}(i,j,k) = p_1^*\mathcal{O}(i) \otimes p_2^*\mathcal{O}(j) \otimes p_3^*\mathcal{O}(k)$. Denote by Γ_u, F_u, X_u, $H_u = \mathbb{P}^4_u$ the fibers over $u \in U$ of $\mathbf{\Gamma}$, $\mathbf{F}$, $\mathbf{X}$, resp. Π.

Lemma 3.3. *There is a canonical isomorphism*

$$\mathcal{T}_{\mathbf{F}/U} = (\mathcal{O}_{\check{\mathbb{P}}^5}(-1) \boxtimes \mathcal{S}_G)|_{\mathbf{F}},$$

where the operation $\boxtimes$ yields a sheaf on the variety $\check{\mathbb{P}}^5 \times G$, containing $\mathbf{F}$ in a natural way.

Proof. The natural inclusions $\mathbf{\Gamma} \subset \mathbf{X} \times_U \mathbf{F} \subset \Pi \times_U \mathbf{F}$ give the following exact sequences:

$$0 \longrightarrow \mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} \longrightarrow \mathcal{N}_{\mathbf{\Gamma}/\Pi\times_U\mathbf{F}} \longrightarrow \mathcal{N}_{\mathbf{X}\times_U\mathbf{F}/\Pi\times_U\mathbf{F}}|_{\mathbf{\Gamma}} \longrightarrow 0, \tag{25}$$

$$0 \longrightarrow \mathcal{T}_{\mathbf{\Gamma}/\mathbf{F}} \longrightarrow (p_1 \times p_2)^*\mathcal{T}_{\Pi/U}|_{\mathbf{\Gamma}} \longrightarrow \mathcal{N}_{\mathbf{\Gamma}/\Pi\times_U\mathbf{F}} \longrightarrow 0.$$

We have $\mathcal{T}_{\mathbf{\Gamma}/\mathbf{F}} = (p_1 \times p_3)^*\mathcal{T}_{\Gamma_G/G}$. The Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow q_G^*\mathcal{O}_{\mathbb{P}^5}(1) \otimes p_G^*\mathcal{S}_G \longrightarrow \mathcal{T}_{\Gamma_G/G} \longrightarrow 0$$

implies $\det \mathcal{T}_{\Gamma_G/G} = q_G^*\mathcal{O}_{\mathbb{P}^5}(1) \otimes p_G^*\mathcal{O}_G(-1)$. We find successively the determinants: $\det \mathcal{T}_{\mathbf{\Gamma}/\mathbf{F}} = \mathcal{O}_{\mathbf{\Gamma}}(2,0,-1)$, $\det(p_1 \times p_2)^*\mathcal{T}_{\Pi/U}|_{\mathbf{\Gamma}} = \mathcal{O}_{\mathbf{\Gamma}}(5,-1,0)$, $\det \mathcal{N}_{\mathbf{\Gamma}/\Pi\times_U\mathbf{F}} = \mathcal{O}_{\mathbf{\Gamma}}(3,-1,1)$, $\mathcal{N}_{\mathbf{X}\times_U\mathbf{F}/\Pi\times_U\mathbf{F}}|_{\mathbf{\Gamma}} = p_1^*\mathcal{N}_{Y/\mathbb{P}^5}|_{\mathbf{\Gamma}} = \mathcal{O}_{\mathbf{\Gamma}}(3,0,0)$, $\det \mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} = \mathcal{O}_{\mathbf{\Gamma}}(0,-1,1)$. Dualizing (25) and tensoring by $\det \mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}}$, we obtain the exact triple

$$0 \longrightarrow \mathcal{O}_{\mathbf{\Gamma}}(-3,-1,1) \longrightarrow \mathcal{N}^\vee_{\mathbf{\Gamma}/\Pi\times_U\mathbf{F}}(0,-1,-1) \longrightarrow \mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} \longrightarrow 0,$$

which gives the canonical isomorphism

$$\boldsymbol{p}_*\mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} \xrightarrow{\sim} R^1\boldsymbol{p}_*\mathcal{O}_{\mathbf{\Gamma}}(-3,-1,1).$$

As $\boldsymbol{p}_*\mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} = \mathcal{T}_{\mathbf{F}/U}$ and by relaive Serre duality, we obtain the canonical isomorphism

$$\mathcal{T}_{\mathbf{F}/U} = (\boldsymbol{p}_*\mathcal{O}_{\mathbf{\Gamma}}(1,1,0))^\vee.$$

The natural identification $\mathcal{S}_G = (p_{G*}q_G^*\mathcal{O}_{\mathbb{P}^5}(1))^\vee$ induces the canonical isomorphism $(\boldsymbol{p}_*\mathcal{O}_{\mathbf{\Gamma}}(1,1,0))^* = (\mathcal{O}_{\check{\mathbb{P}}^5}(-1) \boxtimes \mathcal{S}_G)|_{\mathbf{F}}$, which implies the assertion of the lemma.

□

The next lemma relativizes the isomorphism $\mathcal{N}_{F_u/Z} \simeq \mathcal{S}^\vee_G|_{F_u}$ coming from $\mathcal{N}_{G(1,\mathbb{P}^4_u)/G(1,\mathbb{P}^5)} = \mathcal{S}^\vee_G|_{G(1,\mathbb{P}^4_u)}$.

Lemma 3.4. *There exist canonical isomorphisms*

$$\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z} = \mathcal{O}_{\check{\mathbb{P}}^5}(1) \boxtimes \mathcal{S}^\vee_G|_{\mathbf{F}} = \Omega^1_{\mathbf{F}/U}.$$

Moreover, applying π_*, one obtains the isomorphism

$$\pi_*\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z} = \pi_*\Omega^1_{\mathbf{F}/U},$$

which coincides with that of Lemma 3.2 if one uses the natural identification $\pi_*\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z} = \mathcal{T}_U$, following from the interpretation of $\mathbf{F}/U$ as the universal family over the open set $U \subset \mathrm{Hilb}(Z)$.

