Analytic Torsion Forms and Torsion-Signature Invariants

Donggeng Gong*

Abstract

We define a torsion-signature invariant on the total space of a compact fiber bundle. The analytic torsion form of the fiber bundle is used to find an adiabatic limit formula of the torion-signature.

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1 Introduction

The analytic torsion form of Bismut-Lott[BL] arises naturally from a transgression formula of the smooth version of the Riemann-Roch-Grothendieck theorem. This torsion form is further extended to certain non-compact fiber bundles in [GR]. It is natural to study the contribution of the analytic torsion form of degree ≥ 2 to adiabatic limits of some invariants on the total space of the fiber bundle. Such a nontrivial contribution does not exist for the analytic torsion invariant simply because of the top dimensionality of the Euler form on the base space of the fiber bundle. Hence, we need to consider the adiabatic limits of other invariants.

The purpose of this paper is to introduce a new invariant, the torsion-signature on the total space, and to prove an adiabatic limit formula of this invariant, which involves both

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the analytic torsion form and a modified torsion form. This torsion-signature invariant is motivated by the following question: for a compact fiber bundle $Z \to M \stackrel{\pi}{\to} B$ and its analytic torsion form $\mathcal{T}(T^HM, g^Z, g^E)$ associated with a flat complex vector bundle E over M, what homotopy invariances does the pairing $< \mathcal{L}(B)\mathcal{T}(T^HM, g^Z, g^E), [B] >$ have, provided $\mathcal{T}(T^HM, g^Z, g^E)$ is closed? We refer to Section 2 for the detail of this question. We are led to consider an operator on the total space M which combines the signature operator on the base space B and the Euler-de Rham operator on the fiber. Instead of explicitly producing such an operator, we apply a new mixed super trace to the Euler-de Rham operator on M and get our invariant. The idea is based on the fact that applying two super traces, which correspond to the two gradings given by the degree of forms and by the Hodge star operator on a closed Riemannian manifold, to the Euler-de Rham operator, we obtain the Euler characteristic number and the signature of the manifold (cf. [BGV], Chapter 4), respectively. The mixed super trace is just the tensor product of these two super traces on the exterior cotangent bundle $\wedge(T^*M) \simeq \pi^*(\wedge(T^*B))\hat{\otimes} \wedge (T^*Z)$. The advantage of this trace approach is that we can directly use many existing formulas. For instance, we can quickly get a local mixed index formula. Because of these new features, we have rised more questions than what we have solved in this paper.

This paper is arranged as follows. In Section 2 we recall the analytic torsion form of Bismut-Lott. Section 3 is devoted to a modified torsion form that we need in the adiabatic limit formula. We introduce in Section 4 the mixed super trace and prove a local mixed index formula. We define the torsion-signature invariant in Section 5. This invariant reduces to the analytic torsion invariant when $B = \{pt\}$. Finally, we use in Section 6 the analytic and modified torsion forms to prove an adiabatic limit formula of the torsion-signature invariant.

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2 Analytic Torsion Forms

Let B and Z be closed Riemannian manifolds and $Z \to M \stackrel{\pi}{\to} B$ a smooth compact fiber bundle over B with the fiber Z. Suppose E is a flat complex vector bundle over M with a flat connection ∇^E and a Hermitian metric g^E. Denote by TZ the vertical tangent bundle of M consisting of tangent vectors of M to the fibers. We choose a connection on the fiber bundle M. Then $TM = TZ \oplus T^H M$, where $T^H M$ is the bundle of horizontal vectors, $T^H M \simeq \pi^*(TB)$. From this we can write the exterior cotangent bundle as $\wedge^*(T^*M) \simeq \pi^*(\wedge^*(T^*B))\hat{\otimes} \wedge^* (T^*Z)$.

Let $\mathcal{E}$ be the infinite dimensional $\mathbb{Z}$-graded vector bundle over B whose fiber over $x \in B$ is isomorphic to the space of $E|_{M_x}$-valued forms $\Omega(M_x, E|_{M_x})$ on M_x, where $M_x = \pi^{-1}(x) \simeq Z$. There is a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ given by

$$\nabla^{\mathcal{E}}_U \phi = L_{U^H}\phi,$$

where $\phi \in C^\infty(B, \mathcal{E})$, $U^H \in C^\infty(M, T^H M)$ is the horizontal lift of a vector field U on B, $\pi_*(U^H) = U$, and L_{U^H} is the Lie differentiation on $C^\infty(B, \mathcal{E})$. We have $\Omega(M, E) \simeq \Omega(B, \mathcal{E})$. Let T be the curvature of the fiber bundle M given by

$$T(U_1, U_2) = -P^Z[U_1^H, U_2^H] \in C^\infty(M, TZ),$$

where $P^Z : TM = TZ \oplus T^H M \to TZ$ is the projection. T is a TZ-valued horizontal 2-form on M. Denote by i_T the interior multiplication by T. Then the exterior differential d_M on M defines a flat super connection on $\mathcal{E}$ of total degree 1,

$$d_M = d_Z + \nabla^{\mathcal{E}} + i_T. \tag{1}$$

The terms of (1) are degree-0, 1, and 2 parts of d_M, respectively. Let N be the number operator of $\mathcal{E}$ given by multiplication by j on $C^\infty(M, \wedge^j(TZ) \otimes E)$. Let g^B and g^Z be the Riemannian metrics on B and Z, respectively. Using these two metrics, we can define an L^2-inner product $g^{\mathcal{E}}$ on $\mathcal{E}$, and hence an adjoint d_M^* of d_M as a super connection on $\mathcal{E}$. For $t > 0$, let

$$\begin{aligned} D_t &= \frac{1}{2}(t^{-N/2} d_M^* t^{N/2} - t^{N/2} d_M t^{-N/2}) \\ &= \frac{\sqrt{t}}{2}(d_Z^* - d_Z) + \frac{1}{2} w(\mathcal{E}, g^{\mathcal{E}}) - \frac{1}{2\sqrt{t}}(T \wedge + i_T), \end{aligned} \tag{2}$$

where $w(\mathcal{E}, g^{\mathcal{E}}) = (\nabla^{\mathcal{E}})^* - \nabla^{\mathcal{E}} = (g^{\mathcal{E}})^{-1}(\nabla^{\mathcal{E}} g^{\mathcal{E}})$. Take an odd function $f(z) = z\exp(z^2)$, $z \in \mathbb{C}$, and define $\varphi(\phi) = (2\pi i)^{-\frac{k}{2}}\phi$ for $\phi \in \Omega^k(B)$. We can use the usual super trace Tr^s on $\mathcal{E}$ to define real odd and even differential forms on B, respectively,

$$f(D_t, g^{\mathcal{E}}) = (2\pi i)^{\frac{1}{2}} \varphi Tr^s(f(D_t)), \tag{3}$$

and

$$f'(D_t, g^{\mathcal{E}}) = \varphi Tr^s(\frac{N}{2} f'(D_t)). \tag{4}$$

Let $H(Z, E|_Z)$ be the $\mathbb{Z}$-graded complex vector bundle over B whose fiber over $x \in B$ is isomorphic to the cohomology $H^*(M_x, E|_{M_x})$. By Hodge theory, the metric g^Z on Z induces a metric $g^{H(Z,E|_Z)}$ on $H(Z, E|_Z)$. Furthermore, d_M induces a canonical flat connection $\nabla^{H(Z,E|_Z)}$ on $H(Z, E|_Z)$. Let $o(TZ)$ be the orientation line bundle of TZ, and $e(TZ)$ be the Euler class of Z represented by an $o(TZ)$-valued closed form $e(TZ, \nabla^Z)$ on M,

$$e(TZ, \nabla^Z) = \begin{cases} Pf(\frac{R^Z}{2\pi}) \equiv T(\exp \frac{-R^Z}{2\pi}), & \dim(Z) \ \ even \\ 0, & \dim(Z) \ \ odd, \end{cases}$$

where $\nabla^Z = P^Z \nabla^M$ with ∇^M the Levi-Civita connection on TM associated with metric $g^M = \pi^*(g^B) \oplus g^Z$, T is the Berezin integral from $\wedge(T^*Z)$ to $\mathbb{R}$ [BGV], and R^Z is the curvature of ∇^Z. Define

$$f(\nabla^E, g^E) = (2\pi i)^{\frac{1}{2}} \varphi Tr^s(f(\frac{1}{2}((\nabla^E)^* - \nabla^E))),$$

where $w(E, g^E) = (\nabla^E)^* - \nabla^E = (g^E)^{-1}(\nabla^E g^E)$. Similarly, we can define $f(\nabla^{H(Z,E|_Z)}, g^{H(Z,E|_Z)})$. Set

$$\chi'(Z, E) = \sum_{j=0}^{\dim(Z)} (-1)^j j \dim(H^j(Z, E|_Z)).$$

Let $\chi(Z)$ and $rk(E)$ be the Euler characteristic number of Z and the rank of E, respectively. The properties of the two forms in $(2) - (3)$ are summarized in the following lemma [BL].

Lemma 1 *1)* $\frac{\partial}{\partial t} f(D_t, g^{\mathcal{E}}) = \frac{1}{t} df'(D_t, g^{\mathcal{E}}), \ \ t > 0.$

2) As $t \to 0$,

$$f(D_t, g^{\mathcal{E}}) = \begin{cases} \int_Z e(TZ, \nabla^Z) f(\nabla^E, g^E) + O(t), & if \ \ \dim(Z) \ \ even, \\ O(\sqrt{t}), & if \ \ \dim(Z) \ \ odd, \end{cases}$$

$$f'(D_t, g^{\mathcal{E}}) = \begin{cases} \frac{1}{4} \dim(Z) rk(E) \chi(Z) + O(t), & if \ \ \dim(Z) \ \ even, \\ O(\sqrt{t}), & if \ \ \dim(Z) \ \ odd. \end{cases}$$

3) As $t \to \infty$,

$$
\begin{aligned}
f(D_t, g^{\mathcal{E}}) &= f(\nabla^{H(Z,E|_Z)}, g^{H(Z,E|_Z)}) + O(\frac{1}{\sqrt{t}}), \\
f'(D_t, g^{\mathcal{E}}) &= \frac{1}{2}\chi'(Z,E) + O(\frac{1}{\sqrt{t}}).
\end{aligned}
$$

This lemma enables us to define an analytic torsion form $\mathcal{T}(T^H M, g^Z, g^E)$ of the fiber bundle M as

$$
\begin{aligned}
\mathcal{T}(T^H M, g^Z, g^E) &= \int_0^\infty [f'(D_t, g^{\mathcal{E}}) - \frac{1}{2}\chi'(Z,E) - (\frac{1}{4}\dim(Z) rk(E)\chi(Z) - \\
&- \frac{1}{2}\chi'(Z,E)) f'(\frac{i\sqrt{t}}{2})] \frac{dt}{t}.
\end{aligned} \tag{5}
$$

Note that this torsion form differs from that of [BZ] by a negative sign. Since the closed form $f(D_t, g^{\mathcal{E}})$ is independent of t, we get from Lemma 1 the following transgression formula [BL].

Lemma 2

$$
-d\mathcal{T}(T^H M, g^Z, g^E) = \int_Z e(TZ, \nabla^Z) f(\nabla^E, g^E) - f(\nabla^{H(Z,E|_Z)}, g^{H(Z,E|_Z)}). \tag{6}
$$

The Riemann-Roch-Grothendieck theorem asserts that the two closed forms on the right hand side of (6) represent the same cohomology class. Lemma 2 gives explicitly the difference of these two forms. Formula (6) is also important in determining when the analytic torsion form is a closed even form. Indeed, we get by (6) the first part of the following lemma [BL].

Lemma 3 1) *If $\dim(Z)$ is odd and $H(Z, E|_Z) = 0$, then the analytic torsion form is closed.*

2) *If g^E is covariantly constant with respect to ∇^E and $\dim(Z)$ is even, then $\mathcal{T}(T^H M, g^Z, g^E) = 0$.*

The property that $\mathcal{T}(T^H M, g^Z, g^E)$ is closed in certain cases is important in the application. So we want to discuss another case where the analytic torsion form is closed. To this aim, let $\{e_i\}$ and $\{f_\alpha\}$ be local orthonormal bases of the vertical tangent bundle TZ and TB, respectively, with the corresponding dual bases $\{e^i\}$ and $\{f^\alpha\}$. We also use the same symbols $\{f_\alpha\}$ and $\{f^\alpha\}$ to denote the liftings of $\{f_\alpha\}$ and $\{f^\alpha\}$ to $\pi^*(TB)$ and $\pi^*(T^*B)$, respectively. Denote by m^i (resp. m^α) the exterior multiplication by e^i (resp. f^α). Let

$c(X) = (X\wedge) - i_X$ and $c_v(X) = (X\wedge) + i_X$ for X a tangent vector or 1-form. In particular, let $c^i = c(e_i), c^\alpha = c(f_\alpha), c_v^i = c_v(e_i), c_v^\alpha = c_v(f_\alpha)$. Then

$$D_t = \frac{\sqrt{t}}{2}(-c_v^j \nabla_{e_j}^{Z\otimes E} + \frac{1}{2} i_{e_j}\psi_j) + \frac{1}{2} w(\mathcal{E}, g^{\mathcal{E}}) - \frac{1}{2\sqrt{t}} c_v(T), \tag{7}$$

where $\psi_j = (g^E)^{-1}(\nabla_{e_j}^E g^E), \psi_\alpha = (g^E)^{-1}(\nabla_{f_\alpha}^E g^E)$, and $w(\mathcal{E}, g^{\mathcal{E}}) = m^\alpha(w_{\alpha jk} c^j c_v^k + \psi_\alpha)$ with $w_{\alpha jk}$ given by

$$w_{abc} = X^a(\nabla_{X_c} X_b)$$

for $X_a = e_j$ or f_α.

