Quasi-symmetric line bundles on abelian varieties

Eyal Z. Goren

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn
GERMANY

MPI 95-106

Quasi - Symmetric Line bundles on Abelian varieties

EYAL Z. GOREN

We study a line bundle L over an abelian variety X and an isogeny $f: X \longrightarrow X$ satisfying $f^*L \cong L^n$. We study the problem of explicitly describing the action of f on global sections of powers of L and we determine the relations imposed by f on the 'thetanulwerte' coming from such sections. In addition, the representation theory of finite and adelic Heisenberg groups is discussed.

0. INTRODUCTION

Let X be an abelian variety over an algebraically closed field k of characteristic $p \geq 0$. Let L be an ample line bundle of separable type on X (i.e. $\deg(L)$ is prime to p if $p > 0$). Assume that there exist an isogeny $f: X \longrightarrow X$ such that $f^*L \cong L^n$ for some n and $(deg(f), p) = 1$ if $p > 0$. We say then that f is *quasi symmetry* of L and if $f \neq 1$ that L is *quasi - symmetric*. If $n = 1$ we say that f is a *symmetry* of L. This phenomenon is interesting in the context of the general theory of abelian varieties. One can also motivate the interest in such line bundles by the following observations:

(i) As explained in detail below, automorphisms of a curve $\mathcal{C}$ induce automorphisms of Jac($\mathcal{C}$) and the second power of a carefully chosen line bundle L (inducing the natural polarization of Jac($\mathcal{C}$)) is stable under all these automorphisms.

(ii) If f only satisfies $f^*\phi_L = n\ \phi_L$, where $\phi_L : X \longrightarrow X^\vee$ is the polarization induced by L, that is, if f is an isogeny of the polarized abelian variety (X, ϕ_L) then, as explained in detail below, if L is symmetric, we have $f^*L^2 = (L^2)^n$. This shows that the situation we are dealing with is quite common, and in fact there exist whole families in appropriate moduli spaces characterized by this property.

(iii) Sections of ample line bundles are given over the complex numbers by Riemann's theta functions with characteristics. There are various methods by which one can determine a field containing the values of these functions at points corresponding, for example, to abelian varieties with complex multiplication, of the moduli space of abelian varieties with principal polarization. It is of interest to understand these values as closely as possible.

Before stating some of the results of this paper we recall briefly the definition and basic properties of finite and adelic Heisenberg groups. For a complete account see [Mum1], [Mum3].

HEISENBERG GROUPS.

DEFINITION. A *finite Heisenberg group* G is a group for which there exists an exact sequence

$$1 \longrightarrow k^{\times} \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

such that $k^{\times}$ is the center of G and H is a finite abelian group.

It follows from the definition that the *commutator pairing* from $H \times H$ to $k^{\times}$ is a non-degenerate bimultiplicative skew-symmetric pairing and this implies that if $p > 0$ then $(p, \#H) = 1$ and that the elementary divisors of H appear in pairs (We denote the number of elements of H by d^2). In fact these are the only restrictions on H.

G always contains a finite group, denoted by G^c, characterized as the set of elements whose d^2-th power is trivial. G^c sits in the exact sequence

$$1 \longrightarrow \mu_{d^2} \longrightarrow G^c \longrightarrow H \longrightarrow 1$$

where μ_{d^2} is the group of d^2-th roots of 1 in $k^{\times}$.[1]

We say that a subgroup K of G is a *level subgroup* if π induces an isomorphism between K and $\pi(K)$, and we say then that *K lies above* $\pi(K)$. If F is a subgroup of H then F has a level group above it if and only if F is totally isotropic with respect to the commutator pairing. In fact, any level subgroup of G is contained in G^c, because any level subgroup is of exponent d. We can always find two maximal isotropic subgroups F, F' of H such that $H = F \oplus F'$ - one says that F *has an orthogonal complement* - but it is not true that every maximal isotropic subgroup has an orthogonal complement. This decomposition enables one to prove that G is determined up to an isomorphism by H.

If H is a finite Heisenberg group then Mumford has proved in [Mum1], in analogy with the Stone Von-Neumann theorem, that there exists a unique irreducible representation of H on which $k^{\times}$ acts through its natural character. A complete description of the basic representation theory of finite and adelic Heisenberg groups appears in the appendix.

Finite Heisenberg groups arise as follows :

DEFINITION. Let X be an abelian variety over an algebraically closed field k of characteristic $p \geq 0$. Let L be an ample line bundle on X. We say that L is a line bundle of *separable type* if $(\deg(L), p) = 1$ if $p > 0$ (for $p = 0$ every line bundle is of separable type). Given a line bundle of separable type L we define the *Heisenberg group G(L) associated to it* by

$$G(L) = \{ \phi : L \longrightarrow L \mid \phi \text{ is an automorphism of } L \text{ covering translation by } x \text{ on the base} \}$$

That is $G(L)$ is the group of automorphisms ϕ of L for which there exists some x such that ϕ fits into a commutative diagram

---

[1] Let e be the exponent of H, then one may define such subgroups where d^2 is replaced by e^2, $2e$ (or even e itself if e is odd). The particular choice d^2 is both canonical and convenient. These remarks follow from the formula $(xy)^n = x^n y^n \cdot [y,x]^{n(n-1)/2}$.

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L \\ \downarrow & & \downarrow \\ X & \xrightarrow{T_x} & X, \end{array}$$

where we denote by T_x the translation map $T_x(y) = x + y$.

There is a short exact sequence

$$1 \longrightarrow k^{\times} \longrightarrow G(L) \longrightarrow H(L) \longrightarrow 0,$$

where $H(L) = \operatorname{Ker} \phi_L$, $\phi_L(x) = T_x{}^*L \otimes L^{-1}$. $G(L)$ is a finite Heisenberg group. We refer the reader for a proof of this fact as well for a general discussion of these groups to [Mum1] and [Mum3].

DEFINITION. Let $\mathbb{A}_f$ denote the adele ring of $\mathbb{Q}$ with the component corresponding to p omitted if $p > 0$. An *adelic Heisenberg group* is a group G fitting into an exact sequence

$$1 \longrightarrow k^{\times} \longrightarrow G \longrightarrow \mathbb{A}_f^{2g} \longrightarrow 0,$$

such that $k^{\times}$ is precisely the center of G or, equivalently, such that $k^{\times}$ is contained in the center of G and the commutator pairing

$$\mathbb{A}_f^{2g} \times \mathbb{A}_f^{2g} \longrightarrow k^{\times}$$

is non degenerate.

The uniqueness of a skew - symmetric non degenerate pairing on $\mathbb{A}_f^{2g}$ implies that every adelic Heisenberg group is isomorphic to the group

$$k^{\times} \times \mathbb{A}_f^{g} \times \mathbb{A}_f^{g}$$

with the group law

$$(\alpha, x_1, x_2)(\beta, y_1, y_2) = (\alpha\beta\, \underline{\mathbf{e}}(\tfrac{1}{2}({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)), x_1 + y_1, x_2 + y_2)$$

where

$$\underline{\mathbf{e}} : \mathbb{A} / \hat{\mathbb{Z}} \longrightarrow (k^{\times})_{tor}$$

is a fixed isomorphism. We will usually denote this group by G and call it *the standard adelic Heisenberg group.*

In contrast to finite Heisenberg groups the representation theory of adelic Heisenberg groups is simple. We call a representation of G a *representation of order* n if $k^{\times}$ acts through the character $\alpha \longmapsto \alpha^n$. In analogy with the theory of real Heisenberg groups, there exists a unique continuous irreducible representation of order n for every $n \neq 0$ (see appendix).

Adelic Heisenberg groups arise from abelian varieties by a ' limiting process ' :

DEFINITION. Let X be an abelian variety over an algebraically closed field k of characteristic $p \geq 0$. Let L be an ample line bundle on X of separable type. Let $T(X)$ be the 'separable' Tate module of X, that is,

$$T(X) = \lim_{\substack{\longleftarrow \\ (n,p)=1}} X[n]$$

(if $p = 0$ there is no condition in the limit). Let $V(X) = T(X) \otimes \mathbb{Q}$.

Define the *adelic Heisenberg group associated with* L , $\hat{G}$ (L), as follows :

$\hat{G}\ (L)$ = set of sequences of the form $(x_n, \phi_n)_{n\in \mathbb{N}}$, where : $(x_n)_{n\in \mathbb{N}}$ is an element of $V(X)$, the maps ϕ_n are defined if and only if $x_n \in H(n^*L)$, and then $\phi_n \in G(n^*L)$. If both ϕ_n and ϕ_m are defined and $m = nd$, then $d^*\phi_n = \phi_m$ where the pull back is with respect to $d\, x_m = x_n$ [2]. The group law is given by

$$(x_n, \phi_n)_{n\in \mathbb{N}}\ (y_n, \psi_n)_{n\in \mathbb{N}} = (x_n + y_n, \phi_n \circ \psi_n)_{n\in \mathbb{N}}.$$

$\hat{G}\ (L)$ is an adelic Heisenberg group and there is an exact sequence

$$1 \longrightarrow k^\times \longrightarrow \hat{G}\ (L) \xrightarrow{\pi} V(X) \longrightarrow 0.$$

(For these facts as well as others stated below see [Mum1], [Mum2], [Mum3]). There is a canonical homomorphic section

$$\sigma^L : T(X) \longrightarrow \hat{G}\ (L)$$

given by $(x_n)_n \longmapsto (x_n, \phi_n)_n$ where $(x_n, \phi_n)_n$ is the *unique* element of $\hat{G}\ (L)$ such that ϕ_1 is the identity map. In general $\sigma^L\,(T(X))$ is not a maximal level subgroup. Actually there is a natural isomorphism for every n

$$Normalizer\ (\ \sigma^L(n\,T(X))\ )\ /\ \sigma^L(n\,T(X)) \ \cong\ G(n^*L).$$

<u>Suppose from now on that L is symmetric.</u> Then there is a canonical section

$$\tau^L : V(X) \longrightarrow \hat{G}\ (L)$$

constructed as follows : given $x \in V(X)$ choose some $y \in \hat{G}\ (L)$ such that $2\,\pi(y) = x$, and put $\tau^L(x) = y \cdot \delta_{-1}(y)^{-1}$ (for the definition of δ_{-1}, which is an automorphism of $\hat{G}\ (L)$, inducing multiplication by -1 on $V(X)$ and the identity on $k^\times$, deduced from multiplication by -1, see section II). This definition does not depend on y and defines a section, though not homomorphic, to π.

REPRESENTATIONS AND BASES. There is a natural action of $G(L)$ on $\Gamma(X, L)$:

Let $\phi \in G(L)$ cover translation by x and let $s \in \Gamma(X, L)$ be a global section of L. Then

$$U_\phi(s) = \phi \circ s \circ T_{-x}$$

defines an action of $G(L)$ where the center of $G(L)$ acts naturally. In fact this representation is irreducible. It follows from the discussion of the appendix that choosing a maximal level subgroup K, a K invariant vector v_1 (which is unique up to a scalar) and a section Σ for the commutator map $\chi : G(L) \longrightarrow K^*$ (K^* = the characters of K) we get a basis for $\Gamma(X, L)$. Namely, $\{\ U_{\Sigma(\xi)}(v_1) \mid \xi \in K^*\ \}$. We shall always assume that $\Sigma(1) = 1$. We have also a decomposition of $\Gamma(X, L)$ into eigenspaces of K :

$$\Gamma(X, L) = \bigoplus_{\xi \in K^*} \Gamma(X, L)_\xi$$

where each $\Gamma(X, L)_\xi$ is one dimensional and spanned by $U_{\Sigma(\xi)}(v_1)$. When K has an orthogonal complement K' we can do better. We can choose Σ as the unique isomorphism

[2] In general if $f : X \longrightarrow Y$ is an isogeny and L is a line bundle on Y, then for every isomorphism $\phi : L \longrightarrow L$ covering translation by y on the base and for every $x \in X$ such that $f(x) = y$ there is a unique isomorphism $f^*\phi : f^*L \longrightarrow f^*L$ covering translation by x on the base which is obtained from ϕ. We call this isomorphism - which depends on x - the pull back $f^*\phi$ with respect to $f(x) = y$.

with K' induced by the commutator pairing.

The theory of descent shows (loc. cit.) that there is one - one correspondence between level subgroups K over a fixed subgroup F of $H(L)$ and isomorphism classes of line bundles M on X / F such that $p^*M \cong L$, where $p : X \longrightarrow X / F$ is the natural projection.

Given such M one associates to it all the automorphisms of L that are of the form p^*Id. with respect to all $x \in F$. In that case one can show that $\Gamma(X, L)$ is the direct sum $\bigoplus_{M : p^*M \cong L} p^*\Gamma(X / F, M)$ and in fact fixing some level subgroup K, that corresponds to one of these M's, this is exactly the decomposition to eigenspaces of K.

The analogue for adelic Heisenberg groups is as follows : define

$$\hat{\Gamma}(X, L) = \lim_{\rightarrow} \Gamma(X, n^*L)$$

where the limit runs over all n prime to p if $p > 0$, and is taken with respect to the injections

$$d^* : \Gamma(X, n^*L) \hookrightarrow \Gamma(X, d^*n^*L).$$

Given $s \in \hat{\Gamma}(X, L)$ and $(x_n, \phi_n)_n \in \hat{G}(L)$ define

$$U_{(x_n, \phi_n)_n}(s) = \phi_m \circ s \circ T_{-x_m}$$

where $s \in \Gamma(X, m^*L)$. This is a well defined group action of $\hat{G}(L)$ on $\hat{\Gamma}(X, L)$ and the fundamental fact is that it is irreducible. There is a one to one correspondence between maximal level subgroups containing $\sigma^L(m\, T(X))$ for some m and line bundles of degree one on abelian varieties rationally isogenous to X whose pull back to X is rationally isomorphic to L (see [Mum3] p. 62 ff for the definition and properties of rational isogenies). We call such level subgroups *commensurable with* $\sigma^L(T(X))$. To any such commensurable maximal level subgroup K and a section Σ to the commutator map $\chi : \hat{G}(L) \longrightarrow K^*$ (continuous characters) one can associate a basis $\{ U_{\Sigma(\xi)}(v_1) \mid \xi \in K^* \}$, where $v_1 \in \hat{\Gamma}(X, L)$ is a fixed K invariant vector (which is unique up to a scalar). We have also a decomposition to eigenspaces

$$\hat{\Gamma}(X, L) = \bigoplus_{\xi \in K^*} \hat{\Gamma}(X, L)_\xi$$

and each $\hat{\Gamma}(X, L)_\xi$ is a one dimensional space spanned by $U_{\Sigma(\xi)}(v_1)$. One should notice that for general level subgroups there is no such decomposition although there are certain families of related bases that one can construct from some other level subgroups. We will develop this idea in §3.

CONVENTION. Generally we will denote basis elements with respect to a general level subgroup (it may or may not have an orthogonal complement) by the letters v or δ. We will denote them by the letter s if the subgroup has an orthogonal complement.

MAIN RESULTS.

In general, the content of this paper is as follows:

§1 contains the general geometric background of quasi - symmetric line bundle and the definition of the homomorphisms associated with them.

§2 is devoted to the proof of

THE SYMMETRY THEOREM (PRELIMINARY FORM). *Let X be an abelian variety over an algebraically closed field k. Let L be an ample line bundle of separable type on X. Let $\mathcal{A}(L) = \{g \in Aut(X) \mid g^*L \cong L\}$. Assume that there exists a maximal isotropic subgroup Z of $H(L)$ which is $\mathcal{A}(L)$ - invariant and choose some level subgroup $\mathcal{Z}$ above Z. Choose some non zero vector $v_1 \in \Gamma(X, L)_1 = \Gamma(X, L)^{\mathcal{Z}}$ and choose a section*

$$G(L)^c \xleftarrow{\Sigma} \mathcal{Z}^*, \ \Sigma(1) = 1.$$

Given $g \in \mathcal{A}(L)$, there exists an automorphism δ_g of $G(L)$ lifting the action of g on $H(L)$ (see §1 below). Let $\phi : g^*L \longrightarrow L$ be an isomorphism (determined up to a scalar). Let $T = \phi_* g^* \in \mathrm{End}(\Gamma(X, L))$.
Then :

(*a*) *There exists a unique character $\gamma_g \in \mathcal{Z}^*$, characterized by either :*

(i) $\gamma_g(z)^{-1}\delta_g(z) \in \mathcal{Z}$ *for all* $z \in \mathcal{Z}$, *or by*

(ii) *If M is the line bundle on X / Z corresponding to $\mathcal{Z}$ then $g^*M \otimes M^{-1}$ corresponds to γ_g under*

$$\mathcal{Z}^* \cong Z^* \cong \mathrm{Ker}\,(\,\mathrm{Pic}^0(X / Z) \longrightarrow \mathrm{Pic}^0(X)\,).$$

(*b*) *There exists scalars $b_{g,\chi} \in \mu_{d^2}$ ($\chi \in \mathcal{Z}^*$), determined by the equation*

$$b_{g,\chi}\, U_{\Sigma(\gamma_g \cdot g\chi)}\, v_1 = U_{\delta_g^{-1}(\Sigma(\chi))} \cdot U_{\Sigma(\gamma_g)}\, v_1 \quad ,$$

such that

$$T\left(\sum_{\chi \in \mathcal{Z}^*} a_\chi v_\chi\right) = c(g) \sum_{\chi \in \mathcal{Z}^*} a_\chi b_{g,\chi} v_{\gamma_g \cdot g\chi} \quad ,$$

where $\{ v_\chi = U_{\Sigma(\chi)} v_1 \}$ is a basis for $\Gamma(X, L)$ and $c(g)$ is a scalar determined by the equality $Tv_1 = c(g)\, v_{\gamma_g}$. In particular the matrix describing T, which is given explicitly by the $b_{g,\chi}$'s, is monomial and unitary.

REMARKS. 1) Note that the underlying permutation of T is $\chi \longmapsto \gamma_g \cdot g\chi$. Note also that $b_{g,1} = 1$.

2) For L very ample, T is actually *writing the automorphism g by coordinates.* Note that the indeterminacy up to a scalar of T disappears in projective coordinates.

3) Given any finite automorphism g of X we can create an ample line bundle for which g is a symmetry by taking the ' norm ' of any ample line bundle with respect to g. Since the resulting line bundle is ample our method applies. In particular we see that for every automorphism $g \in \mathrm{Aut}(X)$ of finite order there exists a projective embedding such that the action of g on X is given by a monomial unitary matrix.

4) Both in the case of $k = \mathbb{C}$ and in the general case we can get from the symmetry theorem identities between *functions*. Over $\mathbb{C}$ this could be done by trivializing the pull-back of our line bundle to the universal covering space. In general we may trivialize the pull-back of our line bundle to $V(X)$ á la Mumford ([Mum3]).

§3 contains several topics. We discuss the functorial behavior of adelic Heisenberg groups, projection operators on finite Heisenberg groups and the construction of 'compatible' bases to

$\Gamma(X, n^*L)$. Finally we prove a result analogous to the symmetry theorem for isogenies $f: X \longrightarrow X$ such that $f^*L \cong L^n$ and such that $(deg(f), p) = 1$ if $p > 0$. The precise formulation requires too much preparatory work to be stated here. Along the way we state another elegant version of the symmetry theorem and the section closes with explaining how, under mild restrictions, we can extend our results to isogenies $f : X \longrightarrow Y$ with appropriate line bundles.

§4 contains some examples illustrating the theory.

§5 consists of three topics. The extension of the simultaneous construction of bases for $\Gamma(X, n^*L)$ to all $\Gamma(X, L^n)$. The extension of our results to $\mathbb{Q}$ - isogenies.A concise dictionary between the analytical and algebraic theory.

In the appendix we classify all the irreducible representation of finite Heisenberg groups and determine the decomposition of tensor products of such representations, hence giving an explicit description of the representation ring. The same results are obtained for continuous representations of adelic Heisenberg groups. Although for finite Heisenberg groups there might be many non isomorphic irreducible representations of order n, for adelic Heisenberg groups there is a unique irreducible continuous representation of order n for any $n \in \hat{\mathbb{Z}} - \{0\}$. This is in complete analogy to the well known case of real Heisenberg groups.

The results are explicit enough to easily determine for example the decomposition of $\Gamma(X, L^n)$ as a module of $G(L)$ acting via the natural homomorphism $\varepsilon_n : G(L) \longrightarrow G(L^n)$, or of $Sym^2(\Gamma(X, L))$ yielding in this case a new interpretation of the notion of even and odd theta functions and sheding more light, so we believe, on the multiplication map

$$\Gamma(X, L)^{\otimes n} \longrightarrow \Gamma(X, L^n).$$

ACKNOWLEDGMENTS. It is my pleasure to thank EHUD DE SHALIT for many valuable comments. The final version of this paper was written while enjoying the warm hospitality of the Max Planck Institue at Bonn.

1. QUASI - SYMMETRIC LINE BUNDLES ON ABELIAN VARIETIES.

We retain the notation fixed in the introduction. Thus X is an abelian variety over an algebraically closed field k of characteristic $p \geq 0$. L is an ample line bundle of separable type on X etc.

DEFINITION. Define

$$\mathcal{A}(L) = \{\ g \in Aut(X) \mid L \cong g^*L\ \},$$
$$\mathcal{A}^+(L) = \{\ (g, \phi) \mid \phi : L \rightarrow g^*L \text{ an isomorphism }\}.$$

Note that according to our terminology, L is *quasi - symmetric* if $\mathcal{A}(L) \neq \{1\}$. As customary L is called *symmetric* if $-1 \in \mathcal{A}(L)$.

REMARKS. 1) $\mathcal{A}(L)$ is a finite group.

2) In the sequel we will often confuse divisors, line bundles and invertible sheaves. While this might cause some confusion it has the advantage of making some arguments more transparent. In this connection we remark that one can define $\mathcal{A}^+(L)$ for L = divisor, line bundle, invertible sheaf (in the obvious way) such that under the usual transition between the different concepts the definitions of the various $\mathcal{A}^+(L)$ agree.

LEMMA 1. *There is an exact sequence of groups*

$$1 \longrightarrow k^\times \longrightarrow \mathcal{A}^+(L) \longrightarrow \mathcal{A}(L) \longrightarrow 1,$$

where we define

$$(g, \phi)(h, \psi) = (gh, h^*\phi \circ \psi).$$

This sequence always splits.

Proof. The assertion about the exact sequence is easy to check. To prove the second assertion we note that it is enough to prove that there exists some divisor D on X, such that $L \cong \mathcal{O}_X(D)$ and such that $g^{-1}(D) = D$ for all $g \in \mathcal{A}(L)$. Indeed, given such D, let

$$\alpha : \mathcal{O}_X(D) \longrightarrow L$$

be an isomorphism. Then for every $g \in \mathcal{A}(L)$ we have

$$L \xrightarrow{\alpha^{-1}} \mathcal{O}_X(D) = \mathcal{O}_X(g^{-1}D) \xrightarrow{\sim} g^*\mathcal{O}_X(D) \xrightarrow{g^*\alpha} g^*L$$

which gives us a splitting homomorphism

$$\mathcal{A}(L) \longrightarrow \mathcal{A}^+(L), \quad g \longmapsto (g, g^*\alpha \circ \alpha^{-1}).$$

To find such D start with any divisor F such that $L \cong \mathcal{O}_X(F)$ and such that $0 \notin supp(F)$. The isomorphism $g^*L \cong L$ implies that for all $g \in \mathcal{A}(L)$ there exists a function f_g such that

$$g^*F = F + (f_g).$$

Since $0 \notin supp(F)$ we also have $0 \notin supp(g^*F)$ and therefore $0 \notin supp(f_g)$ and we may normalize the functions f_g for all $g \in \mathcal{A}(L)$ by requiring that $f_g(0) = 1$ and that determines each f_g uniquely.

It is easy to check that

$$g \longmapsto f_g$$

is a 1 - cocycle in $Z^1(\mathcal{A}(L)^{op}, k(X)^\times)$, where $f_g^h(x) = f_g(h(x))$ defines the left action of $\mathcal{A}(L)^{op}$ on $k(X)$. By Hilbert's 90 there exists $\Omega \in k(X)^\times$ such that for all $g \in \mathcal{A}(L)$

$$f_g = \Omega^g / \Omega.$$

We take $D = F - (\Omega)$. Q.E.D.

COROLLARY 1 (of proof). For every line bundle L there exists an $\mathcal{A}(L)$ - invariant divisor D, such that $L \cong \mathcal{O}_X(D)$.

REMARK. Assume that every simple component of X is of dimension at least 2. Define $K_{\mathcal{A}(L)} = X / \mathcal{A}(L)$ and let $\pi : X \longrightarrow K_{\mathcal{A}(L)}$ be the natural quotient map. Then there exists a a divisor F on $K_{\mathcal{A}(L)}$ such that $\pi^*F = D$ where D is a divisor defining L. Indeed, if we define

$$\mathcal{R}(L) = \{ x \in X \mid stab_{\mathcal{A}(L)}(x) \neq \{1\} \}.$$

Since

$$\mathcal{R}(L) = \bigcup_{1 \neq g \in \mathcal{A}(L)} Ker(1-g)$$

it follows that codim($\mathcal{R}(L)$) ≥ 2. Therefore, letting $X^f = X - \mathcal{R}(L)$, $K^f = X^f / \mathcal{A}(L)$ we have natural isomorphisms $\mathrm{Cl}(X) = \mathrm{Cl}(X^f)$, $\mathrm{Cl}(K) = \mathrm{Cl}(K^f)$ and we reduce to proving the same assertion for $D|_{X^f}$ and proving the existence of such F on K^f. But, now the map

$$\pi : X^f \longrightarrow K^f$$

is étale and finite. For such maps, descent theory tells us that such a divisor F exists. Note, however, that we can not conclude that there exists a line bundle M on $K_{\mathcal{A}(L)}$ such that $\pi^*M \cong L$ (the precise conditions for $\mathcal{A}(L) = \{\pm 1\}$ were given in [Mum1]) . The essential reason for that is that on a singular varieyt the concepts of Weil and Cartier divisors diverge. A concrete example is given by any ample symmetric $L(H, \chi)$ such that χ is non-trivial. (see §5 for terminology). Q.E.D.