Proof. One can represent $\mathbf{F}$ as the complete intersection $\boldsymbol{\sigma_{11}} \cap (U \times Z)$, where

$$\boldsymbol{\sigma_{11}} = \{(u,l) \in \check{\mathbb{P}}^5 \times G \mid l \subset H_u\}$$

is the universal Schubert variety σ_{11} over $\check{\mathbb{P}}^5$. It is the scheme of zeros of $(q_G \times \mathrm{id})_*(p_G \times \mathrm{id})^*\tau_0$, where $\tau_0 \in H^0(\mathbb{P}^5 \times \check{\mathbb{P}}^5, \mathcal{O}_{\mathbb{P}^5 \times \check{\mathbb{P}}^5}(1,1))$ is the equation of Π. As $(q_G \times \mathrm{id})_*(p_G \times \mathrm{id})^*\mathcal{O}_{\mathbb{P}^5 \times \check{\mathbb{P}}^5}(1,1) = \mathcal{O}_{\check{\mathbb{P}}^5}(1) \boxtimes \mathcal{S}_G^\vee$, we have proved the first statement.

The second one follows immediately from the Clemens–Welters computation of the differential of the Abel–Jacobi map (see Sect. 2 in [We], or diagram (42) in Sect. 5 below). □

4. Symplectic structure on the Fano variety Z of the cubic 4-fold Y

4.1. **Construction of the symplectic structure.** Let $\overline{\mathbf{F}}$, resp. $\overline{\mathbf{\Gamma}}$ be the closure of $\mathbf{F}$ in $\check{\mathbb{P}}^5 \times G$, resp. of $\mathbf{\Gamma}$ in $\mathbb{P}^5 \times \check{\mathbb{P}}^5 \times G$.

We have the diagram of natural embeddings

$$\begin{array}{ccccc} & & Y \times \overline{\mathbf{F}} & \subset & \mathbb{P}^5 \times \overline{\mathbf{F}} \\ & & \cup & & \cup \\ \overline{\mathbf{\Gamma}} & \subset & \mathbf{X} \times_{\check{\mathbb{P}}^5} \overline{\mathbf{F}} & \subset & \Pi \times_{\check{\mathbb{P}}^5} \overline{\mathbf{F}} \end{array} \tag{26}$$

Remark, that $\overline{\mathbf{F}}$ and $\overline{\mathbf{\Gamma}}$ are nonsingular of dimensions 7 and 8 respectively. This follows from the fact that $\overline{\mathbf{F}}$ admits a natural smooth morphism onto Z with fibers $\mathbb{P}^3$, and $\overline{\mathbf{\Gamma}}/\overline{\mathbf{F}}$ is smooth with fibers $\mathbb{P}^1$. Write down the exact sequence of normal bundles:

$$0 \longrightarrow \mathcal{N}_{\overline{\mathbf{\Gamma}}/Y \times \overline{\mathbf{F}}} \longrightarrow \mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbb{P}^5 \times \overline{\mathbf{F}}} \longrightarrow \mathcal{N}_{Y \times \overline{\mathbf{F}}/\mathbb{P}^5 \times \overline{\mathbf{F}}}|_{\overline{\mathbf{\Gamma}}} \longrightarrow 0 \tag{27}$$

Applying $\wedge^2$ and using the isomorphism $\mathcal{N}_{Y \times \overline{\mathbf{F}}/\mathbb{P}^5 \times \overline{\mathbf{F}}}|_{\overline{\mathbf{\Gamma}}} \simeq \mathcal{O}_{\overline{\mathbf{\Gamma}}}(3,0,0)$, we obtain the exact triple

$$0 \longrightarrow \wedge^2\mathcal{N}_{\overline{\mathbf{\Gamma}}/Y \times \overline{\mathbf{F}}} \longrightarrow \wedge^2\mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbb{P}^5 \times \overline{\mathbf{F}}} \longrightarrow \mathcal{N}_{\overline{\mathbf{\Gamma}}/Y \times \overline{\mathbf{F}}}(3,0,0) \longrightarrow 0 \tag{28}$$

Now, from the proof of Lemma 3.3, it follows $\det \mathcal{N}_{\overline{\mathbf{\Gamma}}/\check{\mathbb{P}}^5 \times \overline{\mathbf{F}}} = \det \mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbf{X} \times_{\check{\mathbb{P}}^5} \overline{\mathbf{F}}} \otimes \det \mathcal{N}_{Y \times \overline{\mathbf{F}}/\mathbb{P}^5 \times \overline{\mathbf{F}}}|_{\overline{\mathbf{\Gamma}}} = \mathcal{O}_{\overline{\mathbf{\Gamma}}}(1,0,1), \omega_{\overline{\mathbf{\Gamma}}/\overline{\mathbf{F}}} = \mathcal{O}_{\overline{\mathbf{\Gamma}}}(-2,0,1)$. Using these formulas and twisting the last exact triple by $\mathcal{O}_{\overline{\mathbf{\Gamma}}}(-3,0,0)$, we obtain:

$$0 \longrightarrow \mathcal{N}^\vee_{\overline{\mathbf{\Gamma}}/Y \times \overline{\mathbf{F}}} \otimes \omega_{\overline{\mathbf{\Gamma}}/\overline{\mathbf{F}}} \longrightarrow \wedge^2\mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbb{P}^5 \times \overline{\mathbf{F}}}(-3,0,0) \longrightarrow \mathcal{N}_{\overline{\mathbf{\Gamma}}/Y \times \overline{\mathbf{F}}} \longrightarrow 0 \tag{29}$$

Remark 4.2. Consider the commutative square

$$\begin{array}{ccc} \overline{\mathbf{\Gamma}} & \xrightarrow{\tilde{\kappa}} & \Gamma \\ \downarrow \boldsymbol{p} & & \downarrow p \\ \overline{\mathbf{F}} & \xrightarrow{\kappa} & Z, \end{array} \tag{30}$$

where $\tilde{\kappa}$ is the restriction to $\overline{\mathbf{\Gamma}}$ of the natural projection $\mathbb{P}^5 \times \check{\mathbb{P}}^5 \times G \to \mathbb{P}^5 \times G$. Then (29) is obtained by applying $\tilde{\kappa}^*$ to the triple

$$0 \to \mathcal{N}^\vee_{\Gamma/Y \times Z} \otimes \omega_{\Gamma/Z} \to \wedge^2\mathcal{N}_{\Gamma/\mathbb{P}^5 \times Z}(-3,0) \to \mathcal{N}_{\Gamma/Y \times Z} \to 0. \tag{31}$$