Let $\Gamma = \pi_1(B)$ be the fundamental group of B, $\rho_1 : \Gamma \to Diff(Z)$ be a homomorphism, and $\tilde{B}$ be the universal classifying space of B, where $Diff(Z)$ is the group of diffeomorphisms on Z. Let $M = \tilde{B} \times_\Gamma Z \xrightarrow{\pi} B$ be a fiber bundle over B with the fiber Z. Suppose $\rho_2 : \Gamma \to GL(\mathbb{C}^k)$ is a representation of Γ on $\mathbb{C}^k$. Then $E = \tilde{B} \times_\Gamma (Z \times \mathbb{C}^k)$ is a flat vector bundle over M and its restriction to each fiber is a trivial bundle $\tilde{E} = Z \times \mathbb{C}^k$ over Z. Let g^E be a Hermitian metric on E such that g^E is locally independent of Z. One can construct such a metric as follows. Choose a finite open covering $\{U_i\}_{i=1}^l$ of B together with smooth cross-sections $\beta_i : U_i \to \tilde{B}$ for the projection $\tilde{B} \xrightarrow{\pi} B$. For $x \in \beta_i(U_i)$ we define a Hermitian metric g_i on $\mathbb{C}_x^k = \mathbb{C}^k$. The $g_i's$ may be different for $i \neq j$. Then for $\nu x \in \nu\beta_i(U_i), \nu \in \Gamma$, we define a metric $\rho_2(\nu^{-1}) g_i$ on $\mathbb{C}_{\nu x}^k$. We thus have a Γ-invariant Hermitian metric $\tilde{g}_i$ on $\Gamma\beta_i(U_i) \times \mathbb{C}^k$. Let $\{\sigma_i\}$ be a partition of unity subordinate to $\{U_i\}$ and $\{\tilde{\sigma}_i\}$ be the pull back of $\{\sigma_i\}$ to $\tilde{B}$. Then $\tilde{g} = \sum_i \tilde{\sigma}_i \tilde{g}_i$ is a Γ-invariant metric on the trivial vector bundle $\tilde{B} \times \mathbb{C}^k$. We consider $\tilde{g}$ as a Γ-invariant metric on the vector bundle $\tilde{B} \times Z \times \mathbb{C}^k$ which induces a required metric g on E. Let the flat connection ∇^E on E be locally given by the usual exterior differentiation. Since the fiber bundle M is locally a product, the curvature tensor T of M is zero. The operator D_t in (7) is now given by

$$D_t = \frac{\sqrt{t}}{2}(-c_v^j \nabla_{e_j}^{Z\otimes E}) + \frac{1}{2} w(\mathcal{E}, g^{\mathcal{E}}). \tag{8}$$

Since $\psi_j = (g^E)^{-1}(\nabla_{e_j}^E g^E) = 0$, $w(E, g^E) = m^\alpha \psi_\alpha$ and $w(\mathcal{E}, g^{\mathcal{E}}) = w(E, g^E) + m^\alpha w_{\alpha jk} \tilde{c}^j \tilde{c}_v^k$.

Lemma 4 *If $w_{\alpha jk} = 0$ and g^E is independent of Z, then the analytic torsion form $\mathcal{T}(T^H M, g^Z, g^E)$ is closed on B.*

Proof. By (6), it suffices to prove that

$$\int_Z e(TZ,\nabla^Z)f(\nabla^E,g^E)-f(\nabla^{H(Z,E|z)},g^{H(Z,E|z)})=0.$$

Recall that $f(\nabla^E,g^E)=(2\pi i)^{\frac{1}{2}}\varphi Tr^s(f(\frac{1}{2}w(E,g^E)))$ and

$$\begin{aligned} f(\nabla^{H(Z,E|z)},g^{H(Z,E|z)}) &= (2\pi i)^{\frac{1}{2}}\varphi Tr^s(f(\frac{1}{2}w(H,g^H))) \\ &= (2\pi i)^{\frac{1}{2}}\varphi Tr^s(f(\frac{1}{2}P_Z w(E,g^E)P_Z)), \end{aligned}$$

where P_Z is the orthogonal projection of $\mathcal{E}$ onto the kernel of $(d_Z^*-d_Z)^2$. Since g^E is independent of Z, $w(E,g^E)=m^\alpha(g^E)^{-1}(\nabla^E_{e_\alpha}g^E)$ is a form on B which is independent of Z. Note that $E|_Z$ is trivial. We get

$$(2\pi i)^{\frac{1}{2}}\varphi Tr^s(f(\frac{1}{2}P_Z w(E,g^E)P_Z))=\chi(Z)(2\pi i)^{\frac{1}{2}}\varphi tr(f(\frac{1}{2}w(E,g^E))),$$

and

$$\int_Z e(TZ,\nabla^Z)f(\nabla^E,g^E)=\chi(Z)(2\pi i)^{\frac{1}{2}}\varphi tr(f(\frac{1}{2}w(E,g^E))).$$

These two identities prove the result. Q.E.D.

The condition that $w_{\alpha jk}=0$ can be satisfied in the following case. let $q:\tilde{B}\times Z\to M=\tilde{B}\times_\Gamma Z$ be the natural projection. We have that $q_*(T\tilde{B}\oplus T(Z))=TM$. Here $T(Z)$ denotes the tangent bundle of Z in order to distinguish the vertical bundle TZ. In particular, the vertical bundle TZ is equal to $q_*(T(Z))$. Assume that there is a Γ-equivariant Riemannian metric on $T(Z)$. Let the Riemannian metric g^Z on TZ be the image under q_* of the Γ-invariant metric on $T(Z)$. Thus g^Z is locally independent of B. Let $T^HM=q_*(T\tilde{B})$. It follows that a horizontal vector $\tilde{U}\in T^HM$ is locally independent of fibers. This implies that $L_{\tilde{U}}g^Z=0$. Hence $w_{\alpha jk}=<S(\tilde{e}_j)\tilde{e}_k,f_\alpha>=\frac{1}{2}(L_{\tilde{U}}g^Z)(e_j,e_k)=0$ (cf. [BC]). Thus $w(\mathcal{E},g^{\mathcal{E}})=m^\alpha\psi_\alpha=w(E,g^E)$.

We now return to the general case. Let $Z\to M\xrightarrow{\pi}B$ be a fiber bundle over B with a flat vector bundle E. When the analytic torsion form is closed, we define

$$\mathcal{T}s(B,M,E)=<L(B)\mathcal{T}(T^HM,g^Z,g^E),[B]>, \qquad (9)$$

with the Hirzebruch L-class $L(B)$ of B and the fundamental class $[B]$ of B.

$\mathcal{T}s(B,M,E)$ is a secondary invariant in the sense that the analytic torsion form appears in the Riemann-Roch-Grothendieck theorem as a secondary class form. Since $\mathcal{T}(T^HM, g^W, g^E)$ is an even form, $\mathcal{T}s(B,M,E)$ is zero for the odd dimensional B. Obviously, if $M = B \times Z$, $E = B \times (\tilde{Z} \times_\rho \mathbb{C}^k)$ with $E|_Z = \tilde{Z} \times_\rho \mathbb{C}^k$ and $\rho : \pi_1(Z) \to GL(\mathbb{C}^k)$. Then

$$\mathcal{T}s(B,M,E) = sig(B)\mathcal{T}(Z, E|_Z),$$

where $sig(B)$ is the signature of B, and $\mathcal{T}(Z, E|_Z)$ is the analytic torsion of Z associated with the bundle $E|_Z$. For convention, for $Z = \{pt\}$, $M = B$, we understand $\mathcal{T}s(B,M,E) = sig(B)$. But for $B = \{pt\}$, M is equal to Z. We interpret $\mathcal{T}s(B,M,E) = \mathcal{T}(Z, E|_Z)$.

One open question is whether $\mathcal{T}s(B,M,E)$ has certain homotopy property. Namely, if $h : B_1 \to B$ is certain homotopy equivalence, does the following equality hold

$$< L(B_1)h^*(\mathcal{T}(T^HM, g^Z, g^E)), [B_1] >=< L(B)\mathcal{T}(T^HM, g^Z, g^E), [B] >?$$

If M is the trivial fiber bundle $B \times Z$ over B, and $E|_Z$ is independent of B, then $\mathcal{T}s(B,M,Z)$ is clearly a homotopy invariant. In view of the Kahn theorem [Ka], $\mathcal{T}s(B,M,Z)$ is unlikely a homotopy invariant in general. Now we list some elementary properties of the invariant $\mathcal{T}s(B,M,E)$.

Proposition 1 *(i) $\mathcal{T}s(B,M,E)$ is a diffeomorphism invariant.*
(ii) Let $Z_1 \to M_i \stackrel{\pi_i}{\to} B_i$ be a fiber bundle and E_i a flat bundle over M_i such that the analytic torsion form $\mathcal{T}(T^HM_i, g^{Z_i}, g^{E_i})$ is closed, $i = 1, 2$. Let $Z_1 \times Z_2 \to M_1 \times M_2 \to B_1 \times B_2$ be the product of the fiber bundles M_1 and M_2, and $P_i : M_1 \times M_2 \to M_i$ be the projection. Then

$$\begin{aligned}\mathcal{T}s(B_1 \times B_2, M_1 \times M_2, P_1^*(E_1) \otimes P_2^*(E_2)) &= rk(E_1)\chi(Z_1)sig(B_1)\mathcal{T}s(B_2, M_2, E_2) \\ &+ rk(E_2)\chi(Z_2)sig(B_2)\mathcal{T}s(B_1, M_1, E_1).\end{aligned}$$

Proof. Since $L(B)$ and $[B]$ are diffeomorphism invariants, it suffices to prove that the analytic torsion form $\mathcal{T}(T^HM, g^Z, g^E)$ is independent of the metric g^B. But this is obvious from the definition of the analytic torsion form.

Part (ii) follows from the product formula of the analytic torsion form (see Theorem 3.28 [BL] for the case $B_1 = B_2$) . Q.E.D.

3 Modified Analytic Torsion Forms

In this section we introduce a modified analytic torsion form of the compact fiber bundle M. This torsion form will be used in the adiabatic limit in Section 6.

We use the notation of Section 2. Let for $t > 0$

$$\begin{aligned} C_t &\equiv \frac{1}{2}(t^{-\frac{N}{2}}(d_M)^* t^{\frac{N}{2}} + t^{\frac{N}{2}} d_M t^{-\frac{N}{2}}) \\ &= \frac{\sqrt{t}}{2}(d_Z^* + d_Z) + \nabla^{\mathcal{E}} + \frac{1}{2} w(\mathcal{E}, g^{\mathcal{E}}) - \frac{1}{2\sqrt{t}} c(T). \end{aligned} \tag{10}$$

Let δ_t be the automorphism of $\Omega(B,\mathcal{E})$ given by $\delta_t(\xi) = t^{-\frac{j}{2}}\xi, \xi \in \Omega^j(B,\mathcal{E})$. Then $C_t = t^{\frac{1}{2}}\delta_t((d_M)^* + d_M)\delta_t^{-1}$ and $D_t = t^{\frac{1}{2}}\delta_t((d_M)^* - d_M)\delta_t^{-1}$. Hence, $C_t^2 = -D_t^2$. As shown in [BGV], $Tr^s(e^{-C_t^2})$ is a closed form on B.

Lemma 5 1) *As $t \to \infty$,*

$$e^{-C_t^2} = e^{(\frac{1}{2}w(H(Z,E|_Z), g^{H(Z,E|_Z)}))^2} + O(\frac{1}{\sqrt{t}}),$$

in the sense of all C^l-norms.