THEOREM 3. *Let* L *be an ample line bundle on* X. *Define*

$$\mathcal{A}_{\blacksquare}(L) = \{ g \in Aut(X) \mid \text{there exists } y(g) \in X \text{ s.t. } g^*L \cong T_{y(g)}{}^*L \}.$$

Then : 1) $\mathcal{A}_{\blacksquare}(L)$ *is a finite group.*

2) *Let* S *be a subgroup of* $\mathcal{A}_{\blacksquare}(L)^{op}$ *of order* s. *Then*

$$g \longmapsto y(g) \ (mod\ H(L)) \qquad \text{for } g \in S,$$

is a 1 - *cocycle representing a class in* $\mathrm{H}^1(S, X / H(L))$ *of order* $m \mid (s, 2)$.

3) *There exists* $\kappa \in X$ *such that*

$$g^* T_\kappa{}^*L^m \cong T_\kappa{}^*L^m \qquad \text{for all } g \in S,$$

that is

$$S \subseteq \mathcal{A}(T_\kappa{}^*L^m).$$

Further, κ *is unique up to an element of*

$$\bigcap_{g \in \mathcal{A}_{\blacksquare}(L)} (1-g)^{-1} H(L^m).$$

Proof. 1) Note that since L is ample $\mathcal{A}_{\blacksquare}(L)$ is precisely the group of automorphisms preserving the algebraic equivalence class of L. That is, precisely the group of automorphisms preserving the polarization ϕ_L which is finite.

2) First note that $y(g)$ is unique $mod\, H(L)$ and that $\mathcal{A}_{\blacksquare}(L)$ preserves $H(L)$. Then

$$\begin{aligned}(gh)^*L &\cong h^*g^*L \\ &\cong h^*T_{y(g)}{}^*L \\ &\cong T_{h^{-1}(y(g))}{}^*\, h^*L \\ &\cong T_{h^{-1}(y(g))+y(h)}{}^*\, L.\end{aligned}$$

This implies that

$$y(gh) = h*y(g) + y(h),$$

where we put $h* t = h^{-1}(t)$ (this is an action of $\mathcal{A}_{\blacksquare}(L)^{op}$). This shows that for $g \in S$ we get a cocycle in

$$\mathrm{H}^1(S, X / H(L)),$$

which is killed by s and therefore it is of order $m \mid s$. Moreover, we may assume, after translating L which amounts to changing everything by a coboundary, that L is symmetric. Therefore g^*L is symmetric for every g which implies that $2y(g) \in H(L)$. Explicating these remarks we see that there exists some $\overline{\kappa_0} \in X / H(L)$ such that

$$m\overline{y(g)} = (1 - g)*\overline{\kappa_0} \qquad \text{for all } g \in S$$

(we denote elements of $X / H(L)$ by $\bar{x}, \bar{y}$ etc.). Choose some $\kappa \in X$ such that $m\bar{\kappa} = \overline{\kappa_0}$. Then there exists an element $\overline{r(g)} \in (X / H(L))[m] = H(L^m) / H(L)$, such that

$$\overline{y(g)} = (1 - g)*\overline{\kappa_0} + \overline{r(g)}.$$

We have :

$$(1) \qquad g^*L^m \cong T_{y(g)}{}^*L^m \cong T_{(1-g)*\kappa + r(g)}{}^*L^m \cong T_{(1-g)*\kappa}{}^*L^m.$$

$$(2) \qquad g^*T_\kappa{}^*L^m \cong T_{g^{-1}(\kappa)}{}^*g^*L^m \cong T_{g^{-1}(\kappa)+(1-g)*\kappa}{}^*L^m \cong T_\kappa{}^*L^m.$$

That proves the first part of 3). To get the uniqueness assertion we note that if

$$g^*T_{\kappa'}{}^*L^m \cong T_{\kappa'}{}^*L^m,$$

then the second and fourth expressions of (2) shows that

$$g^*L^m \cong T_{(1-g)*\kappa'}{}^*L^m,$$

hence,

$$(1 - g)*\kappa' = y(g) \ mod\, H(L^m).$$

Since κ satisfies the same equality we see that

$$(1 - g)*m(\kappa - \kappa') \in H(L),$$

therefore,

$$\kappa - \kappa' \in \bigcap_{g \in \mathcal{A}_{\blacksquare}(L)} ((1 - g)*\mathrm{m})^{-1} H(L)$$

which implies the uniqueness assertion. Q.E.D.

PROPOSITION 4. *Let $\pi : \mathcal{C} \longrightarrow \mathcal{B}$ be a Galois covering of smooth complete curves with Galois group $\mathcal{P}$ of elements. Fix some base point $c \in \mathcal{C}$, and let $\Theta = \Theta_c$ be the theta divisor with respect to the embedding determined by the base point c,*

$$\mathcal{C} \hookrightarrow \mathrm{Jac}(\mathcal{C}).$$

Choose some $\kappa \in \mathrm{Jac}(\mathcal{C})$ such that $T_\kappa{}^*\Theta_c$ is symmetric and let $L = \mathcal{O}_{\mathrm{Jac}(\mathcal{C})}(T_\kappa{}^*\Theta_c)$. Then

$$\mathcal{P} \subseteq \mathcal{A}(L^2).$$

Proof. We could have used Theorem 3 but it is better to argue directly using the same rational. Let L be defined as above. Then for every $g \in \mathcal{P}$ we have

$$g^*L \cong T_{y(g)}{}^*L$$

for some unique $y(g)$. Since L is symmetric so is g^*L and therefore $2y(g) \in H(L) = \{0\}$. That is $y(g) \in \mathrm{Jac}(\mathcal{C})[2]$. Thus,

$$g^*L^2 \cong (T_{y(g)}{}^*L)^2 \cong T_{y(g)}{}^*(L^2) \cong L^2. \qquad \text{Q.E.D.}$$

REMARKS. 1) Proposition 4 shows that there are many examples of line bundles L such that $\mathcal{A}(L) \supsetneq \{\pm 1\}$. Other examples may be constructed using the theory of complex multiplication.

2) Note that in $\mathrm{Pic}_{g-1}(\mathcal{C})$ the theta divisor is certainly invariant. The problem Proposition 4 deals with is essentially the problem of non - existence of a common fixed point for $\mathcal{P}$. Example II in §4 shows that the curve $y^2 = x^6 - 1$ has no point fixed by all its automorphisms. (Indeed, by Lefschetz fixed point formula, the number of fixed points for the automorphism π given there is 2 and these are the points $\{(0, i), (0, -i)\}$. On the other hand the fixed points of the hypereliptic involution are $\{(0, \zeta) \mid \zeta^6 = 1\}$). If there were a point p such that the theta divisor with respect to p - denoted by Θ_p - is invariant under all these automorphisms then $\Theta_p = \Theta_{g(p)} = T_{g(p)-p}\Theta_p$. Since the theta divisor is of degree 1 we conclude that $g(p) = p$ for all g which is impossible. Therefore, Proposition 4 is the best we can hope for in general.

DEFINITION. Let L be a symmetric ample line bundle. Define

$$\mathcal{E}(L) = \{ f \in \mathrm{End}(X) \mid f^*L \cong L^n \textit{ for some } n,\ (p, deg(f)) = 1 \textit{ if } p > 0 \},$$
$$\mathcal{E}^0(L) = \{ f \in \mathrm{End}(X) \mid f^*L \cong L^{n^2} \textit{ for some } n,\ (p, deg(f)) = 1 \textit{ if } p > 0 \}.$$

REMARK. The condition $f^* L \cong L^n$ is not too strong. Indeed, if f^*L is only algebraically equivalent to L^n, then the same considerations as in Proposition 4 show that $f^*L^2 \cong L^{2n}$.

Consider now an isogeny $f: Y \longrightarrow X$ and let L be an ample line bundle on X. The following lemma describes the basic functoriality of the adelic Heisenberg groups associated with line bundles.

LEMMA - DEFINITION 5. 1) *There exists a canonical isomorphism*

$$j(f, L) : \hat{G}(f^*L) \longrightarrow \hat{G}(L)$$

fitting into the following commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(f^*L) & \longrightarrow & V(Y) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Id.} & & \downarrow{\scriptstyle j(f,L)} & & \downarrow{\scriptstyle V(f)} & & \\ 1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(L) & \longrightarrow & V(X) & \longrightarrow & 0. \end{array}$$

2) *If $f \in \mathcal{A}(L)$ then the same holds for $G(f^*L)$ and $G(L)$ with the obvious modifications. Assume that $f \in \mathcal{E}(L), f^*L \cong L^n$, then :*

3) *There is a canonical surjective homomorphism*

$$\delta_f : \hat{G}(L) \longrightarrow \hat{G}(L)$$

fitting into the following commuting diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(L) & \longrightarrow & V(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \delta_f} & & \downarrow{\scriptstyle V(f)} & & \\ 1 & \longrightarrow & k^\times & \longrightarrow & \hat{G}(L) & \longrightarrow & V(X) & \longrightarrow & 0. \end{array}$$

where $\alpha_n : k^\times \longrightarrow k^\times$ *is given by* $\alpha_n(t) = t^n$.

4) *If* $f \in \mathcal{A}(L)$ *then the same holds for* $G(L)$ *with the needed modifications.*

Proof. The definition and the stated properties of $j(f, L)$ appear in [Mum3] Proposition 4.9. To prove 3) and 4) let

$$\phi : f^*L \longrightarrow L^n$$

be an isomorphism and

$$\phi^* : G(L^n) \longrightarrow G(f^*L)$$

be the induced isomorphism (which is independent of the choice of ϕ) and denote by the same symbol ϕ^* the induced isomorphism

$$\phi^* : \hat{G}(L^n) \longrightarrow \hat{G}(f^*L).$$

The definition of δ_f is given in either the finite or adelic case by

$$\delta_f = j(f, L) \circ \phi^* \circ \varepsilon_n ,$$

where $\varepsilon_n : \hat{G}(L) \longrightarrow G(L^n)$ is given by $\varepsilon_n(\phi) = \phi^{\otimes n}$ and $\varepsilon_n : \hat{G}(L) \longrightarrow \hat{G}(L^n)$ is the induced homomorphism. The verification of the stated properties of this homomorphism is immediate. Q.E.D.

REMARKS. 1) The homomorphisms δ_f, δ_g satisfy

$$\delta_f \circ \delta_g = \delta_{fg} .$$

This follows easily from the definitions.

2) For f multiplication by n (n any integer) one can check that our δ_n is equal to the homomorphism δ_n defined in [Mum1] p. 308.

II. THE SYMMETRY THEOREM.

From now until the end of this section fix a maximal isotropic subgroup Z of $H(L)$ and a maximal level subgroup $\mathcal{Z}$ above Z. Later we will put further conditions on these subgroups.

Given $g \in \mathcal{A}(L)$ we can define a new action U^g of $G(L)$ on $\Gamma(X, L)$ by

$$U^g_z(s) = U_{\delta_g(z)}(s).$$

Since the scalars still act naturally we have a unique, up to a scalar, intertwining linear operator

$$T : \Gamma(X, L) \longrightarrow \Gamma(X, L),$$

satisfying

$$U_z \circ T = T \circ U_{\delta_g(z)}, \qquad \text{for all } g \in G(L), s \in \Gamma(X, L).$$

This follows from the fact, analogous to the Stone - Von - Neumann theorem for real Heisenberg groups, that $G(L)$ has a unique irreducible representation of order 1 (see [Mum1] and appendix). T is determined, up to a scalar, by the fact that it takes a $\mathcal{Z}$- invariant vector to a $\delta_{g^{-1}}(\mathcal{Z})$ - invariant vector and by its equivariance property.

CENTRAL OBSERVATION. Let $\phi : g^*L \longrightarrow L$ be an isomorphism. Then

$$\phi_* \, g^* : \Gamma(X, L) \longrightarrow \Gamma(X, L)$$

is a linear isomorphism. I claim that this too is an intertwining operator, therefore is equal, up to a scalar to T :

Denote by $\mathcal{T}$ the map $\phi_* \, g^*$. We have to prove that

$$\mathcal{T}(U_{\delta_g(z)} s) = U_z(\mathcal{T}(s)).$$

<u>Claim</u> : For every $r \in G(g^*L)$, $s \in \Gamma(X, L)$ we have $g^*(U_{j(g, L)(r)} s) = U_r(g^*s)$.

Proof (of claim). Let $r = (x, \phi)$, and $j(g, L)(r) = (gx, \psi)$. Then

$$\begin{aligned} g^*(U_{(gx, \psi)} s) &= g^*(\psi \circ s \circ T_{gx}^{-1}) \\ &= g^*\psi \circ g^*s \circ T_x^{-1} \\ &= U_r(g^*s). \end{aligned}$$

Then, using the claim, we get

$$\begin{aligned} \mathcal{T}(U_{\delta_g(z)} s) &= \mathcal{T}(U_{j(g, L)(\phi^{-1} z \phi)} s) \\ &= \phi \, g^*(U_{j(g, L)(\phi^{-1} z \phi)} s) \\ &= \phi \, \phi^{-1} U_z \, \phi(g^*s) \\ &= U_z(\mathcal{T}(s)). \end{aligned}$$

Q.E.D.

Our goal is to describe the map T as explicitly as possible. We start with the following ad hoc but convenient definition :

DEFINITION. Given an algebraic subgroup $A \subsetneq X$ define $\mathrm{Pic}(X)^A$ to be the image of $\mathrm{Pic}(X / A)$ under the natural pull-back homomorphism

$$\mathrm{Pic}(X / A) \longrightarrow \mathrm{Pic}(X)$$

(Note that if A is finite then $\mathrm{Pic}(X)^A$ is of finite index in $\mathrm{Pic}(X)$).

<u>Until the end of this section we assume that Z is a maximal isotropic subgroup of $H(L)$ which is $\mathcal{A}(L)$ - characteristic.</u>[3]

EXAMPLE. If M is of degree 1 and $L = M^{n^2}$, then $X[n] \subsetneq H(L) = X[n^2]$ is always maximal isotropic and an $\mathcal{A}(L)$ - characteristic subgroup.

Consider the exact sequence

$$(1) \qquad 0 \longrightarrow Z \longrightarrow X \xrightarrow{\pi} X / Z \longrightarrow 0$$

which yields the dual exact sequence

$$(2) \qquad 0 \longrightarrow Z^\vee \longrightarrow (X / Z)^\vee \xrightarrow{\pi^*} X^\vee \longrightarrow 0 ,$$

[3] Actually everything we would prove works equally well for any subgroup of $\mathcal{A}(L)$, in particular for cyclic subgroups. The assumption is made only for convenience of presentation.

where $Z^\vee$ is by definition the kernel of π^*. This sequence expands to an exact sequence

$$(3) \qquad 0 \longrightarrow Z^\vee \longrightarrow \mathrm{Pic}(X/Z) \longrightarrow \mathrm{Pic}(X)^Z \longrightarrow 0.$$

All these sequences are sequences of $\mathscr{A}(L)^{op}$ - modules. Taking group cohomology we get

$$(4) \qquad (\mathrm{Pic}(X)^Z)^{\mathscr{A}(L)^{op}} \longrightarrow \mathrm{H}^1(\mathscr{A}(L)^{op}, Z^\vee) \longrightarrow \mathrm{H}^1(\mathscr{A}(L)^{op}, (X/Z)^\vee).$$

In particular the sheaf L gives us a cocycle $\{ g \longmapsto N_g \} \in \mathrm{H}^1(\mathscr{A}(L)^{op}, Z^\vee)$. By its definition it is obtained as follows :

Choose some $M \in \mathrm{Pic}(X/Z)$ such that $\pi^*M \cong L$. Then

$$N_g = g^*M \otimes M^{-1}.$$

Changing the choice of M amounts to changing the cocycle by a coboundary. Now, by the general theory of descent there is a natural choice of M; Since we have already fixed a level subgroup $\tilde{Z}$ there is a unique M corresponding to it, namely, the one that $\tilde{Z}$ is the descent data for it. Let M denote this particular sheaf.

Using the canonical isomorphism $Z^\vee \cong \mathrm{Hom}(Z, k^\times) = Z^*$, we have a cocycle

$$\{ g \longmapsto \gamma_g \} \in \mathrm{Z}^1(\mathscr{A}(L)^{op}, Z^*),$$

obtained from $\{ g \longmapsto N_g \}$.

On the other hand, for every $z \in \tilde{Z}$ there is a unique scalar $\beta_g(z)$ such that

$$\beta_g(z)\, \delta_g(z) \in \tilde{Z}.$$

For a fixed g, β_g is a character of Z, and it is easy to check that

$$\{ g \longmapsto \beta_g \} \in \mathrm{Z}^1(\mathscr{A}(L)^{op}, Z^*).$$

PROPOSITION 6. $\beta_g = \gamma_g^{-1}$.

Proof. Let us first recal the description of the injection $Z^* \hookrightarrow \mathrm{Pic}(X/Z)$. We use [Ser] as a reference for this. Let $\eta \in Z^*$ and consider the following diagram, where N_η is <u>defined</u> as the push out of the first square :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & X & \xrightarrow{\pi} & X/Z & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & N_\eta & \xrightarrow{p} & X/Z & \longrightarrow & 0.
\end{array}$$

$N_\eta = \mathbb{G}_m \times X / \{ (\eta(-z), z) \mid z \in Z \}$. The map π^* is the pull back operation

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \pi^*D & \xrightarrow{P} & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \pi} & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & D & \xrightarrow{p} & X/Z & \longrightarrow & 0.
\end{array}$$

$\pi^*D = D \times_{X/Z} X = \{ (g, x) \mid g \in D,\ x \in X,\ p(g) = \pi(x) \}$. In particular :

$$\pi^*N_\eta = \{ [(\alpha, y), x] \mid (\alpha, y) \in \mathbb{G}_m \times X / \{(\eta(-z), z)\}_{z \in Z},\ x \in X,\ \pi(x) = \pi(y) \}.$$

We have an isomorphism

$$\phi_\eta : \pi^* N_\eta \longrightarrow \mathbb{A}^1 \times X$$

given by

$$[(\alpha, y), x] \longmapsto (\alpha / \eta(x - y), x).$$

(The inverse is just $[(\alpha, x), x] \longleftarrow (\alpha, x)$).

We have fixed M and $\mathcal{Z}$. The connection between them is as follows : choose an isomorphism $\alpha : \pi^*M \longrightarrow L$, then

$$\mathcal{Z} = \{ (x, T_x{}^*\alpha \circ \alpha^{-1}) \mid x \in Z \}.$$

Put $\alpha_\eta = \alpha \otimes \phi_\eta$,

$$\alpha_\eta : \pi^*(M \otimes N_\eta) = \pi^*M \otimes \pi^*N_\eta \longrightarrow L.$$

We get an explicit description of the level subgroup belonging to $M \otimes N_\eta$, which we denote by $\mathcal{Z}_\eta$:

$$\mathcal{Z}_\eta = \{ (x, T_x{}^*\alpha_\eta \circ \alpha_\eta^{-1}) \mid x \in Z \}.$$

There is a unique character β_η such that for all $x \in Z$,

$$(x, \beta_\eta(x) \cdot T_x{}^*\alpha_\eta \circ \alpha_\eta^{-1}) = (x, T_x{}^*\alpha \circ \alpha^{-1}).$$

LEMMA. $\beta_\eta = \eta$.

Let us assume this lemma for a moment and show that it implies the proposition. If η is such that $N_\eta = N_g$ (equivalently $\gamma_g = \eta$), then

$$M \otimes N_\eta = M \otimes N_g = g^*M,$$

hence $\mathcal{Z}_\eta$ is the level subgroup corresponding to g^*M in $G(L)$. By functoriality, the subgroup corresponding to g^*M in $G(g^*L)$ is $j(g, M)^{-1}(\mathcal{Z})$, and under the isomorphism

$$\phi : G(L) \longrightarrow G(g^*L)$$

it corresponds to $(\delta_g)^{-1}(\mathcal{Z}) = \delta_{g^{-1}}(\mathcal{Z})$.

Therefore, if $y = gx$ and $(y, T_y{}^*\alpha \circ \alpha^{-1}) \in \mathcal{Z}$ we have

$$\beta_{g^{-1}}(y)\, \delta_{g^{-1}}(y, T_y{}^*\alpha \circ \alpha^{-1}) = \beta_{g^{-1}}(gx)\,(x, T_x{}^*\alpha_\eta \circ \alpha_\eta^{-1}) \in \delta_{g^{-1}}(\mathcal{Z}).$$

However, since $\delta_{g^{-1}}(\mathcal{Z}) = \mathcal{Z}_\eta$ we have also

$$\beta_\eta(x)\,(x, T_x{}^*\alpha_\eta \circ \alpha_\eta^{-1}) \in \delta_{g^{-1}}(\mathcal{Z}).$$

Therefore

$$\beta_\eta = g\beta_{g^{-1}}$$

and using the Lemma we get

$$\gamma_g = \eta = \beta_\eta = g\beta_{g^{-1}}$$

whence

$$\gamma_g^{-1} = (g\beta_{g^{-1}})^{-1} = \beta_g$$

(using the cocycle relation).

Proof (of the Lemma). The way to prove that $\beta_\eta = \eta$ is to compute the action at the fiber of L at zero, L_0, of the maps $T_x{}^*\alpha \circ \alpha^{-1}$ and $T_x{}^*\alpha_\eta \circ \alpha_\eta^{-1}$. Consider the following diagrams :

$$
\begin{array}{ccc}
L_0 & \xleftarrow{\ \alpha\ } & (\pi^*M)_0 \\
 & & \| \\
(T_x^*L)_0 & \xleftarrow{\ T_x^*\alpha\ } & (T_x^*\pi^*M)_0
\end{array}
\qquad
\begin{array}{ccc}
L_0 & \xleftarrow{\ \alpha\cdot\phi_\eta\ } & \pi^*(M\otimes N_\eta)_0 = (\pi^*M)_0\otimes(\pi^*N_\eta)_0 \\
 & & \| \\
(T_x^*L)_0 & \xleftarrow{\ T_x^*\alpha\cdot\ T_x^*\phi_\eta\ } & (T_x^*\pi^*M)_0\otimes(T_x^*\pi^*N_\eta)_0
\end{array}
$$

(all the maps in this diagram are the specialization to the fiber at zero. This is omitted for typographical reasons). We see that

$$\beta_\eta(x) = \phi_\eta(0) / (T_x^*\,\phi_\eta)(0).$$

This scalar is described by the following diagram

$$
(5)\qquad
\begin{array}{ccc}
\mathbb{A}^1\times X & \xleftarrow{\ \phi_\eta\ } & \pi^*N_\eta \\
 & & \| \\
T_x^*(\mathbb{A}^1\times X) & \xleftarrow{\ T_x^*\phi_\eta\ } & T_x^*\pi^*N_\eta
\end{array}
$$

Recall that

$$\pi^*N_\eta = \{\ [(\alpha, y), x]\ \mid\ (\alpha, y)\in \mathbb{G}_m\times X / \{(\eta(-z), z)\}_{z\in Z}\ , x\in X,\ \pi(x) = \pi(y)\ \},$$

and therefore

$$T_x^*\pi^*N_\eta = \{\ (y, s)\mid y\in X, s\in \pi^*N_\eta,\ T_x(y) = P(s)\ \}.$$

The isomorphism $\pi^*N_\eta \longrightarrow T_x^*\pi^*N_\eta$ is given by [4]

$$[(\alpha, z_1), z_2] \longmapsto (z_2\,, [(\alpha, z_1), z_2 + x]).$$

Similarily,

$$\mathbb{A}^1\times X = \{(\alpha, x)\mid \alpha\in k^\times,\ x\in X\},$$

$$T_x^*(\mathbb{A}^1\times X) = \{[y, (\alpha, z)]\ \mid\ y\in X, (\alpha, z)\in \mathbb{A}^1\times X,\ T_x^*(y) = z\}.$$

$T_x^*(\mathbb{A}^1\times X)_0$ is naturally identified with $(\mathbb{A}^1\times X)_x$ by $[0, (\alpha, x)] \longleftrightarrow (\alpha, x)$. Hence diagram (5) at the fibers at zero looks like

$$
\begin{array}{ccccc}
 & & (\alpha, 0) & \xleftarrow{\ \phi_\eta\ } & [(\alpha, 0), 0] \\
 & & & & \downarrow \\
(\alpha/\eta(x), x) & \longleftrightarrow & [0, (\alpha/\eta(x), x)] & \xleftarrow{\ T_x^*\phi_\eta\ } & (0, [(\alpha, 0), x])
\end{array}
$$

Q.E.D.

Let us construct an example showing that $\{g \longmapsto \gamma_g\}\in \mathrm{H}^1(\mathcal{A}(L)^{op}, Z^\vee)$ is not trivial. Recall first (see (4)), the exact sequence

$$(6)\qquad (\mathrm{Pic}(X)^Z)^{\mathcal{A}(L)^{op}} \longrightarrow \mathrm{H}^1(\mathcal{A}(L)^{op}, Z^\vee) \longrightarrow \mathrm{H}^1(\mathcal{A}(L)^{op}, (X/Z)^\vee).$$

[4] In general if $f: Z\to Y$, $g: Y\to X$, and $p: N\to X$ is a line bundle then

$g^*N = \{(y, l)\mid y\in Y, l\in N, p(l) = g(y)\}$ with morphism $p': g^*N\to Y$,

$f^*g^*N = \{(z, s)\mid z\in Z, s\in g^*N,\ p'(s) = f(z)\}$, and

$(g\circ f)^*N = \{(z, l)\mid z\in Z, l\in N, p(l) = (g\circ f)(z)\}$.

We have $f^*g^*N \cong (g\circ f)^*N$ by $(z, (y, l)) \longmapsto (z, l)$.

LEMMA 7. *Let* S *be a finite cyclic group of order* s, $S = <\sigma>$. *Put*

$${}_sX = \{x \in X \mid x + \sigma x + \ldots + \sigma^{s-1} x = 0\}.$$

Then

$$\mathrm{H}^1(S, X) \cong {}_sX \,/\, ({}_sX)^0,$$

where $({}_sX)^0$ *is the connected component of* ${}_sX$. *In particular, if* ${}_sX$ *is connected then* $\mathrm{H}^1(S, X) = \{0\}$.

Proof. The kernel of the map

$$(1 - \sigma) : {}_sX \longrightarrow {}_sX$$

is contained in $X[s]$ and in particular is finite. That implies that $(1 - \sigma)(X)$ which is connected is equal to the connected component ${}_sX^0$. Therefore, by the well known description of H^1 for cyclic groups,

$$\mathrm{H}^1(S, X) \cong {}_sX \,/\, {}_sX^0,$$

and in particular is trivial if ${}_sX$ is connected. Q.E.D.