For any point $z \in \overline{\mathbf{F}}$, respectively, $l \in Z$, we have $\wedge^2\mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbb{P}^5\times\overline{\mathbf{F}}}(-3,0,0)|\boldsymbol{p}^{-1}(z) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}$, , respectively, $\wedge^2\mathcal{N}_{\Gamma/\mathbb{P}^5\times Z}(-3,0)|p^{-1}(l) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6}$, so that the base change implies

$$R^i\boldsymbol{p}_*(\wedge^2\mathcal{N}_{\overline{\mathbf{\Gamma}}/\mathbb{P}^5\times\overline{\mathbf{F}}}(-3,0,0)) = 0, \quad R^ip_*(\wedge^2\mathcal{N}_{\Gamma/\mathbb{P}^5\times Z}(-3,0)) = 0, \quad i \geq 0, \tag{32}$$

Applying R^ip_* to (31) and, respectively, $R^i\boldsymbol{p}_*$ to (29), and using (32), relative Serre duality, Remark 4.2 and the isomorphism $p_*\mathcal{N}_{\Gamma/\mathbb{P}^5\times Z} \simeq \mathcal{T}Z$, we obtain the isomorphisms

$$\wedge_Z: \quad \mathcal{T}Z \ \tilde{\to}\ \Omega_Z, \tag{33}$$

and, respectively,

$$\kappa^*\wedge_Z: \quad \kappa^*\mathcal{T}Z \ \tilde{\to}\ \kappa^*\Omega_Z. \tag{34}$$

Since, by [B-D] Z is an irreducible symplectic variety, the Bochner principle for irreducible symplectic manifolds [B] implies that (33) is a symplectic form on Z. Thus we obtain an explicit construction of the symplectic structure on Z, which is stated in the following theorem:

Theorem 4.3. *The symplectic structure on Z is the connecting homomorphism (33) in the long exact sequence of the functors R^ip_*, associated to the exact triple (31).*

Now recall that, by the results of the previous section,

$$\boldsymbol{p}_*\mathcal{N}_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} = \mathcal{T}\mathbf{F}/U, \quad \boldsymbol{p}_*\mathcal{O}_{\mathbf{\Gamma}}(1,1,0) = \mathcal{N}_{\mathbf{F}/U\times Z}, \tag{35}$$

By relative Serre duality,

$$R^1\boldsymbol{p}_*(\mathcal{N}^\vee_{\mathbf{\Gamma}/\mathbf{X}\times_U\mathbf{F}} \otimes \omega_{\mathbf{\Gamma}/\mathbf{F}}) = \Omega_{\mathbf{F}/U}, \quad R^1\boldsymbol{p}_*(\mathcal{O}_{\mathbf{\Gamma}}(1,1,0)^\vee \otimes \omega_{\mathbf{\Gamma}/\mathbf{F}}) = \mathcal{N}^\vee_{\mathbf{F}/U\times Z}. \tag{36}$$

Thus, using (35), (36), (34) and applying $R^i\boldsymbol{p}_*$ to (29), we obtain the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}\mathbf{F}/U & \longrightarrow & \kappa^*\mathcal{T}Z & \longrightarrow & \mathcal{N}_{\mathbf{F}/U\times Z} & \longrightarrow & 0 \\ & & \downarrow & & \kappa^*\wedge_Z\downarrow & & \sigma_0\downarrow & & \\ 0 & \longrightarrow & \mathcal{N}^\vee_{\mathbf{F}/U\times Z} & \longrightarrow & \kappa^*\Omega_Z & \longrightarrow & \Omega_{\mathbf{F}/U} & \longrightarrow & 0 \end{array} \tag{37}$$

By Theorem 4.3, $\wedge_Z$ is a symplectic form; besides, by construction, σ_0 is the isomorphism of Lemma 3.4. Thus we obtain

Theorem 4.4. *The commutative diagram (37) is antiselfdual and the isomorphism σ_0 in this diagram coincides with the canonical isomorphism of Lemma 3.4.*

In particular, the composition $\mathcal{T}\mathbf{F}/U \to \kappa^*\mathcal{T}Z \xrightarrow{\kappa^*\wedge_Z} \kappa^*\Omega_Z \to \Omega_{\mathbf{F}/U}$ in this diagram is the zero map. This gives another proof of the result of Voisin [V] that Fano surfaces of hyperplane sections of Y are the Lagrangian subvarieties of the symplectic structure on Z.

5. Relation between symplectic structures on $\mathfrak{M}_Y$ and $\mathbf{J}_Y$

5.1. **Codifferential of the Abel-Jacobi map.** We recall here the following Clemens-Welters interpretation of the codifferential $d\Phi^\vee$ of the Abel-Jacobi map $\Phi: B \to J(X)$, where B is the base of a certain family of curves on a smooth hyperplane section $X = Y \cap \mathbb{P}^4$ of the cubic Y (see [We, Sect. 2]). Let $[C]$ be a point in B corresponding to a locally complete intersection (l.c.i.) curve $C \subset X$ and let $y = \Phi(C)$. Applying $\wedge^3$ to the exact triple $0 \to \mathcal{N}^\vee_{X/\mathbb{P}^4} \to \Omega^1_{\mathbb{P}^4}|X \to \Omega^1_X \to 0$ and twisting by $\mathcal{N}_{X/\mathbb{P}^4} = \mathcal{O}_X(3)$, we obtain the exact triple

$$0 \to \Omega^2_X \to \Omega^3_{\mathbb{P}^4} \otimes \mathcal{N}_{X/\mathbb{P}^4} \to \mathcal{O}_X(1) \to 0. \tag{38}$$

Passing to the cohomology of this sequence, we obtain the coboundary isomorphism

$$R: \quad H^0(\mathcal{O}_X(1)) \quad \tilde{\to} \quad H^1(\Omega^2_X). \tag{39}$$

Remark that, by construction, R is the restriction to a fiber $\mathbb{P}^4$ of the isomorphism $\mathrm{pr}_{2*}\,\mathcal{O}_{\mathbf{X}}(1,1) = R^1\,\mathrm{pr}_{2*}\,\Omega^2_{\mathbf{X}/U}$ from Lemma 3.2.