2) *As $t \to 0$, the heat kernel $e^{-C_t^2}(x,x)$ satisfies*

$$e^{-C_t^2}(x,x) \sim (4\pi t)^{-\frac{\dim(Z)}{2}} \sum_{j=0}^{\infty} t^j K_j(x), \tag{11}$$

where $K_j(x) \in \sum_{2l \le 2j} \Omega^{2l}(M, End_{C(T^*Z)}(\wedge(T^*Z) \otimes E|_Z))$.

Proof. See Theorems 9.19 and 10.21 in [BGV]. Q.E.D.

By Lemma 5, we can define the modified analytic torsion form of M as follows.

Definition 1. The modified analytic torsion form $\mathcal{T}_1(T^H M, g^Z, g^E)$ of M associated with the flat complex vector bundle E is

$$\begin{aligned} \mathcal{T}_1(T^H M, g^Z, g^E) &= \int_\tau^\infty \varphi Tr^s(\frac{N}{2}(e^{-C_{4t}^2} - e^{(\frac{1}{2}w(H(Z,E|_Z), g^{H(Z,E|_Z)}))^2})\frac{dt}{t} \\ &+ \frac{d}{dr}(\frac{1}{\Gamma(r)}\int_0^\tau t^{r-1}\varphi Tr^s(\frac{N}{2}(e^{-C_{4t}^2} - e^{(\frac{1}{2}w(H(Z,E|_Z), g^{H(Z,E|_Z)}))^2})dt)_{r=0} \end{aligned} \tag{12}$$

where τ is any small positive number.

$\mathcal{T}_1(T^H M, g^Z, g^E)$ is an even form on B. It shares some properties of the analytic torsion form $\mathcal{T}(T^H M, g^Z, g^E)$. We list two properties in the following.

Proposition 2 *Let B and Z_i be closed Riemannian manifolds. Suppose $Z_i \to M_i \stackrel{\pi_i}{\to} B$ is a compact fiber bundle, and E_i is a flat complex vector bundle over M_i with Hermitian metric g^{E_i}, $i = 1, 2$. Let $Z = Z_1 \times Z_2$ with the product metric $g^Z = g^{Z_1} \times g^{Z_2}$, and $Z \to M \to B$ be the product fiber bundle of M_1 and M_2. Form $E = P_1^*(E_1) \otimes P_2^*(E_2)$ and $g^E = P_1^*(g^{E_1}) \otimes P_2^*(g^{E_2})$, where $P_i : M \to M_i$ is the natural projection. Then*

$$\mathcal{T}_1(T^H M, g^Z, g^E) = rk(E_1)\chi(Z_1)\mathcal{T}_1(T^H M_2, g^{Z_2}, g^{E_2}) + rk(E_2)\chi(Z_2)\mathcal{T}_1(T^H M_1, g^{Z_1}, g^{E_1}). \quad (13)$$

Proof. Note that

$$\begin{aligned}
\Omega(M, E) &\simeq \Omega(B)\hat{\otimes} C^\infty(M, E) \\
&\simeq \Omega(B)\hat{\otimes} C^\infty(B, (\wedge(T^* Z_1) \otimes E_1|_{Z_1})\hat{\otimes}(\wedge(T^* Z_2) \otimes E_2|_{Z_2})), \\
N &= N_1 \hat{\otimes} I + I \hat{\otimes} N_2,
\end{aligned}$$

and

$$D_t^2 = D_{1,t}^2 \hat{\otimes} I + I \hat{\otimes} D_{2,t}^2.$$

Hence,

$$\begin{aligned}
Tr^s(\frac{N}{2} e^{D_t^2}) &= Tr^s(\frac{N}{2} e^{D_{1,t}^2} e^{D_{2,t}^2}) \\
&= Tr^s(\frac{N_1}{2} e^{D_{1,t}^2}) Tr^s(e^{D_{2,t}^2}) + Tr^s(\frac{N_2}{2} e^{D_{2,t}^2}) Tr^s(e^{D_{1,t}^2}).
\end{aligned}$$

By local index theory techniques, one can check (see [BL]) that

$$Tr^s(e^{D_{i,t}^2}) = rk(E_i)\chi(Z_i).$$

Now (13) follows clearly from (12). Q.E.D.

Recall that the analytic torsion invariant $\mathcal{T}(M_x, E_x)$ of each fiber M_x associated with vector bundle $E_x = E|_{M_x}$, $x \in B$, is defined to be

$$\mathcal{T}(M_x, E_x) = e^{\frac{1}{2}\frac{\partial}{\partial u} Tr^s(N(-P_{V_x}^\perp (V_x)^2 P_{V_x}^\perp))^{-u}|_{u=0}}, \quad (14)$$

where $V_x = (d_Z^* - d_Z)|_{M_x}$, $P_{V_x}^\perp = 1 - P_{V_x}$, and P_{V_x} is the orthogonal projection onto the kernel of V_x. The following lemma asserts that the zero-form component $\mathcal{T}_1(T^H M, g^Z, g^E)_{[0]}$ of $\mathcal{T}_1(T^H M, g^Z, g^E)$ at $x \in B$ is equal to $\log \mathcal{T}(M_x, E_x)$.

Proposition 3 $\mathcal{T}_1(T^HM, g^Z, g^E)_{[0]}(x) = \log \mathcal{T}(M_x, E_x)$.

Proof. This is obvious, since $w(H(Z, E|_Z), g^{H(Z,E|_Z)})$ is a one-form on B, by (12) and (14),

$$\begin{aligned}\mathcal{T}_1(T^HM, g^Z, g^E)_{[0]}(x) &= \int_\tau^\infty \varphi Tr^s(\frac{N}{2}(e^{V_x^2} - P_{V_x}))\frac{dt}{t} \\ &+ \frac{d}{dr}(\frac{1}{\Gamma(r)}\int_0^\tau t^{r-1}\varphi Tr^s(\frac{N}{2}(e^{V_x^2} - P_{V_x}))dt)_{r=0} \\ &= \log \mathcal{T}(M_x, E_x).\end{aligned}$$

Q.E.D.

The modified torsion form $\mathcal{T}_1(T^HM, g^Z, g^E)$ is still quite mystery. For instance, we do not know how this form depends on its arguments and when it is a closed form on B. The difficulty is that we do not have at moment (2) and (3) of Lemma 1 for the complementary error function $erfc(x)$. From this point of view, the analytic torsion form of Bismut-Lott has advantage over the modified torsion form. However, it is much more convenient to use the modified torsion form in the adiabatic limit. This is the main reason that we discuss the modified torsion form here.

4 Local Mixed Index Theorem

The main purpose of this section is to introduce a mixed super trace on $\wedge(T^*M) \simeq \pi^*(\wedge(T^*B)) \hat{\otimes}\wedge(T^*Z)$, which combines the super trace on $\pi^*(\wedge(T^*B))$ corresponding to the grading given by the Hodge star operator and the super trace on $\wedge(T^*Z)$ associated with the grading given by the degree of forms. We apply this mixed super trace to the Euler-de Rham operator on the total space M and get a local mixed index theorem.

Let $Z \to M \xrightarrow{\pi} B$ be the compact fiber bundle as in Section 2. Recall that the Hodge operator $*_M$ on M is given by the chirality operator on $\wedge^*(T^*M)$, namely, $*_M(\xi) = i^p c(x^1)\cdots c(x^{\dim(M)})\xi$, where $\{x^i\}$ is a local orthonormal basis of T^*M, $c(x) = (x\wedge) - i_x$, $p = \frac{\dim(M)}{2}$ for $\dim(M)$ even, and $p = \frac{\dim(M)+1}{2}$ for $\dim(M)$ odd. The definition of $*_M$ is slightly different from the usual one since $*_M^2 = 1$. Let $C(T^*M)$ be the bundle of Clifford algebras over M whose fiber over $x \in M$ is the Clifford algebra $C(T_x^*M)$ of the Euclidean space T_x^*M. Then $\wedge^*(T^*M)$ is a Z_2-graded Clifford-bundle over M with a $C(T^*M)$-action given by $v\xi = c(v)\xi$, $v \in T_x^*M$ and $\xi \in \wedge^*(T_x^*M)$, and with a grading given by the degree of forms. We have a natural Z_2-grading on $End(\wedge^*(T^*M))$ and a super trace tr^s on $C^\infty(M, End(\wedge^*(T^*M)))$ given by

$$tr^s(a) = tr(a_{11}) - tr(a_{22})$$

for $a = (a_{ij})_{i,j=1}^2$ on $\wedge^*(T^*M) = \wedge^{ev}(T^*M) \oplus \wedge^{odd}(T^*M)$. There is a super trace Tr^s on $L^2(M, \wedge^*(T^*M))$ which is given for an operator A with a smooth kernel $K_A(x,y)$ by

$$Tr^s(A) = \int_M tr^s(K_A(x,x)).$$

We assume for the moment that M is even dimensional. Then $End(\wedge^*(T^*M)) = C(T^*M) \otimes End_{C(T^*M)}(\wedge^*(T^*M))$, and for $a(x) \in C(T^*M), b \in End_{C(T^*M)}(\wedge^*(T^*M))$,

$$tr^s(a(x) \otimes b(x)) = (-2i)^{\frac{\dim(M)}{2}} \sigma_{\dim(M)}(a(x)) tr_r^s(b(x)),$$

where $\sigma_{\dim(M)} : C(T^*M) \to \wedge^{\dim(M)}(T^*M)$ is the top dimensional piece of the symbol map, $(-2i)^{\frac{\dim(M)}{2}} \sigma_{\dim(M)}$ is the super trace on $C(T^*M)$ and tr_r^s is the relative trace given by

$$tr_r^s(b) = 2^{-\frac{\dim(M)}{2}} tr^s(*_M b).$$

tr_r^s extends to a linear map from $\Omega(M, End_{C(T^*M)}(\wedge^*(T^*M)))$ to $\Omega(M)$. Thus,

$$tr^s(K_A(x,x)) = (-2i)^{\frac{\dim(M)}{2}} tr_r^s(\sigma_{\dim(M)}(K_A(x,x))). \tag{15}$$

On the other hand, we can also use the Hodge star operator $*_M$ to grade $\wedge^*(T^*M)$ and define a usual trace $tr_* = 2^{-\frac{\dim(M)}{2}} tr_{\wedge^*(T^*M)}$ on $End_{C(T^*M)}(\wedge^*(T^*M))$, which can be extended to a linear map from $\Omega(M, End_{C(T^*M)}(\wedge^*(T^*M)))$ to $\Omega(M)$. We have thus another pointwise super trace

$$tr_*^s(K_A(x,x)) = (-2i)^{\frac{\dim(M)}{2}} tr_*(\sigma_{\dim(M)}(K_A(x,x))) \tag{16}$$

and then another super trace

$$Tr_*^s(A) = \int_M tr_*^s(K_A(x,x)). \tag{17}$$

The difference between the two traces Tr^s and Tr_*^s is that we use a relative super trace in (15) and a usual trace in (16). Let $D_M = \bar{d}_M + d_M$, where $\bar{d}_M$ is the adjoint of d_M with respect to the Hermitian structure on M. Applying these two traces to the operator $A = e^{-tD_M^2}$ and letting $t \to 0$, we obtain the index formulas for the Euler characteristic number and the signature of M, respectively. See ([BGV] Chapter 4) for details.

Now for the total space of the fiber bundle M, we have $T^*M = (T^H M)^* \oplus T^*Z$ and $\wedge^*(T^*M) = \wedge^*((T^H M)^*) \hat{\otimes} \wedge^* (T^*Z)$. We may use a tensor product of the above two traces tr^s and tr^s_* to get a new trace. More specifically, we assume throughout this section that the dimension of B is even. Let tr^s_M be the tensor product of the trace tr^s_* on $\wedge^*((T^H M)^*)$ and the super trace tr^s on $\wedge^*(T^*Z)$. For an even dimensional fiber Z, tr^s_M acts on the smooth kernel $K_A(x,y)$ of an operator A as $tr^s_M(K_A(x,x)) = (-2i)^{\frac{\dim(M)}{2}} tr^s_{*,r}(\sigma_{\dim(M)}(K_A(x,x)))$, where $tr^s_{*,r}$ is the tensor product of the pointwise traces tr^s_r and tr_*. The trace tr^s_M together with an integral on M produces a super trace Tr^s_M on $L^2(M, \wedge^*(T^*M))$. If we tensor $\wedge^*(T^*M)$ with the flat vector bundle E over M and use the usual trace on E, we get a trace as above which is still denoted by Tr^s_M. Thus if A is of Tr^s_M-trace class operator on $L^2(M, \wedge^*(T^*M) \otimes E)$ with a smooth kernel $K_A(x,y)$, then

$$Tr^s_M(A) = \int_M tr^s_M(K_A(x,x)).$$

We first apply the trace Tr^s_M to the operator $D_M = \bar{d}_M + d_M$ and consider the following invariant for an even dimensional Z,

$$ind_w(D_M) = \lim_{t \to 0} Tr^s_M(e^{-tD_M^2}). \tag{18}$$

$ind_w(D_M)$ is well defined by Theorem 4.1 [BGV]. The geometric meaning of $ind_w(D_M)$ is that it represents a mixture of the Euler characteristic number and the signature of M.