Let us consider now a generic principally polarized abelian variety X. Let M be an ample symmetric line bundle on X, and take $L = M^4$. Then $\mathscr{A}(L) = \{\pm 1\}$, $H(L) = X[4]$, and we take $Z = X[2]$. In that case ${}_{\mathscr{A}(L)}X = X$ is connected, and thus

$$\mathrm{H}^1(\mathscr{A}(L)^{op}, (X/Z)^{\vee}) = \{0\},$$

further

$$\begin{aligned} \mathrm{H}^1(\mathscr{A}(L)^{op}, Z^{\vee}) &= \mathrm{H}^1(\mathscr{A}(L), X[2]) \\ &\cong \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^{2g}) \\ &\cong (\mathbb{Z}/2\mathbb{Z})^{2g}. \end{aligned}$$

From this we conclude that

$$(\mathrm{Pic}(X)^Z)^{\mathscr{A}(L)^{op}} \longrightarrow \mathrm{H}^1(\mathscr{A}(L)^{op}, Z^{\vee})$$

is surjective with a non trivial image, which shows that the cocycles $\{g \longmapsto \gamma_g\}$ appearing above are generally non - trivial.

<u>Back to intertwining maps</u> :

Let us review the situation. X is an abelian variety, L an ample line bundle on X of degree d, $Z \subsetneq H(L)$ is a maximal isotropic subgroup which is $\mathscr{A}(L)$ - characteristic, and $\mathscr{Z}$ is a maximal level subgroup over Z. We denote by χ: $G(L) \longrightarrow \mathscr{Z}^*$ the commutator map, $\chi^{\kappa}(z) = [\kappa, z]$, $\kappa \in G(L)$, $z \in \mathscr{Z}$.

Decompose $\Gamma(X, L)$ according to characters of $\mathscr{Z}$:

$$\Gamma(X, L) = \bigoplus_{\psi \in \mathscr{Z}^*} \Gamma(X, L)_{\psi}.$$

Since we are dealing with a representation of order 1 each component is 1 dimensional (see appendix). Define

$$\Psi = \frac{1}{d} \sum_{z \in \mathscr{Z}} U_{\delta_{g^{-1}}(z)}.$$

Ψ is a projection operator on the one dimensional subspace of the $\delta_{g^{-1}}(\mathscr{Z})$ invariants.

Claim : *If* $v \in \Gamma(X, L)_\sigma$ *then* $U_\kappa v \in \Gamma(X, L)_{\chi^\kappa \cdot \sigma}$, *for all* $\kappa \in G(L)$.

Proof. Let $z \in \mathcal{Z}$, then

$$U_z U_\kappa v = U_{[z,\kappa]} U_\kappa U_z v = \chi^\kappa(z) \cdot \sigma(z) \cdot U_\kappa v.$$

Therefore, since both $\mathcal{Z}$ and $\delta_{g^{-1}}(\mathcal{Z})$ are above Z, and the kernel of $\kappa \longmapsto \chi^\kappa$ is $k^\times \mathcal{Z}$, we conclude that for all $\kappa \in \delta_{g^{-1}}(\mathcal{Z})$, $v \in \Gamma(X, L)_\sigma$ if and only if $U_\kappa v \in \Gamma(X, L)_\sigma$. Thus, there exists a unique τ such that $\Psi(\Gamma(X, L)_\tau) \neq \{0\}$, and for that τ for every $v \in \Gamma(X, L)_\tau$ we have $\Psi(v) = v$, that is

$$\Gamma(X, L)_\tau = \Gamma(X, L)^{\delta_g^{-1}(\mathcal{Z})}.$$

However, let $\kappa \in \delta_{g^{-1}}(\mathcal{Z})$, say $\kappa = \delta_{g^{-1}}(z)$, then, identifying $\mathcal{Z}^*$ with Z^*,

$$\begin{aligned} v &= U_\kappa v \\ &= U_{\delta_g^{-1}(z)}\, v \\ &= U_{(\beta_{g^{-1}}(z))^{-1}}\, U_{\beta_{g^{-1}}(z)\,\delta_{g^{-1}}(z)}\, v \\ &= \left(\beta_{g^{-1}}(z)\right)^{-1} \tau(g^{-1}z)\quad v \\ &= \left(\beta_{g^{-1}}^{-1} \cdot g^{-1}\tau\right)(z)\, v. \end{aligned}$$

That implies that for every $z \in \mathcal{Z}$, $\left(\beta_{g^{-1}}^{-1} \cdot g^{-1}\tau\right)(z) = 1$. Therefore $\beta_{g^{-1}}^{-1} \cdot g^{-1}\tau = 1$, or ,

$$\tau = g\beta_{g^{-1}} = \beta_g^{-1} = \gamma_g.$$

Choose some section

$$G(L)^c \overset{\Sigma}{\leftarrow} \mathcal{Z}^*, \quad 1 = \Sigma(1),$$

to the commutator map

$$G(L) \longrightarrow \mathcal{Z}^*, \quad \kappa \longmapsto \chi^\kappa.$$

Choose some non zero $v_1 \in \Gamma(X, L)_1$. Then

$$\{\, v_\chi = U_{\Sigma(\chi)}\, v_1 \mid \chi \in \mathcal{Z}^* \,\}$$

is a basis for $\Gamma(X, L)$. We have

$$T(v_1) = c(g) \cdot v_{\gamma_g},$$

for some $c(g) \in k^\times$. Let $c_{g,\chi} \in \mu_{d^2}\mathcal{Z}$ be defined by the equation

$$c_{g,\chi}\, \Sigma(\gamma_g \cdot g\chi) = \delta_{g^{-1}}(\Sigma(\chi)) \cdot \Sigma(\gamma_g)) \quad .$$

There are scalars $b_{g,\chi} \in \mu_{d^2}$ such that

$$b_{g,\chi}\, U_{\Sigma(\gamma_g \cdot g\chi)}\, v = U_{\delta_{g^{-1}}(\Sigma(\chi))} \cdot U_{\Sigma(\gamma_g)}\, v \quad . \tag{7}$$

These scalars appear when computing $T(v_\chi)$:

$$\begin{aligned} T(v_\chi) &= T(U_{\delta_g(\delta_g^{-1}(\Sigma(\chi)))} v_1) \\ &= U_{\delta_g^{-1}(\Sigma(\chi))}\, T(v_1) \\ &= c(g) \cdot U_{\delta_g^{-1}(\Sigma(\chi))}\, v_{\gamma_g} \\ &= c(g) \cdot b_{g,\chi}\, v_{\gamma_g \cdot g\chi}\,. \end{aligned}$$

We have proved

THE SYMMETRY THEOREM (PRELIMINARY FORM). *Let X be an abelian variety over an algebraically closed field k. Let L be an ample line bundle of separable type on X. Let $\mathcal{A}(L) = \{g \in Aut(X) \mid g^*L \cong L\}$. Assume that there exists a maximal isotropic subgroup Z of $H(L)$ which is $\mathcal{A}(L)$ - invariant and choose some level subgroup $\tilde{Z}$ above Z. Choose some non zero vector $v_1 \in \Gamma(X, L)_1 = \Gamma(X, L)^{\tilde{Z}}$ and choose a section*

$$G(L)^c \xleftarrow{\Sigma} \tilde{Z}^*, \ \Sigma(1) = 1.$$

Given $g \in \mathcal{A}(L)$, let $\phi : g^*L \to L$ be an isomorphism (determined up to a scalar). Let $T = \phi_* g^* \in \mathrm{End}(\Gamma(X, L))$.

Then :

(*a*) *There exists a unique character $\gamma_g \in \tilde{Z}^*$, characterized by either :*

(i) $\gamma_g(z)^{-1}\delta_g(z) \in \tilde{Z}$ *for all* $z \in \tilde{Z}$, *or by*

(ii) *If M is the line bundle on X / Z corresponding to $\tilde{Z}$ then $g^*M \otimes M^{-1}$ corresponds to γ_g under*

$$\tilde{Z}^* \cong Z^* \cong \mathrm{Ker}(\mathrm{Pic}^0(X/Z) \longrightarrow \mathrm{Pic}^0(X)).$$

(*b*) *There exists scalars $b_{g,\chi} \in \mu_{d^2}$ ($\chi \in \tilde{Z}^*$), determined by the equation*

$$b_{g,\chi} U_{\Sigma(\gamma_g \cdot g\chi)} v_1 = U_{\delta_g^{-1}(\Sigma(\chi))} \cdot U_{\Sigma(\gamma_g)} v_1 ,$$

such that

$$T\Big(\sum_{\chi \in \tilde{Z}^*} a_\chi v_\chi\Big) = c(g) \sum_{\chi \in \tilde{Z}^*} a_\chi b_{g,\chi} v_{\gamma_g g\chi} ,$$

where $\{v_\chi = U_{\Sigma(\chi)} v_1\}$ is a basis for $\Gamma(X, L)$ and $c(g)$ is a scalar determined by the equality $Tv_1 = c(g)\, v_{\gamma_g}$. In particular the matrix describing T, which is given explicitly by the $b_{g,\chi}$'s, is monomial and unitary.

COROLLARY 1. Assume that $L = \mathcal{O}_X(D)$. Then ϕ_* is multiplication by a function f_g. Suppose further that $0 \notin supp(D)$, then there exists a function Ω (independent of g) such that $f_g = \Omega^g / \Omega$ and $0 \notin supp(\Omega)$. We get then that for <u>all x</u>

$$\frac{\Omega(gx)}{\Omega(x)} v_\chi(gx) = c(g)\, b_{g,\chi}\, v_{\gamma_g g\chi}(x) , \tag{9}$$

where $c(g)$ is a non zero constant. In particular

$$v_\chi(0) = c(g)\, b_{g,\chi} v_{\gamma_g \cdot g\chi}(0) . \tag{10}$$

REMARKS. 1) As remarked in the introduction, for L very ample T is actually *writing the automorphism g by coordinates.*

2) Consider in Corollary 1 a special case where $\gamma_g = 1$, which is a kind of ' total symmetry ' with respect to g. Note that for every g we have $b_{g,1} = 1$. Therefore, if $v_1(0) \neq 0$ we conclude that $c(g) = 1$. Note now that for every χ such that $g\chi = \chi$ we get the obvious conclusion

$$c(g)\, b_{g,\chi} \neq 1 \Rightarrow v_\chi(0) = 0.$$

That is, we get a vanishing result for certain theta constants (see §5 for a classical interpretation).

Our next task is to give an explicit expression to the coefficients $b_{g,\chi}$ appearing in the

. Symmetry Theorem. Although the method can be carried out in complete generality we will assume that $L = n^*M$ for some ample line bundle M of degree 1 and that

$$Z = X[n], \quad \mathcal{Z} = K(n, M)$$

(by $K(n, M)$ we mean the level subgroup associated to data n, L, M).

The particular choices made below are not the best from the algebraic point of view. They are made so that over the complex numbers one gets ' the most classical ' sections and actions. Other choices will be developed in the next section.

Choose a theta structure

$$\Delta : G(L) \cong G(\delta),$$

where

$$G(\delta) = k^\times \times \left(\bigoplus_{i=1}^{g} \tfrac{1}{n} \mathbb{Z} / n\mathbb{Z} \right) \times \left(\bigoplus_{i=1}^{g} \tfrac{1}{n} \mathbb{Z} / n\mathbb{Z} \right)$$

with the group law

$$(\alpha, x, l)(\beta, y, m) = (\alpha\beta\, \underline{e}\,({}^t(nx) \cdot (nm)),\, x + y,\, l + m),$$

where for any $a \in \mathbb{Z}$, $\underline{e}\,(a) = \zeta^a$ for ζ a fixed n^2-th root of 1. Define 'the half commutator'

$$F\left(\begin{pmatrix} x \\ l \end{pmatrix}, \begin{pmatrix} y \\ m \end{pmatrix}\right) = \underline{e}\,({}^t(nx) \cdot (nm)).$$

Define

$$S = \{ (x, l) \mid x, l \in (\mathbb{Z}/n\mathbb{Z})^g \}, \quad \mathcal{S} = \{(1, x, l) \mid x, l \in (\mathbb{Z}/n\mathbb{Z})^g \}.$$

We can always choose Δ such that $\Delta(\mathcal{Z}) = \mathcal{S}$.

Let us choose a section to the commutator map

$$G(\delta) \longrightarrow \mathcal{S}^*$$

by prescribing a set of representatives to $G(\delta) / k^\times \mathcal{S}^*$,

$$Rep = \{ (1, \tfrac{1}{n} \begin{pmatrix} x \\ l \end{pmatrix}) \mid x = (x_1, \ldots, x_g),\ l = (l_1, \ldots, l_g),\ 0 \le x_i, l_i < n \},$$

thereby getting a section

$$\psi \longmapsto \Sigma(\psi) = (1, \alpha(\psi)), \quad \forall\ \psi \in \mathcal{S}^*.$$

Via Δ, each automorphism δ_g of $G(L)$ induces an automorphism, still denoted δ_g, on $G(\delta)$, hence induces an action on $H(\delta)$ given by a genuine symplectic matrix $M_g \in \mathrm{Sp}(2g, \mathbb{Z}/n^2\mathbb{Z})$ - the n^2-adic representation of g. Therefore we may write, for $(\alpha, w) = (\alpha, w_1, w_2) \in G(\delta)$,

$$\delta_g(\alpha, w) = (\alpha \cdot m_g(w), M_g w),$$

where m_g satisfies the identity

$$\frac{m_g(w_1 + w_2)}{m_g(w_1) \cdot m_g(w_2)} = \frac{F(M_g w_1, M_g w_2)}{F(w_1, w_2)}.$$

Using $\{\cdot\}$ to denote fractional part, we get

$$\Sigma(\chi\psi) = (1, \{\alpha(\chi) + \alpha(\psi)\}),$$

$$\delta_{g^{-1}}(\Sigma(\chi)) = (m_{g^{-1}}(\alpha(\chi)), M_{g^{-1}}\, \alpha(\chi)).$$

Since

$$\chi^{(1, M_{g^{-1}}(\alpha(\chi)))}(y) = [\overline{y}, M_{g^{-1}}(\alpha(\chi))] = [M_g\, \overline{y}, \alpha(\chi)] = g\chi(\overline{y}),$$

we conclude that

$\Sigma(g\chi) = (1, \{M_{g^{-1}}\alpha(\chi)\})$.

Therefore

$\Sigma(\gamma_g \cdot g\chi) = (1, \{\alpha(\gamma_g) + M_{g^{-1}}\alpha(\chi)\})$,

$\delta_{g^{-1}}(\Sigma(\chi)) \cdot \Sigma(\gamma_g) = (m_{g^{-1}}(\alpha(\chi)) \cdot F(M_{g^{-1}}(\alpha(\chi)), \alpha(\gamma_g)), M_{g^{-1}}(\alpha(\chi)) + \alpha(\gamma_g))$.

Whence (using $[\cdot]$ to denote integral part),

$$c_{g,\chi} = \left(\frac{m_{g^{-1}}(\alpha(\chi)) \cdot F(M_{g^{-1}}(\alpha(\chi)), \alpha(\gamma_g))}{F([M_{g^{-1}}(\alpha(\chi)) + \alpha(\gamma_g)], \{M_{g^{-1}}(\alpha(\chi)) + \alpha(\gamma_g)\})}, [M_{g^{-1}}(\alpha(\chi)) + \alpha(\gamma_g)] \right).$$

Put

$$v_g(\chi) = M_{g^{-1}}(\alpha(\chi)) + \alpha(\gamma_g),$$

then, after some simple calculations we get

$$b_{g,\chi} = \frac{m_{g^{-1}}(\alpha(\chi)) F(v_g(\chi) - \alpha(\gamma_g), \alpha(\gamma_g))}{F(\{v_g(\chi)\}, [v_g(\chi)])}.$$

THE SYMMETRY THEOREM (EXPLICIT FORM I). *Under the hypothesis of the Symmetry theorem and the additional hypothesis made above we have*

(11)

$$T\left(\sum_{\chi \in \mathbb{Z}^*} a_\chi v_\chi\right) = c(g) \sum_{\chi \in \mathbb{Z}^*} a_\chi \frac{m_{g^{-1}}(\alpha(\chi)) F(v_g(\chi) - \alpha(\gamma_g), \alpha(\gamma_g))}{F(\{v_g(\chi)\}, [v_g(\chi)])} v_{\gamma_g g\chi}$$

If $\gamma_g = 1$, *then*

$$T\left(\sum_{\chi \in \mathbb{Z}^*} a_\chi v_\chi\right) = c(g) \sum_{\chi \in \mathbb{Z}^*} a_\chi \frac{m_{g^{-1}}(\alpha(\chi))}{F(\{M_{g^{-1}}(\alpha(\chi))\}, [M_{g^{-1}}(\alpha(\chi))])} v_{g\chi} \quad (12)$$

COROLLARY 1(EXPLICIT FORM I).

$$\frac{\Omega(gx)}{\Omega(x)} v_\chi(gx) = c(g) \frac{m_{g^{-1}}(\alpha(\chi)) F(v_g(\chi) - \alpha(\gamma_g), \alpha(\gamma_g))}{F(\{v_g(\chi)\}, [v_g(\chi)])} v_{\gamma_g g\chi}(x), \quad (13)$$

where $c(g)$ *is a non zero constant. In particular*

(14) $$v_\chi(0) = c(g) \cdot \frac{m_{g^{-1}}(\alpha(\chi))\, F(v_g(\chi) - \alpha(\gamma_g), \alpha(\gamma_g))}{F(\{v_g(\chi)\}, [v_g(\chi)])} v_{\gamma_g g\chi}(0).$$

If $\gamma_g = 1$ *then, if* $v_1(0) \neq 0$, *these formulas reduce to*

(15) $$\frac{\Omega(gx)}{\Omega(x)} v_\chi(gx) = \frac{m_{g^{-1}}(\alpha(\chi))}{F(\{M_{g^{-1}}(\alpha(\chi))\}, [M_{g^{-1}}(\alpha(\chi))])} v_{g\chi}(x)$$

and

(16) $$v_\chi(0) = \frac{m_{g^{-1}}(\alpha(\chi))}{F(\{M_{g^{-1}}(\alpha(\chi))\}, [M_{g^{-1}}(\alpha(\chi))])} v_{g\chi}(0).$$

3. THE QUASI SYMMETRY THEOREM.

We start with a discussion of Göpel structures and the system of bases constructed by them. We keep the notation used so far. We assume through out this section that L is a symmetric even ample line bundle of degree 1 (although some of the definitions and results still hold if the degree is greater). We refer to [Mum3] pp. 60 -61 for the definition of ' even '.

DEFINITION. A Göpel structure on $V(X)$ with respect to L is a pair of maximal isotropic subgroups V_1, V_2 of $V(X)$ such that :

(i) $V(X) = V_1 \oplus V_2$.
(ii) $T(X) = T_1 \oplus T_2$ where $T_i = T(X) \cap V_i$.
(iii) $\sigma^L = \tau^L$ on each T_i. (See introduction for the definitions of σ^L, τ^L).

Given a Göpel structure we can define a system of bases for the vector spaces $\Gamma(X, n^*L)$. To do this we need the following

LEMMA 8. *For every* n *there is a canonical isomorphism*

$$\text{Normalizer}(\sigma^L(n\,T(X)))/(\sigma^L(n\,T(X))) \cong G(n^*L)$$

given by

$$(x_m, \phi_m)_m \longmapsto (x_n, \phi_n).$$

Proof. An easy generalization of [Mum3] Proposition 4.13. Q.E.D.

DEFINITION. Let $Z(n) = n\,T(X)$, $\mathcal{Z}(n) = \sigma^L(n\,T(X))$ and $N(n) = \text{Normalizer}(\mathcal{Z}(n))$. Let

$$K(n)_1 = \tau^L(V_1) \cap N(n), \quad K(n)_2 = \tau^L(V_2) \cap N(n).$$

Let $\Phi_n : N(n) \longrightarrow G(n^*L)$ be the homomorphism inducing the isomorphism of Lemma 8. Let

$$L(n)_1 = \Phi_n(K(n)_1), \; L(n)_2 = \Phi_n(K(n)_2).$$

LEMMA 9. *$L(n)_1$ and $L(n)_2$ are maximal level subgroups of $G(n*L)$ which are orthogonal complements of each other.*

Proof. Clear. Q.E.D.

We stray from our main course to study projection operators on finite Heisenberg groups: Let G be a finite Heisenberg group

$$1 \longrightarrow k^\times \longrightarrow G \longrightarrow H \longrightarrow 0,$$

and let K be a maximal level subgroup of G, $\#K = d$. Decompose the unique irreducible representation of G where the center acts naturally, denoted by Γ, as

$$\Gamma = \bigoplus_{\psi \in K^*} \Gamma_\psi$$

and fix some non zero $\delta_1 \in \Gamma_1$. Let $\{ \delta_\psi \mid \psi \in K^* \}$ be a basis for Γ with $\delta_\psi \in \Gamma_\psi$ Denote by

$$\chi : G \longrightarrow K^*$$

the ' commutator ' map

$$y \longmapsto \chi^y , \; \chi^y(z) = z y z^{-1} y^{-1} .$$

Given another maximal level subgroup S, let P_S be the projection operator on the one dimensional space Γ^S of the S - invariants given by

$$P_S = \frac{1}{d} \sum_{s \in S} U_s .$$

Finally, let

$$A_K(S) = \{ \psi \in K^* \mid P_S(\delta_\psi) \neq 0 \}.$$

LEMMA 10. 1) *Let $s \in S$, $\psi \in K^*$ then*

$$U_s \delta_\psi \in \Gamma_{\psi \cdot \chi^s} .$$

Define

$$U_s \delta_\psi = a(s, \psi) \cdot \delta_{\psi \cdot \chi^s} .$$

Then

$$a(s_1 s_2 , \psi) = a(s_1, \psi \cdot \chi^{s_2}) \cdot a(s_2, \psi).$$

2) *Assume that K has an orthogonal complement K'. There is an isomorphism*

$$\Xi : K^* \longrightarrow K'$$

determined by the commutator pairing. Let

$$\delta_\psi = U_{\Xi(\psi)} \delta_1 .$$

then

$$a(s, \psi_1) = \frac{\psi_1(s)}{\psi_2(s)} a(s, \psi_2),$$

where, by definition, $\psi(s) = [s, \Xi(\psi)]$ for any $\psi \in K^*$.

3) *Let*

$$\mathfrak{S} = \operatorname{Ker} \chi \cap S$$

and let $\alpha \in \mathfrak{S}^*$ be the unique character such that for all $s \in \mathfrak{S}$

$$\alpha(s) \cdot s \in K$$

Then, regarding characters of level subgroups B as characters of their projection $\bar{B}$ to H we have

$$A_K(S) = \{ \psi \in K^* \mid \psi|_{\bar{S}} = \alpha \}.$$

Proof. 1) for every $k \in K$ we have

$$\begin{aligned} U_k U_s \delta_\psi &= U_s U_k U_{[k,s]} \delta_\psi \\ &= \psi(k) \cdot \chi^s(k) \cdot U_s \delta_\psi \\ &= (\psi \cdot \chi^s)(k) \cdot U_s \delta_\psi. \end{aligned}$$

$$\begin{aligned} a(s_1 s_2, \psi) \delta_{\psi \cdot \chi^{s_1 s_2}} &= U_{s_1} U_{s_2} \delta_\psi \\ &= U_{s_1} a(s_2, \psi) \delta_{\psi \cdot \chi^{s_2}} \\ &= a(s_1, \psi \cdot \chi^{s_2}) \cdot a(s_2, \psi) \delta_{\psi \cdot \chi^{s_1} \cdot \chi^{s_2}} . \end{aligned}$$

.

2) Let us compute $a(s, \psi)$.

$$\begin{aligned} a(s, \psi) \delta_{\psi\chi^s} &= U_s \delta_\psi \\ &= U_s U_{\Xi(\psi)} \delta_1 \\ &= U_{\Xi(\psi)} U_s U_{[s, \Xi(\psi)]} \delta_1 \\ &= \psi(s) U_{\Xi(\psi)} U_s \delta_1 \\ &= \psi(s) a(s, 1) U_{\Xi(\psi)} \delta_{\chi^s} \\ &= \psi(s) a(s, 1) \delta_{\psi\chi^s} . \end{aligned}$$

From this follows the general formula.

3)
$$\begin{aligned} P_S(\delta_\psi) &= \frac{1}{d} \sum_{s \in S} U_s \delta_\psi \\ &= \frac{1}{d} \sum_{s \in S} a(s, \psi) \delta_{\psi \cdot \chi^s} \\ &= \frac{1}{d} \sum_{\gamma \in \chi(S)} \left(\sum_{s \in \chi^{-1}(\gamma)} a(s, \psi) \right) \delta_{\psi \cdot \gamma} . \end{aligned}$$

Choose some representatives $s_\gamma \in \chi^{-1}(\gamma)$, then

$$\begin{aligned} P_S(\delta_\psi) &= \frac{1}{d} \sum_{\gamma \in \chi(S)} \left(\sum_{s \in \operatorname{Ker}\chi \cap S} a(s_\gamma s, \psi) \right) \delta_{\psi \cdot \gamma} \\ &= \frac{1}{d} \sum_{\gamma \in \chi(S)} \left(a(s_\gamma, \psi) \sum_{s \in \operatorname{Ker}\chi \cap S} a(s, 1) \cdot \psi(s) \right) \delta_{\psi \cdot \gamma} \end{aligned}$$

using part 1), 2). Notice that this expression is zero if and only if each sum

$$a(s_\gamma, \psi) \sum_{s \in \operatorname{Ker}\chi \cap S} a(s, 1) \cdot \psi(s)$$

is zero and that the vanishing of this sum does not depend on γ. Notice also that $a(s, 1)$ is a character on $\mathfrak{S}$ and therefore this sum vanishes if and only if $\psi|_{\mathfrak{S}} \neq a(\cdot, 1)^{-1}|_{\mathfrak{S}}$. To finish the proof we need only to check that $a(\cdot, 1) = \alpha^{-1}$. Let $s \in \mathfrak{S}$, then $\alpha(s)\, s \in K$ whence

$$\begin{aligned} a(s, 1)\, \delta_1 &= U_s\, \delta_1 \\ &= \alpha(s)^{-1}\, U_{\alpha(s)\, s}\, \delta_1 \\ &= \alpha(s)^{-1}\, \delta_1 . \end{aligned}$$

Q.E.D.