Next, the triple (38) restricted to C clearly fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^2_X|C & \longrightarrow & \Omega^3_{\mathbb{P}^4}\otimes\mathcal{N}_{X/\mathbb{P}^4}|C & \longrightarrow & \mathcal{O}_C(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^3_X\otimes N_{C/X} & \longrightarrow & \Omega^3_{\mathbb{P}^4}\otimes N_{C/\mathbb{P}^4} & \longrightarrow & \Omega^3_X\otimes N_{X/\mathbb{P}^4}|C & \longrightarrow & 0
\end{array} \tag{40}$$

Remark that, for a l.c.i. curve $C\subset X$, $\Omega^3_X\otimes N_{C/X} = \omega_X\otimes N_{C/X} = \omega_X\otimes\det N_{C/X}\otimes N^\vee_{C/X} = \omega_C\otimes N^\vee_{C/X}$, so that by Serre duality $H^1(\Omega^3_X\otimes N_{C/X}) = H^0(N_{C/X})^\vee$.

Hence, passing to the cohomology in the bottom triple, we obtain the map

$$\beta_C: \quad H^0(\mathcal{O}_C(1)) \quad\to\quad H^0(N_{C/X})^\vee. \tag{41}$$

According to loc. cit., the codifferential $d\Phi^\vee|_{[C]}$ at the point $[C]$ of the Abel-Jacobi map $\Phi: B \;\to\; J(X)$ makes the following diagram commutative:

$$\begin{tikzcd}
H^0(\mathcal{O}_X(1)) \arrow[rr, "R", "\sim"'] \arrow[d, "\otimes\mathcal{O}_C"] & & H^1(\Omega^2_X) \arrow[d, "\wr"] \\
H^0(\mathcal{O}_C(1)) \arrow[r, "\beta_C"] & H^0(\mathcal{N}_{C/X})^\vee & \Omega^1_{J(X)}|y. \arrow[l, "d\Phi^\vee|\{C\}"']
\end{tikzcd} \tag{42}$$

Now, as in Section 3, let $U = \{\mathbb{P}^4\in\check{\mathbb{P}}^5 | X = Y\cap\mathbb{P}^4 \text{ is smooth}\}$, $h:\mathbf{J}\to U$ and $\pi:\mathbf{F}\to U$ the projections and $\sigma = \pi_*\sigma_0 : \pi_*\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z}|U \;\tilde{\to}\; \pi_*\Omega^1_{\mathbf{F}/\check{\mathbb{P}}^5}|U \simeq h_*\Omega^1_{\mathbf{J}/U}$ the canonical isomorphism (see Lemmas 3.2 and 3.4). Take a point $y\in h^{-1}(U)$ and let $X = \mathbb{P}^4\cap Y$, where $\mathbb{P}^4 = h(y)$. Remark that, by Lemma 3.4, there is a natural identification $H^0(\mathcal{O}_X(1)) \simeq h^*\pi_*\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z}|y$. These isomorphisms together with (42) and the corresponding base change isomorphisms give the commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{O}_X(1)) \arrow[rrr, "\simeq"] \arrow[d, "R"', "\wr"] & & & h^*\pi_*\mathcal{N}_{\mathbf{F}/\check{\mathbb{P}}^5\times Z}|y \arrow[d, "\wr"', "h^*\sigma"] \\
H^1(\Omega^2_X) & \simeq \quad \Omega^1_{\mathbf{J}/U}|y & \simeq & h^*h_*\Omega^1_{\mathbf{J}/U}|y
\end{tikzcd} \tag{43}$$

Here $H^0(\mathcal{O}_X(1)) = h^*\mathcal{T}U|y$ and the isomorphism R globalizes to

$$\mathcal{R}: \quad h^*\mathcal{T}U \quad\tilde{\to}\quad \Omega^1_{\mathbf{J}/U}. \tag{44}$$

Now consider the symplectic structure of Donagi-Markman $\wedge_{\mathbf{J}} : \mathcal{T}\mathbf{J} \;\tilde{\to}\; \Omega^1_{\mathbf{J}}$ induced on $\mathbf{J}$ via the symplectic structure $\wedge_Z$ on Z. Then putting together (43) and the right hand commutative square of (37), we obtain

Theorem 5.2. *There is an antiselfdual commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}\mathbf{J}/U & \longrightarrow & \mathcal{T}\mathbf{J} & \longrightarrow & h^*\mathcal{T}U & \longrightarrow & 0 \\ & & \wr| & & \wedge_{\mathbf{J}}\ \wr| & & \mathcal{R}\ \wr| & & \\ 0 & \longrightarrow & h^*\Omega^1_U & \longrightarrow & \Omega^1_{\mathbf{J}} & \longrightarrow & \Omega^1_{\mathbf{J}/U} & \longrightarrow & 0 \end{array} \tag{45}$$