Theorem 1 *Let $Z \to M \xrightarrow{\pi} B$ be a fiber bundle with Z and B closed Riemannian manifolds. Let B be even dimensional and E be a flat vector bundle over M with a flat connection ∇^E. (i) If Z is even dimensional, then*

$$ind_w(D_M) = (2\pi i)^{-\frac{\dim(M)}{2}} rk(E) \int_B \hat{A}(B) \int_Z \hat{A}(Z) tr^s_{*,r}(e^{-\hat{R}^M}), \tag{19}$$

where $\hat{A}(B)$ is the $\hat{A}$-class of B, $\hat{R}^M = \frac{1}{4} R^M_{jk} c^j_v c^k_v$, $R^M_{jk} =< e_j, R^M e_k >$ with the curvature R^M of the Levi-Civita connection ∇^M. Let $\bar{N}$ be the number operator of the total space M.

As $t \to 0$,

$$(ii) \qquad Tr^s_M(e^{-tD^2_M}) = \begin{cases} ind_w(D_M) + O(t), & \dim(Z) \ \ even, \\ O(\sqrt{t}), & \dim(Z) \ \ odd. \end{cases}$$

$$(iii) \qquad Tr^s_M(\bar{N}e^{-tD^2_M}) = \begin{cases} \frac{dim\ M}{2} ind_w(D_M) + O(t), & \dim(Z) \ \ even, \\ \frac{1}{\sqrt{t}} rk(E) \int_M a_M + O(\sqrt{t}), & \dim(Z) \ \ odd, \end{cases}$$

where a_M *is an odd form on* M.

(iv) *Let* $V_M = \bar{d}_M - d_M$.

$$Tr^s_M(\sqrt{t}V_M e^{-tD^2_M}) = \begin{cases} O(t), & \dim(Z) \ \ even, \\ O(\sqrt{t}), & \dim(Z) \ \ odd. \end{cases}$$

$$(v) \qquad Tr^s_M((1-2tD^2_M)e^{-tD^2_M}) = \begin{cases} ind_w(D_M) + O(t), & \dim(Z) \ \ even, \\ O(\sqrt{t}), & \dim(Z) \ \ odd. \end{cases}$$

$$(vi) \qquad Tr^s_M(\frac{\bar{N}}{2}(1-2tD^2_M)e^{-tD^2_M}) = \begin{cases} \frac{\dim(M)}{4} ind_w(D_M) + O(t), & \dim(Z) \ \ even, \\ O(\sqrt{t}), & \dim(Z) \ \ odd. \end{cases}$$

Proof. (i) Since (19) is local, we can assume B and Z are oriented and spin manifolds. By Theorem 4.1 [BGV], the heat kernel $K_t(x,y)$ of D_M has the following expansion,

$$K_t(x,x) \sim (4\pi t)^{-\frac{\dim(M)}{2}} \sum_{i=0}^{\infty} t^i K_i(x), \tag{20}$$

with $K_i(x) \in C^\infty(M, C_{2i}(T^*M) \otimes End_{C(T^*M)}(\wedge^*(T^*M)))$. Here $C_{2i}(T^*M)$ is the subset of elements in the Clifford bundle $C(T^*M)$ with degree $\leq 2i$. The symbol of sum of the $K_i^{'}s$ is

$$\sigma(K) = \sum_{i=0}^{\frac{\dim(M)}{2}} \sigma_{2i}(K_i) = \hat{A}(M)\exp(-\hat{R}^{M\otimes E,u}) \in \Omega(M, End_{C(T^*M)}(\wedge^*(T^*M))).$$

Here $\hat{R}^{M\otimes E,u} = \hat{R}^M \otimes I + I \otimes R^{E,u}$, and $R^{E,u}$ is the curvature of $\nabla^{E,u} = \nabla^E + \frac{1}{2}w(\nabla^E, g^E)$, $R^{E,u} = -\frac{1}{4}w(\nabla^E, g^E)^2$. For $a \in C_{2i}(T^*M), i < \frac{\dim(M)}{2}$, we know that its super trace is zero. Thus,

$$\begin{aligned} ind_w(D_M) &= \lim_{t\to 0}(4\pi t)^{-\frac{\dim(M)}{2}} \sum_{i \geq \frac{\dim(M)}{2}} t^i \int_M tr^s_M(K_i(x)) \\ &= (4\pi)^{-\frac{\dim(M)}{2}} \int_M tr^s_M(K_{\frac{\dim(M)}{2}}(x)) \\ &= (4\pi)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M) tr^s_M(e^{-\hat{R}^{M\otimes E,u}}) \\ &= (2\pi i)^{-\frac{\dim(M)}{2}} \int_M \hat{A}(M) tr^s_{*,r}(e^{-\hat{R}^M}) tr(e^{-R^{E,u}}). \end{aligned}$$

We obtain from ([BL], (3.77)) that $tr^s(e^{-R^{E,u}}) = rk(E)$. This implies (19).

(ii) If Z is even dimensional, the result follows from Part (i) and (20). For odd dimensional Z, we still have Formula (20). However, the heat kernel in this case contains only the half-integer powers of t. Also $tr^s_M(a) = 0$ for $a \neq c^1 \cdots c^{\dim(M)} c^1_v \cdots c_v^{\dim(M)}$, since both tr^s_* and tr^s have such a property. Thus, by local index theory techniques, we get that as $t \to 0$,

$$\lim_{t\to 0} Tr^s_M(e^{-tD^2_M}) = \lim_{t\to 0} \sum_{i > \frac{\dim(M)}{2}} t^{i-\frac{\dim(M)}{2}} \int_M tr^s_M(K_i(x)) = 0,$$

since $tr^s_M(K_i(x))$ is an even form and $\dim(M)$ is odd.

(iii) Let $\{e_a\}$ and $\{e^a\}$ be local orthonormal bases of TM and T^*M, respectively. The operator $\bar{N}$ can be written as

$$\bar{N} = \frac{1}{2}\sum_{i=1}^{\dim(M)} c(e_{a_i})c_v(e_{a_i}) + \frac{\dim(M)}{2},$$

where $c(e_{a_i})$ and $c_v(e_{a_i})$ are the Clifford variables associated with e_{a_i} [BZ]. Using the Getzler rescaling $G_{\sqrt{t}}$, we get that as $t \to 0$,

$$G_{\sqrt{t}}(\frac{\sqrt{t}}{2}\sum_{i=1}^{\dim(M)} c(e_{a_i})c_v(e_{a_i})) \to \frac{1}{2}\sum_{i=1}^{\dim(M)} e^{a_i}\wedge c_v(e_{a_i})$$

and

$$G_{\sqrt{t}}(tD^2_M) \to (\partial_a - \frac{1}{4}R^M_{ab}x_b)^2 + \hat{R}^{M\otimes E,u}.$$

Therefore, for the even dimensional Z,

$$Tr^s_M(\frac{\sqrt{t}}{2}\sum_{i=1}^{\dim(M)} c(e_{a_i})c_v(e_{a_i})e^{-tD^2_M}) \to$$
$$\frac{(2\pi i)^{-\frac{\dim(M)}{2}}}{2}\int_M \hat{A}(M) tr^s_{*,r}(\sum_{i=1}^{\dim(M)} e^{a_i}c_v(e_{a_i})e^{-\hat{R}^M} tr^s(e^{-R^{E,u}})).$$

As shown in ([BZ], XI) with the usual trace replaced by the mixed one, we obtain that the above limit is zero. Hence by Part (ii), we have

$$\lim_{t\to 0} Tr^s_M(\bar{N}e^{-tD^2_M}) = \lim_{t\to 0} Tr^s_M(\frac{\dim(M)}{2}e^{-tD^2_M}) = \frac{\dim(M)}{2} ind_w(D_M).$$

The approximate rate is clear. Similarly, one can prove the odd dimensional case.

(iv) Let $\dim(Z)$ be even. We may apply the proof of Theorem 3.16 [BL] to the situation here. Indeed, let z be an odd Grassmann variable which anticommutes with all other Grassmann variables, $z^2 = 0$. For $\xi \in \Omega(B)\otimes \mathbb{C}[z], \xi = \xi_0 + z\xi_1, \xi_i \in \Omega(B)$, we define $\xi^z = \xi_1$.

Then

$$\begin{aligned}\lim_{t\to 0} Tr^s_M(\sqrt{t}V_M e^{-tD_M^2}) &= \lim_{t\to 0} Tr^s_M(\frac{1}{\sqrt{t}}e^{-t(D_M^2 - zV_M)})^z \\ &= (2\pi i)^{-\frac{\dim(M)}{2}}\int_M \hat{A}(M)tr^s_{*,r}(e^{-\hat{R}^M})tr(e^{-R^{E,u}+\frac{z}{2}\psi})^z \\ &= (2\pi i)^{-\frac{\dim(M)}{2}}\int_M \hat{A}(M)tr^s_{*,r}(e^{-\hat{R}^M})tr(\frac{1}{2}\psi e^{-(\frac{1}{2}\psi)^2}) = 0,\end{aligned}$$

since $tr(\frac{1}{2}\psi e^{-(\frac{1}{2}\psi)^2})$ is an odd form. Similarly, by the argument of Part (ii), we get the result for the odd dimensional case.

(v) By local index theory techniques, we have that for the even dimensional Z,

$$\begin{aligned}&\lim_{t\to 0} Tr^s_M(-2tD_M^2 e^{-tD_M^2}) = \lim_{t\to 0} Tr^s_M(e^{-t(1+2z)D_M^2})^z \\ =\ & \{\int_M (2\pi i(1+2z))^{-\frac{\dim(M)}{2}} \det{}^{\frac{1}{2}}(\frac{(1+2z)\frac{R^M}{2}}{\sinh(1+2z)\frac{R^M}{2}})tr^s_{*,r}(e^{-(1+2z)\hat{R}^{M\otimes E,u}})\}^z \\ =\ & (2\pi i)^{-\frac{\dim(M)}{2}} 2\int_M \hat{A}(M)tr^s_{*,r}(\hat{R}^M e^{-\hat{R}^M})tr(R^{E,u}e^{-R^{E,u}}) = 0.\end{aligned}$$

The last step follows from the fact that $tr(R^{E,u}e^{-R^{E,u}}) = 0$ by Proposition 1.3 in [BL]. Hence by Part (ii), we obtain the result for the even dimensional Z. If Z is odd dimensional, we use the above argument together with the proof of Part (ii) to get the assertion.

(vi) This part essentially follows from Part (v) and the proof of Theorem 3.21 [BL]. Indeed, consider the following trivial fiber bundle $Z' \to M' = M\times\mathbb{R}^*_+ \xrightarrow{\pi'} \mathbb{R}^*_+$ over $\mathbb{R}^*_+$, where $\pi'(x,s) = s$ and $Z' \simeq M$. Let $\pi'' : M' \to M$ be the natural projection. Then $TZ' = (\pi'')^*(TM)$ and $(\pi'')^*E$ is a flat vector bundle over M'. We define a metric $g^{Z'}$ on TZ' such that $g^{Z'}|_{M\times\{s\}}$ is equal to $\frac{(\pi'')^*g^M}{s}$. Let the horizontal distribution T^HM' of M' be given by $T^HM' = (\pi')^*T\mathbb{R}^*_+$. We have that $\nabla^{Z'} = (\pi'')^*\nabla^M + ds(\frac{\partial}{\partial s} - \frac{1}{2s})$, $R^{Z'} = (\pi'')^*R^M$ and $R^{Z'}(\frac{\partial}{\partial s},\cdot) = 0$. Clearly,

$$d_{M'} = d_M + ds\partial_s,$$

and the fiberwise adjoint $d^*_{M'}$ of $d_{M'}$ is

$$\begin{aligned}d^*_{M'} &= s^{-(\bar{N}-\frac{dim\ M}{2})}(\bar{d}_M + ds\partial_s)s^{(\bar{N}-\frac{\dim(M)}{2})} \\ &= s\bar{d}_M + ds(\partial_s + \frac{1}{s}(\bar{N} - \frac{\dim(M)}{2})).\end{aligned}$$

Here $\bar{d}_M$ is the adjoint of d_M. Using the notation of Section 2, we get

$$2D'_t = t^{-\frac{\bar{N}}{2}}d^*_{M'}t^{\frac{\bar{N}}{2}} - t^{\frac{\bar{N}}{2}}d_{M'}t^{-\frac{\bar{N}}{2}}$$

$$= \quad s^{-\frac{\bar N}{2}} 2D_{st} s^{\frac{\bar N}{2}} + \frac{ds}{s}(\bar N - \frac{\dim(M)}{2}).$$

By the Taylor expansion,

$$\begin{aligned} Tr^s_M(f(2D'_t))(s) &= Tr^s_M(s^{-\frac{\bar N}{2}} f(2D_{st}) s^{\frac{\bar N}{2}}) + \frac{2ds}{s} Tr^s_M(s^{-\frac{\bar N}{2}} \frac{\bar N}{2} f'(2D_{st}) s^{\frac{\bar N}{2}}) \\ &- \frac{\dim(M)}{2s} ds Tr^s_M(s^{-\frac{\bar N}{2}} f'(2D_{st}) s^{\frac{\bar N}{2}}). \end{aligned}$$

Using the proof of Part (iv) and the fact that $R^{Z'}_v(\frac{\partial}{\partial s}, \cdot) = 0$, we obtain that the limits of $Tr^s_M(f(2D'_t))(s)$ and $Tr^s_M(s^{-\frac{\bar N}{2}} f(2D_{st}) s^{\frac{\bar N}{2}})(1)$ as $t \to 0$ do not contain a ds-term. Hence the ds term of the above identity together with Part (v) proves the assertion. Q.E.D.