Recall that L is an ample symmetric line bundle of degree 1 on X. The maximal level subgroup $\mathcal{Z}(1)$ induces maximal level subgroups $M(n)$ on each $G(n^*L)$.

We would like to mention two reasons for introducing these groups: The first one is that we can decompose $\hat{\Gamma}(X, L)$ according to characters of $\mathcal{Z}(1)$. This can not be done with respect to $\tau^L(V_1)$.

The second reason is that, as we have already commented above, in the complex case the sections giving the decomposition with respect to $\mathcal{Z}(1)$ are the classical theta functions $\theta\begin{bmatrix} * \\ * \end{bmatrix}$ for certain characteristics multiplied by a certain trivial exponent. We should remark that the sections giving the decomposition with respect to the level subgroup $L(n)_1$ are no less noble. They are of the form $\theta\begin{bmatrix} * \\ 0 \end{bmatrix}$ for certain characteristics multiplied by a certain trivial exponent (a classical example which also demonstrates the relations between the bases to be obtained below, turns out to be, after some algebro - analytic dictionary has been built, Proposition 1.3 p.124 , Mumford / Tata lectures on theta I).

Choose some non zero section $\Theta \in \Gamma(X, L)$. It is unique up to a scalar. For every n $n^*\Theta$ is the unique up to a scalar invariant section of $M(n)$. Choose a section

$$\Sigma_n : M(n)^* \longrightarrow G(n^*L)$$

to the commutator map

$$\chi : G(n^*L) \longrightarrow M(n)^*$$

and let

$$\delta_1 = \Theta,\ \ \delta_\psi = U_{\Sigma_n(\psi)}\, \delta_1.$$

Later on we will choose the Σ_n's more carefully and then we will baptize these bases. The notation $a(x, \psi)$ appearing below is the one used in Lemma 10 1) for $K = M(n)$, $S = L(n)_1$ (so K has no orthogonal complement but S does). In the case $K = L(n)_1$, $S = M(n)$ (which is dealt in Lemma 10 2), 3))we will not need a notation for the scalars of Lemma 10.

LEMMA 11. *Put* $s(n)_1 = P_{L(n)_1}(n^*\Theta)$. *Then* $s(n)_1 \neq 0$.

Proof. In the proof of Lemma 10 the following expression was obtained

$$P_S(\delta_\psi) = \frac{1}{n^{2g}} \sum_{\gamma \in \chi(S)} \left(\sum_{s \in \operatorname{Ker}\chi \cap S} a(s_\gamma s, \psi) \right) \delta_{\psi \cdot \gamma} .$$

The derivation of it did not use any orthogonality assumption. Using part 1) of Lemma 10, we get for $S = L(n)_1$, $K = M(n)$, $\psi = 1$, $\delta_\psi = n^*\Theta$, χ: $G(n^*L) \longrightarrow M(n)^*$,

$$P_{L(n)_1}(n^*\Theta) = \frac{1}{n^{2g}} \sum_{\gamma \in \chi(L(n)_1)} \left(a(s_\gamma, 1) \sum_{s \in \operatorname{Ker}\chi \cap L(n)_1} a(s,1) \right) \delta_\gamma .$$

In this case $\operatorname{Ker}\chi = k^\times M(n) = \Phi_n(k^\times \mathbb{Z}(1))$ and we see that since $\sigma^L = \tau^L$ on T_1 we actually have $\operatorname{Ker}\chi \cap L(n)_1 \subset M(n)$. Therefore if $s \in \operatorname{Ker}\chi \cap L(n)_1$ then

$$a(s,1)\,\delta_1 = U_s\,\delta_1 = \delta_1.$$

Thus

$$P_{L(n)_1}(n^*\Theta) \neq 0. \qquad \text{Q.E.D.}$$

DEFINITION. For every n define a basis

$$\mathcal{B}(n) = \{\, s(n)_\psi \mid \psi \in L(n)_1{}^* \,\}$$

of $\Gamma(X, n^*L)$ as follows : Let

$$s(n)_1 = P_{L(n)_1}(n^*\Theta),$$

let

$$\Xi_n : L(n)_1{}^* \longrightarrow L(n)_2$$

be the isomorphism determined by the commutator pairing of $G(n^*L)$ and let

$$s(n)_\psi = U_{\Xi_n(\psi)}\, s(n)_1 .$$

COROLLARY. Let $L(n)_1[n] = \{$ elements of order n in $L(n)_1$ $\}$, then

$$s(n)_\psi = \frac{1}{n^g} \sum_{u \in L(n)_1 / L(n)_1[n]} a(\Xi_n(\psi)u, 1)\, \delta_{\chi^u \cdot \chi^{\Xi_n(\psi)}} ,$$

(where $\chi : G(n^*L) \longrightarrow M(n)^*$).

Proof. Note that χ induces an isomorphism $L(n)_1 / L(n)_1[n] \cong \chi(L(n)_1)$. Therefore, for $\psi = 1$ the corollary follows from the proof of Lemma 11. The general case follows by applying $U_{\Xi_n(\psi)}$ and using part 1) of Lemma 10. Q.E.D.

Let us also record the following

LEMMA 12.
$$n^*\Theta = \sum_{\psi \in L(n)_1^*[n]} s(n)_\psi .$$

Proof. Let us first check that this true up to a scalar. By Lemma 10 $n^*\Theta = P_{M(n)}(s(n)_1)$ at least up to a scalar. But

$$\begin{aligned} P_{M(n)}(s(n)_1) &= \frac{1}{n^{2g}} \sum_{m \in M(n)} U_m s(n)_1 \\ &= \frac{1}{n^{2g}} \sum_{m_2 \in M(n) \cap L(n)_2} \; \sum_{m_1 \in M(n) \cap L(n)_1} U_{m_2} U_{m_1} s(n)_1 \\ &= \frac{1}{n^g} \sum_{m_2 \in M(n) \cap L(n)_2} U_{m_2} s(n)_1 \quad . \end{aligned}$$

$$= \frac{1}{n^g} \sum_{\psi \in L(n)_1^*[n]} s(n)_\psi .$$

To check the constant apply $P_{L(n)_1}$ to this sum. By linearity and Lemma 10 used for $K = S = L(n)_1$ we get

$$P_{L(n)_1}\Big(\frac{1}{n^g} \sum_{\psi \in L(n)_1^*[n]} s(n)_\psi\Big) = P_{L(n)_1}\Big(\frac{1}{n^g} s(n)_1\Big) = \frac{1}{n^g} s(n)_1 . \qquad \text{Q.E.D.}$$

LEMMA 13. *For every d the bases $\mathcal{B}(n)$ and $\mathcal{B}(dn)$ are related as follows: Let $\alpha : dL(dn)_2 / L(dn)_2[d] \longrightarrow L(n)_2$ be the natural isomorphism. Denoting by α the induced isomorphism $dL(dn)_1^* / L(dn)_1^*[d] \longrightarrow L(n)_1^*$. We have*

$$d^*(s(n)_\psi) = \sum_{\substack{\tau \in dL(dn)_1^* \\ \alpha(\tau) = \psi}} s(dn)_\tau .$$

Proof. The formula up to a scalar follows from the isogeny theorem of Mumford, [Mum1] p.302. The translation to the notation appearing there is as follows :

Mumford	X, Y	π	M	L	$\tilde{K}$
Us	X	d	n^*L	d^*n^*L	$nM(dn)$

The level structures for every m are completely determined by choosing a free $\hat{\mathbb{Z}}$ basis for T_2. This yields for $L(dn)_2$ an isomorphism $L(dn)_2 \cong K(\delta_{dn})$ and $L(n)_2 \cong K(\delta_n)$ where δ_{dn} and δ_n are the types $(d^2n^2, d^2n^2, ..., d^2n^2)$ and $(n^2, n^2, ..., n^2)$ respectively. We

get then natural isomorphisms $L(dn)_1 \cong K(\delta_{dn})^*$, $L(n)_1 \cong K(\delta_n)^*$ and therefore uniquely determined theta structures j_{dn}, j_n.

Mumford	δ_L, δ_M	j_L, j_M	K_1	K_2
Us	$(d^2n^2, ..., d^2n^2), (n^2, ..., n^2)$	j_{dn}, j_n	$(\mathbb{Z}/d^2n^2\mathbb{Z})^g[d]$	$(\mathbb{Z}/d^2n^2\mathbb{Z})^{g*}[d]$

Mumford	$K_1^\perp$	$K_2^\perp$	$\sigma : K_1^\perp / K_1 \rightarrow K(\delta_M)$
Us	$d(\mathbb{Z}/d^2n^2\mathbb{Z})^g$	$d(\mathbb{Z}/d^2n^2\mathbb{Z})^{g*}$	natural isom.

The conclusion is that up to a scalar is that $A\,\delta_u = \sum_{v\,;\,\sigma(v) = u} \delta_v$ †where δ_t denotes the delta function at t, and A is as in [Mum1] loc. cit.. We need only verify now that the functions $s(n)_\psi$ are the functions corresponding to the delta functions at points of $K(\delta_n)$ with the right nornalizations of the isomorphisms β_M appearing there. That is easily checked since the delta functions are characterized by the way the Heisenberg group acts on them. This proves our claim up to a scalar. The rest follows by comparing this with the formula for $n^*\Theta$ and $d^*n^*\Theta$ given by Lemma 12. Q.E.D.

Given an ample even symmetric line bundle L on X and an isogeny $f \in \mathcal{E}^{0}(L)$, say

$$f^*L \cong L^{n^2}, \quad \phi : f^*L \longrightarrow L^{n^2} \text{ an isomorphism,}$$

define a new action U^f of $\hat{G}(L)$ on $\hat{\Gamma}(X, L)$ by

$$U^f_z(s) = U_{\delta_f(z)}(s)$$

and an action of $\hat{G}(L)$ on $\hat{\Gamma}(X, L^{n^2})$ by

$$U^n_z(s) = U_{\varepsilon_{n^2}(z)}(s).$$

Then, both $\hat{\Gamma}(X, L)$ and $\hat{\Gamma}(X, L^{n^2})$ are irreducible representations of $\hat{G}(L)$ of order n^2 and there exists therefore (see appendix) a unique intertwining map

$$T : \hat{\Gamma}(X, L) \longrightarrow \hat{\Gamma}(X, L^{n^2}).$$

As in the case of automorphisms we have a

CENTRAL OBSERVATION. The linear isomorphism

$$\phi_* f^* : \hat{\Gamma}(X, L) \longrightarrow \hat{\Gamma}(X, L^{n^2})$$

is an intertwining operator for these two actions, therefore equal up to a scalar to T [5].

Proof. Let $\mathcal{T} = \phi_* f^*$. Then

(1) $$\mathcal{T}(U_{\delta_f(z)}(s)) = \mathcal{T}(U_{j(f,L)(\phi^{-1}\varepsilon_n(z)\phi)}(s))$$

<u>Claim</u>. For every $r \in \hat{G}(f^*L)$, $s \in \hat{\Gamma}(X, L)$ we have

$$f^*(U_{j(f,L)(r)}(s)) = U_r(f^*(s)).$$

Proof (of claim). Let $r = (r_m, \phi_m)_m$ then $j(f, L)(r) = (y_m, \psi_m)_m$ where ϕ_m is the pull back by f of ψ_m with respect to $f(r_m) = y_m$.

$$\begin{aligned} f^*[U_{(y_m, \psi_m)_m}(s)] &= f^*[\psi_k \circ s \circ T_{f(r_k)}^{-1}] \qquad \text{for every } k \text{ divisible enough} \\ &= \phi_k \circ f^*s \circ T_{r_k}^{-1} \\ &= U_{(r_m, \phi_m)_m}(f^*s) \end{aligned}$$

Using the claim we get from (1) that

$$\begin{aligned} \mathcal{T}(U_{\delta_f(z)}(s)) &= \phi_* U_{\phi^{-1}\varepsilon_n(z)\phi}(f^*s) \\ &= U_{\varepsilon_n(z)}(\phi_* f^*s) \\ &= U_{\varepsilon_n(z)} \mathcal{T}(s). \end{aligned}$$

Q.E.D.

The next thing we have to find is some subgroup of $\hat{G}(L)$ whose image under each of the maps ε_{n^2} and δ_f is a maximal level subgroup. The operator T is then determined up to a scalar by the condition that it must take invariant vector of the second level subgroup to an invariant vector of the first.

[5] There is a Schur lemma for adelic Heisenberg groups. The reason is that every intertwining operator must take the invariants of a maximal level subgroup to themselves and this is a one dimensional space if the level subgroup is chosen right. Further, the operator is determined as usual by its action on a single non zero vector.

LEMMA 14. *We have the following functorial properties* :

1) $\varepsilon_n\, \sigma^L(x) = \sigma^{L^n}(x)$; $\varepsilon_n\, \tau^L(x) = \tau^{L^n}(x)$.

2) $\eta_n\, \sigma^{L^n}(x) = \sigma^L(nx)$; $\eta_n\, \tau^{L^n}(x) = \tau^L(nx)$. (*see* [Mum1] *for the definition of* η_n).

3) *If* $\phi : L \longrightarrow M$ *is an isomorphism then*
$\phi_*\, \sigma^L(x) = \sigma^M(x)$; $\phi_*\, \tau^L(x) = \tau^M(x)$.

4) $j(f, L)\, \sigma^{f^*L}(x) = \sigma^L(V(f)x)$; $j(f, L)\, \tau^{f^*L}(x) = \tau^L(V(f)x)$.

5) $\delta_f\, \sigma^L(x) = \sigma^L(V(f)\, x)$; $\delta_f\, \tau^L(x) = \tau^L(V(f)\, x)$.

6) *Writing every element of* $\hat{G}\ (L)$ *as* $\lambda \cdot \tau^L(x)$ *we have for* $f^*L \cong L^n$,

$$\delta_f(\lambda \cdot \tau^L(x)) = \lambda^n\, \tau^L(V(f)\, x).$$

The proof is completely straightforward and therefore omitted.

DEFINITION. Assume that $\phi : f^*L \cong L^{n^2}$, L ample even symmetric of degree 1. Define

$$\mathcal{Z} = \{\ \sigma^L(x) \mid x \in T(X)\ \} \subset \hat{G}(L),$$
$$f^*(\mathcal{Z}) = j(f, L)^{-1}(\mathcal{Z}) \subset \hat{G}(f^*L),$$
$$f^*(\mathcal{Z})_\phi = \phi_* f^*(\mathcal{Z}) \subset \hat{G}(L^{n^2}),$$
$$\mathcal{Z}(f) = \varepsilon_{n^2}^{-1}\,(f^*(\mathcal{Z})_\phi) = \delta_f^{-1}(\mathcal{Z}) \subset \hat{G}(L).$$

Let $e_*^L : \frac{1}{2} \dagger T(X) / T(X)$ be the quadratic form defined by

$$\sigma^L(2x) = e_*^L(x)\, \tau^L(2x).$$

(see [Mum3] p.59 ff.).

LEMMA 15. *We have*

$$f^*(\mathcal{Z})_\phi = \{\, e_*^L(x/2)\ \tau^{L^{n^2}}(V(f)^{-1}(x)) \mid x \in T(X)\ \},$$

and the projection of $f^*(\mathcal{Z})_\phi$ *to* $G(n^*L)$ *is* $K(f, L)$ *- the level subgroup corresponding to the descent data* $L^{n^2} \cong f^*L$.

Proof. The first assertion follows imediately from Lemma 14 and the definitions. To get the second, one considers the preimage of $K(f, L)$ in $\hat{G}\ (L^{n^2})$ under the homomorphism

$$Normalizer\ (\ \sigma^{L^{n^2}}(T(X))\ ) \longrightarrow G(n^*L).$$

It is a level subgroup which must be maximal by index consideration. But it clearly has the same invariant vector as $f^*(\mathcal{Z})_\phi$ does, namely, $\phi_* f^* \Theta$ where Θ is a generator of the one dimensional vector space $\Gamma(X, L)$. This implies equality (to ease the argument note that both cover the same maximal isotropic subgroup). Q.E.D.

We keep the assumption $f \in \mathcal{E}^0(L)$, $f^*L \cong L^{n^2}$. Let $\omega \in A_{L(n)_1}(K(f, L))$ where the fixed level subgroup with orthogonal complement is $L(n\ )_1$, the maximal level subgroup of $G(n^*L)$ constructed before. Then $P_{K(f,\,L)}\ (s(n)_\omega\ )$ is the unique, up to a scalar, $f^*(\mathcal{Z})_\phi$ invariant vector.

Decompose $\hat{\Gamma}\ (X, L)$ according to eigen spaces of $\mathcal{Z}$,

$$\hat{\Gamma}(X, L) = \bigoplus_{\psi \in \mathcal{Z}^*} \hat{\Gamma}(X, L)_\psi$$

Choose a section $\Sigma : \mathcal{Z} \longrightarrow \hat{G}\ (L)$ and let $\Theta_\psi = U_{\Sigma(\psi)}(\Theta)$. Then $\{\ \Theta_\psi \mid \psi \in \mathcal{Z}^*\ \}$ is a basis for $\hat{\Gamma}(X, L)$, each Θ_ψ spans $\hat{\Gamma}(X, L)_\psi$ and for every n,

$$\mathcal{D}(n) = \{ \Theta_\psi \mid \psi \in \mathcal{Z}^*[n] \}$$

is a basis for $\Gamma(X, n^*L)$. Therefore if we had some nice choice of a section we could explicitly write the vector $s(n)_\omega$ and thus solve the problem of writing the pull back $\phi_* f^*$ explicitly. That is our next objective.

Before plunging into details, let us explain what we are about to do. We start by choosing a good theta structure for the big group $\hat{G}(L)$ and a section for $\hat{G}(L) \longrightarrow \mathcal{Z}^*$ (We will assume that the Göpel structure is obtained from this theta structure. There seems to be no point in generalizing). We work with some ' indeterminate ' in our section saving its specialization to the end. That makes the generalization of the case treated below to the general case easier.

Since we have assumed that L is an ample even symmetric line bundle of degree 1 on X [6] , that means that there exists, by [Mum3] Proposition 4.20, a theta structure

$$\Delta: \hat{G}(L) \longrightarrow G,$$

where G is the standard Heisenberg group as in the introduction, having the following properties :

If we define $V_1, V_2 \subsetneq V(X)$ by

$$\Delta(\tau^L(V_1)) = \{ (1, z, 0) \mid z \in \mathbb{A}_f^g \}, \quad \Delta(\tau^L(V_2)) = \{ (1, 0, z) \mid z \in \mathbb{A}_f^g \}.$$

Then,

$$\Delta(\sigma^L(T_1)) = \{ (1, z, 0) \mid z \in \hat{\mathbb{Z}}_f^g \}, \quad \Delta(\sigma^L(T_2)) = \{ (1, 0, z) \mid z \in \hat{\mathbb{Z}}_f^g \},$$

and e_*^L goes over to the function e_* where :

$$e_*(x/2, y/2) = (-1)^{{}^t x \cdot y}.$$

That implies that V_1, V_2 is a Göpel structure and that

$$\Delta(\mathcal{Z}) = \{ (e_*(x/2, y/2), x, y) \mid x, y \in \hat{\mathbb{Z}}_f^g \},$$

$$\Delta(\tau^L(V(X))) = \{ (1, x, y) \mid x, y \in \mathbb{A}_f^g \}.$$

Let $\mathcal{S} = \Delta(\mathcal{Z})$. We have

$$\Delta(K(n)_1) = \{ (1, z, 0) \mid nz \in \hat{\mathbb{Z}}_f^g \}, \quad \Delta(K(n)_2) = \{ (1, 0, z) \mid nz \in \hat{\mathbb{Z}}_f^g \},$$

whence we get finite theta structures

$$\Delta_n : G(n^*L) \longrightarrow G(\delta_{n^2})$$

where

$$G(\delta_n) = k^\times \times \left(\tfrac{1}{n} \hat{\mathbb{Z}}_{ff}{}^g / n \hat{\mathbb{Z}}_{ff}{}^g\right) \times \left(\tfrac{1}{n} \hat{\mathbb{Z}}_{ff}{}^g / n \hat{\mathbb{Z}}_{ff}{}^g\right)$$

or simply (under the canonical isomorphism)

$$G(\delta_n) = k^\times \times \left(\tfrac{1}{n} \mathbb{Z}^g / n \mathbb{Z}^g\right) \times \left(\tfrac{1}{n} \mathbb{Z}^g / n \mathbb{Z}^g\right)$$

with the usual group law,

$$(\alpha, x, l)(\beta, y, m) = (\alpha\beta\, \zeta_{n^2}^{{}^t(nx) \cdot (nm)}, x + y, l + m)$$

by demanding the natural homomorphism between $\Delta(K(n)_i)$ and $\left(\tfrac{1}{n} \hat{\mathbb{Z}}_{ff}{}^g / \hat{\mathbb{Z}}_{ff}{}^g\right)$ (we have to choose ζ_{n^2} so that the commutator pairings agree. Therefore we take $\zeta = \underline{e}(1/n^2)$). That is, denoting by

[6] This assumption is not essential but it does simplify the calculations below. We remark that every symmetric line bundle becomes even symmetric after a translation by a torsion point of order 2. Since no new idea is involved in treating the more general case we make this assumption.

$$p_n : k^\times \Delta(K(n)_1)\Delta(K(n)_2) = \Delta(N_{\hat{G}(L)}\ (\sigma^L(nT(X)))) \longrightarrow G(\delta_n)$$

the natural projection, we have that p_n induce the natural projection on $\Delta(K(n)_i)$. The element $(1, x, y)$ goes under this homomorphism to

$$\begin{aligned} p_n((1,x,y)) &= p_n((\underline{\mathbf{e}}(\tfrac{1}{2}\,{}^t y\cdot x), 0, y)(1, x, 0)) \\ &= (\underline{\mathbf{e}}(\tfrac{1}{2}\,{}^t y\cdot x), x, y). \end{aligned}$$

Therefore, the image of $\mathcal{S}$ under these theta structures is

$$p_n(\mathcal{S}) = \{ (1, x, y) \mid x, y \in \mathbb{Z}^g / n\,\mathbb{Z}^g \}$$

which is precisely the same sort of theta structure used in the Symmetry Theorem.

Define

$$E(x, y) = \underline{\mathbf{e}}(\tfrac{1}{2}\,{}^t y\cdot x).$$

Choose as a set of representatives to the cosets $G / k^\times \Delta(\mathcal{Z})$ defined by the commutator homomorphism

$$G \longrightarrow \mathcal{S}^*$$

the set

$$\mathrm{REP} = \{ (l(x, y), x, y) \mid x, y \in \mathbb{Q}^g \cap [0, 1) \}$$

where $l : \mathbb{A}_f^g \times \mathbb{A}_f^g \longrightarrow k^\times$ is a function with the property

$$l(x, 0) = l(0, x) = 1 \qquad \text{for all } x \in \mathbb{A}_f^g.$$

This gives us a section

$$\Sigma : \mathcal{S}^* \longrightarrow G$$

which we write as

$$\Sigma(\psi) = (l(\alpha(\psi)), \alpha(\psi)),$$

and sections

$$\Sigma_n : (\mathcal{S} / n\mathcal{S})^* \longrightarrow G(\delta_n)$$

which we can write, identifying $(\mathcal{S} / n\mathcal{S})^*$ with $\mathcal{S}^*[n]$, as

$$\Sigma_n(\psi) = (l(\alpha(\psi))\cdot E(\alpha(\psi)), \alpha(\psi))$$

These sections induce sections

$$\Sigma : \mathcal{Z}^* \longrightarrow \hat{G}(L)$$
$$\Sigma_n : M(n)^* \longrightarrow G(n^*L).$$

One advantage of such theta structures is the simple form which the homomorphisms δ_f now have :

By lemma 14, 6) we have

$$\delta_f(\lambda\cdot \tau^L(x)) = \lambda^{n^2}\cdot \tau^L(V(f)\,x),$$

denoting by M_f the matrix representing V(f) on $\mathbb{A}_f^{2g}$ and by D_f the induced homomorphism on G we get

$$D_f\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\lambda^{n^2}, M_f\begin{pmatrix} x \\ y \end{pmatrix}\right),$$

where $\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = (\lambda, x, y)$.

In the case where $f \in \mathcal{A}(L)$ we get

$$d_f : G(\delta_n) \longrightarrow G(\delta_n)$$

$$d_f\left(\lambda, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \left(\lambda\cdot m_f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right), M_f\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where

$$m_f\left(\binom{x}{y}\right) = \frac{E\left(M_f\binom{x}{y}\right)}{E\binom{x}{y}} \quad .$$

The advantage of this expression is that it is completely explicit once the adic representation of f is known. Because of the usefulness of this result we record it as

LEMMA 16.
$$D_f\left(\lambda, \binom{x}{y}\right) = \left(\lambda^{n^2}, M_f\binom{x}{y}\right),$$
$$d_f\left(\lambda, \binom{x}{y}\right) = \left(\lambda \cdot \frac{E\left(M_f\binom{x}{y}\right)}{E\binom{x}{y}}, M_f\binom{x}{y}\right).$$

Digression - The Symmetry Theorem.