5.3. **Local lifts of $\wedge_{\mathbf{J}}$ to $\mathfrak{M}$.** Consider the open subset $\mathcal{H}' = \{c \in \mathrm{Hilb}^{5n}_Y | \ C$ is a smooth projectively normal quintic $\}$ of Hilb^{5n}_Y, with its projection $\rho' : \mathcal{H}' \to \check{\mathbb{P}}^5 : C \mapsto < C >$. Let Y be general enough, so that, by [Ma-Ti], $\mathcal{H}'$ is an irreducible smooth open subset of Hilb^{5n}_Y, $\dim \mathcal{H}' = 15$, $\rho(\mathcal{H}')$ is dense open in $\check{\mathbb{P}}^5$ and there exists a dense open subset $U \subset \check{\mathbb{P}}^5$, respectively, a dense open subset $\mathcal{H}$ of $\mathcal{H}'$, $\mathcal{H} \subset {\rho'}^{-1}(U)$, such that any $X \in U$ is a smooth hyperplane section of Y and $\rho = \rho'|\mathcal{H} : \mathcal{H} \to U$ is a smooth morphism of relative dimension 10. For any point $C \in \mathcal{H}$, there exists a local analytic (or local in the étale topology) section of the projection ρ passing through C, i.e. an analytic (resp. étale) neighbourhood $\mathcal{U} \ni \rho(C)$ and a holomorphic (resp. regular) map $s = s_{\mathcal{U}} : \mathcal{U} \to \mathcal{H}$ such that $s(\mathcal{U}) \ni C$. Let $\mathcal{H}_{\mathcal{U}} = \rho^{-1}(\mathcal{U})$, $\mathbf{J}_{\mathcal{U}} = h^{-1}(\mathcal{U})$, $\mathfrak{M}_{\mathcal{U}} = p^{-1}(\mathcal{U})$, where $p : \mathfrak{M} \to \check{\mathbb{P}}^5 : \mathcal{E} \to < \mathrm{Supp}(\mathcal{E}) >$ is the projection, and denote $p_{\mathcal{U}} = p|_{\mathfrak{M}_{\mathcal{U}}}$. The section s defines a relative Abel-Jacobi map $\Phi_s : \mathcal{H}_{\mathcal{U}} \to \mathbf{J}_{\mathcal{U}} : C \mapsto \Phi_{X,s}(C)$, where $\Phi_{X,s} : \rho^{-1}(X) \to J_X : C \mapsto \Phi(C - s(X))$ is the usual Abel-Jacobi map, $J_X = h^{-1}(X)$. By [Ma-Ti, Theorem 5.6] the map $\Phi_{\mathcal{U}}$ has the Stein factorization $\Phi_{\mathcal{U}} = \Psi_{\mathcal{U}} \circ \phi_{\mathcal{U}}$, where $\phi_{\mathcal{U}} : \mathcal{H}_{\mathcal{U}} \to \mathfrak{M}_{\mathcal{U}}$ is a smooth holomorphic map of relative dimension 5 and $\Psi_{\mathcal{U}} : \mathfrak{M}_{\mathcal{U}} \to \mathbf{J}_{\mathcal{U}}$ is an étale holomorphic map.

Next, passing to $\wedge^2$ in the lower triple of (45), we obtain the diagram

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & h^*\Omega^2_{\mathcal{U}} & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \Omega^2_{\mathbf{J}} & \xrightarrow{e_{\mathbf{J}}} & \Omega^2_{\mathbf{J}/\mathcal{U}} & \longrightarrow & 0 \\ & & \varepsilon_{\mathbf{J}} \downarrow & & & & & & \\ & & \mathcal{H}om(h^*\mathcal{T}\mathcal{U}, \Omega^1_{\mathbf{J}_{\mathcal{U}}/\mathcal{U}}) & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{46}$$

Similarly we have on $\mathfrak{M}_{\mathcal{U}}$

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & p_{\mathcal{U}}^{*}\Omega^2_{\mathcal{U}} \arrow[d] & & & \\
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\varepsilon"'] & \Omega^2_{\mathfrak{M}_{\mathcal{U}}} \arrow[r, "e_{\mathfrak{M}}"] & \Omega^2_{\mathfrak{M}_{\mathcal{U}}/\mathcal{U}} \arrow[r] & 0, \\
 & \mathcal{H}om(p_{\mathcal{U}}^{*}\mathcal{T}\mathcal{U}, \Omega^1_{\mathfrak{M}_{\mathcal{U}}}) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
\tag{47}
$$

and since $\Psi_{\mathcal{U}} : \mathfrak{M}_{\mathcal{U}} \to \mathbf{J}_{\mathcal{U}}$ is étale, the diagram (47) can be obtained from (46) by applying $\Psi_{\mathcal{U}}^{*}$:

$$
(47) = \Psi_{\mathcal{U}}^{*}(46). \tag{48}
$$

Now let $\Lambda_{\mathbf{J}} \in H^0(\Omega^2_{\mathbf{J}})$ be the Donagi-Markman symplectic structure on $\mathbf{J}$ and $\Lambda_{\mathbf{J}_{\mathcal{U}}} = \Lambda_{\mathbf{J}}|\mathfrak{M}_{\mathcal{U}}$ its restriction to $\mathbf{J}_{\mathcal{U}}$, and let $\Lambda_{\mathfrak{M}_{\mathcal{U}}} = \Lambda|\mathfrak{M}_{\mathcal{U}}$, where Λ is the Yoneda quasi-symplectic form introduced in Proposition 2.1 above). By Proposition 2.1, for any $[\mathcal{E}] \in \mathfrak{M}_{\mathcal{U}}$ the vector space $T_{[\mathcal{E}]}\mathfrak{M}_{\mathcal{U}}/\mathcal{U} \subset T_{[\mathcal{E}]}\mathfrak{M}_{\mathcal{U}}$ is isotropic w.r.t. $\Lambda_{\mathfrak{M}_{\mathcal{U}}}$, hence

$$
e_{\mathfrak{M}}(\Lambda_{\mathfrak{M}_{\mathcal{U}}}) = 0. \tag{49}
$$

By the same reason

$$
e_{\mathbf{J}}(\Lambda_{\mathbf{J}_{\mathcal{U}}}) = 0. \tag{50}
$$

The conditions (49) and (50) imply that $\Lambda_{\mathfrak{M}_{\mathcal{U}}} \in H^0(\mathcal{G})$, respectively, $\Lambda_{\mathbf{J}_{\mathcal{U}}} \in H^0(\mathcal{F})$. Now prove that

$$
\varepsilon(\Lambda_{\mathfrak{M}_{\mathcal{U}}}) = \Psi_{\mathcal{U}}^{*}\varepsilon_{\mathbf{J}}(\Lambda_{\mathbf{J}_{\mathcal{U}}}), \tag{51}
$$

or equivalently, according to (48), that

$$
\varepsilon(\Lambda_{\mathfrak{M}_{\mathcal{U}}} - \Psi_{\mathcal{U}}^{*}\Lambda_{\mathbf{J}_{\mathcal{U}}}) = 0. \tag{52}
$$