Zhang [Zh] has recently obtained a similar index formula for the general case where $T^H M$ could be any sub-bundle E_1 of TM. In fact, for the even rank E_1, our method works in this general case as well, since $\wedge^*(T^*M) \simeq \wedge^*(E_1^*) \hat\otimes \wedge^*((E_1^\perp)^*)$ and a super trace could similarly be defined. The point of our method is that we can directly use the usual formulas since we only change the super trace. See also [Go] for the Euler-signature type operator.

For the use of the next section, we list the following proposition whose proof is obvious.

Proposition 4 *With the assumption of Theorem* 1*, there is a constant* $\lambda_0 > 0$ *such that for* $t \to \infty$*,*

1) $Tr^s_M(e^{-tD_M^2}) = Tr^s_M(P_{Ker\ D_M}) + O(e^{-\lambda_0 t})$,

2) $Tr^s_M(\bar N e^{-tD_M^2}) = Tr^s_M(\bar N P_{Ker\ D_M}) + O(e^{-\lambda_0 t})$,

3) $Tr^s_M(\bar N((1 - 2tD_M^2) e^{-tD_M^2}) = Tr^s_M(\bar N P_{Ker\ D_M}) + O(e^{-\lambda_0 t})$.

5 Torsion-Signature Invariants

In this section we use the mixed super trace to define a torsion type invariant for the operator D_M. Denote by $D_M^\perp$ the restriction of D_M to the orthogonal complement of $KerD_M$. As usual, we have that $Tr^s_M(\frac{\bar N}{2} e^{-t(D_M^\perp)^2}) \sim O(e^{-\lambda_0 t})$ as $t \to \infty$ and for $t \to 0$, $Tr^s_M(\frac{\bar N}{2} e^{-t(D_M^\perp)^2})$ has an asymptotic expansion in t which is the integral powers of t for $\dim(M)$ even and the half-integer powers of t for $\dim(M)$ odd. Thus

$$\zeta(r) = \frac{1}{\Gamma(r)} \int_0^\infty Tr^s_M(\frac{\bar N}{2} e^{-t(D_M^\perp)^2}) t^r \frac{dt}{t}, \quad \Re r > \frac{\dim(M)}{2}, \tag{21}$$

extends to a meromorphic function of $r \in \mathbb{C}$ such that $\zeta(r)$ is holomorphic near the zero.

Definition 2. The torsion-signature $\mathcal{T}_w(M,E)$ of M associated with the flat vector bundle E is defined as

$$\log \mathcal{T}_w(M,E) = \frac{d\zeta(r)}{dr}|_{r=0}.$$

$\mathcal{T}_w(M,E)$ is very special for $M = B \times Z$ and $E = \pi_1^*(E_B) \otimes \pi_2^*(E_Z)$, where $\pi_2 : B \times Z \to Z$ is the natural projection and E_B and E_Z are two flat vector bundles over B and Z, respectively. In this case, $D_M^2 = D_B^2 \otimes 1 + 1 \otimes D_Z^2$ and

$$Tr_M^s(\frac{\bar{N}}{2}e^{-t(D_M^2)^\perp}) = Tr_*^s(P_{KerD_B})Tr^s(\frac{N}{2}e^{-t(D_Z^2)^\perp}) + Tr^s(P_{KerD_Z})Tr_*^s(\frac{N'}{2}e^{-t(D_B^2)^\perp}),$$

where N' is the number operator of B. By the McKean-Singer index formula,

$$Tr_*^s(\frac{N'}{2}e^{-t(D_B^\perp)^2}) = const.Tr(*_B\frac{N'}{2}e^{-t(D_B^\perp)^2}) = const.\frac{\dim(B)}{4}Tr(*_Be^{-t(D_B^\perp)^2}) = 0,$$

where we used $*_B\frac{N'}{2} + \frac{N'}{2}*_B = \frac{\dim(B)}{2}*_B$. We get

$$\log \mathcal{T}_w(M,E) = sig(B, E_B) \log \mathcal{T}(Z, E_Z).$$

Hence $\mathcal{T}_w(M,E)$ is not trivial in general. Because of this, we call $\mathcal{T}_w(M,E)$ the torsion-signature invariant of M associated with E. In particular, we consider it for the fiber bundle $M = \tilde{B} \times_\Gamma Z$ and the flat vector bundle E associated with two representations ρ_i of $\Gamma = \pi_1(B)$ as we discussed in Section 2. In this case, we denote $\mathcal{T}_w(M,E)$ by $\mathcal{T}_w(B, Z, \rho_1, \rho_2)$.

Proposition 5 *(i) For $B = \{pt\}$, $\mathcal{T}_w(\{pt\}, Z, id, id)$ is equal to the analytic torsion $\mathcal{T}(Z,E)$ of Z associated with the trivial vector bundle E.*
(ii) For $Z = \{pt\}$, $\log \mathcal{T}_w(B, \{pt\}, id, id) = 0$.
(iii) $\log \mathcal{T}_w(B, Z, id, id) = sig(B, E_B) \log \mathcal{T}(Z, E_Z)$,
where id means the trivial representation of Γ.

Proof. (i) For $B = \{pt\}$, Tr_M^s is the usual super trace on $L^2(Z, \wedge^*(T^*Z) \otimes E)$. The result is obviously true.

(ii) Since $Tr_M^s = Tr_*^s$ on $L^2(B, \wedge^*(T^*B) \otimes E)$ and $Tr_*^s(\frac{N'}{2}e^{-t(D_B^2)^\perp}) = 0$ by the above argument, the assertion clearly holds.

(iii) This is the trivial case that we just discussed above. Q.E.D.

Part (iii) shows that $\mathcal{T}_w(M,E)$ might be independent of the metric in general only when the cohomology $H^*(Z,E|_Z)=0$. An unsolved problem is to find a formula for the dependence of $\mathcal{T}_w(M,E)$ on the metric.

The following version of $\mathcal{T}_w(M,E)$ will be used in the next section. Let $u(x)=(1-2x)e^{-x}$.

Proposition 6

$$\begin{aligned}\log\mathcal{T}_w(M,E) &= \int_0^\infty [Tr^s_M(\frac{\bar N}{2}u(tD_M^2))-Tr^s_M(\frac{\bar N}{2}P_{KerD_M}) \\ &- (\frac{\dim(M)}{4}ind_w(D_M)-Tr^s_M(\frac{\bar N}{2}P_{KerD_M}))u(t)]\frac{dt}{t}. \end{aligned} \tag{22}$$

Proof. By Theorem 1 and Proposition 4, the right hand side of (22) is well defined. For an even dimensional Z, we have

$$\begin{aligned} RHS\ \ of\ \ (22) &= \int_0^\infty [Tr^s_M(\frac{\bar N}{2}u(tD_M^2)^\perp)-(\frac{\dim(M)}{4}ind_w(D_M)-Tr^s_M(\frac{\bar N}{2}P_{KerD_M}))u(t)]\frac{dt}{t} \\ &= \frac{d}{dr}\{\frac{1}{\Gamma(r)}\int_0^\infty [Tr^s_M(\frac{\bar N}{2}e^{-t(D_M^2)^\perp})-(\frac{\dim(M)}{4}ind_w(D_M) \\ &- Tr^s_M(\frac{\bar N}{2}P_{KerD_M}))u(t)]t^r\frac{dt}{t}\}_{r=0}+\int_0^\infty 2\frac{d}{dt}Tr^s_M(\frac{\bar N}{2}e^{-tD_M^2})dt \\ &= \log\mathcal{T}_w(M,E)+Tr^s_M(\bar N P_{KerD_M})-\frac{\dim(M)}{2}ind_w(D_M) \\ &- \frac{d}{dr}\{\frac{1}{\Gamma(r)}\int_0^\infty(\frac{\dim(M)}{4}ind_w(D_M)-Tr^s_M(\frac{\bar N}{2}P_{KerD_M}))u(t)]t^r\frac{dt}{t}\}_{r=0} \\ &= \log\mathcal{T}_w(M,E). \end{aligned}$$

Here we used Theorem 1 (iii). The above proof is also valid for the odd dimensional case except that we have to consider

$$\begin{aligned} &\frac{d}{dr}\{\frac{1}{\Gamma(r)}\int_0^\infty Tr^s_M(\frac{\bar N}{2}(-2tD_M^2)e^{-tD_M^2})t^r\frac{dt}{t}\}_{r=0} \\ = &\int_\varepsilon^\infty \frac{d}{dt}Tr^s_M(\bar N e^{-tD_M^2})dt+\frac{d}{dr}\{\frac{1}{\Gamma(r)}\int_0^\varepsilon t^r\frac{d}{dt}Tr^s_M(\bar N e^{-tD_M^2})dt\}_{r=0} \\ = &Tr^s_M(\bar N P_{KerD_M}). \end{aligned}$$

Q.E.D.

We will link the $\mathcal{T}_w(M,E)$ to the analytic torsion form and the invariant $\mathcal{T}s(B,M,E)$ by an adiabatic limit formula in the next section. We refer to [GP] for the corresponding invariant in the operator situation.

6 Adiabatic Limits of Torsion-Signature Invariants

Our purpose of this section is to find an adiabatic limit formula of the invariant $\mathcal{T}_w(M,E)$ which is related to the analytic torsion form and the invariant $\mathcal{T}s(B,M,E)$ defined in Section 2. Unless specifically stated, we assume throughout this section that B is an even dimensional closed Riemannian manifold, M is a compact fiber bundle over B with a closed Riemannian manifold Z as typical fiber, and that E is a flat vector bundle over M.