We want to reconsider the Symmetry Theorem. As in the derivation of the explicit form of the symmetry theorem we take the symmetric line bundle to be n^*L. Moreover, we assume that L itself is even symmetric. Recall that the intertwining operator

$$T: \Gamma(X, n^*L) \longrightarrow \Gamma(X, n^*L)$$

was determined by

$$T(\Theta) = \Theta_{\gamma_g},$$

where now we take the basis

$$\Theta_\psi = U_{\Sigma_n(\psi)}\, \Theta_1, \quad \Theta = \Theta_1.$$

We have

$$\Sigma_n(\psi) = (\, l(\alpha(\psi)) \cdot E(\alpha(\psi)), \alpha(\psi)\,).$$

We want to determine $T(\Theta_\psi)$. The intertwining property implies that

$$\begin{aligned} T(\Theta_\psi) &= T(U_{\Sigma_n(\psi)}\, \Theta_1) \\ &= U_{\delta_{g^{-1}}((\, l(\alpha(\psi))\, E(\alpha(\psi)), \alpha(\psi)))}\; U_{(\, l(\alpha(\gamma_g)) \cdot E(\alpha(\gamma_g)), \alpha(\gamma_g))}\, \Theta_1 \end{aligned}$$

By Lemma 16

$$\delta_{g^{-1}}(\, l(\alpha(\psi)) \cdot E(\alpha(\psi)), \alpha(\psi)\,) = (\, l(\alpha(\psi)) \cdot E(M_{g^{-1}}\, \alpha(\psi)), M_{g^{-1}}\, \alpha(\psi)\,)$$

Therefore, using $[\cdot]$, $\{\cdot\}$, to denote integral and fractional parts, we get

$$\begin{aligned} &\delta_{g^{-1}}(\, l(\alpha(\psi)) \cdot E(\alpha(\psi)), \alpha(\psi)\,)\,(\, l(\alpha(\gamma_g)) \cdot E(\alpha(\gamma_g)), \alpha(\gamma_g)\,) \\ &= (\, l(\alpha(\psi)) \cdot E(M_{g^{-1}}\, \alpha(\psi)), M_{g^{-1}}\, \alpha(\psi)\,)\,(\, l(\alpha(\gamma_g)) \cdot E(\alpha(\gamma_g)), \alpha(\gamma_g)\,) \\ &= (\, l(\alpha(\psi))\, l(\alpha(\gamma_g)) \cdot E(M_{g^{-1}}\, \alpha(\psi))\, E(\alpha(\gamma_g))\, F(M_{g^{-1}}\, \alpha(\psi), \alpha(\gamma_g)\,),\; M_{g^{-1}}\, \alpha(\psi) + \alpha(\gamma_g)\,) \\ &= \Big(\, l(\alpha(\psi))\;\, l(\alpha(\gamma_g))\;\, E(M_{g^{-1}}\, \alpha(\psi))\;\, E(\alpha(\gamma_g))\;\, F(M_{g^{-1}}\, \alpha(\psi), \alpha(\gamma_g)\,) \\ &\qquad F(\{M_{g^{-1}}\, \alpha(\psi) + \alpha(\gamma_g)\}, [M_{g^{-1}}\, \alpha(\psi) + \alpha(\gamma_g)]\,)^{-1},\quad \{M_{g^{-1}}\, \alpha(\psi) + \alpha(\gamma_g)\}\Big) \\ &\qquad\quad \times \Big(1,\; [M_{g^{-1}}\, \alpha(\psi) + \alpha(\gamma_g)]\Big) \end{aligned}$$

Using the fact that $p_n(\mathcal{S}) \subset \hat{G}(L)$ corresponds under the theta structure Δ_n to $M(n) = K(n, L)$, we conclude that

$$U_{\delta_{g^{-1}}((\, l(\alpha(\psi))\, E(\alpha(\psi)), \alpha(\psi)))}\; U_{(\, l(\alpha(\gamma_g)) \cdot E(\alpha(\gamma_g)), \alpha(\gamma_g))}\, \Theta_1 = b_{g,\psi}\; \Theta_{\gamma_g \cdot g\psi},$$

where

$$\begin{aligned} b_{g,\psi} &= l(\alpha(\psi))\ l(\alpha(\gamma_g))\ \ l(\{M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)\})^{-1} \\ &\times E(M_{g^{-1}}\alpha(\psi))\ E(\alpha(\gamma_g))\ E(\{M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)\})^{-1} \\ &\times F(M_{g^{-1}}\alpha(\psi),\alpha(\gamma_g))\ \ F(\{M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)\},[M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)])^{-1} \end{aligned}$$

Clearly a good choice of l is $l(x, y) = E(x, y)^{-1}$. Making this choice we get

$$\begin{aligned} b_{g,\psi} &= E(\alpha(\psi))^{-1}\ E(M_{g^{-1}}\alpha(\psi))\ F(M_{g^{-1}}\alpha(\psi),\alpha(\gamma_g)) \\ &\times\ F(\{M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)\},[M_{g^{-1}}\alpha(\psi)+\alpha(\gamma_g)])^{-1} \end{aligned}$$

Put as before

$$v_g(\psi) = M_{g^{-1}}\alpha(\psi) + \alpha(\gamma_g)$$

then we get the familiar expression

$$b_{g,\psi} = m_{g^{-1}}(\alpha(\psi))\, F(v_g(\psi) - \alpha(\gamma_g),\ \alpha(\gamma_g))\, F(\{\alpha(\gamma_g)\},[\alpha(\gamma_g)])^{-1}.$$

Finally, note that

$$\gamma_g = \beta_g^{-1}\ ;\ \beta_g = (m_g \mid_{\mathcal{Z}})^{-1}\ .$$

That is

$$\gamma_g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = E\left(M_g\begin{pmatrix} x \\ y \end{pmatrix}\right) / E\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

THE SYMMETRY THEOREM (EXPLICIT FORM II). *Let L be an ample even symmetric line bundle of degree one on X. Let $Z = X[n]$ and let $\mathcal{Z}$ be the maximal level subgroup of $G(n^*L)$ lying above Z and corrseponding to the descent data n, L ($\mathcal{Z} = M(n)$ in our new terminology). Choose some non zero section $\Theta_1 \in \Gamma(X, n^*L)_1$ and choose a section Σ to the commutator map as described above*

$$G(n^*L) \leftarrow^{\Sigma} \text{---}\ \mathcal{Z}^*.$$

Given $g \in \mathcal{A}(n^*L)$ let $\phi : g^*n^*L \rightarrow n^*L$ be an isomorphism. Let $T = \phi_* g^*$. Then there exists a constant $c(g)$ such that

$$T\left(\sum_{\psi\in\mathcal{Z}^*} a_\psi \Theta_\psi\right) = c(g) \sum_{\psi\in\mathcal{Z}^*} a_\psi b_{g,\psi}\Theta_{\gamma_g\cdot g\psi} \qquad ,$$

where $\{\Theta_\psi = U_{\Sigma(\psi)}\Theta_1\}$ is a basis for $\Gamma(X, n^*L)$. We have

$$\gamma_g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = E\left(M_g\begin{pmatrix} x \\ y \end{pmatrix}\right) / E\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Further, the scalars $b_{g,\psi}$ are given by

$$b_{g,\psi} = m_{g^{-1}}(\alpha(\psi))\, F(v_g(\psi) - \alpha(\gamma_g),\ \alpha(\gamma_g))\, F(\{v_g(\psi)\},[v_g(\psi)])^{-1},$$

where

$$v_g(\psi) = M_{g^{-1}}\alpha(\psi) + \alpha(\gamma_g)$$

and

$$m_{g^{-1}}(\alpha(\psi)) = E(M_{g^{-1}}\alpha(\psi))\ E(\alpha(\psi))^{-1}\ ,$$
$$E(x, y) = \underline{e}(\tfrac{1}{2}\, {}^t y \cdot x)\,.$$

COROLLARY 1. Assume that $n^*L = \underline{O}_X(D)$, then ϕ_* is multiplication by a function f_g. Suppose further that $0 \notin supp(D)$, then there exists a function Ω such that $f_g = \Omega^g / \Omega$ and

$0 \notin supp(\Omega)$. We get then that for all x

$$\frac{\Omega(gx)}{\Omega(x)}\,\Theta_\psi(gx) = c(g)\,b_{g,\psi}\,\Theta_{\gamma_g\cdot g\psi}(x) \quad ,$$

where $c(g)$ is a non zero constant which is independent of ψ. In particular,

$$\Theta_\psi(0) = c(g)\,b_{g,\psi}\,\Theta_{\gamma_g\cdot g\psi}(0) \quad .$$

Assume that $\gamma_g = 1$ then, if $\Theta_1(0) \neq 0$ and therefore $c(g) = 1$, we get

$$\frac{\Omega(gx)}{\Omega(x)}\,\Theta_\psi(gx) = \frac{E(M_{g^{-1}}(\alpha(\psi)))}{E(\alpha(\psi))}\,\Theta_{g\psi}(x) \quad ,$$

$$\Theta_\psi(0) = \frac{E(M_{g^{-1}}(\alpha(\psi)))}{E(\alpha(\psi))}\,\Theta_{g\psi}(0) \quad .$$

EXAMPLE. To illustrate this theorem take the simplest case $g = -1$. Then by our assumption :

$$\gamma_{-1} = 1, \quad \alpha(\gamma_{-1}) = 0, \quad m_{-1} \equiv 1, \quad (-1)\psi = \psi^{-1} \text{ for every } \psi.$$

It follows that $b_{g,\psi} = 1$ for every ψ. Therefore, there exists a constant $c(-1)$, independent of ψ, such that

$$T(\Theta_\psi) = \Theta_{\psi^{-1}}$$

for every ψ. This implies that we may normalize ϕ such that

$$\phi_*(-1)^*(\Theta_\psi) = c(-1)\,\Theta_{\psi^{-1}}$$

for every ψ where $c(-1) = \{\pm 1\}$.

If the conditions of Corollary 1 hold, then $c(-1) = 1$.

Using this one can get, up to ± 1, [Mum1] ' Inverse Formula' p.331, [Mum3] Cor. 6.21 p.114, or [Kem] Theorem 4 p.71. We can get the exact constant which is 1 under the conditions of Corollary 1.

One should now proceed to the detailed study of some classical examples, e.g. the examples obtained from Prym varieties, factors of the Fermat's curve, modular curves etc. This is a subject for another paper but few simple examples are given in the next section.

Back to the general discussion !

Define a basis for $\hat{\Gamma}(X, L)$ by

$$\{\ \Theta_\psi = U_{\Sigma(\psi)}\,\Theta \mid \psi \in \mathcal{Z}^*\ \}.$$

Where now Σ is the particular section we have specified. As before, this basis has the property that

$$\mathcal{D}(n) = \{\ \Theta_\psi \mid \psi \in \mathcal{Z}^*[n]\ \}$$

is a basis for $\Gamma(X, n^*L)$ for every n.

Recall that we had the sections

$$\Xi_n : L(n)_1{}^* \longrightarrow L(n)_2$$

and the bases $\mathcal{B}(n)$ to $\Gamma(X, n^*L)$ were defined using Ξ_n :

$$\mathcal{B}(n) = \{\ s(n)_\psi = U_{\Xi_n(\psi)}\,s(n)_1 \mid \psi \in L(n)_1{}^*\ \}.$$

Write

$$\Xi_n(\psi) = (1, 0, \beta(\psi)).$$

LEMMA 17. 1) $$s(n)_1 = \sum_{\psi \in M(n)_2^*} \Theta_\psi ,$$

where $M(n)_i = M(n) \cap L(n)_i$. *(Note that this is an equality in* $\hat{\Gamma}(X, L)$*).*

2) *Given* $\rho \in L(n)_1{}^*$ *define* $\rho^\dagger \in M(n)^*$ *as*

$$\rho \mid_{M(n)_1} \in M(n)_1{}^* \hookrightarrow M(n)^* = M(n)_1{}^* \oplus M(n)_2{}^*.$$

Put for $\psi \in M(n)^*$, $\rho \in L(n)_1{}^*$ *and* l *as above*

$$\Phi(l, \psi, \rho) = E(\alpha(\psi)) \cdot l\,(\alpha(\psi)) \cdot E(\{\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}\})^{-1}\; l(\{\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}\})^{-1}$$
$$\cdot\; F(\{\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}\},\ [\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}])^{-1}$$

where $[\cdot]$, $\{\cdot\}$ *denote integral and fractional parts respectively. Then*

$$s(n)_\rho = \frac{1}{n^g} \sum_{\psi \in M(n)_2^*} \Phi\,(l, \psi, \rho)\, \Theta_{\psi \cdot \rho} \quad .$$

3) *Choose* $l\,(x, y) = E(x, y)^{-1}$ *then, putting* $\Phi(\psi, \rho) = \Phi(l, \psi, \rho)$, *we have*

$$\Phi(\psi, \rho) = F(\{\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}\},\ [\alpha(\psi) + \begin{pmatrix} 0 \\ \beta(\rho) \end{pmatrix}])^{-1}$$

and

$$s(n)_\rho = \frac{1}{n^g} \sum_{\psi \in M(n)_2^*} \Phi\,(\psi, \rho)\, \Theta_{\psi \cdot \rho} \quad .$$

Proof. We defined $s(n)_1 = P_{L(n)_1}(\Theta)$. Since under the theta structure Δ_n we have

$$\Delta_n\,(L(n)_1) = \{\ (1, x, 0) \mid x \in \left(\tfrac{1}{n}\,\mathbb{Z} / n\,\mathbb{Z}\right)^g \ \},$$

we may take as representatives to $\Delta_n\,(L(n)_1)\ /\ \Delta_n\,(M(n)_1)$ elements of REP which are of course the image under Σ of $\Delta_n\,(M(n)_2)^*$. Therefore (recall that $\Theta = n^*\Theta$ in $\hat{\Gamma}(X, L)$)

$$\begin{aligned}
P_{L(n)_1}(\Theta) &= \frac{1}{n^{2g}} \sum_{z \in L(n)_1} U_z\, \Theta \\
&= \frac{1}{n^{2g}} \sum_{\Sigma(\psi)\,;\ \psi \in M(n)_2^*}\ \sum_{z \in M(n)_1} U_{\Sigma(\psi)} U_z\, \Theta \\
&= \frac{1}{n^g} \sum_{\Sigma(\psi)\,;\ \psi \in M(n)_2^*} U_{\Sigma(\psi)}\, \Theta \\
&= \frac{1}{n^g} \sum_{\psi \in M(n)_2^*} \Theta_\psi \quad .
\end{aligned}$$

2) By definition $s(n)_\rho = U_{(1,0,\beta(\rho))}\, s(n)_1$. Thus

$$\begin{aligned}
s(n)_\rho &= U_{(1,0,\beta(\rho))}\, s(n)_1 \\
&= U_{(1,0,\beta(\rho))} \frac{1}{n^g} \sum_{\Sigma(\psi)\,;\ \psi \in M(n)_2^*} U_{\Sigma(\psi)}\, \Theta \\
&= \frac{1}{n^g} \sum_{\Sigma(\psi)\,;\ \psi \in M(n)_2^*} U_{(1,0,\beta(\rho))\,(\,l(\alpha(\psi)\cdot E(\alpha(\psi)),\ \alpha(\psi)\,)}\, \Theta \quad .
\end{aligned}$$

To get the formula we want, we calculate (using that $F\left(\left(\begin{smallmatrix}0\\ *\end{smallmatrix}\right),\left(\begin{smallmatrix}*\\ *\end{smallmatrix}\right)\right)=1$)

$$
\begin{aligned}
&(1, 0, \beta(\rho))\,(\, l(\alpha(\psi))\, E(\alpha(\psi)),\ \alpha(\psi)\,)\\
&\quad = (\, E(\alpha(\psi))\, l\,(\alpha(\psi)),\ \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,)\\
&\quad = (E(\alpha(\psi))\, l\,(\alpha(\psi))\, F(\{\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\}, [\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,])^{-1}, \{\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\})\\
&\qquad\qquad \times\ (1, [\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,]\,)\\
&\quad = (\Phi(l,\, \psi, \rho)\, l\,(\{\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\})\, E(\{\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\}),\ \{\, \alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\})\\
&\qquad\qquad \times\ (\,1\,,\ [\alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,]\,).
\end{aligned}
$$

Whence

$$s(n)_\rho = \frac{1}{n^g} \sum_{\psi \in M(n)_2^*} \Phi(l,\, \psi, \rho)\, \Theta_{\psi\cdot\rho} \quad .$$

Q.E.D.

REMARK. A classical case of the transformation formulas we have just proved is (for the right choice of l) the change of basis inverse to the change of basis given in Mumford / Tata lectures on theta I p. 124

THE QUASI SYMMETRY THEOREM. *Let X be an abelian variety over an algebraically closed field k of characteristic $p \geq 0$. Let L be an ample even symmetric line bundle of degree 1 on X. Let $f : X \longrightarrow X$ be a quasi symmetry of L, $\phi : f^*L \longrightarrow L^{n^2}$ an isomorphism. Fix a theta structure as above and let the groups $L(n)_i$, and the bases $\mathcal{B}(n)$, be defined as above. Let Δ, Δ_n be the system of theta structures obtained and $\Sigma : \sigma^L(T(X))^* \longrightarrow \hat{G}\,(L)$ be the section constructed by the set of representatives REP.*

Let $K(f, L)$ be the maximal level subgroup associated with the descent data $f^*L \cong L^{n^2}$, and let $A(K(f, L))$ be defined with respect to $L(n)_1$. Then the map

$$\phi_* f^* :\ \hat{\Gamma}\,(X, L) \longrightarrow \hat{\Gamma}\,(X, n^*L)$$

is an intertwining operator with respect to the δ_f and ε_{n^2} action and therefore equal up to a scalar to the intertwining map determined by the equality in $\hat{\Gamma}\,(X, L)$

$$\phi_* f^*\, \Theta = P_{K(f,\,L)}\, s(n)_\omega$$

where $\omega \in A(K(f, L))$ is arbitrary. We have

$$s(n)_\rho = \frac{1}{n^g} \sum_{\psi \in M(n)_2^*} \Phi(\psi, \rho)\, \Theta_{\psi\cdot\rho}$$

(for $l(x, y) = E(x, y)^{-1}$), where

$$\Phi(\psi, \rho) = F(\{\alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,\},\ [\alpha(\psi) + \begin{pmatrix}0\\ \beta(\rho)\end{pmatrix}\,]\,)^{-1}.$$

REMARKS. 1) We regard here $K(f, L)$ as an ' atom '. We do not decompose this data further and explicate $P_{K(f,\,L)}\, s(n)_\omega$. The same attitude is manifested with regard to writing the general formula for $\phi_* f^*$. Both of these details clearly could be supllied by some tedious computations. However, over the complex numbers, using the Appell - Humbert formalism we shall give in §5 an explicit description of $K(f, L)$ just to illustrate the technique.

2) The natural idea would be to give a criterion stating when $P_{K(f,L)}\,\Theta_\psi$ is non zero and save the detour of going through the groups $L(n)_i$. However, this does not seem to exists. The reason is that $\mathcal{Z}$ is very far from having an orthogonal complement, which enables us to decompose $\hat{\Gamma}(X, L)$ with respect to it but a nice Lemma as Lemma 10 does not exist. When we take a maximal level subgroup with an orthogonal complement as $\tau^L(V_1)$, we have Lemma 10 but we can not decompose $\hat{\Gamma}(X, L)$ with respect to it. However, *we can do it on finite levels* with the groups $L(n)_i$. The resulting bases are well enough connected to the base with respect to $\mathcal{Z}$ and nicely related (by the isogeny theorem) to make sense, and to be practical for an explicit computation. Some interesting examples, where these computations may be of interest are given by Prym varieties and Fermat curves (for automorphisms) and by Humbert surfaces and elliptic curves with C.M. (for isogenies). A detailed study of such examples is a topic for another paper.

It is very important to deal also with a more general situation than considered above. Consider the following situation :

X, Y are abelian varieties over k, and

$$f: X \longrightarrow Y$$

an isogeny of degree prime to p. Suppose that there exists an ample even symmetric line bundle L on Y of degree 1, and an ample even symmetric line bundle M on X of degree 1, such that

$$f^*L \cong M^n$$

(n is determined, of course, by f). In this situation the question is how to write the sections f^*s, $s \in \Gamma(Y, L)$ as sections of M relative to the bases that we have constructed. The situation we have dealt with above is when $X = Y$, $L = M$ and is therefore a special case of this more general setting. It is amusing to note that the more general case is, under mild restrictions on f, a special case of the special case. Indeed, consider the line bundle $N = p_1{}^*M \otimes p_2{}^*L$ on $X \times Y$, and let

$$\xi : X \times Y \longrightarrow X \times Y$$

be the composition

$$X \times Y \xrightarrow{Id.\times\,\phi_L} X \times \mathrm{Pic}^0(Y) \xrightarrow{f\,\times f^\vee} Y \times \mathrm{Pic}^0(X) \xrightarrow{Id.\,\times\,\phi_M^{-1}} Y \times X \xrightarrow{per.} X \times Y$$

where $per.(y, x) = (x, y)$.

LEMMA. *Assune that either $\deg f$ is odd or that for some g, $f = g \circ 2$, then* $\xi^*N \cong N^n$.

Proof. $\xi(x, y) = ((\phi_M^{-1} \circ f^\vee \circ \phi_L)(y), f(x))$. Therefore

$$\xi^*N \cong p_1{}^*f^*L \otimes p_2{}^*(\phi_M^{-1} \circ f^\vee \circ \phi_L)^*M$$
$$\cong p_1{}^*M^n \otimes p_2{}^*(\phi_M^{-1} \circ f^\vee \circ \phi_L)^*M.$$

Hence, it is sufficient to prove that

$$(\phi_M^{-1} \circ f^\vee \circ \phi_L)^*M \cong L^n.$$

First of all

$$\begin{aligned} f^*(\phi_M^{-1} \circ f^{\vee} \circ \phi_L)^* M &\cong (\phi_M^{-1} \circ f^{\vee} \circ \phi_L \circ f)^* M \\ &\cong n^* M \\ &\cong M^{n^2}. \end{aligned}$$

We have also $f^*L^n \cong M^{n^2}$. It follows that L^n and $M_1 = (\phi_M^{-1} \circ f^{\vee} \circ \phi_L)^* M$ are algebraically equivalent. Therefore

$$e^{L^n} = e^{M_1}$$

and since both are symmetric it is enough to prove ([Mum3] Lemma 4.25) that $e_*^{L^n} = e_*^{M_1}$. Now the functorial properties of the sections σ, τ stated in Lemma 14 show that universally

$$e_*^{g^*D}(x) = e_*^{D}(V(g)x)$$

(where g is an isogeny between abelian varieties, D an ample symmetric line bundle etc.). Whence,

$$e_*^{M_1}(y) = e_*^{M}(V(\phi_M^{-1} \circ f^{\vee} \circ \phi_L)(y)).$$

If $f = g \circ 2$ then n satisfies $n^g = \deg f$ and therefore is even. Hence $e_*^{L^n} \equiv 1$. We also have that $f^{\vee} = g^{\vee} \circ 2$ and therefore $V(\phi_M^{-1} \circ f^{\vee} \circ \phi_L)$ kills $T(Y)[2]$ whence $e_*^{M_1} \equiv 1$.

If, on the other hand $\deg f$ is odd then the map $V(f) : T(X)[2] \longrightarrow T(Y)[2]$ is surjective. Take some $x \in T(X)[2]$ satisfying $V(f)(x) = y$. Then,

$$\begin{aligned} e_*^{M_1}(y) &= e_*^{M}(V(\phi_M^{-1} \circ f^{\vee} \circ \phi_L)(V(f)(x))) \\ &= e_*^{M}(V(\phi_M^{-1} \circ f^{\vee} \circ \phi_L \circ f)(x)) \\ &= e_*^{M}(nx) \\ &= e_*^{n^*M}(x) \\ &= e_*^{M^{n^2}}(x) \\ &= e_*^{(f^*L)^n}(x) \\ &= (e_*^{f^*L}(x))^n \\ &= (e_*^{L}(V(f)x))^n \\ &= e_*^{L^n}(y). \end{aligned}$$

Q.E.D.

REMARK. The conditions of the Lemma are necessary. For a typical example let E be an elliptic curve, $\omega \in E[2]$ a non zero point and let $Y = E / \langle \omega \rangle$. Denote by f the natural projection $E \longrightarrow Y$. Let α, β be generators of $E[4]$ such that $2\alpha = \omega$.

Let L be the line bundle of degree 1 on Y defined by the 2 torsion point $f(\alpha)$. Then f^*L is defined by the divisor $\{\alpha, \alpha + \omega\}$ and we can take as a symmetric line bundle M such that $f^*L \cong M^2$ the line bundle defined by the point $0 \in E$. Both M and L are symmetric. If we identify E and Y with their duals in tha canonical way we find that the dual map $f^{\vee} : Y \longrightarrow E$ is the composition

$$E / \langle \omega \rangle \longrightarrow E / E[2] \longrightarrow E$$

where the last arrow is the isomorphism induced from multiplication by 2 on E. $f^{\vee *}M$ is defined by the divisor $\{0, 0 + t\}$, t is the non zero point of $E[2] / \langle \omega \rangle$. $f^{\vee *}M$ is not isomorphic to L^2 and ones see (following the proof of the Lemma) that the Lemma does not hold in this case.

Now consider the diagram

$$X \hookrightarrow X \times Y \longrightarrow X \times Y,$$

where $i(x) = (x, 0)$. Then, of course,

$$i^*\xi^*N \cong M^n,$$

but note that this requires a choice of a trivialization of $i^*p_2{}^*L^n$, which is determined up to a scalar and whose effect on global sections is *evaluation at zero.*

(i) By Kunneth formula $\Gamma(X \times Y,\ p_1{}^*M \otimes p_2{}^*L) = \Gamma(X, M) \otimes \Gamma(Y, L)$.

(ii) The map

$$\xi^* : \Gamma(X \times Y,\ p_1{}^*M \otimes p_2{}^*L) \longrightarrow \Gamma(X \times Y,\ p_1{}^*M^n \otimes p_2{}^*L^n) = \Gamma(X, M^n) \otimes \Gamma(Y, L^n),$$

is well understood by the Quasi Symmetry Theorem (More on that below).

(iii) The map i^* is just evaluating t in $s \otimes t$ at zero.

(iv) The composition $i^*\xi^* = (0, f(x))$ is ' the map we seek '.

Note that the map ξ^* which evantually gives us the map f^* uses the descent data of both f and $f^{\vee}$. In general the whole structure needed for the study of ξ is obtained as the product of the structures for L and M. Or, to use another sloppy formulation theory of theta functions is multiplicative. That means, e.g., that

$$\hat{G}(p_1^*M \otimes p_2^*L) \equiv \hat{G}(M) \times \hat{G}(L) / \{(\alpha, \alpha^{-1}) \mid \alpha \in k^{\times}\}$$

and therefore that Göpel structures for L and M induce a Göpel structure for $p_1{}^*M \otimes p_2{}^*L$. Moreover the two systems of level subgroups $K_L(n)_i$, $L_L(n)_i$, $K_M(n)_i$, $L_M(n)_i$ can be multiplied to get such system for $p_1{}^*M \otimes p_2{}^*L$ etc.