To prove (51), take any $\mathcal{E} \in \mathfrak{M}_{\mathcal{U}}$ and let $y = \Psi_{\mathcal{U}}([\mathcal{E}])$, $X = h(y)$. By virtue of the identifications $T_{[\mathcal{E}]}\mathfrak{M}_{\mathcal{U}}/\mathcal{U} = H^1(\mathcal{E}\otimes\mathcal{E})$, $p_{\mathcal{U}}^{*}\mathcal{T}\mathcal{U}|_{[\mathcal{E}]} = H^0(\mathcal{O}_X(1))$ we have an isomorphism $^{\sharp}\Lambda := \varepsilon(\Lambda_{\mathfrak{M}_{\mathcal{U}}})|_{[\mathcal{E}]} : H^0(\mathcal{O}_X(1)) \xrightarrow{\sim} H^1(\mathcal{E}\otimes\mathcal{E})^{\vee}$. By Theorem 5.2 together with the identifications, $T_y\mathbf{J}_{\mathcal{U}}/\mathcal{U} = H^1(\Omega^2_X)^{\vee}$, $h^{*}\mathcal{T}\mathcal{U}|_y = H^0(\mathcal{O}_X(1))$ we have an isomorphism $R = \varepsilon_{\mathbf{J}}(\Lambda_{\mathbf{J}_{\mathcal{U}}})|_y : H^0(\mathcal{O}_X(1)) \xrightarrow{\sim} H^1(\Omega^2_X)$. Thus, by (42), the equality (51) is reduced to the commutativity of the diagram

$$
\begin{tikzcd}
H^1(\Omega^2_X) \arrow[r, "d\Psi_{\mathcal{U}}^{\vee}"] & H^1(\mathcal{E}\otimes\mathcal{E})^{\vee} \\
H^0(\mathcal{O}_X(1)) \arrow[u, "R", "\wr"'] \arrow[ur, "^{\sharp}\Lambda"'] &
\end{tikzcd}
\quad , \tag{53}
$$

i.e., after dualizing and using(42), to that of

$$\begin{array}{ccc} H^0(\mathcal{N}_{C/X}) & \xrightarrow{d\Phi_{X,s}|_{[C]}} & H^1(\mathcal{E}\otimes\mathcal{E}) \\ {\scriptstyle \beta_C^\vee}\big\downarrow & & \big\downarrow{\scriptstyle ({}^\sharp\Lambda)^\vee} \\ H^0(\mathcal{O}_C(1))^\vee & \xrightarrow{\sim} & H^0(\mathcal{O}_X(1))^\vee \end{array} \quad , \tag{54}$$

where C is any curve in $\phi_{\mathcal{U}}^{-1}([\mathcal{E}])$.

By Lemma 2.2 and (11), the last diagram is equivalent to

$$\begin{array}{ccc} H^0(\mathcal{N}_{C/X})\otimes H^0(\mathcal{O}_C(1)) & \xrightarrow{m} & H^0(\mathcal{N}_{C/X}(1)) \\ {\scriptstyle d\Phi_{X,s}|_{[C]}\otimes(\mathrm{res}^{-1})}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\delta}} \\ H^1(\mathcal{E}\otimes\mathcal{E})\otimes H^0(\mathcal{O}_X 1)) & \xrightarrow{m} & H^1(\mathcal{E}\otimes\mathcal{E}(1)), \end{array} \tag{55}$$

where m is the ordinary multiplication, $\tilde{\delta}$ the composition $\tilde{\delta}:\ H^0(\mathcal{N}_{C/X}(1)) \overset{\delta}{\twoheadrightarrow} H^1(\mathcal{I}^2_{C,X}(3)) \underset{\sim}{\xleftarrow{\gamma_1}} H^1(\mathcal{I}_{C,X}(2)\otimes\mathcal{E}) \underset{\sim}{\xleftarrow{\gamma_2}} H^1(\mathcal{E}\otimes\mathcal{E}(1))$, δ the connecting map in the cohomology of the exact triple $0\to\mathcal{I}^2_{C,X}(3))\to\mathcal{I}_{C,X}(3))\to\mathcal{N}_{C/X}(1)\to 0$, and the isomorphisms $\gamma_1,\ \gamma_2$ are obtained from the triples $0\to\mathcal{O}_X(1)\to\mathcal{I}_{C,X}(2)\otimes\mathcal{E}\to\mathcal{I}^2_{C,X}(3)\to 0,\ \ 0\to\mathcal{E}\to\mathcal{E}\otimes\mathcal{E}(1)\to\mathcal{I}_{C,X}(2)\otimes\mathcal{E}\to 0$ in using the equalities $h^i(\mathcal{O}_X(1))=0,\ \ h^i(\mathcal{E})=0,\ i=1,2$. The commutativity of (55) is obvious. Hence, (51) follows.

Denote $\Lambda_{\mathcal{U},s}=\Lambda_{\mathfrak{M}_{\mathcal{U}}}-\Psi_{\mathcal{U}}^*\Lambda_{\mathbf{J}_{\mathcal{U}}}$. ¿From (52) and (46), it follows that $\Lambda_{\mathcal{U},s}\in H^0(p_{\mathcal{U}}^*\Omega^2_{\mathcal{U}})$. [3] Thus, we obtain the following theorem relating the quasi-symplectic structures $\Lambda_{\mathfrak{M}_{\mathcal{U}}}$ and $\Lambda_{\mathbf{J}_{\mathcal{U}}}$.

Theorem 5.4. *For the data $(\mathcal{U},s)$ specified above, there exists a 2-form $\Lambda_{\mathcal{U},s}\in H^0(p_{\mathcal{U}}^*\Omega^2_{\mathcal{U}})$ such that*

$$\Lambda_{\mathfrak{M}_{\mathcal{U}}}=\Psi_{\mathcal{U}}^*\Lambda_{\mathbf{J}_{\mathcal{U}}}+\Lambda_{\mathcal{U},s} \tag{56}$$

5.5. **Symplectic structure on $\mathfrak{M}$.** Only at this point we will use the result of [Il-Ma]:

Theorem 5.6. *For any nonsingular cubic threefold X, the map $\Psi: M_X\longrightarrow J^2(X)$ is an open embedding.*