Recall the exterior differential d_M of M with coefficients in E is equal to

$$d_M = d_Z + \nabla^{\mathcal{E}} + i_T.$$

Let $\bar{d}_M$, $\bar{d}_Z$, $\bar{\nabla}^{\mathcal{E}}$ and $\bar{i}_T$ be the adjoints of these operators with respect to the Hermitian structure on the total space M. We know $\Omega(M,E) \simeq \Omega(B)\hat{\otimes}C^\infty(B,\wedge^*(T^*Z,E|_Z))$. As before, denote by $\bar{N}$ and N' the number operators on $\Omega(M,E)$ given by $\bar{N}\xi = j\xi$ and $N'\xi = k\xi$ for $\xi \in \Omega^k(B)\hat{\otimes}C^\infty(B,\wedge^l(T^*Z,E|_Z))$, $k+l=j$. Now the adiabatic limit amounts to blowing up the metric of the base space B by considering the metric $g_\varepsilon^M = \frac{\pi^*(g^B)}{\varepsilon} \oplus g^Z$ on M and then taking the limit as $\varepsilon \to 0$. With respect to this new metric, the adjoint of the exterior differential $d_{M,\varepsilon}$ of M is

$$\bar{d}_{M,\varepsilon} = \varepsilon^{-N'}\bar{d}_M\varepsilon^{N'} = \bar{d}_Z + \varepsilon^{-1}\bar{\nabla}^{\mathcal{E}}\varepsilon + \varepsilon^{-2}\bar{i}_T\varepsilon^2.$$

Consider

$$C_\varepsilon = \frac{1}{2}(\varepsilon^{-N'}\bar{d}_M\varepsilon^{N'} + d_M)$$

and

$$D_\varepsilon = \frac{1}{2}(\varepsilon^{-N'}\bar{d}_M\varepsilon^{N'} - d_M).$$

We have that with a local orthonormal frame $\{e_i, e_\alpha\}$ of TM as in Section 2 where we used f_α for e_α,

$$\begin{aligned} C_\varepsilon &= \frac{1}{2}[(m^j\nabla^{M\otimes E}_{e_j,\varepsilon} + m^\alpha\sqrt{\varepsilon}\nabla^{M\otimes E}_{e_\alpha,\varepsilon})^* + (m^j\nabla^{M\otimes E}_{e_j,\varepsilon} + m^\alpha\sqrt{\varepsilon}\nabla^{M\otimes E}_{e_\alpha,\varepsilon})] \\ &= \frac{1}{2}(c^j\nabla^{M\otimes E,u}_{e_j,\varepsilon} + \sqrt{\varepsilon}c^\alpha\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}) - \frac{1}{4}c_v^j\psi_j - \frac{\sqrt{\varepsilon}}{4}c_v^\alpha\psi_\alpha, \end{aligned} \tag{23}$$

where $\nabla^{M\otimes E}_{,\varepsilon} = \nabla^M_{,\varepsilon}\otimes 1 + 1\otimes\nabla^E_{,\varepsilon}$, $\nabla^{M\otimes E,u}_{,\varepsilon} = \nabla^{M\otimes E}_{,\varepsilon} + \frac{1}{2}\psi$, and $\nabla^M_{,\varepsilon}$ is the Levi-Civita connection of M with respect to the metric g_ε^M. Using the usual Lichnerowicz formula

([BGV], P.126), we get

$$\begin{aligned}
C_\varepsilon^2 &= \frac{1}{4}\{(c^j\nabla^{M\otimes E,u}_{e_j,\varepsilon}+\sqrt{\varepsilon}c^\alpha\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon})^2-\frac{1}{2}[c^j\nabla^{M\otimes E,u}_{e_j,\varepsilon}+\sqrt{\varepsilon}c^\alpha\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon},c_v^j\psi_j+\sqrt{\varepsilon}c_v^\alpha\psi_\alpha]\\
&+ \frac{1}{4}(c_v^j\psi_j+\sqrt{\varepsilon}c_v^\alpha\psi_\alpha)^2\}\\
&= \frac{1}{4}\{-\Delta_\varepsilon^{M\otimes E,u}+\frac{K_\varepsilon}{4}+\frac{1}{2}c^ic^jR_\varepsilon(e_i,e_j)+\sqrt{\varepsilon}c^ic^\alpha R_\varepsilon(e_i,e_\alpha)\\
&+ \frac{1}{2}\varepsilon c^\alpha c^\beta R_\varepsilon(e_\alpha,e_\beta)-\frac{1}{2}[c^j(\nabla^{M\otimes E,u}_{e_j,\varepsilon}c_v^l)\psi_l+\sqrt{\varepsilon}c^j(\nabla^{M\otimes E,u}_{e_j,\varepsilon}c_v^\alpha)\psi_\alpha\\
&+ c^j(c_v^l(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_l)+\sqrt{\varepsilon}c_v^\alpha(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_\alpha))\\
&+ \sqrt{\varepsilon}c^\alpha((\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}c_v^l)\psi_l+\sqrt{\varepsilon}(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}c_v^\beta)\psi_\beta)\\
&+ \sqrt{\varepsilon}c^\alpha(c_v^l(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_l)+\sqrt{\varepsilon}c_v^\beta(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_\beta))]\\
&+ \frac{1}{4}(c_v^jc_v^k\psi_j\psi_k+\sqrt{\varepsilon}c_v^jc_v^\alpha\psi_j\psi_\alpha+\sqrt{\varepsilon}c_v^\alpha c_v^k\psi_k\psi_\alpha+\varepsilon c_v^\alpha c_v^\beta\psi_\alpha\psi_\beta)\}, \qquad (24)
\end{aligned}$$

where $\Delta_\varepsilon^{M\otimes E,u}=\sum_a((\nabla^{M\otimes E,u}_{e_a,\varepsilon})^2-\nabla^{M\otimes E,u}_{\nabla^M_{e_a,\varepsilon}e_{a,\varepsilon}})$, K_ε is the scalar curvature of (M,g_ε^M), and $R_\varepsilon=R_\varepsilon^M\otimes 1+1\otimes R^{E,u}$.To further compute the terms in (24), let $\nabla^\oplus=\pi^*(\nabla^B)\oplus\nabla^Z$ and $S=\nabla^M-\nabla^\oplus$. The torsion S has the following useful properties.

$(i)2<S(X_1)X_2,X_3>+<T(X_1,X_2),X_3>+<T(X_3,X_1),X_2>-<T(X_2,X_3),X_1>=0$,

where $T(X_1,X_2)=\nabla^\oplus_{X_1}X_2-\nabla^\oplus_{X_2}X_1-[X_1,X_2]$ is the torsion of $\nabla^\oplus$.

(ii) S is a one-form with values in antisymmetric elements of $End(TM)$. If $X\in TM$, then $S(X)$ maps TZ to T^HM. If $X\in T^HM$, then $S(X)$ maps T^HM to TZ.

(iii) $S(\tilde U)\tilde U=0$, for $\tilde U\in T^HM$.

(iv) For $\tilde U_i\in T^HM$ and $X_i\in TZ$,

$$\begin{aligned}
<S(X_1)X_2,\tilde U_1> &= -<S(X)\tilde U_1,X_2>,\\
<S(\tilde U_1)X_1,\tilde U_2> &= -<S(\tilde U_1)\tilde U_2,X_1>=<S(X_1)\tilde U_1,\tilde U_2>.
\end{aligned}$$

All other components $<S(\cdot),\cdot>$ vanish. These numbers are related to w_{abc} in Section 2.

$(v)<S(\cdot),\cdot>$ is independent of the metric on B.

We refer to [Bi][BC] for more details about these facts. Note that T takes values in TZ and $T(X_1,X_2)=0$ for $X_i\in TZ$ and $T(\tilde U,\tilde U)=0$ for $\tilde U\in T^HM$. In view of these facts, we have the following.

$$\nabla^{M\otimes E,u}_{e_i,\varepsilon} = \nabla^{Z\otimes E,u}_{e_i}+\frac{1}{2}<S(e_i)e_j,e_\alpha>c^j\sqrt{\varepsilon}c^\alpha+\frac{1}{4}<S(e_i)e_\alpha,e_\beta>\varepsilon c^\alpha c^\beta,$$

$$
\begin{aligned}
\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon} &= \sqrt{\varepsilon}(\nabla^{\oplus\otimes E,u}_{e_\alpha} + \frac{1}{2} < S(e_\alpha)e_i, e_\beta > c^i\sqrt{\varepsilon}c^\beta),\\
\nabla^{M\otimes E,u}_{\nabla^M_{e_i,\varepsilon}e_i,\varepsilon} &= (\nabla^{Z\otimes E,u} + S^\varepsilon)_{\nabla^Z_{e_i}e_i} + (\nabla^{Z\otimes E,u} + S^\varepsilon)_{\varepsilon S(e_i)e_i},
\end{aligned}
$$

and

$$
\nabla^{M\otimes E,u}_{\nabla^M_{e_\alpha,\varepsilon}e_\alpha,\varepsilon} = \sqrt{\varepsilon}(\nabla^{M\otimes E,u} + S^\varepsilon)_{\sqrt{\varepsilon}\nabla^M_{e_\alpha}e_\alpha}.
$$

Here $S^\varepsilon = \nabla^M_\varepsilon - \nabla^\oplus_\varepsilon$. We use the convention that $(\nabla_{e_a} + A(e_a))^2 = \sum_a(\nabla_{e_a} + A(e_a))^2_r - (\nabla + A)_{\sum_a \nabla_{e_a}e_a}$, where $(.)_r$ means an honest square. Then

$$
\begin{aligned}
\Delta^{M\otimes E,u}_\varepsilon &= (\nabla^{Z\otimes E,u}_{e_i} + \frac{1}{2} < S(e_i)e_j, e_\alpha > c^j\sqrt{\varepsilon}c^\alpha + \frac{1}{4} < S(e_i)e_\alpha, e_\beta > \varepsilon c^\alpha c^\beta)^2\\
&- \nabla^{Z\otimes E,u}_{\sum_i \varepsilon S(e_i)e_i} - S^\varepsilon(\sum_i \varepsilon S(e_i)e_i)\\
&+ (\sqrt{\varepsilon}(\nabla^{\oplus\otimes E,u}_{e_\alpha} + \frac{1}{2} < S(e_\alpha)e_i, e_\beta > c^i\sqrt{\varepsilon}c^\beta))^2. \qquad (25)
\end{aligned}
$$

By (24) and (25), we obtain the following important Lichnerowicz type formula.

Theorem 2

$$
\begin{aligned}
C^2_\varepsilon &= \frac{1}{4}\{-(\nabla^{Z\otimes E,u}_{e_i} + \frac{1}{2} < S(e_i)e_j, e_\alpha > c^j\sqrt{\varepsilon}c^\alpha + \frac{1}{4} < S(e_i)e_\alpha, e_\beta > \varepsilon c^\alpha c^\beta)^2\\
&+ \nabla^{Z\otimes E,u}_{\sum_i \varepsilon S(e_i)e_i} + \frac{\varepsilon}{2} < S(S(e_i)e_i)e_j, e_\alpha > c^j\sqrt{\varepsilon}c^\alpha\\
&- (\sqrt{\varepsilon}(\nabla^{\oplus\otimes E,u}_{e_\alpha} + \frac{1}{2} < S(e_\alpha)e_i, e_\beta > c^i\sqrt{\varepsilon}c^\beta))^2\\
&+ \frac{K_\varepsilon}{4} + \frac{1}{2}c^ic^jR_\varepsilon(e_i, e_j) + \sqrt{\varepsilon}c^ic^\alpha R_\varepsilon(e_i, e_\alpha) + \frac{1}{2}\varepsilon c^\alpha c^\beta R_\varepsilon(e_\alpha, e_\beta)\\
&- \frac{1}{2}[c^jc_v(\nabla^M_{e_j,\varepsilon}e_l)\psi_l + \sqrt{\varepsilon}c^jc_v(\nabla^M_{e_j,\varepsilon}e_\alpha)\psi_\alpha + c^jc^l_v(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_l)\\
&+ \sqrt{\varepsilon}c^jc^\alpha_v(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_\alpha) + \sqrt{\varepsilon}c^\alpha c_v(\nabla^M_{e_\alpha,\varepsilon}e_l)\psi_l + \varepsilon c^\alpha c_v(\nabla^M_{e_\alpha,\varepsilon}e_\beta)\psi_\beta\\
&+ \sqrt{\varepsilon}c^\alpha c^l_v(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_l) + \varepsilon c^\alpha c^\beta_v(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_\beta)]\\
&+ \frac{1}{4}(c^j_vc^k_v\psi_j\psi_k + \varepsilon c^\alpha_vc^\beta_v\psi_\alpha\psi_\beta)\}. \qquad (26)
\end{aligned}
$$

Q.E.D.