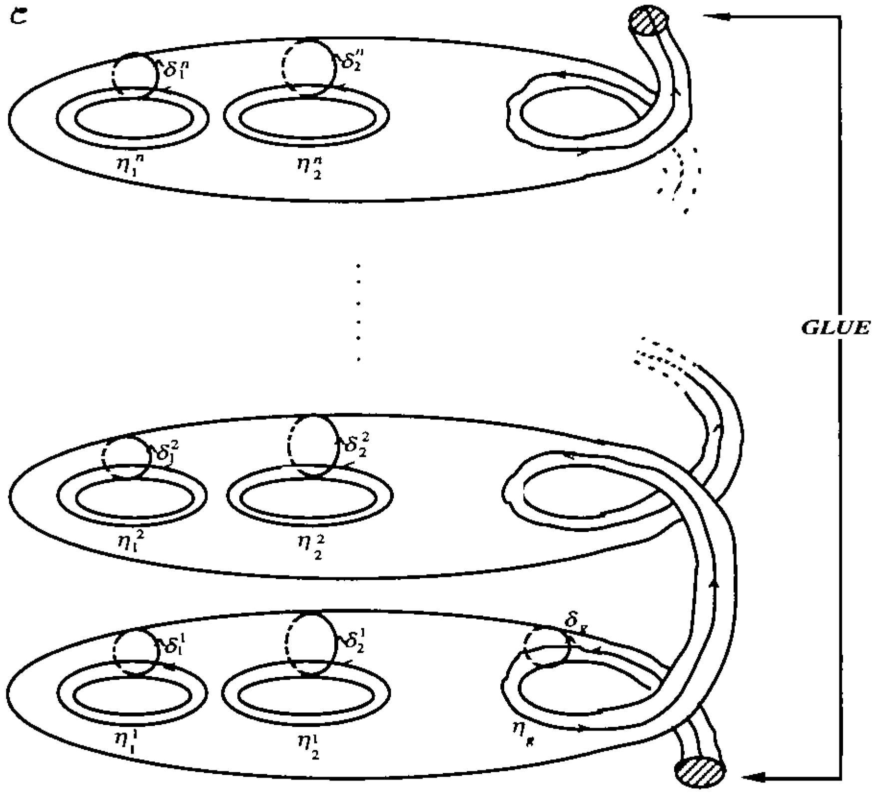
I do not go further into this presently to keep this exposition at a reasonable length, but I intend to deal with some interesting examples in the future.

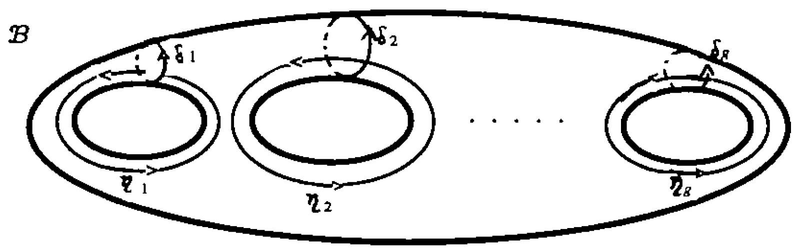
4. EXAMPLES

In this section we give three examples. All of them concern Riemann surfaces. The first one is the case of a cyclic unramified covering of a Riemann surface. The second is the curves $y^2 = x^{2g+2} - 1$ and the third is $y^2 = x^p - 1$. We work over the complex numbers and use topological arguments and pictures. However, there should be no difficulty writing everything in arbitrary characteristic. We start by determining the representation on the first homology group.

I. A CYCLIC COVERING.

Let $\mathcal{B}$ be a Riemann surface of genus $g \geq 1$. Let $\delta_1, \ldots, \delta_g, \eta_1, \ldots, \eta_g$ be a symplectic basis for $H_1(\mathcal{B}, \mathbb{Z})$. Let $\mathcal{C}$ be the cyclic covering of order n of $\mathcal{B}$ obtained by ' unwinding η_g n times '.





With respect to the symplectic basis for $H_1(\mathcal{C}, \mathbb{Z})$ given by

$$\{ \delta_g,\ \delta_1^{\,1}, \delta_1^{\,2}, \ldots, \delta_1^{\,n},\ \ \delta_2^{\,1}, \delta_2^{\,2}, \ldots, \delta_2^{\,n},\ \ldots ,\ \delta_{g-1}^{\,1},\ \delta_{g-1}^{\,2}, \ldots, \delta_{g-1}^{\,n},$$
$$\eta_g,\ \eta_1^{\,1}, \eta_1^{\,2}, \ldots, \eta_1^{\,n},\ \ \eta_2^{\,1}, \eta_2^{\,2}, \ldots, \eta_2^{\,n}, \ldots,\ \ \eta_{g-1}^{\,1}, \eta_{g-1}^{\,2}, \ldots, \eta_{g-1}^{\,n} \},$$

the generator of the cover automorphism group, π, which is 'raising one level' is given by

$$\begin{pmatrix} 1 & & & & & & \\ & E & & & & & \\ & & \ddots & & & & \\ & & & E & & & \\ & & & & 1 & & \\ & & & & & E & & \\ & & & & & & \ddots & \\ & & & & & & & E \end{pmatrix}, \quad E = \begin{pmatrix} 0\,0 & \cdots & 1 \\ 1\,0 & \cdots & 0 \\ 0\,1 & & 0 \\ \vdots & \ddots & \vdots \\ & \cdots & 0\,0 \end{pmatrix}.$$

II. $y^2 = x^{2g+2} - 1$.

In every hyperelliptic curve $\mathcal{C}$: $y^2 = f(x)$, the automorphism ι

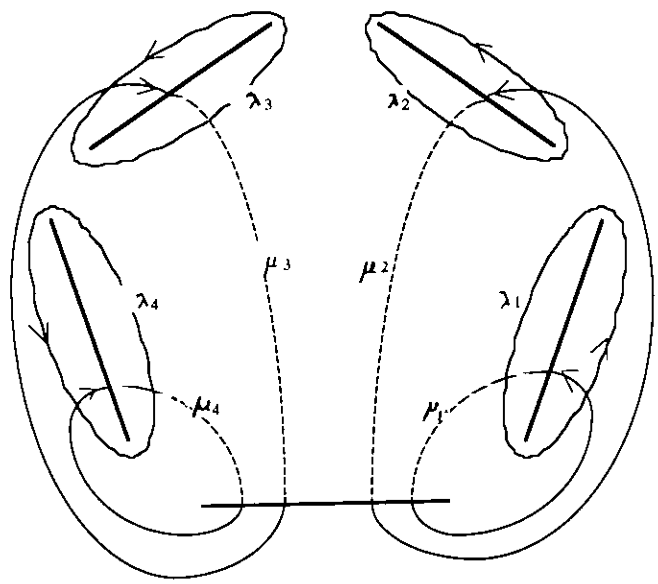
$$y \longmapsto -y, \quad x \longmapsto x$$

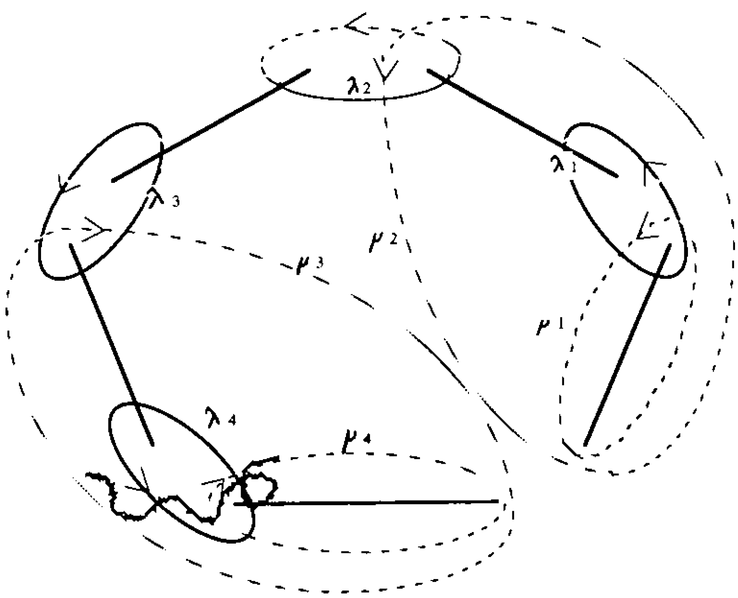
induces multiplication by -1 on $H_1(\mathcal{C}, \mathbb{Z})$. Therefore we examine only the automorphism π determined by

$$y \longmapsto y, \quad x \longmapsto \zeta x,$$

$\zeta = exp(2\pi i / (2g+2))$.

We consider this curve, which is of genus g, as a two - sheet covering of $\mathbb{P}^1(\mathbb{C})$, obtained by branch cuts. The following diagram demonstrates this as well as giving a basis for $H_1(\mathcal{C}, \mathbb{Z})$ - dotted lines denote curves on the lower sheet while whole lines denote curves on the upper sheet. For simplicity we demonstrate only the case $g = 4$, the other cases being similar.





The second picture describes the images of the basis elements after applying the automorphism π. That is, for example, the label λ_1 in the lower picture denotes the <u>image</u> of λ_1 under π. The matrix representing π with respect to the basis $\lambda_1, \lambda_2, \mu_1, \mu_2$ is

$$\begin{pmatrix} 0 & \begin{matrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{matrix} \\ \begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{matrix} & 0 \end{pmatrix}.$$

In general, choosing the bases similarily, we get that the matrix representing π is given by

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where

$$B = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & -1 & 1 & 0 \end{pmatrix}.$$

III. $y^2 = x^p - 1$.

This curve is of genus $(p-1)/2$ and for p prime its Jacobian is a simple abelian variety with complex multiplication by $\mathbb{Z}[\zeta]$. Using similar description of the curve and picking a similar basis for $H_1(\mathcal{C}, \mathbb{Z})$ one gets that the automorphism π determined by

$$y \longmapsto y, \; x \longmapsto \zeta x,$$

where $\zeta = exp(2\pi i / p)$ is given for $p = 3$ by

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

and for $p > 3$ by

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

In each one of these examples we take as the even symmetric line bundle on the appropriate Jacobian the line bundle determined by the period matrix of these homology bases with respect to the standard map by some ' dual basis ' of $\Gamma(\mathcal{C}, \Omega_{\mathcal{C}}^1)$. By this we mean that the image is of the form (τ, I), $\tau \in \mathfrak{H}$ - the appropriate Siegel upper space. This gives also a decomposition of the period lattice $(\tau, I)\mathbb{Z}^{2g}$ to $\tau\,\mathbb{Z}^g \oplus \mathbb{Z}^g$ and thus a very specific line bundle, the one obtained from the trivial line bundle on $\mathbb{C}^g$ by dividing by the factor of automorphy

$$a(\lambda, v) = \chi_0(\lambda)\, exp(\pi\, H(v, \lambda) + \tfrac{\pi}{2}\, H(\lambda, \lambda))$$

where

$$H = (\mathrm{Im}\ \tau)^{-1},$$

$$\chi_0(\lambda) = exp(\pi i\ Im H(\lambda_1, \lambda_2)),$$

using the decomposition $\lambda = \lambda_1 + \lambda_2$ obtained from $\mathbb{C}^g = \mathbb{R} \otimes \tau\mathbb{Z}^g \oplus \mathbb{R} \otimes \mathbb{Z}^g$ (see [LB]).

In order to derive the explicit form of the Symmetry theorem in these examples, let us compute the expressions appearing in it. We will compute the formulas for π^{-1}. Recall that

$$m_\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = E(M_\pi\begin{pmatrix} x \\ y \end{pmatrix}) / E(\begin{pmatrix} x \\ y \end{pmatrix}),$$

where

$$E(\begin{pmatrix} x \\ y \end{pmatrix}) = \underline{\mathbf{e}}(\tfrac{1}{2}\ {}^t x \cdot y).$$

I. The matrix M_π is of the form

$$\begin{pmatrix} K & 0 \\ 0 & {}^tK^{-1} \end{pmatrix}$$

and therefore $m_\pi \equiv 1$. This implies that $\gamma_\pi = 1$, because $\gamma_\pi = m_\pi|_{\mathcal{Z}}$. Thus $v_{\pi^{-1}}(\psi) = M_\pi \alpha(\psi)$, which has no integral part. Therefore

$$b_{\pi^{-1}, \psi} = 1$$

for all ψ.

This implies that up to a scalar $\phi_* g^* : \Gamma(X, L) \longrightarrow \Gamma(X, L)$ is given by

$$T\left(\sum_{\psi \in \mathcal{Z}^*} a_\psi\, \Theta_\psi\right) = \sum_{\psi \in \mathcal{Z}^*} a_\psi\, \Theta_{g\psi}$$

a simple permutation matrix. Note that, by the corollary to the Symmetry Theorem, in many cases we can get that this scalar is 1.

II. The matrix M_π is of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

and $B = -{}^tC^{-1}$. In this case that

$$E(M_\pi\begin{pmatrix} x \\ y \end{pmatrix}) = E(\begin{pmatrix} x \\ y \end{pmatrix})^{-1},$$

and

$$m_\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = E(\begin{pmatrix} x \\ y \end{pmatrix})^{-2}.$$

However, note that for $\begin{pmatrix} x \\ y \end{pmatrix}$ in Z we get

$$m_\pi(\begin{pmatrix} x \\ y \end{pmatrix}) = \underline{\mathbf{e}}(-{}^t y \cdot x) = 1.$$

That implies that $\gamma_\pi = 1$.

The constants $b_{\pi,\psi}$ are given by

$$b_{\pi,\psi} = E(\alpha(\psi))^{-2} F\left(\left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \alpha(\psi) \right\}, \left[\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \alpha(\psi) \right] \right)^{-1}$$

which are completely explicit roots of unity.

III. The matrix M_π is of the form
$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$
where A, B, C are as defined above. In this case

$$\begin{aligned} E\left(M_\pi\begin{pmatrix} x \\ y \end{pmatrix}\right) &= E\left(\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right) \\ &= E\left(\begin{pmatrix} Ax+By \\ Cx \end{pmatrix}\right) \\ &= \underline{\mathbf{e}}\left(\tfrac{1}{2} \left({}^t x \cdot {}^t C(Ax+By) \right) \right) \\ &= \underline{\mathbf{e}}\left(-\tfrac{1}{2} \left({}^t x \cdot y + x_g^2 \right) \right). \end{aligned}$$

Thus

$$m_\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underline{\mathbf{e}}\left(-\tfrac{1}{2} \left({}^t x \cdot y + x_g^2 \right) \right)$$

which is not trivial on Z. In fact

$$\gamma_\pi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \underline{\mathbf{e}}\left(\tfrac{1}{2} x_g^2 \right).$$

One can write now the symmetry formula explicitly.

5. CONCLUDING REMARKS.

This section is devoted to three topics. The first is the construction of compatible theta structures allowing one the simultaneous construction of bases for $\Gamma(X, L^n)$ for all n. This construction furnishes the necessary background for extending our results to isogenies f such that $f^*L \cong L^n$ where n needs not be a square. The second topic is the extension of our results to $\mathbb{Q}$-isogenies. That is to elements of $\mathbb{Q}\otimes\mathrm{End}(X)$. The third topic is a short dictionary between the algebraic and analytic languages in case the ground field is the complex numbers.

I. SIMULTANEOUS BASES.

One of the main points of our approach to the problem of writing quasi symmetries by explicit formulae is the possibility of a simultaneous construction of bases for all the spaces $\Gamma(X, n^*L)$. Conceptually, their nice behavior is a result of a certain compatability of the theta structures defining them. Our purpose now is to define and explain this compatibility and extend it in a way that allows us to extend our simultaneous construction to all the spaces $\Gamma(X, L^n)$.

DEFINITION. Let L be an ample symmetric line bundle of separable type on an abelian abelian variety X. A *symmetric theta structure* for L is an isomorphism

$$\Delta : G(L) \longrightarrow G(\delta)$$

such that

$$\Delta \circ \delta_{-1} = D_{-1} \circ \Delta$$

where D_{-1} is the automorphism of $G(\delta)$ defined by

$$D_{-1}(\alpha, x, l) = (\alpha, -x, -l)$$

(See [Mum1] pp.316-7). A *symmetric theta structure for* (L, L^n) (or a *compatible pair of theta structures*) is a pair of symmetric theta structures

$$\Delta_n : G(L^n) \longrightarrow G(n\delta)$$
$$\Delta_l : G(L) \longrightarrow G(\delta)$$

such that

$$\Delta_n \circ \varepsilon_n = E_n \circ \Delta_1,$$
$$\Delta_1 \circ \eta_n = H_n \circ \Delta_n.$$

For a definition of ε_n, η_n, E_n, H_n see loc. cit. pp. 309-310, 316.

In general the existence of such a compatible pair is not a trivial matter (I do not know when it exists). The case n = 2 is discussed thoroughly by Mumford, loc. cit. §2. However, in the case we are considering it is easy to prove the existence.

Let L be an ample even symmetric line bundle on X. Define three homomorphisms for the standard adelic Heisenberg group G:

$$\mathcal{D}_{-1}, \mathcal{E}_n, \mathcal{H}_n$$

$\mathcal{D}_{-1}(\alpha, x, l) = (\alpha, -x, -l);$
$\mathcal{E}_n(\alpha, x, l) = (\alpha^n, nx, l);$
$\mathcal{H}_n(\alpha, x, l) = (\alpha^n, x, nl).$

Define a theta structure

$$\Delta_n : \hat{G}(L^n) \longrightarrow G$$

by

$$\Delta_n(x) = \mathcal{E}_n(\Delta_1(z))$$

where Δ_1 is a symmetric theta structure whose existence is guaranteed by our assumptions on L and z is an element such that $\varepsilon_n(z) = x$. Since $\text{Ker } \mathcal{E}_n = \mu_n = \text{Ker } (\varepsilon_n : \hat{G}(L) \longrightarrow \hat{G}(L^n))$ this is well defined. It is easy to check that we have indeed defined a symmetric theta structure.

Consider the pair (Δ_n, Δ_{nm}) of symmetric theta structures for $(\hat{G}(L^n), \hat{G}(L^{nm}))$. It is obviously compatible for ε_m, $\mathcal{E}_n$. Since it is enough to verify the η_m, $\mathcal{H}_m$ compatibility for elements of the form $\tau^M(x)$ $M = L^n, L^{nm}$, and since $\Delta_n(\tau^M(x)) = \Delta_n(\varepsilon_n \tau^L(x)) = \mathcal{E}_n(\Delta_1(\tau^L(x)))$ (for $M = L^n$, and similarily if $M = L^{nm}$) employing the identity $\eta_n(\tau^{L^{nm}}(x)) = \tau^{L^m}(nx)$ (which is part of the content of Lemma 14) one can easily establish the desired compatiblity.

Further, for any n we have an induced isomorphism

$$G(L^n) \cong N_{\hat{G}(Ln)}(\sigma^L(T(X))) / (\sigma^L(T(X)))$$
$$\cong N_G(\mathcal{E}_n(\mathcal{S})) / (\mathcal{E}_n(\mathcal{S}))$$
$$\cong G(n\delta)$$

($\mathcal{J}$ was defined in § 3 p. 30), where $\delta = (1, 1, \dots, 1)$. The last isomorphism is the following:

$$N_G(\mathcal{E}_n(\mathcal{J})) = \{(\alpha, x, l) | x \in (\hat{\mathbb{Z}}_{ff})^g, \, l \in (\frac{1}{n}\hat{\mathbb{Z}}_{ff})^g\}$$

and the isomorphism is established by sending the maximal level subgroups

$$\{(1, x, 0) | x \in (\hat{\mathbb{Z}}_{ff})^g\}, \, \{(1, 0, l) | l \in (\frac{1}{n}\hat{\mathbb{Z}}_{ff})^g\}$$

to their obvious images in $G(n\delta)$ (the group law on $G(n\delta)$ is defined using the $n-th$ root of unity $\underline{e}(1/n)$ - compare p. 30). Further, the maps between the various $G(n\delta)$ induced from the maps $D_{-1}, \mathcal{E}_n, \mathcal{H}_n$, are precisely the maps D_{-1}, E_n, H_n and therefore we have succeeded in constructing compatible pairs of theta structures for (L^n, L^{nm}) for all n, m. At last, one can also verify that under the isomorphism $G(L^{n^2}) \cong G(n^* L)$ the theta structure just constructed for $G(L^{n^2})$ agrees with the one constructed previously for $G(n^* L)$.

II. VIRTUAL SYMMETRIES.

Suppose that $f \in \mathbb{Q} \otimes End(X)$. Then usually for $x \in X$ $f(x)$ does not make sense. Thus it is not clear what, if all, $f^* s, s \in \Lambda(X, L)$, should mean. However, when X is a complex abelian variety, say $X = \mathbb{C}^g / \Lambda$ then we may identify $\Lambda(X, L)$ with certain holomoprhic functions on $\mathbb{C}^g$ and then we may define $f^* s$ by $f^* s(x) = s(f(x))$ (using the same notation for the complex representation of f).
On the other hand f is not far from being an isogeny. In fact there exists a natural number n such that $nf \in End(X)$. For such an n, nf is an isogeny, for every $x \in X$ $nf(x)$ is meaningful and $(nf)^* s$ is defined with the usual meaning. Now, it is not difficult to convince oneself that the natural embedding

$$\hat{\Gamma}(X, n^* L) \hookrightarrow \hat{\Gamma}(X, L)$$

is the right way to define the action of $\frac{1}{n}$. This will be further justified when we discuss theta *functions* below.

DEFINITION. Let $f \in (\mathbb{Q} \otimes End(X))^x$. Let n be such that $nf \in End(X)$ and assume that $(nf)^* L \cong n^* L$ (such an f will be henceforth called a *virtual symmetry* of L). Define the map

$$f^* : \hat{\Gamma}(X, L) \to \hat{\Gamma}(X, L)$$

as the composition of the maps $(nf)^* : \hat{\Gamma}(X, L) \to \hat{\Gamma}(X, n^* L)$ and the natural embedding $I : \hat{\Gamma}(X, n^* L) \to \hat{\Gamma}(X, L)$.

Recalling the definition of $\hat{\Gamma}(X, L)$ it is easy to see that this is well defined and generalizes the usual natural definition for symmetries $f \in End(X)^x$ of L. Note that any automorphism f preserving the polarization determined by L is a virtual symmetry for L (in virtue of the symmetry of L). Therefore we also obtain somewhat more flexibility in the treatment of automorphisms as well. We remark that it is easy to generalize this definition for certain other $f \in \mathbb{Q} \otimes End(X)$ (obtaining thereby also the same kind of flexibility) or to the relative situation discussed in the end of § 3. We leave that to the reader.

The map $(nf)^*$ is an intertwining operator for the action of $\hat{G}(L)$ via ε_{n^2} and δ_{nf} as we have already seen. Recall (Lemma 14) that $\delta_{nf}\tau^L(x) = \tau^L((nf)(x))$. Therefore we make the following

DEFINITION. For a virtual symmetry f define

$$\delta_f : \hat{G}(L) \to \hat{G}(L)$$

by

$$\delta_f(\lambda \cdot \tau(x)) = \lambda \cdot \tau(f(x)).$$

Let us verify that this is an automorphism. Since $\tau(x)\tau(y) = e^L(x, y/2)\tau(x+y)$, we need only to check that $e^L(x, y/2) = e^L(f(x), f(y)/2)$. Changing variables to nx, ny and using that $(nf)^*L \cong n^*L$ we get

$$\begin{aligned} e^L(f(nx), f(ny)/2) &= e^L((nf)(x), (nf)(y)/2) \\ &= e^{(nf)^*L}(x, y/2) \\ &= e^{n^*L}(x, y/2) \\ &= e^L(nx, ny/2). \end{aligned}$$

For any virtual symmetry f we may twist the action of $\hat{G}(L)$ by $\delta_f : U^f_z = U_{\delta_f(z)}$.

LEMMA. f^* is an intertwining operator:

$$f^* \circ U^f = U \circ f^*.$$

PROOF. For scalars $\lambda \in k^x$ the assertion is clear. Therefore it is enough to check it for elements of the form $\tau^L(nx)$, $x \in V(X)$.

$$\begin{aligned} f^* \circ U^f_{\tau^L(nx)} &= f^* \circ U_{\tau^L(nfx)} \\ &= I \circ (nf)^* \circ U^{nf}_{\tau^L(x)} \\ &= I \circ U_{\varepsilon_n 2(\tau^L(x))} \circ (nf)^* \\ &= I \circ U_{\tau^{n^*L}(x)} \circ (nf)^*. \end{aligned}$$

Therefore we need only to check that

$$I \circ U_{\tau^{n^*L}(x)} = U_{\tau^L(nx)} \circ I.$$

This is essentially Lemma 14,4). Q.E.D.

In characteristic $p \neq 0$ we do not have a universal covering space for X. However, Mumford had shown that to a large extent $V(X)$ is the right substitute. For any $s \in \hat{\Gamma}(X, L)$ he defined a *function* Θ_s on $V(X)$. Therefore we should check that

$$\Theta_{f^*s}(x) = \Theta_s(f(x)).$$

Having verified this we may safely claim that our definitions are right. Let us recall the definition of Θ_s (a full discussion appears in [Mum3]):

Fix an isomorphism

$$\varepsilon : L(0) \to k,$$

thereby fixing for any isogeny h (and in particular for h equal multiplication by n) an isomorphism

$$\varepsilon : h^*L(0) \underset{can.}{\cong} L(0) \to k.$$

Given $x \in V(X)$, let $\tau^L(x) = (x_n, \phi_n)_n$ and $s \in \hat{\Gamma}(X, L)$ define

$$\Theta_s(x) = \varepsilon(\phi_n^{-1} s(x_n)).$$

More precisely, choose n divisible enough such that both s is represented by some $s_n \in \Gamma(X, n^*L)$ and $\phi_n \in G(n^*L)$. Then $\Theta_s(x) = \varepsilon(\phi_n^{-1} s_n(x_n))$. It is easily verified that this is well defined.