Proof. See Theorem 3.2 and Corollaries 3.3, 5.1 in [Il-Ma]. □

This theorem implies that the above maps $\Psi_{\mathcal{U}}:\mathfrak{M}_{\mathcal{U}}\longrightarrow\mathbf{J}_{\mathcal{U}}$ defined by local sections $s_{\mathcal{U}}:\mathcal{U}\longrightarrow\mathcal{H}$ of $\rho:\mathcal{H}\longrightarrow U$ are also open embeddings. Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open covering of U in the classiacal or étale topology such that there exist local sections $s_\alpha:\mathcal{U}_\alpha\longrightarrow\mathcal{H}$ of ρ, and let $\sigma_\alpha:\mathcal{U}_\alpha\longrightarrow\mathfrak{M}_{\mathcal{U}}$ be the corresponding sections of Ψ. Then $c=(\sigma_{\alpha\beta})_{\alpha,\beta\in I}$, where $\sigma_{\alpha\beta}=\sigma_\beta-\sigma_\alpha$, is a 1-cocycle of the covering $\{\mathcal{U}_\alpha\}$ with values in the sheaf $\underline{\mathbf{J}}$ of holomorphic (or regular) sections of $\mathbf{J}$, and $\mathfrak{M}$ is naturally identified with an open subset in the torsor $\mathbf{J}^c$ constructed from c. The following lemma summarizes some easy facts about the relations between structures of Lagrangian fibrations on an abelian scheme and on its torsors.

Lemma 5.7. *Let B be a nonsingular variety of dimension n and $f: A\longrightarrow B$ a smooth family of connected commutative algebraic (or complex Lie) groups such that the generic fiber A_b is an abelian variety (or a compact complex torus) of dimension n. Let c be any cohomology class from $H^1(B,\underline{A})$ in the étale (or classical) topology, and $g: X=A^c\longrightarrow B$ the torsor constructed from c.*

[3] Clearly, the form $\Lambda_{\mathcal{U},s}$ depends on the choice of the section $s:\mathcal{U}\to\mathcal{H}$.

Then the following assertions hold:

(i) Assume that X possesses a quasi-symplectic structure Λ_X such that g is a Lagrangian fibration with respect to Λ_X. Then A possesses also a quasi-symplectic structure Λ_A for which f is Lagrangian, and for any local section $s : V \longrightarrow X$ of g inducing the isomorphism $i_s : X_V = g^{-1}(V) \xrightarrow{\sim} A_V = f^{-1}(V)$, there exists a 2-form $\omega_V \in H^0(V, \Omega^2)$ such that $\Lambda_X - i_s^* \Lambda_A = g^* \omega_V$ on V. Say that such quasi-symplectic structures Λ_X, Λ_A are associated to each other.

(ii) If Λ_X, Λ_A are associated quasi-symplectic structures, then any pair $\Lambda_{1,X} \in \Lambda_X + g^* H^0(B, \Omega^2)$, $\Lambda_{1,A} \in \Lambda_A + f^* H^0(B, \Omega^2)$ is also a pair of associated quasi-symplectic structures. Moreover, there always exists such a $\Lambda_{1,A}$ which is symplectic.

In particular, if $H^0(B, \Omega^2) = 0$ and if there exists a pair Λ_X, Λ_A of associated quasi-symplectic structures, then it is unique and Λ_A is in fact symplectic.

Λ_A associated to a given Λ_X is also unique and symplectique if we impose the additional condition that the restriction of Λ_A to the section of neutral elements of the fibers of f is identically 0.

(iii) Conversely, assume that A possesses a quasi-symplectic structure Λ_A such that f is a Lagrangian fibration with respect to Λ_A. Assume also that at least one of the following two conditions is verified: $H^1(B, \Omega^2) = 0$, or c is of finite order. Then X has a quasi-symplectic structure Λ_X associated to Λ_A. In the case when c is of finite order, Λ_X can be chosen symplectic.

(iv) In the algebraic category, $H^1_{ét}(B, \underline{A})$ is torsion, and any torsor with a quasi-symplectic Lagrangian structure has also a symplectic one.

Proof. For (i), (ii) see Propositions 2.3, 2.6 in [Ma]. In fact, the assumptions of these propositions are formulated there for symplectic structures, but the proofs remain valid also for quasi-symplectic ones. Proposition 2.6 in loc. cit. treats the case of part (iii) when $H^1(B, \Omega^2) = 0$. We need to introduce some settings of this proof to explain the case when c is of finite order.

For any $b \in B$ and for a sufficiently small neighborhood V of b, there exist local parameters $u_1, \dots, u_n$ on V and a local section s_V of $g_V : X_V \longrightarrow V$ identifying X_V with A_V in such a way that s_V becomes the section e of neutral elements of the fibers of f. One can realize A_V as the quotient $\Omega^1_V / \mathcal{L}$ of the cotangent bundle of V by a local system $\mathcal{L}$ generically of rank $2n$. Let $z_1, \dots, z_n$ be the coordinates on Ω^1_B, dual to $du_1, \dots, du_n$. Then the restriction of Λ_A to A_V can be written in the form

$$\Lambda_X = \sum_p du_p \wedge dz_p + \beta, \quad \beta = \sum_{p,q} \beta_{pq}(u) du_p \wedge du_q. \tag{57}$$

Here β does not depend on z, as was proved in loc. cit. Let $\Lambda_{0,A} = \sum_p du_p \wedge dz_p = \Lambda_X - \beta$. The invariance of $\Lambda_{0,A}$ under the translations by sections from $\mathcal{L}$ is equivalent to

$$d(\sum_p \gamma_p du_p) = 0 \ , \tag{58}$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ is any local section of $\mathcal{L}$. Hence we have also the invariance under translations by sections from $\mathbb{Q} \otimes \mathcal{L}$, and the finite order case follows.

The part (iv) is obtained by the successive application of (i), (iii). ☐

Now we can prove

Theorem 5.8. *There exists a symplectic structure $\tilde{\Lambda}_{\mathfrak{M}}$ on $\mathfrak{M}$ such that the following two conditions are verified:*

(a) $\tilde{\Lambda}_{\mathfrak{M}} - \Lambda_{\mathfrak{M}} \in H^0(U, p^* \Omega^2_U)$, hence p is Lagrangian w. r. t. $\tilde{\Lambda}_{\mathfrak{M}}$, and $\tilde{\Lambda}_{\mathfrak{M}}$ also satisfies the conclusion of Theorem 5.4.

(b) $\tilde{\Lambda}_{\mathfrak{M}}$ extends to a symplectic structure $\Lambda_{\mathbf{J}^c}$ associated to $\Lambda_{\mathbf{J}}$.