More generally, let z be an auxiliary odd Grassmann variable which anticommutes with all other odd Grassmann variables and $z^2 = 0$. We want to compute $4tC^2_\varepsilon - 2zt^{\frac{1}{2}}D_\varepsilon$. We have

$$
\begin{aligned}
D_\varepsilon &= \frac{1}{2}[(m^j\nabla^{M\otimes E}_{e_j,\varepsilon} + m^\alpha\sqrt{\varepsilon}\nabla^{M\otimes E}_{e_\alpha,\varepsilon})^* - (m^j\nabla^{M\otimes E}_{e_j,\varepsilon} + m^\alpha\sqrt{\varepsilon}\nabla^{M\otimes E}_{e_\alpha,\varepsilon})]\\
&= \frac{1}{2}[-c^j_v\nabla^{M\otimes E,u}_{e_j,\varepsilon} - \sqrt{\varepsilon}c^\alpha_v\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon} + \frac{1}{2}c^j\psi_j + \frac{\sqrt{\varepsilon}}{2}c^\alpha\psi_\alpha]. \qquad (27)
\end{aligned}
$$

Let

$$\mathcal{D}_{i,\varepsilon}=\nabla^{Z\otimes E,u}_{e_i}+\frac{1}{2}<S(e_i)e_j,e_\alpha>c^j\sqrt{\varepsilon}c^\alpha+\frac{1}{4}<S(e_i)e_\alpha,e_\beta>\varepsilon c^\alpha c^\beta,$$

and

$$\mathcal{D}_{\alpha,\varepsilon}=\sqrt{\varepsilon}(\nabla^{\oplus\otimes E,u}_{e_\alpha}+\frac{1}{2}<S(e_\alpha)e_i,e_\beta>c^i\sqrt{\varepsilon}c^\beta).$$

$$\begin{aligned}
&\mathcal{D}_{j,\varepsilon}\frac{z}{2\sqrt{t}}c^j_v+\frac{z}{2\sqrt{t}}c^j_v\mathcal{D}_{j,\varepsilon}-\frac{z}{2\sqrt{t}}c_v(\nabla^Z_{e_j}e_j)\\
=&\ \frac{z}{2\sqrt{t}}[c_v(\nabla^Z_{e_j}e_j)+2c^j_v\nabla^{Z\otimes E,u}_{e_j}+\frac{1}{2}<S(e_j)e_l,e_\alpha>\sqrt{\varepsilon}(c^j_vc^l-c^lc^j_v)c^\alpha\\
+&\ \frac{1}{4}<S(e_j)e_\alpha,e_\beta>\varepsilon 2c^\alpha c^\beta c^j_v-c_v(\nabla^Z_{e_j}e_j)]. \qquad (28)
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}_{\alpha,\varepsilon}\frac{z}{2\sqrt{t}}c^\alpha_v+\frac{z}{2\sqrt{t}}c^\alpha_v\mathcal{D}_{\alpha,\varepsilon}-\frac{z}{2\sqrt{t}}c_v(\sqrt{\varepsilon}\nabla^B_{e_\alpha}e_\alpha)\\
=&\ \frac{z\sqrt{\varepsilon}}{2\sqrt{t}}[c_v(\nabla^B_{e_\alpha}e_\alpha)+2c^\alpha_v\nabla^{\oplus\otimes E,u}_{e_\alpha}+\frac{1}{2}<S(e_\alpha)e_i,e_\beta>\sqrt{\varepsilon}2c^ic^\beta c^\alpha_v\\
-&\ c_v(\sqrt{\varepsilon}\nabla^B_{e_\alpha}e_\alpha)]. \qquad (29)
\end{aligned}$$

Hence by (27) – (29), we obtain

$$\begin{aligned}
&t[\mathcal{D}_{j,\varepsilon}\frac{z}{2\sqrt{t}}c^j_v+\frac{z}{2\sqrt{t}}c^j_v\mathcal{D}_{j,\varepsilon}-\frac{z}{2\sqrt{t}}c_v(\nabla^Z_{e_j}e_j)+\mathcal{D}_{\alpha,\varepsilon}\frac{z}{2\sqrt{t}}c^\alpha_v+\frac{z}{2\sqrt{t}}c^\alpha_v\mathcal{D}_{\alpha,\varepsilon}\\
-&\ \frac{z}{2\sqrt{t}}c_v(\sqrt{\varepsilon}\nabla^B_{e_\alpha}e_\alpha)]=-2z\sqrt{t}\mathcal{D}_\varepsilon+z\sqrt{t}(\frac{1}{2}c^j\psi_j+\frac{\sqrt{\varepsilon}}{2}c^\alpha\psi_\alpha). \qquad (30)
\end{aligned}$$

Therefore, we have proved by (26) and (30) the following Lichnerowicz type formula, which is useful in many situations.

Theorem 3

$$\begin{aligned}
4tC^2_\varepsilon-2z\sqrt{t}D_\varepsilon =&\ t\{-(\nabla^{Z\otimes E,u}_{e_i}+\frac{1}{2}<S(e_i)e_j,e_\alpha>c^j\sqrt{\varepsilon}c^\alpha+\frac{1}{4}<S(e_i)e_\alpha,e_\beta>\varepsilon c^\alpha c^\beta\\
-&\ \frac{z}{2\sqrt{t}}c^j_v)^2+\nabla^{Z\otimes E,u}_{\sum_i\varepsilon S(e_i)e_i}+\frac{\varepsilon}{2}<S(S(e_i)e_i)e_j,e_\alpha>c^j\sqrt{\varepsilon}c^\alpha\\
-&\ (\sqrt{\varepsilon}(\nabla^{\oplus\otimes E,u}_{e_\alpha}+\frac{\sqrt{\varepsilon}}{2}<S(e_\alpha)e_i,e_\beta>c^i\sqrt{\varepsilon}c^\beta-\frac{z}{2\sqrt{t}}c^\alpha_v))^2\\
+&\ \frac{K_\varepsilon}{4}+\frac{1}{2}c^ic^jR_\varepsilon(e_i,e_j)+\sqrt{\varepsilon}c^ic^\alpha R_\varepsilon(e_i,e_\alpha)+\frac{1}{2}\varepsilon c^\alpha c^\beta R_\varepsilon(e_\alpha,e_\beta)\\
-&\ \frac{1}{2}[c^jc_v(\nabla^M_{e_j,\varepsilon}e_l)\psi_l+\sqrt{\varepsilon}c^jc_v(\nabla^M_{e_j,\varepsilon}e_\alpha)\psi_\alpha+c^jc^l_v(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_l)\\
+&\ \sqrt{\varepsilon}c^jc^\alpha_v(\nabla^{M\otimes E,u}_{e_j,\varepsilon}\psi_\alpha)+\sqrt{\varepsilon}c^\alpha c_v(\nabla^M_{e_\alpha,\varepsilon}e_l)\psi_l+\varepsilon c^\alpha c_v(\nabla^M_{e_\alpha,\varepsilon}e_\beta)\psi_\beta\\
+&\ \sqrt{\varepsilon}c^\alpha c^l_v(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_l)+\varepsilon c^\alpha c^\beta_v(\nabla^{M\otimes E,u}_{e_\alpha,\varepsilon}\psi_\beta)]+\frac{1}{4}(c^j_vc^k_v\psi_j\psi_k+\varepsilon c^\alpha_vc^\beta_v\psi_\alpha\psi_\beta)\}\\
+&\ \frac{z}{2\sqrt{t}}(c^j\psi_j+\sqrt{\varepsilon}c^\alpha\psi_\alpha). \qquad (31)
\end{aligned}$$

Q.E.D.

We now apply the Getzler rescaling $G_{(\varepsilon t)^{\frac{1}{2}}}$ to $4tC_\varepsilon^2$. Namely, we make the following changes: $\nabla_{e_\alpha} \to (\varepsilon t)^{-\frac{1}{2}}\nabla_{e_\alpha}, c^\alpha \to ((\varepsilon t)^{-\frac{1}{2}}e_\alpha\wedge) - (\varepsilon t)^{\frac{1}{2}}\iota_{e_\alpha}$ and $c_v^\alpha \to c_v^\alpha$. Then by (26),

$$\begin{aligned}
&\lim_{\varepsilon\to 0} G_{(\varepsilon t)^{\frac{1}{2}}}(4tC_\varepsilon^2) = t\{-(\nabla_{e_i}^{Z\otimes E,u} + \frac{1}{2} < S(e_i)e_j, e_\alpha > c^j\frac{e_\alpha}{\sqrt{t}} + \frac{1}{4t} < S(e_i)e_\alpha, e_\beta > e_\alpha e_\beta)^2 \\
-&\frac{1}{t}(\partial_\alpha - \frac{1}{4}R^B_{\alpha\beta}x_\beta)^2 + \frac{K}{4} + \frac{1}{2}c^ic^jR(e_i,e_j) + \frac{c^im^\alpha}{\sqrt{t}}R(e_i,e_\alpha) + \frac{1}{2t}m^\alpha m^\beta R_0(e_\alpha,e_\beta) \\
-&\frac{1}{2}[c^jc_v^l(\nabla_{e_j,0}^{M\otimes E,u}\psi_l) + \frac{m^\alpha}{\sqrt{t}}c_v(\nabla^M_{e_\alpha,0}e_l)\psi_l + \frac{m^\alpha}{\sqrt{t}}c_v^l(\nabla^{M\otimes E,u}_{e_\alpha,0}\psi_l)] + \frac{1}{4}c_v^jc_v^k\psi_j\psi_k\}. \qquad (32)
\end{aligned}$$

Here we used the facts that with appropriate local orthonormal bases of TM and T^*M, $\nabla^M_{e_j,\varepsilon}e_l = \varepsilon(\nabla^M_{e_j}e_l)^H$ (horizontal component) and $\nabla^M_{e_j,\varepsilon}e_\alpha = (\nabla^M_{e_j}e_\alpha)^v + \varepsilon(\nabla^M_{e_j}e_\alpha)^H$. $(\cdot)^v$ is the vertical component. Note that $\nabla^{M\otimes E,u}_{,0} = \lim_{\varepsilon\to 0}\nabla^{M\otimes E,u}_{,\varepsilon} = \nabla^{M\otimes E,u} - P^HS = \nabla^{Z\otimes E,u} + \nabla^B$, where $P^H : TM = T^HM \oplus TZ \to T^HM$ is the projection. $R_0(e_\alpha,e_\beta) = R(e_\alpha,e_\beta) + \hat{R}^B(e_\alpha,e_\beta)$. Let $C_t = \frac{1}{2}(t^{-N/2}d_M^*t^{N/2} + t^{N/2}d_Mt^{-N/2})$. Here d_M^* is the fiberwise adjoint of d_M. Theorem 3.11 [BL] together with (32) proves

$$\begin{aligned}
\lim_{\varepsilon\to 0} G_{(\varepsilon t)^{\frac{1}{2}}}(4tC_\varepsilon^2) &= C_{4t}^2 - \frac{\sqrt{t}m^\alpha}{2}c_v^l(\nabla^B_{e_\alpha}\psi_l) \\
&- (\partial_\alpha - \frac{1}{4}R^B_{\alpha\beta}x_\beta)^2 + \frac{1}{2}m^\alpha m^\beta\hat{R}^B(e_\alpha,e_\beta) \\
&= C_{4t}^2 - \sqrt{t}C' - (\partial_\alpha - \frac{1}{4}R^B_{\alpha\beta}x_\beta)^2 + \frac{1}{2}m^\alpha m^\beta\hat{R}^B(e_\alpha,e_\beta) \\
&\overset{def}{\equiv} B_{4t}, \qquad (33)
\end{aligned}$$

where

$$C' = \frac{m^\alpha}{2}c_v^l(\nabla^B_{e_\alpha}\psi_l). \qquad (34)$$

C' comes from the term $m^\alpha c_v^l(\nabla^{M\otimes E,u}_{e_\alpha,0}\psi_l) = m^\alpha c_v^l(\nabla^{Z\otimes E,u}_{e_\alpha}\psi_l + \nabla^B_{e_\alpha}\psi_l)$ in (32). Note that if the metric g^E on E is either locally independent of B or covariantly-constant with respect to ∇^E, then $C' = 0$. In particular, if ∇^E is fiberwise unitary, then $C' = 0$. This is the case where the adiabatic limit of η-invariants is discussed [BC].

We now come to one of our main results. Let M_ε be the manifold M with the metric g_ε^M.

Theorem 4 *Let* $\dim(B)$ *be even,* $C' = 0$ *and* $H^*(Z, E|_Z) = 0$. *If* Z *is even dimensional, assume further* $ind_w(D_M) = 0$. *Then*

$$\lim_{\varepsilon\to 0}\log\mathcal{T}_w(M_\varepsilon, E) = 2^{\frac{\dim(B)}{2}}\int_B \varphi\mathcal{L}(B)tr(-\frac{R^B}{2}\coth\frac{R^B}{2})\mathcal{T}_1(T^HM, g^Z, g^E)$$

$$+ \quad 2^{\frac{\dim(B)}{2}} \int_B \varphi \mathcal{L}(B) \mathcal{T}(T^H M, g^Z, g^E) + \int_B \varphi \hat{A}(B) tr_*(-2\hat{R}^B e^{-\hat{R}^B}) \mathcal{T}_1(T^H M, g^Z, g^E), \quad (35)$$

where $\mathcal{L}(B) = \det^{\frac{1}{2}}(\frac{R^B/2}{\tanh R^B/2})$ *is the stable Hirzebruch L-class of* B*.*

Proof. Since $H^*(Z, E|_Z) = 0$, the operator $d_Z^* + d_Z$ is invertible. We can apply Proposition 4.41 [BC] to conclude that D_{M_ε} is invertible for ε very close to zero. Thus, $P_{KerD_M} = 0$. Hence $Tr^s_M(\frac{\bar{N}}{2} P_{KerD_M}) = 0$. By Proposition 6, this implies

$$\lim_{\varepsilon \to 0} \log \mathcal{T}_w(M_\varepsilon, E) = \lim_{\varepsilon \to 0} \int_0^\infty Tr^s_M(\frac{\bar{N}}{2} f'(i\sqrt{t} D_{M_\varepsilon})) \frac{dt}{t}. \quad (36)$$

Recall that N' is the number operator of B. $N' = \frac{1}{2} \sum_{i=0}^{\dim(B)} c(e_{a_i}) c_v(e_{a_i}) + \frac{\dim(B)}{2}$. Then

$$\bar{N} = (N + \frac{\dim(B)}{2}) + \frac{1}{2} \sum_{i=0}^{\dim(B)} c(e_{a_i}) c_v(e_{a_i}).$$