An important interpretation of this definition is as follows: for every n "evaluation at zero" defines a linear functional

$$\ell_0 : \Gamma(X, n^*L) \to n^*L(0) \cong L(0) \to k.$$

They induce a linear functional

$$\ell_0 : \hat{\Gamma}(X, L) \to k.$$

One can prove (loc. cit.) that

$$\Theta_s(x) = \ell_0(U_{\tau(-x)} s).$$

Given an isogeny $h : X \to X$ we have an induced

$$\ell_0 : \hat{\Gamma}(X, h^*L) \to k,$$

and it is clear that

$$\ell_0(h^*s) = \ell_0(s).$$

Therefore we see that $\Theta_{h^*s}(x) = \ell_0(U_{\tau^{h^*L}(-x)} h^*s) = \ell_0(h^* U_{\tau^L(-h(x))} s) = \ell_0(U_{\tau^L(-h(x))} s) = \Theta_s(h(x))$. In particular

$$\Theta_{(nf)^*s}(x) = \Theta_s((nf)(x)).$$

I claim that the embedding $I : \hat{\Gamma}(X, n^*L) \to \hat{\Gamma}(X, L)$ has the effect

$$\Theta_{I(s)}(nx) = \Theta_s(x).$$

(This justifies further our previous remark that $I : \hat{\Gamma}(X, n^*L) \to \hat{\Gamma}(X, L)$ shouls be considered as the effect of $\frac{1}{n}$.) From the definitions it follows that $\Theta_{I(s)}(0) = \Theta_s(0)$. Let us use the identity

$$\Theta_{U_{\lambda \cdot \tau(y)}(s)}(x) = \lambda \cdot e(y, x/2) \cdot \Theta_s(x - y)$$

(loc. cit. Lemma 5.7) to conclude that

$$\begin{aligned}\Theta_s(x) &= \Theta_{U_{\tau^{n^*L}(-x)}(s)}(0)\\ &= \Theta_{I(U_{\tau^{n^*L}(-x)}(s))}(0)\\ &= \Theta_{U_{\tau^L(-nx)}(I(s))}(0)\\ &= \Theta_{I(s)}(nx).\end{aligned}$$

Finally, we get

$$\Theta_{f^* s}(x) = \Theta_{I((nf)^* s)}(x) = \Theta_{(nf)^* s}\left(\frac{1}{n}x\right) = \Theta_s(f(x)).$$

We summarize all this by

Theorem. *Let f be a virtual symmetry. The map $f^* : \hat{\Gamma}(X, L) \longrightarrow \hat{\Gamma}(X, L)$ defined as the composition of the maps $(nf)^* : \hat{\Gamma}(X, L) \longrightarrow \hat{\Gamma}(X, n^*L)$ and the natural embedding $I : \hat{\Gamma}(X, n^*L) \longrightarrow \hat{\Gamma}(X, L)$ is an intertwining map for the usual action and the δ_f-twisted action of $\hat{G}(L)$ where δ_f is the automorphism of $\hat{G}(L)$ defined by $\delta_f(\lambda \cdot \tau^L(x)) = \lambda \cdot \tau^L(f(x))$.*

We have an identity of theta functions

$$(*) \qquad \Theta_{f^*(s)}(x) = \Theta_s(f(x)).$$

Remarks. 1) Over the complex numbers, for the line bundle $L(H, \chi)$, f is a virtual symmetry if and only if $f^*H = H$. This is because we are dealing with symmetric line bundles only. In the case where X is an abelian variety with C. M. by a C. M. field K and f is a unit of K, the condition $f^*H = H$ is equivalent to the condition $N_{K/K^+}(f) = 1$. This new abundance of virtual symmetries is one of the motivations for the introduction of this concept.

2) The relation $(*)$ would not be true for a general twist coming from the general symplectic group. For example, given $M \in Sp(2g, \mathbf{A}_{ff})$ we can define an automorphism

$$\delta_M : G \longrightarrow G$$

by

$$\delta_M(\alpha, x) = (\alpha, Mx).$$

This would yield as usual an equivariant map taking Θ to some Θ'. In general we would not have $\Theta'(x) = \Theta(Mx)$.

III. OVER $\mathbb{C}$

This section follows in terminology and notation the book [LB]. Since our general description of the theory over $\mathbb{C}$ is well known and most of our new contributions are easily proved we will not offer any proofs to the assertions below.
Let Λ be a lattice (i.e. discrete maximal rank subgroup) in $\mathbb{C}^g$. Let (H,χ) be an Apple-Humbert data. Recall that this means that H is a non-degenerate Hermitian form on $\mathbb{C}^g$, and that χ is a semi-character on Λ. That is

$$\chi : \Lambda \to \mathbb{C}_1,\ \chi(\lambda_1+\lambda_2) = \chi(\lambda_1)\chi(\lambda_2)exp(\pi i E(\lambda_1,\lambda_2))$$

where $E = Im\, H$.
Given such a data define a factor of automorphy $a = a_{(H,\chi)}$ on $\Lambda \times \mathbb{C}^g$ by

$$(1) \qquad a(\lambda, V) = \chi(\lambda)exp(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)).$$

The lattice Λ acts on the trivial bundle $\mathbb{C}^g \times \mathbb{C}$ over $\mathbb{C}^g$ by

$$\lambda^*(v,t) = (v+\lambda,\ a(\lambda,v)\cdot t),$$

defining an ample line bundle $L = L(H,\chi)$ of degree $\sqrt{det\,E}$ on $X = \mathbb{C}^g/\Lambda$. The global sections of this bundle, $\Gamma(X,L)$, are identified with holomorphic functions Θ on $\mathbb{C}^g$ satisfying

$$\Theta(V+\lambda) = a(\lambda,v)\Theta(v).$$

The Apple-Humbert data satisfies

- $L(H_1,\chi_1)\otimes L(H_2,\chi_2) \cong L(H_1+H_2,\chi_1\chi_2)$
- $f^*L(H,\chi) = L(f^*H, f^*\chi)$
- $L(H,\chi)$ is symmetric iff $Im\,\chi \subseteq \{\pm 1\}$.

One can extend the definition verbatim to degenerate matrices H and in particular get $Pic^0(X) = \{L(0,\chi) | \chi \in \Lambda^*\}$.
Define a group structure on $\mathbb{C}^x \times \mathbb{C}^g$ by

$$[\alpha,w][\beta,x] = [\alpha\beta\, exp(\pi H(x,w)),\ x+w].$$

We call this group $\hat{G}(L)_{\mathbb{C}}$ and denote by $\hat{G}(L)_{\mathbb{Q}}$ its subgroup consisting of all elements $[\alpha,w]$ where $w \in \mathbb{Q}\otimes\Lambda$.

Define also

$$G(L)^+ = \{[\alpha,w] | \alpha \in \mathbb{C}^x, w \in \Lambda^\perp\},$$
$$T(L) = \{[a(w,0),w] | w \in \Lambda\}$$

($\Lambda^\perp$ is with respect to E).

Note that $\hat{G}(L)_{\mathbb{C}}$ acts on $\mathbb{C}^g \times \mathbb{C}$ by

$$[\alpha, w][v, t] = [v + w, \alpha exp(\pi H(v, w))t].$$

Now, one easily proves that $G(L)^+/T(L)$ is cononically isomorphic to $G(L)$ and that the formula

$$\lambda * [v, t] = [a(\lambda, 0), \lambda][v, t]$$

holds. We also remark that the commutator pairing e^L of $G(L)$ is given by

$$e^L([\alpha, w], [\beta, x]) = exp(-2\pi i E(w, x)).$$

If $L = L(H, \chi)$ is symmetric one verifies that we have $e_*^L(x) = \chi(2x)$ for $x \in \frac{1}{2}\Lambda$.

The action of $G(L)$ at the global sections of L is

$$([\alpha, w]\Theta)(x) = \alpha exp(\pi H(x - w, w))\Theta(x - w).$$

Given a totally isotropic decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ with respect to E, we define a semi-character χ^0 on Λ by

$$\chi^0(V) = exp(\pi i E(v_1, v_2)),$$

where the decomposition $v = v_1 + v_2$ is deduced from $\mathbb{C}^g \cong (\mathbb{R} \otimes \Lambda_1) \oplus (\mathbb{R} \otimes \Lambda_2)$. Then $L = L(H, \chi^0)$ is a symmetric line bundle. Given any other semi-character χ for H, we can find some $c \in \mathbb{C}^g$ such that

$$\chi(v) = \chi^0(V) exp(2\pi i E(c, v)). \tag{2}$$

We note that c is not unique, but once it is chosen we can extend the definition of the factor of automorphy $a(H, \chi)$ given by (1) to elements of $\Lambda^\perp$ by using (2) as the definition of the extended semicharacter. We define the theta function $\Theta = \Theta^c_{L(H,\chi)}$ on $\mathbb{C}^g$ by

$$\begin{aligned}\Theta(v) =& exp\left(-\pi H(v, c) - \frac{\pi}{2}H(c, c) + \frac{\pi}{2}B(v + c, v + c)\right) \\ &\times \sum_{\lambda \in \Lambda_1} exp\left(\pi(H - B)(v + c, \lambda) - \frac{\pi}{2}(H - B)(\lambda, \lambda)\right)\end{aligned}$$

where B is the $\mathbb{C}$ linear extendion of $H|_{R\otimes\Lambda_2}$. The decomposition of Λ induces a decomposition $\Lambda^\perp = \Lambda_1^\perp \oplus \Lambda_2^\perp$.

Define

$$K_i = \{[a(w.0), w] | w \in \Lambda_i^\perp\} \quad , i = 1, 2.$$

(Note the " hidden " defendence on the choice of c).

One can verify the following statements:

- $\Theta \in \Gamma(\mathbb{C}^g/\Lambda, L(H, \chi))$.
- K_i are mutually orthogonal maximal level subgroups of $G(L)$.
- Θ is K_2 invariant.

- Define for $U \in \Lambda_1^{\perp}/\Lambda_1$,

$$\Theta_u = [a(-u,0), -u]\Theta,$$

then $\{\Theta_u | u \in \Lambda_1^{\perp}/\Lambda_1\}$ is a basis for $\Gamma(\mathbb{C}^g/\Lambda, L(H,\chi))$ and for $[\alpha, w] \in G(L)$

$$[\alpha, w]\Theta_u = \alpha \cdot e^L(u - w_1, w_2) \cdot a(w,0)^{-1}\Theta_{u-w_1}.$$

A special important case is as follows:
Given $\tau \in \mathfrak{h}_g$, we define:

- a lattice $\Lambda_\tau = (\tau I)\mathbb{Z}^{2g}$,
- an abelian variety $X_\tau = \mathbb{C}^g/\Lambda_\tau$,
- a principal polarization $H_\tau = (Im\tau)^{-1}$,
- a decomposition $\Lambda_{\tau 1} = \tau\mathbb{Z}^g, \Lambda_{\tau 2} = I\mathbb{Z}^g$,
- a symmetric form B_τ- the $\mathbb{C}$-linear extension of $H_\tau|_{\mathbb{R}\otimes\Lambda_{\tau 2}}, B_\tau(v,w) = {}^t v(Im\tau)^{-1}w$,
- a semicharacter $\chi_\tau^0(v) = exp(\pi i E_\tau(v_1, v_2)), E_\tau = ImH_\tau$,
- a basis for Λ_τ consisting of the columns of (τI)
- a line bundle $L_\tau = L(H_\tau, \chi_\tau)$ on X_τ.

We define the functions $\Theta_{n,w,\tau}$ for $w \in \tau\left(\frac{1}{n^2}\mathbb{Z}\right)^g$ by

$$\Theta_{n,w,\tau} = [a_{(n^*H_\tau, n^*\chi_\tau^0)}(-w,0), -w]_{G(n^*L_\tau)}\Theta_{(n^*H, n^*\chi_\tau^0)}$$

.

Then $\{\Theta_{n,w,\tau} | w \in \tau(\frac{1}{n^2}\mathbb{Z}/\mathbb{Z})^g\}$ is a basis for $\Gamma(X_\tau, n^*L_\tau)$.
We have

$$\Theta_{n,w,\tau}(z) = exp\left(\frac{\pi n^2}{2}B_\tau(z,z)\right)\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(n^2 z, n^2\tau)$$

where $w = \tau w^1, w^1 \in \left(\frac{1}{n^2}\mathbb{Z}\right)^g$ and for every $\epsilon, \epsilon^1 \in \mathbb{R}^g$ we have Riemann's theta function

$$\Theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z,\tau) = \sum_{N\in\mathbb{Z}^g} exp\left\{2\pi i\left(\frac{1}{2}\,{}^t(N+\epsilon)\tau(N+\epsilon) + {}^t(N+\epsilon)(z+\epsilon')\right)\right\}$$

Coming back to the more general situation, let L be a symmetric line bundle $L = L(H,\chi)$ on $X = \mathbb{C}^g/\Lambda$. Then given an authomorphism g of X, the Appell-Humbert theorem implies that g is a symmetry of $L(H,\chi)$ if and only if $g^*H = H, g^*\chi = \chi$. Assuming this is the case we can show that

$$\delta_g[\alpha, w] = [\alpha, gw].$$

Also $\epsilon_n[\alpha, w]_{G(L)} = [\alpha^n, w]_{G(L^n)}, \eta_n[\alpha, w]_{G(L^n)} = [\alpha^n, nw]_{G(L)}$, and if $w \in \frac{1}{m^2}\Lambda$ then $n^*[\alpha, w]_{G(m^*L)} = [\alpha, w/n]_{G(n^*m^*L)}$. Picking a symplectic basis $(\omega_1\omega_2)$ of Λ and letting $w_1, \ldots, w_{2g}$ be the columns of $(\omega_1\omega_2)$, $W_i = \left(\frac{W_i}{n}\right)_{n\in\mathbb{N}^+}$, then $W_1, \ldots, W_{2g}$ is a basis of $T(X)$ over $\hat{\mathbb{Z}}$ and a basis for $V(X)$ over $\mathbb{A}_f$. For every $x \in \mathbb{Q}\otimes\Lambda, x^\bullet = \left(\frac{x}{n}\right)_{n\in\mathbb{N}^+}$ is in $V(X)$ and

$$\tau^L(x^\bullet) = \left(\left[exp\left(\frac{\pi}{2}H(x,x)\right), \frac{x}{n}\right]_{G(n^*L)}\right)_{n\in\mathbb{N}^+}$$

. We have a natural embedding

$$i : \hat{G}(L)_{\mathbf{Q}} \to \hat{G}(L)$$

given by

$$[\alpha, x] \to ([\alpha, \frac{x}{n}]_{G(n^*L)})_{n \in \mathbf{N}^+}.$$

The $\epsilon_n, \eta_n, \delta_n$ formulas are immediately deduced from their finite counterparts. For a concrete example of the system of maximal level subgroups consider again the line bundle L_τ on X_τ.

$$G(n^*L_\tau) = \{[\alpha, w] | \alpha \in \mathbf{C}^x, w \in (\tau I)\left(\frac{1}{n^2}\mathbf{Z}\right)^{2g}\}$$

with the group law

$$[\alpha_1, w_1][\alpha_2, w_2] = [\alpha_1\alpha_2 exp(\pi n^2 H(w_2, w_1)), w_1 + w_2].$$

The subgroups of $G(n^*L_\tau)$

$$L(n)_1 = \{[a(w,0)w] | w \in \tau\frac{1}{n^2}\mathbf{Z}^g \oplus I\mathbf{Z}^g\}$$

$$L(n)_2 = \{[a(w,0)w] | w \in \tau\mathbf{Z}^g \oplus I\frac{1}{n^2}\mathbf{Z}^g\}$$

$$M(n) = \{[a(w,0)w] | w \in \tau\frac{1}{n}\mathbf{Z}^g \oplus I\frac{1}{n}\mathbf{Z}^g\},$$

($a = a_{(n^*H, n^*\chi^0_\tau)}$) are maximal level subgroups. $L(n)_1, L(n)_2$ are mutually orthogonal and $M(n)$ is the level subgroup defining L from n^*L.

The subspace $V_1 = \tau\mathbf{Q}^g, V_2 = I\mathbf{Q}^g$ induce a natural Göpel structure on $\hat{G}(L)_{\mathbf{Q}}$ which gives a Göpel structure on $\hat{G}(L)$ via i and extension of scalars. This Göpel structure is inducing the system $\{L(n)_i\}_n$. We have already described how the bases of $\Gamma(X_\tau, n^*L_\tau)$ with respect to $L(n)_1, L(n)_2$ look like. The basis with respect to $M(n)$ and the section used in the symmetry theorem (for the obvious theta structure) is

$$\left\{exp\left(\frac{\pi n^2}{n}B_\tau(z,z)\right)\Theta\begin{bmatrix}\epsilon & /n \\ \epsilon' & /n\end{bmatrix}(nz,\tau) | \epsilon, \epsilon' \in \mathbf{Z}^g, 0 \le \epsilon, \epsilon' < (n, \dots, n)\right\}$$

Finally, consider the following situation: we have two lattices $\Lambda_1 \supseteq \Lambda$, yielding an exact sequence

$$0 \longrightarrow \Lambda_1/\Lambda \longrightarrow \mathbf{C}^g/\Lambda \xrightarrow{i} \mathbf{C}^g/\Lambda_1 \longrightarrow 0$$

Let $L_1 = L(H, \chi)$ be some line bundle on $\mathbf{C}^g/\Lambda_1$. Let $L = i^*L_1$. Let K be the corresponding level subgroup of $G(L)$. Then

(i) $L = L(H, \chi|_\Lambda)$.

(ii) $K = \{[a_{(H,\chi)}(\lambda, 0), \lambda] | \lambda \in \Lambda_1/\Lambda\}$

(iii) Let $L = L(H, \chi)$ be a line bundle on $\mathbf{C}^g/\Lambda$, and let $K = \{[\alpha(\lambda] | \lambda\}$ be a maximal level subgroup of $G(L)$. Let L_1 be the line bundle on $\mathbf{C}^g/\Lambda_1$ defined by K (in particular $i^*L_1 \cong L$). Then $L_1 = L(H, \chi^*)$ where $\chi^*|_\Lambda = \chi$ and $\chi^*(\lambda) = \alpha(\lambda)/exp\left(\frac{\pi}{2}H(\lambda, \lambda)\right)$.

APPENDIX - REPRESENTATIONS OF FINITE AND ADELIC HEISENBERG GROUPS

A1. THE FIRST CLASSIFICATION

This section gives a 'primary' classification of the irreducible representations of finite Heisenberg groups. We fix the following notation :

k - an algebraically closed field.

G - a finite Heisenberg group sitting in the exact sequence

$$1 \longrightarrow k^\times \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

where the order of H is d^2.

K - a maximal level subgroup of G.

α_n - the homomorphism $k^\times \longrightarrow k^\times$ given by $t \longmapsto t^n$.

B^* - the character group of a group B.

$B[n]$, B^n - the subgroup of elements of order n and the subgroup of n-th powers, respectively, of an abelian group B.

DEFINITION. Let $U: G \longrightarrow GL(V)$ be a representation of G on a k - vector space V. We say that (V, U) is *of order n* if $k^\times$ acts through α_n.

Let (V, U) be a fixed irreducible representation of order n of G. Decompose V according to eigenspaces of K

$$V = \bigoplus_{\chi \in K^*} V_\chi$$

and choose some χ_0 such that $V_{\chi_0} \neq \{0\}$.

LEMMA A1.1. $\bigoplus_{\chi \in K^{*n}} V_{\chi_0 \chi}$ *is a non-zero G-invariant subspace of V , hence equal to V.*

Proof. We have a group isomorphism

$$G / k^\times K \xrightarrow{\simeq} K^*, \quad y \longmapsto \chi^y$$

where

$$\chi^y(z) = [z, y],$$

and $[z, y] = zyz^{-1}y^{-1}$. It is easy to check that

$$U_y(V_\psi) = V_{\psi \cdot (\chi^y)^n}\,. \qquad \text{Q.E.D.}$$

Choose a set theoretic section $\sigma : K^* \longrightarrow G$ to the map $y \longmapsto \chi^y$. We shall always assume that the image of σ is contained in G^c. This is possible since $k^\times G^c = G$.

LEMMA A1.2. 1) V_{χ_0} *is an irreducible representation of* $k^\times K\sigma(K^*[n])$ *henceforth denoted by* ρ.

2) $(V, U) \cong Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho)$.

Proof. First of all, $k^{\times}K\sigma(K^*[n])$ or $\mu_{d^2}K\sigma(K^*[n])$ <u>do</u> act on each $V_{\chi_0\chi}$ and we denote this action by $\rho_{\chi_0\chi}$. Thus $\rho_{\chi_0} = \rho$. Second, we note that the representation theory of G is governed by that of G^c and the representation theory of $k^{\times}K\sigma(K^*[n])$ by that of $\mu_{d^2}K\sigma(K^*[n])$. This means for example that we can identify $Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho)$ with $Ind^{G^c}_{\mu_{d^2}K\sigma(K^*[n])}(\rho)$ and the first is irreducible with respect to G if and only if the second is irreducible with respect to G^c. Conveniently, we may work with characters of finite groups. Assuming that 2) is true we get by Frobenius duality:

$$
\text{(A1.1)}\qquad
\begin{aligned}
1 &= <\Phi_U, \Phi_U> \\
&= <\Phi_U, \Phi_{Ind_A^{G^c}\rho}> \\
&= <\Phi_{U|}, \Phi_\rho> \\
&= \sum_{\chi\in K^{*n}} <\Phi_{\rho_{\chi_0\chi}}, \Phi_\rho> \\
&= <\Phi_\rho, \Phi_\rho> + \sum_{\substack{\chi\in K^{*n}\\ \chi\neq 1}} <\Phi_{\rho_{\chi_0\chi}}, \Phi_\rho> .
\end{aligned}
$$

where $A = \mu_{d^2}K\sigma(K^*[n])$, and Φ denotes the character of the appropriate representation. This proves 1) and the obvious fact that the different $\rho_{\chi_0\chi}$ are all non-isomorphic (although they induce the same representation).

To prove 2) we use the well known interpretation of $V' = Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho)$ as

$$V' = \bigoplus g_i V_{\chi_0}$$

where $\{g_i\}$ are representatives of $G/k^{\times}K\sigma(K^*[n])$. An element $g\in G$ acts on a vector $g_i v$ by $g\cdot g_i v = g_j(\rho_{\chi_0}(r)v)$ if $gg_i = g_j r$. Define a G - linear transformation

$$V' \longrightarrow V$$

by

$$g_i v \longmapsto U_{g_i} v, \qquad (v\in V_{\chi_0}).$$

Since $\{V_{\chi_0\chi}\}_{\chi\in K^{*n}}$ are permuted by G transitively, $\dim(V) = [G : k^{\times}K\sigma(K^*[n])] = \dim(V')$. Hence to show our map is an isomorphism we need only prove it is surjective. That follows if we observe that U_{g_i} is an isomorphism from V_{χ_0} to $V_{\chi_0\chi^{d}r}$ and every component $V_{\chi_0\chi}$ of V is of this form for a suitable g_i. Q.E.D.

LEMMA A1.3. *Let (W, ρ) be an irreducible representation of $k^{\times}K\sigma(K^*[n])$ of order n such that $\rho|_{k^{\times}K} = \alpha_n\chi\cdot Id_W$ for some $\chi\in K^*$, then $(V, U) = Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho)$ is an irreducible representation of G of order n.*

Proof. Set

$$V = \oplus\, g_i W$$

where $\{g_i\}$ are representatives of $G / k^{\times}K\sigma(K^*[n])$. An element $b\in K$ acts on $g_i W$ by

$$\rho(g_i^{-1}bg_i) = \alpha_n\chi(g_i^{-1}bg_i)$$
$$= \alpha_n\chi(\chi^{g_i}(b)\cdot b)$$
$$= \chi^{g_i^n}(b)\cdot \chi(b).$$

That means that if we decompose V to eigenspaces of K then $g_iW = V_{\chi^{g_i^n}\chi}$. Each g_iW is a representation of $k^{\times}K\sigma(K^*[n])$, henceforth denoted ρ_{g_i} , which clearly can not contain a representation isomorphic to ρ as can be seen by observing the way K acts. Using the same computation as in (2.1) we get

$$<\Phi_U, \Phi_U> \; = \; <\Phi_\rho, \Phi_\rho> \; + \sum_{\substack{\{g_i\} \\ g_i \notin k^{\times}K\sigma(K^*[n])}} <\Phi_{\rho_{g_i}}, \Phi_\rho>$$
$$= 1. \qquad\qquad \text{Q.E.D.}$$

We summarize all we have proved by

THEOREM A1.4. *Let* G *be a finite Heisenberg group,* K *a maximal level subgroup of* G *and* $\sigma: K^* \longrightarrow G^c$ *a set theoretic section to the map* $y \longmapsto \chi^y$.

(1) *Let* ρ *be an irreducible representation of* $k^{\times}K\sigma(K^*[n])$, *then* $\rho|_{k^{\times}K}$ *is isotypical, equal to* $\alpha_n\chi$ *with some multiplicity and*

$$(V, U) = Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho)$$

is an irreducible representation of order n *of* G*. Further, every irreducible representation of* G *is obtained in this fashion.*

(2) *If* ρ *and* ρ' *are two irreducible representations of order* n *of* $k^{\times}K\sigma(K^*[n])$ *then*

$$Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho) \cong Ind^{G}_{k^{\times}K\sigma(K^*[n])}(\rho')$$

if and only if ρ *and* ρ' *belong to the same orbit under the* G*-action given by*

$$(g, \rho) \longmapsto g\rho\,; \; g\rho(b) = \rho(g^{-1}bg).$$

(3) *Given an irreducible representation* U *of* G *on a* k*-vector space* V *one obtains the full orbit of the representations* ρ *associated to it by* (2) *by letting* $k^{\times}K\sigma(K^*[n])$ *act through* U *on the various* K*-eigenspaces of* V.

(4) *Every irreducible representation of* G^c *is of order* n *for some* n*. Hence these representations for* $1 \le n \le d^2$ *are generators for the representation ring of* G^c.

Proof. We have already proved everything except for the last assertion. To see it is true, decompose an irreducible representation of G^c according to characters of μ_{d^2} and note that since μ_{d^2} is central each one of them is G^c invariant. Hence, there exist a unique eigenspace of μ_{d^2} on which the action is given by some α_n. Q.E.D.

We conclude this section by a lemma that follows immediately from Frobenius duality.

LEMMA A1.5. *Let* ρ *be an irreducible representation of* $k^{\times}K\sigma(K^*[n])$ such that $\rho|_{k^{\times}K} = \alpha_n\chi{\cdot}Id_W$ *for some* $\chi \in K^*$ *, then* ρ *appears* $dim(\rho)$ *times in* $Ind_{k^{\times}K}^{k^{\times}K\sigma(K^*[n])}(\alpha_n\chi)$.

A2. CLASSIFICATION WITH RESPECT TO A MAXIMAL LEVEL SUBGROUP WITH AN ORTHOGONAL COMPLEMENT

We keep the notation of §A1. We shall assume through out this section that K has an orthogonal complement. Thus, we may choose σ as a homomorphism into G^c and henceforth we assume this has been done. In this section we use the results of §2 to get an explicit classification of the irreducible representations of order n of G.