Proof. We are in the algebraic situation, so we can apply (iii) to construct a symplectic structure $\Lambda_{\mathbf{J}^c}$ associated to the Donagi–Markman form $\Lambda_{\mathbf{J}}$. Define $\tilde{\Lambda}_{\mathfrak{M}} = \Lambda_{\mathbf{J}^c}|_{\mathfrak{M}}$, so that (b) is verified. The definition of the relation "associated" in (i) of Lemma 5.7 and Theorem 5.4 imply also (a). $\square$

Remark 5.9. We conjecture that $\wedge_{\mathfrak{M}}$ is symplectic itself.

Remark 5.10. The étale cohomology class c defining $\mathbf{J}^c$ is of order 3, because one can send the curve $3C$ into $J^2(X)$, where $C \subset X = Y \cap H$ is a normal elliptic quintic, in a canonical way in using five times the plane section of Y as the reference curve for the definition of the Abel–Jacobi map. So, besides the trivial torsor $\mathbf{J}$ and $\mathbf{J}^c$, there exists one more interesting torsor $\mathbf{J}^{c^2}$. What geometric interpretation can one give to this torsor?

Remark 5.11. One can extend our construction to a bigger open set $\overline{U}$, such that $\operatorname{codim} \check{\mathbb{P}}^5 \backslash \overline{U} \geq 2$. We will have the same Theorems 5.4, 5.8 for the (quasi)-symplectic structures $\Lambda_{\overline{\mathbf{J}}}$, $\Lambda_{\overline{\mathbf{J}^c}}$, $\Lambda_{\overline{\mathfrak{M}}}$. As $H^0(\overline{U}, \Omega^2) = 0$, the associated quasi-symplectic structures on the extended families $\overline{h} : \overline{\mathbf{J}} \longrightarrow \overline{U}$, $\overline{h}^c : \overline{\mathbf{J}}^c \longrightarrow \overline{U}$ are unique and symplectic.

Proof. Take for $\overline{U}$ the locus of hyperplanes H such that $H \cap Y$ has at most one nondegenerate double point as singularity. Any cubic X with only one isolated singular point contains a normal elliptic quintic curve which does not pass through the singular point. To see this, take a hyperplane section of X which is a nonsingular cubic surface $S \subset \mathbb{P}^3$. It contains a 5-dimensional linear system of quintics of arithmetic genus 1, and its generic member is a non-singular space elliptic curve C. By Lemma 2.7, c) in [Ma-Ti], whose proof uses only the nonsingularity of X at points of S, we have $h^0(\mathcal{N}_{C/X}) = 10, h^1(\mathcal{N}_{C/X}) = 0$. Hence the Hilbert scheme of X is smooth of dimension 10 at $[C]$. The family of quintics of genus 1 in hypeplane sections of X being 9-dimensional, we see that C is deformable to a non-space curve, which is the wanted normal elliptic quintic.

Define now M_X to be the subset in the moduli space of sheaves on X consisting of the vector bundles of rank 2 obtained by Serre's construction from all the normal elliptic quintics in X that do not pass through the singular point of X, and $\overline{\mathfrak{M}}$ the union of the M_X over all $X = Y \cap H$ with $H \in \overline{U}$. All our proofs concerning the dimension, the nonsingularity of M_X, $\mathfrak{M}$, as well as the definition and the proof of the nondegeneracy of $\Lambda_{\mathfrak{M}}$ work without any modification with $\overline{\mathfrak{M}}$ in place of $\mathfrak{M}$ for all $H \in \overline{U}$. Thus we obtain the Yoneda quasi-symplectic structure $\Lambda_{\overline{\mathfrak{M}}}$ on $\overline{\mathfrak{M}}$.

Donagi–Markman [D-M, Example 8.22] remark that their symplectic structure extends to a 2-form $\overline{\Lambda_{\mathbf{J}}}$ on the generalized relative intermediate Jacobian $\overline{h} : \overline{\mathbf{J}} \longrightarrow \overline{U}$. They write that it is not difficult to see that it is nondegenerate over the boundary. This also follows easily from our setting.

Indeed, the nondegeneracy of $\Lambda_{\overline{\mathfrak{M}}}$ implies that the vertical tangent bundle of $\overline{\mathfrak{M}}$ is isomorphic to $\Omega^1_{\overline{U}}$, hence its associated group scheme $\overline{\mathbf{J}}$ is isomorphic to $\Omega^1_{\overline{U}}/\mathcal{L}$, as in the proof of Lemma 5.7. The standard symplectic structure of the cotangent bundle of $\overline{U}$ descends to the quotient by $\mathcal{L}$ over U, hence also over $\overline{U}$, as the descent condition (58) extends by continuity. Thus we obtain a symplectic structure $\Lambda_{\overline{\mathbf{J}}}$ on $\overline{\mathbf{J}}$. As the restrictions of $\Lambda_{\overline{\mathbf{J}}}$, $\overline{\Lambda_{\mathbf{J}}}$ to U are associated to the same quasi-symplectic structure on $\mathbf{J}^c$, they differ by a lift of a 2-form on U. Hence they define the same isomorphism between $\Omega^1_{\overline{U}}$ and $\mathcal{T}_{\overline{\mathbf{J}}/\overline{U}}$ over U, which is the identity one for $\Lambda_{\overline{\mathbf{J}}}$. By continuity, this holds also over $\overline{U}$. Hence $\overline{\Lambda_{\mathbf{J}}}$ is nondegenerate. As $H^0(\overline{U}, \Omega^2) = 0$, we have the uniqueness over $\overline{U}$, hence $\overline{\Lambda_{\mathbf{J}}} = \Lambda_{\overline{\mathbf{J}}}$.

The extension of Theorem 5.4 follows trivially from the following observation: if the regular section $\Lambda_{\overline{\mathfrak{M}}} - \Psi^*_{\mathcal{U}}\Lambda_{\mathbf{J}}$ of the vector bundle $\Omega^2_{\overline{\mathfrak{M}}}$ over $\mathcal{U}$ belongs to the vector subbundle $\overline{p}^*\Omega^2_{\overline{U}}$ over $U \cap \mathcal{U}$, then it does over the whole of $\mathcal{U}$. $\square$

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