We thus break up (36) into two terms which correspond to the two terms of $\bar{N}$. Let us first consider the term associated to $(N + \frac{\dim(B)}{2})$. Let $g(x) = (1 + 2x)e^x$. By the fact that $V^2_{M,\varepsilon} = -C^2_\varepsilon$ and (33), we have

$$\begin{aligned} \lim_{\varepsilon \to 0} Tr^s_M(\frac{2N + \dim(B)}{4} g(4tV^2_{M,\varepsilon})) &= \lim_{\varepsilon \to 0} Tr^s_M(\frac{2N + \dim(B)}{4} g(G_{(\varepsilon t)^{\frac{1}{2}}}(-4tC^2_\varepsilon))) \\ &= Tr^s_M(\frac{2N + \dim(B)}{4} g(-B^2_{4t})) \quad (37) \end{aligned}$$

uniformly for $t \in [\delta, T]$ and arbitrary $\delta, T > 0$. (37) also holds uniformly for $t \in [0, \delta]$. Indeed by (33), $\lim_{\varepsilon \to 0} G_{\varepsilon^{\frac{1}{2}}}(4C^2_\varepsilon) = B_4$. The asymptotic expansion of $Tr^s_M(\frac{2N+\dim(B)}{4} g(-tG_{\sqrt{\varepsilon}}(4C^2_\varepsilon)))$ at the origin depends on the local symbol of the operator $G_{\sqrt{\varepsilon}}(4C^2_\varepsilon)$, and its coefficients approximate to those in the expansion of $Tr^s_M(\frac{2N+\dim(B)}{4} g(-tB_4))$ with remainder bounded by a number independent of ε. Hence we obtain

$$\int_0^T Tr^s_M(\frac{2N + \dim(B)}{4} g(4tV^2_{M,\varepsilon})) \frac{dt}{t} \to \int_0^T Tr^s_M(\frac{2N + \dim(B)}{4} g(-B_{4t})) \frac{dt}{t}, \quad \varepsilon \to 0.$$

To estimate the large time contribution of $Tr^s_M(\frac{2N+\dim(B)}{4} g(4tV^2_{M,\varepsilon}))$, we need the assumption that $H^*(Z, E|_Z) = 0$. Thus there exists a $\lambda_0 > 0$ such that the eigenvalues of $-V^2_{M,\varepsilon}$ are greater than λ_0 (cf. [BC], Prop. 4.41).

$$\begin{aligned} Tr^s_M(\frac{2N + \dim(B)}{2} e^{4tV^2_{M,\varepsilon}}) &\leq const. e^{-(4t-1)\lambda_0} Tr^s_M(e^{V^2_{M,\varepsilon}}) \\ &\leq const. e^{-(4t-1)\lambda_0} \varepsilon^{-\frac{\dim(B)}{2}}, \quad t \to \infty. \quad (38) \end{aligned}$$

Here we used the fact that the metrics g_ε^M have uniformly bounded geometry with volume growth as $\varepsilon^{-\frac{\dim(B)}{2}}$. Therefore,

$$|\int_T^\infty Tr_M^s(\frac{2N+\dim(B)}{4}e^{4tV_{M,\varepsilon}^2})\frac{dt}{t}| \leq const.\varepsilon^{-\frac{\dim(B)}{2}}\frac{1}{4T\lambda_0}e^{-(4T-1)\lambda_0}.$$

By choosing T such that $(4T-1)\lambda_0 = \frac{\dim(B)}{2}|\ln\varepsilon|$, we get

$$\int_T^\infty Tr_M^s(\frac{2N+\dim(B)}{2}e^{4tV_{M,\varepsilon}^2})\frac{dt}{t} \to 0, \quad \varepsilon \to 0. \tag{39}$$

Similarly, one can prove that as $\varepsilon \to 0$,

$$\int_T^\infty Tr_M^s(\frac{2N+\dim(B)}{2}2(4tV_{M,\varepsilon}^2)e^{4tV_{M,\varepsilon}^2})\frac{dt}{t} \to 0.$$

Note that $\int_T^\infty Tr_M^s(\frac{2N+\dim(B)}{4}g(-B_{4t}))\frac{dt}{t} \to 0$ as $T \to \infty$ (see the proof of Lemma 3.2 [GR]). Hence, we obtain

$$\lim_{\varepsilon\to 0}\int_0^\infty Tr_M^s(\frac{2N+\dim(B)}{4}g(4tV_{M,\varepsilon}^2))\frac{dt}{t} = \int_0^\infty Tr_M^s(\frac{2N+\dim(B)}{4}g(-B_{4t}^2))\frac{dt}{t}. \tag{40}$$

We now use local index theory techniques (cf. [BGV], Chapter 4) and get

$$\lim_{\varepsilon\to 0} G_{\varepsilon^{\frac{1}{2}}} tr_M^s(e^{4tV_{M,\varepsilon}^2}) = (2\pi i)^{-\frac{\dim(B)}{2}}\hat{A}(B)tr_*(e^{-\hat{R}^B})tr^s(e^{-C_{4t}^2+\sqrt{t}C'}).$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} G_{\varepsilon^{\frac{1}{2}}} tr_M^s(2(4tV_{M,\varepsilon}^2)e^{4tV_{M,\varepsilon}^2}) &= [\det{}^{\frac{1}{2}}(\frac{(1+2z)\frac{R^B}{2}}{\sinh(1+2z)\frac{R^B}{2}})(2\pi i(1+2z))^{-\frac{\dim(B)}{2}} \\ &\cdot \quad tr_*(e^{-(1+2z)\hat{R}^B})tr^s(e^{(1+2z)(-C_{4t}^2+\sqrt{t}C')})]^z.\end{aligned}$$

On the other hand, by Theorem 3.15 [BL] and the assumption $H^*(Z, E|_Z) = 0$, we have that $\varphi Tr^s(e^{-C_{4t}^2})$ is an exact form. This together with the proof of Theorem 1(v) implies that $\varphi Tr^s((1-2C_{4t}^2)e^{-C_{4t}^2})$ is also exact. Hence, by (40) and $C' = 0$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^\infty Tr_M^s(\frac{2N+\dim(B)}{4}f'(i\sqrt{t}D_{M_\varepsilon}))\frac{dt}{t} = \int_B \varphi\hat{A}(B)tr_*(e^{-\hat{R}^B})\mathcal{T}_1(T^HM, g^Z, g^E) \\ &+ \quad \{\int_B[(1+2z)^{-\frac{\dim(B)}{2}}\det{}^{\frac{1}{2}}(\frac{(1+2z)\frac{R^B}{2}}{\sinh(1+2z)\frac{R^B}{2}})tr_*(e^{-(1+2z)\hat{R}^B}) \\ &\cdot \quad \frac{d}{dr}(\frac{1}{\Gamma(r)}\int_0^\infty(\frac{1}{2})Tr^s(Ne^{(1+2z)(-C_{4t}^2)})t^r\frac{dt}{t})_{r=0}]^z\}(2\pi i)^{-\frac{\dim(B)}{2}}.\end{aligned} \tag{41}$$

Now, since $z^2=0$, $(1+2z)^{-\frac{\dim(B)}{2}}=1-\frac{\dim(B)}{2}2z$. Let $h(x)=\frac{1}{2}\log\frac{\frac{x}{2}}{\sinh\frac{x}{2}}$. $h'(x)=\frac{1}{2x}[1-\frac{\frac{x}{2}}{\tanh\frac{x}{2}}]$.

$$\begin{aligned}\det{}^{\frac{1}{2}}\Big(\frac{(1+2z)\frac{R^B}{2}}{\sinh(1+2z)\frac{R^B}{2}}\Big) &= \exp tr(h((1+2z)R^B))\\ &= \exp tr(h(R^B)+z(1-\frac{\frac{R^B}{2}}{\tanh\frac{R^B}{2}}))\\ &= \exp trh(R^B)(1+ztr(1-\frac{R^B}{2}\coth\frac{R^B}{2})).\end{aligned}$$

Since $\hat{A}(B)tr_*(e^{-\hat{R}^B})=2^{\frac{\dim(B)}{2}}\mathcal{L}(B)$, a computation proves that the second term in the right hand side of (41) is equal to

$$\begin{aligned}&\{\int_B\varphi\mathcal{L}(B)(-\dim(B)-1+tr(1-\frac{R^B}{2}\coth\frac{R^B}{2}))\mathcal{T}_1(T^HM,g^Z,g^E)\\ +&\int_B\varphi\mathcal{L}(B)\int_0^\infty(\frac{1}{2})\varphi Tr^s(N(1+2(-C_{4t}^2))e^{-C_{4t}^2})\frac{dt}{t}\}2^{\frac{\dim(B)}{2}}\\ -&\ 2\int_B\varphi\hat{A}(B)tr_*(\hat{R}^Be^{-\hat{R}^B})\mathcal{T}_1(T^HM,g^Z,g^E).\end{aligned}$$

From this we get that the left hand side of (41) is equal to the right hand side of (35). Therefore, by (36) it remains to check

$$\lim_{\varepsilon\to 0}\int_0^\infty Tr_M^s(\frac{1}{2}\sum_{i=0}^{\dim(B)}c(e_{a_i})c_v(e_{a_i})g(4tV_{M,\varepsilon}^2))\frac{dt}{t}=0.$$

But this clearly follows from the proofs of Theorem 1 (iii) and of (41). Q.E.D.

By the above proof, we get the following corollary.

Corollary 1 *With the condition of Theorem* 4*,*

$$\lim_{\varepsilon\to 0}\log\mathcal{T}_w(M_\varepsilon,E)=2^{\frac{\dim(B)}{2}}\int_B\varphi\mathcal{L}(B)\mathcal{T}_1(T^HM,g^Z,g^E).$$

Proof. By assumption, we have that for $\tau>0$ very small and $\varepsilon\to 0$,

$$\begin{aligned}\log\mathcal{T}_w(M_\varepsilon,E) &= \frac{d}{dr}(\frac{1}{\Gamma(r)}\int_0^\tau t^rTr_M^s(\frac{\bar{N}}{2}e^{-tD_{M_\varepsilon}^2})\frac{dt}{t})_{r=0}\\ &+ \int_\tau^\infty Tr_M^s(\frac{\bar{N}}{2}e^{-tD_{M_\varepsilon}^2})\frac{dt}{t}.\end{aligned}$$

The result follows from the proof of Theorem 4. Q.E.D.

Remark 1 (i) *Formula* (35) *is much more complicated than the formula in Corollary* 1*. This is due to the fact that compared to* $\mathcal{T}_1(T^HM,g^Z,g^E)$*, the analytic torsion form has an* *extra term. Up to a constant, the second term of* (35) *is the invariant* $\mathcal{T}s(B, M, E)$ *in Section* 2.

(ii) The argument above also works for the adiabatic limit of the analytic torsion invariant of the total space. But for this case one can only see the contribution of the degree zero component of the analytic torsion form, since the Euler form $e(TB, \nabla^B)$ *is in the top dimension* $\dim(B)$. *The formula under the assumption of Theorem* 4 *is*

$$\begin{aligned} &\lim_{\varepsilon\to 0}\log\mathcal{T}(M_\varepsilon, E) = (2\pi i)^{\frac{dimB}{2}}\int_B \varphi e(TB,\nabla^B)\mathcal{T}(T^HM, g^Z, g^E) \\ +\ & (2\pi i)^{\frac{dimB}{2}}\int_B \varphi e(TB,\nabla^B) tr(-\frac{R^B}{2}\coth\frac{R^B}{2})\mathcal{T}_1(T^HM, g^Z, g^E) \\ +\ & \int_B \varphi \hat{A}(B) tr_r^s(-2\hat{R}^B e^{-\hat{R}^B})\mathcal{T}_1(T^HM, g^Z, g^E). \end{aligned} \tag{42}$$

Indeed, to apply the proof of Theorem 4 *to the situation here, we need only to change the trace* Tr^s_M *to* Tr^s *and* $ind_w(D_M)$ *to* $ind(D_M)$ *and note that* $\hat{A}(B)tr_r^s(e^{-\hat{R}^B}) = (2\pi i)^{\frac{\dim(B)}{2}} Pf(-\frac{R^B}{2\pi})$. *Similar formula to that in Corollary* 1 *also holds.*

(iii) For $C' \neq 0$, *one needs other forms to get an adiabatic limit formula. The case where* $H^*(Z, E|_Z) \neq 0$ *is much more involved, and is considered by Dai-Melrose [DM] and Forman [Fo] for the analytic torsion invariant. Theorem* 4 *can be extended to the* L^2*-case.*

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Max-Planck-Institut für Mathematik, Gottfried Claren Str. 26, 53225 Bonn, Germany