LEMMA A2.1 *Every irreducible representation ρ of order n of $k^\times K\sigma(K^*[n])$, such that $\rho|_{k^\times K} = \alpha_n\chi$ for some $\chi \in K^*$, is 1-dimensional and is of the form*

$$\rho(\alpha x\sigma(\psi)) = \alpha^n\chi(x)\tau(\psi) \qquad \alpha \in k^\times, x \in K, \psi \in K^*[n],$$

for a suitable $\tau \in (K^*[n])^*$.

Conversely, given any n, $\chi \in K^*$, $\tau \in (K^*[n])^*$ we have an irreducible 1-dimensional representation ρ of order n of $k^\times K\sigma(K^*[n])$ defined as above.

Proof. Let (V, ρ) be an irreducible representation of order n of $k^\times K\sigma(K^*[n])$. We consider it as a representation of $K^*[n]$ via σ and decompose it according to characters

$$V = \bigoplus_{\tau \in K^*[n]^*} a_\tau V_\tau,$$

where a_τ are the multiplicities. It is easy to see that each V_τ is $k^\times K\sigma(K^*[n])$ invariant. That proves the first part of the lemma.

To prove the second part we need only check that ρ as defined is a homomorphism :

$$\rho(\alpha x\sigma(\psi)\alpha_1 x_1\sigma(\psi_1)) = \rho(\alpha\alpha_1[x_1^{-1}, \sigma(\psi)]xx_1\sigma(\psi)\sigma(\psi_1)])$$
$$= (\alpha\alpha_1)^n\chi(xx_1)\tau(\psi\psi_1).$$

where we used $[x_1^{-1}, \sigma(\psi)]^n = [x_1^{-1}, \sigma(\psi^n)] = 1$. Q.E.D.

We may now rephrase and explicate THEOREM A1.4.

THEOREM A2.2. *Let K be a maximal level subgroup with an orthogonal complement. The irreducible representations of order n of G (G^c) are in one to one correspondence with triples (n, χ, τ) ($1 \le n \le d^2$ respectively) where $\chi \in K^*/K^{*n}$, $\tau \in (K^*[n])^*$.*

The representation corresponding to (n, χ, τ) is $Ind^G_{k^\times K\sigma(K^*[n])}(\alpha_n\chi\tau)$. It is of dimension $r(n) = \#K^{*n}$ and is denoted by $(W_{(n,\chi,\tau)}, \rho_{(n,\chi,\tau)})$. Its character, denoted by $\Phi_{(n,\chi,\tau)}$ is given by

(A2.1) $$\Phi_{(n,\chi,\tau)}(\alpha x\sigma(\psi)) = r(n)\alpha^n\chi(x)\tau(\psi)\cdot 1_{K[n]}(x)\cdot 1_{K^*[n]}(\psi)$$

for $\alpha \in k^\times, x \in K, \psi \in K^*[n]$.

Fix n, m. Put $d(n) = \#K^*[n]$, $s = (n, m)$, $a_{n,m} = d\cdot d(s)^2/d(n)d(m)d(n+m)$. Let Φ be the character of $W_{(n,\chi_1,\tau_1)}\otimes W_{(m,\chi_2,\tau_2)}$ and choose some $\beta \in K^*[n+m]^*$ whose restriction to $K^*[s]$ is $\tau_1\tau_2|_{K^*[s]}$. then

(A2.2) $$W_{(n,\chi_1,\tau_1)}\otimes W_{(m,\chi_2,\tau_2)} = \bigoplus_{\substack{\chi\in K^{*x}/K^{*n+m}\\ \tau\in K^*[n+m]^{*s}}} a_{n,m} W_{(n+m,\chi_1\chi_2\chi,\beta\tau)}\,.$$

Proof. While translating Theorem A1.4 one should note that the G action on (n, χ_0, τ) changes only χ_0 to some $\chi_0\chi'$ and leaves τ intact. To see this is the case, use the decomposition appearing in Lemma A1.1 of $W_{(n,\chi_0,\tau)}$ as $\bigoplus_{\chi\in K^{*n}} V_{\chi_0\chi}$. Suppose that $K^*[n]$ acts on V_{χ_0} by τ,

$$U_{\sigma(\eta)}v = \tau(\eta)v \qquad \forall v\in V_{\chi_0}, \forall\eta\in K^*[n].$$

Taking any $y\in G^c$ we need to verify that the same holds for $U_y v$. Indeed,

$$\begin{aligned} U_{\sigma(\eta)}U_y v &= U_y U_{y^{-1}\sigma(\eta)y} v \\ &= U_y U_{\chi^y(\sigma(\eta))\cdot\sigma(\eta)} v \\ &= U_y(\chi^y(\sigma(\eta))^n\cdot\tau(\eta)v) \\ &= \tau(\eta)U_y v\,. \end{aligned}$$

It remains to prove the assertions concerning the characters. It is easy to see, using the decomposition of $W_{(n,\chi,\tau)}$ with respect to K^* that $\dim(W_{(n,\chi,\tau)}) = r(n)$ and that

$\alpha\in k^\times$ acts as $\alpha^n I_{r(n)}$,
$x\in K$ acts as $\chi(x)\cdot \mathrm{diag}[\chi_1(x), \ldots, \chi_{r(n)}(x)]$ where $\{\chi_1, \ldots, \chi_{r(n)}\} = K^{*n}$,
$\psi\in K^*\setminus K^*[n]$ acts as a permutation matrix of a permutation with no fixed points,
$\psi\in K^*[n]$ acts by $\tau(\psi)\cdot I_{r(n)}$.

Hence, if $\Phi_{(n,\chi,\tau)}(\alpha x\sigma(\psi))$ is non-zero we must have $\psi\in K^*[n]$ and then

$$\Phi_{(n,\chi,\tau)}(\alpha x\sigma(\psi)) = \alpha^n\chi(x)\tau(\psi)\cdot\Big(\sum_{\lambda\in K^{*n}}\lambda(x)\Big).$$

Since K^{*n} is dual to $K[n]$ we get (A2.1).

To compute Φ we first note that only characters corresponding to representations of order $n+m$ can appear in the decomposition of Φ. We have thus to compute $<\Phi, \Phi_{(h,\chi,\tau)}>$ only for $h = n+m$, $\chi\in K^*$, $\tau\in K^*[n+m]^*$. Secondly, we note that

$$\Phi(\alpha x\sigma(\psi)) = \alpha^{n+m}r(n)r(m)[\chi_1\chi_2(x)][\tau_1\tau_2(\psi)]\cdot 1_{K[s]}(x)\cdot 1_{K^*[s]}(\psi)$$

by the general formula for tensor products and the equality $K[n]\cap K[m] = K[s]$, $K^*[n]\cap K^*[m] = K^*[s]$. Therefore

$$<\Phi, \Phi_{(n+m,\chi,\tau)}> = \frac{1}{d^4}\sum_{g\in G^c}\Phi(g)\cdot\Phi_{(n+m,\chi,\tau)}(g^{-1})$$

$$= \frac{1}{d^4}\sum_{\alpha,x,\psi}\Phi(\alpha x\sigma(\psi))\cdot\Phi_{(n+m,\chi,\tau)}(\alpha^{-1}[x,\psi]^{-1}x^{-1}\psi^{-1})$$

$$= \frac{r(n)r(m)r(n+m)}{d^2} \cdot \sum_{x,\, \psi} \left\{ [x, \psi]^{-(n+m)} [\chi_1\chi_2(x)] \chi(x^{-1}) [\tau_1\tau_2(\psi)] \tau(\psi^{-1}) \cdot 1_{K[s]}(x) \cdot 1_{K^*[s]}(\psi) \right\}$$

$$= \frac{d}{d(n)d(m)d(n+m)} \left(\sum_{x \in K[s]} \chi_1\chi_2\chi^{-1}(x) \right) \left(\sum_{\psi \in K^*[s]} \tau_1\tau_2\tau^{-1}(\psi) \right).$$

The last expression is zero unless <u>both</u> $\chi_1\chi_2\chi^{-1}|_{K[s]} = 1$ <u>and</u> $\tau_1\tau_2\tau^{-1}|_{K[s]} = 1$. If <u>both</u> happen then this expression is equal to $d \cdot d(s)^2 / d(n)d(m)d(n+m)$. Any such χ is of the form $\chi_1\chi_2\xi$, $\xi \in K^{*s}$. Any such τ is of the form $\beta\eta$ where η is in the dual of $K^*[s]$ in $K^*[n+m]^*$ which is just $K^*[n+m]^{*s}$ and $\beta \in K^*[n+m]^*$ is a character whose restriction to $K^*[s]$ is $\tau_1\tau_2|_{K^*[s]}$. Thus we get

$$\text{(A2.3)} \qquad \Phi = \sum_{\substack{\chi \in K^{*s} / K^{*n+m} \\ \tau \in K^*[n+m]^{*s}}} a_{n,m} \Phi_{(n+m,\, \chi_1\chi_2\chi,\, \beta\tau)},$$

which proves (A2.2). Q.E.D.

A3. REPRESENTATIONS OF ADELIC HEISENBERG GROUPS

NOTATION.

k - an algebraically closed field of characteristic $p \geq 0$.

$\mathbb{A}$ - the adele ring of $\mathbb{Q}$.

$\mathbb{A}_{ff}$ - the subring of $\mathbb{A}$ of finite adeles without the p - component if $p > 0$.

$\mathbf{e}$ - an isomorphism $\mathbb{A}_{ff}/\hat{\mathbb{Z}}_{ff} \longrightarrow k^\times_{tor}$.

G - the set $k^\times \times \mathbb{A}_{ff}^{2g}$ with the multiplication rule, turning G into a group, given by

$$(\lambda, x_1, x_2) \cdot (\mu, y_1, y_2) = (\lambda\mu \cdot \mathbf{e}(\tfrac{1}{2}({}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1)), x_1 + y_1, x_2 + y_2).$$

σ - the group homomorphism $\hat{\mathbb{Z}}_{ff}^{2g} \longrightarrow G$ given by

$$\sigma(x_1, x_2) = ((-1)^{{}^t x_1 \cdot x_2}, x_1, x_2).$$

DEFINITION. Let (V, ρ) be a representation of G. We say that (V, ρ) is a *continuous representation of* G if for every $v \in V$ there exists an m, depending on v, such that for every $x \in m\, \sigma(\hat{\mathbb{Z}}_{ff}^{2g})$ we have $\rho(x)\, v = v$.

Define now $G^c = k^\times_{tor} \times \mathbb{A}_{ff}^{2g}$ with the same group law. We say that a representation (V, ρ) of G^c is of order n, $n \in \mathbb{Z} - \{0\}$ if $k^\times_{tor}$ acts through the character a_n, $a_n(t) = t^n$. We remark that given an ample symmetric line bundle L and a theta structure

$$\Delta : \hat{G}(L) \longrightarrow G$$

we have $\Delta(k^\times_{tor} \times \mathbb{A}_{ff}^{2g}) = k^\times_{tor} \times \mathbb{A}_{ff}^{2g}$ and we may define $\hat{G}(L)^c$ in the obvious way. Therefore one may talk of representations of adelic order of $\hat{G}(L)^c$.

THEOREM A3.1. (1) *There is a unique irreducible continuous representation of G^c of order n, for every $n \in \mathbb{Z} - \{0\}$. This representation is henceforth denoted by V_n.*

(2) *For $n = 0$ there is one to one correspondence between irreducible continuous representations of order 0 of G and continuous characters of $\mathbb{A}_{ff}^{2g}$. The representation corresponding to $\psi \in \mathrm{Hom}_C(\mathbb{A}_f^{2g}, k^\times)$ is denoted by $V_{(0,\psi)}$. We denote by V_0 the representation* $\bigoplus_{\psi \in \mathrm{Hom}_C(\mathbb{A}_f^{2g}, k^\times)} V_{(0,\psi)}$.

(3) i) $V_{(0,\psi)} \otimes V_{(0,\chi)} = V_{(0,\psi\chi)}$.

ii) $V_{(0,\psi)} \otimes V_n = V_n$.

iii) $V_n \otimes V_r = \oplus V_{n+r}$, *an infinite countable sum* *if* $n + r \neq 0$, *and* $V_n \otimes V_r = V_0$, *if* $n + r = 0$.

Proof. The existence and uniqueness for $n = 1$ are proved in [Mum1] Proposition 5.2. To get the clain for a general n define for every $n \in \mathbb{Z} - \{0\}$ a matrix $M(n) \in \mathrm{GSp}(2g, \mathbb{A}_{ff})$ by $M(n) = \begin{pmatrix} I & 0 \\ 0 & nI \end{pmatrix}$. Define a surjective homomorphism

$$\delta(n) : G^c \longrightarrow G', \ \delta(n)(\alpha, x) = (\alpha', M(n)x).$$

Since the multiplier of $M(n)$, $\varpi(M(n)) = n$, this is indeed a homomorphism. Now given a representation (V, ρ) of order n define a new representation ρ' of G^c by the formula

$$\rho'(x) = \rho(\delta(n)^{-1}(x)).$$

One easily verifies that this is a continuous representation of order 1 of G^c. That is $\delta(n)$ gives a bijection between representations of order 1 and order n. That proves 1).

Since 2) is clear we have to prove only 3). Part i) is clear and ii) follows from 1 by tensoring with $V_{(0,\psi^{-1})}$. In part iii) the only question is with what multiplicity does V_{n+r} appear in $V_n \otimes V_r$. It is not difficult to check that in an irreducible representation of order n the dimension of the invariants under a maximal level subgroup is precisely $|n|^g$. Choose the maximal level subgroup $\mathcal{Z} = \sigma(\hat{\mathbb{Z}}_{ff}^{2g})$ and decompose both V_n and V_r with respect to $\sigma(\hat{\mathbb{Z}}_{ff}^{2g})$:

$$V_n = \bigoplus_{\psi \in \mathcal{Z}^*} (V_n)_\psi \quad , \quad V_r = \bigoplus_{\psi \in \mathcal{Z}^*} (V_r)_\psi \quad .$$

Therefore

$$V_n \otimes V_r = \bigoplus_{\psi \in \mathcal{Z}^*} \left(\bigoplus_{\tau \in \mathcal{Z}^*} (V_r)_{\psi\tau} \otimes (V_n)_{\tau^{-1}} \right) \qquad \text{Q.E.D.}$$

REMARKS. 1) Theorem A3.1 has an exact analog in the theory of real Heisenberg groups. See [CR] §2.2.

2) The analog theorem for G and representations of order $n \in \mathbb{Z}$ hold of course.

A4. APPLICATIONS TO THE THEORY OF THETA FUNCTIONS.

This section gives some applications of the theory developed so far. The applications given are mainly decompositions of certain global sections of line bundles on abelian varieties as modules of finite and adelic Heisenberg groups.

1) We have a homomorphism

$$\varepsilon_n : G(L) \longrightarrow G(L^{\otimes n})$$

given by $\phi \longmapsto \phi^{\otimes n}$. ε_n restricts to α_n on $k^\times$ and induces by passing to quotients the natural embedding of $H(L)$ into $H(L^{\otimes n})$. Via ε_n we have a sequence of $G(L)$ modules

$$\Gamma(X, L)^{\otimes n} \longrightarrow Sym^n(\Gamma(X, L)) \longrightarrow \Gamma(X, L^{\otimes n}).$$

2) Assume that L is symmetric. One can define a homomorphism

$$\eta_n : G(L^{\otimes n}) \longrightarrow G(L)$$

(see [Mum1]) which induces the homomorphisms α_n on $k^\times$ and multiplication by n on $H(L^{\otimes n})$. It turns $\Gamma(X, L)$ into a representation of order n of $G(L^{\otimes n})$.

3) For L symmetric we have for every integer n a map $\delta_n : G(L) \longrightarrow G(L)$ which is equal to α_{n^2} on $k^\times$ and induces multiplication by n on $H(L)$. That gives a representation of order n^2 of $G(L)$.

The maps described above satisfy the identities

i) $\delta_n = \eta_n \circ \varepsilon_n$ for $\delta_n : G(L) \longrightarrow G(L)$, $\eta_n : G(L^{\otimes n}) \longrightarrow G(L)$, $\varepsilon_n : G(L) \longrightarrow G(L^{\otimes n})$.

ii) $\delta_n = \varepsilon_n \circ \eta_n$ for $\delta_n : G(L^{\otimes n}) \longrightarrow G(L^{\otimes n})$, $\eta_n : G(L^{\otimes n}) \longrightarrow G(L)$, $\varepsilon_n : G(L) \longrightarrow G(L^{\otimes n})$.

There are analogous maps ε_n, η_n, δ_n for adelic Heisenberg groups. These maps are obtained from the previous ones in a standard fashion and have similar properties.

We get representations of $G(L)$ ($\hat{G}(L)$) of orders n and n^2 via ε_n and δ_n,

respectively. There are also representations of order n of $G(L)$ ($\hat{G}(L)$) on $\Gamma(X, L^{\otimes n})$ and $Sym^n(\Gamma(X, L))$. We now study them.

I. $\Gamma(X, L)^{\otimes n}$ as a $G(L)$ module of order n.

Choose a maximal level subgroup $K(L)$ of $G(L)$ which has an orthogonal complement. We know that there is a unique irreducible representation of order 1 of $G(L)$ - $W_{(1,1,1)}$ in our notation. It is of dimension d, where $d^2 = \#H(L)$. Since $d = \deg(L) = \dim(\Gamma(X, L))$, we conclude Mumford's observation that $\Gamma(X, L)$ is the aforementioned representation. Therefore, if Φ is the character of the $G(L)$ action on $\Gamma(X, L)^{\otimes n}$ then $\Phi = \Phi_{(1,1,1)}{}^n$. Using (3.1) we get

$$\Phi(\alpha x \sigma(\psi)) = r(1)^n \cdot \alpha^n \cdot 1_{K[1]}(x) \cdot 1_{K^*[1]}(\psi),$$

where, as before, $r(n) = \#K^{*n}$, $d(n) = \#K^*[n]$.

Otherwise said :

(A4.1)
$$\Phi(\alpha x\sigma(\psi)) = \begin{cases} d^n \cdot \alpha^n & \text{if } x = 0 \text{ and } \psi = 1. \\ 0 & \text{otherwise.} \end{cases}$$

Now, we simply compute the ' inner products ' :

$$< \Phi, \Phi_{(n,\chi,\tau)} > = \frac{1}{d^4} \sum_{(\alpha x\sigma(\psi)) \in G(L)^c} \Phi(\alpha x\sigma(\psi)) \cdot \Phi_{(n,\chi,\tau)}((\alpha x\sigma(\psi))^{-1})$$

$$= \frac{1}{d^4} \sum_{\alpha \in \mu_{d^2}} \Phi(\alpha) \cdot \Phi_{(n,\chi,\tau)}(\alpha^{-1})$$

$$= \frac{d^{n-1}}{d(n)} .$$

Therefore,

$$\Gamma(X, L)^{\otimes n} = \bigoplus_{\substack{\chi \in K(L)^*/K(L)^{*n} \\ \tau \in K(L)^*[n]^*}} \frac{d^{n-1}}{d(n)} \cdot W_{(n,\chi,\tau)} .$$

II. $\Gamma(X, L^{\otimes n})$ as a $G(L)$ module of order n, via ε_n.

Choosing a set theoretic splitting we may write $G(L) = k^\times \times H(L)$ and the group law is given by

$$(\alpha_1, h_1)(\alpha_2, h_2) = (\alpha_1\alpha_2 \cdot F_L(h_1, h_2), h_1 + h_2).$$

F_L is a normalized 2 - cocycle :

(a) $F_L(h_1, h_2)F_L(h_1 + h_2, h_3) = F_L(h_1, h_2 + h_3)F_L(h_2, h_3)$.

(b) $F_L(0, 0) = 1$.

The homomorphism $\varepsilon_n : G(L) \longrightarrow G(L^{\otimes n})$ can be written as

$$(\alpha, h) \longmapsto (\alpha^n \cdot s(h), h),$$

where $s : G(L) \longrightarrow k^\times$ satisfies

$$\frac{s(h_1, h_2)}{s(h_1)s(h_2)} = \frac{F_{L^n}(h_1, h_2)}{F_L(h_1, h_2)^n} .$$

Note that $s(0) = 1$. Now, the character of the natural action of $G(L^{\otimes n})$ on $\Gamma(X, L^{\otimes n})$ is just $\Phi_{(1,1,1)}$. Thus, we have

$$\Phi(\alpha x\sigma(\psi)) = \begin{cases} \alpha \cdot D & \text{if } x = 0 \text{ and } \psi = 1. \\ 0 & \text{otherwise.} \end{cases}$$

where $D^2 = \#H(L^{\otimes n}) = n^{2g} \cdot \#H(L) = n^{2g} \cdot d^2$, that is $D = d \cdot n^g$.

Therefore, if Φ is the character of the ε_n action of $G(L)$ on $\Gamma(X, L^{\otimes n})$, then :

(A4.2) $$\Phi(\alpha, h) = \begin{cases} \alpha^n \cdot D & \text{if } h = 0. \\ 0 & \text{otherwise.} \end{cases}$$

We can compute now the multiplicity of each $\Phi_{(n,\chi,\tau)}$ in Φ.

$$< \Phi, \Phi_{(n,\chi,\tau)} > = \frac{1}{d^4} \sum_{(\alpha, h) \in G(L)^c} \Phi(\alpha, h) \cdot \Phi_{(n,\chi,\tau)}((\alpha, h)^{-1})$$

$$= \frac{1}{d^4} \sum_{\alpha \in \mu_{d^2}} \Phi(\alpha) \cdot \Phi_{(n,\chi,\tau)}(\alpha^{-1})$$

$$= \frac{n^g}{d(n)} .$$

We conclude that

(A4.3) $$\Gamma(X, L^{\otimes n}) = \bigoplus_{\substack{\chi \in K(L)^*/K(L)^{*n} \\ \tau \in K(L)^*[n]^*}} \frac{n^g}{d(n)} \cdot W_{(n,\chi,\tau)}$$

III. $Sym^2(\Gamma(X, L))$ as a representation of order 2 of G(L).

Recall the basic decomposition :

$$\Gamma(X, L) = \bigoplus_{\chi \in K^*} k \cdot v_\chi ,$$

where v_1 is arbitrary, $v_\chi = U_{\sigma(\chi)} v_1$ and σ is taken to be a homomorphism into $G(L)^c$. $Sym^2(\Gamma(X, L))$ has a basis $\{ v_\chi v_\tau \mid \chi, \tau \in K^* \}$ and is $d(d+1)/2$ dimensional. The action of $G(L)^c$ is given by :

$\alpha \in k^\times$ acts by $\alpha^2 I$, where I is the identity $d(d+1)/2$ matrix.
$x \in K$ acts by $\chi\tau(x)$ on $v_\chi v_\tau$.
$\psi \in K^*$ acts by $v_\chi v_\tau \longmapsto v_{\chi\psi} v_{\tau\psi}$.

Denoting by Φ the character of $Sym^2(\Gamma(X, L))$ we see that

$$\Phi(\alpha x \sigma(\psi)) = \alpha^2 \cdot \sum_{\{\chi, \tau\} \,;\, \{\chi, \tau\} = \{\chi\psi, \tau\psi\}} \chi\tau(x) .$$

If the order of ψ is not 1 or 2 then there are no such couples $\{\chi, \tau\}$. Distuinguishing cases a short computation gives

(A4.4) $$\Phi(\alpha x \sigma(\psi)) = \tfrac{1}{2} \alpha^2 d \cdot 1_{K[2]}(x) \cdot 1_{K^*[2]}(\psi) \cdot \psi(x) \cdot (1_{K^*[2]}(\psi) + d \cdot 1_{K[1]}(x)).$$

Let us compute the multiplicity of which an irreducible representation of order 2, $W_{(n,\chi,\tau)}$ appear in $Sym^2(\Gamma(X, L))$:

$$< \Phi, \Phi_{(2,\chi,\tau)} > = \frac{1}{d^4} \cdot \sum_{\substack{\mu \in \mu_{d^2} \\ x \in K[2] \\ \psi \in K[2]^*}} \Phi(\alpha x \sigma(\psi)) \Phi_{(2,\chi,\tau)}(\alpha^{-1}[x, \sigma(\psi)]^{-1} x^{-1} \sigma(\psi)^{-1})$$

$$= \frac{d^2 \cdot \frac{d(d+1)}{2} \cdot r(2)}{d^4} + \frac{1}{d^4} \cdot \sum_{\substack{\mu \in \mu_{d^2} \\ (x,\psi) \in K[2] \times K[2]^* \setminus \{(1,1)\}}} \frac{d}{2} \cdot \psi(x) \cdot r(2) \cdot \chi(x)^{-1} \tau(\psi)^{-1}$$

(A4.5) $$= \frac{d+1}{2d(2)} + \frac{1}{2d(2)} \left(-1 + \sum_{(x,\psi) \in K[2] \times K[2]^*} \psi(x) \cdot \chi^{-1}(x) \cdot \tau^{-1}(\psi) \right)$$

$$= \frac{1}{2d(2)} \left(d + \sum_{\psi \in K^*[2]} \tau^{-1}(\psi) \left(\sum_{x \in K[2]} \psi \chi^{-1}(x) \right) \right)$$

$$= \frac{1}{2d(2)} \left(d + \sum_{\psi \in K^*[2]} \tau^{-1}(\psi) \left(\sum_{x \in K[2]^{**}} x(\psi \chi^{-1} |_{K[2]}) \right) \right)$$

$$= \frac{1}{2d(2)} \left(d + d(2) \cdot \sum_{\substack{\psi \in K^*[2] \text{ s.t.} \\ \psi|_{K[2]} = \chi|_{K[2]}}} \tau^{-1}(\psi) \right).$$

Consider the last expression of (A4.5). Let us denote by $K^*[2]_{tr}$ the subgroup of all characters $\psi \in K^*[2]$ whose restriction to $K[2]$ is trivial. Then if $\#K^*[2]_{tr} > 1$ then this aforementioned expression is equal to $\frac{1}{2} \cdot r(2)$ no matter what χ is. Let us denote by $K^*[2]_{rs}$ the subgroup of $K[2]^*$ obtained by restricting the elements of $K^*[2]$ to $K[2]$. That is we have an exact sequence

$$1 \longrightarrow K^*[2]_{tr} \longrightarrow K^*[2] \xrightarrow{res.} K^*[2]_{rs} \longrightarrow 1.$$

If $K^*[2]_{tr} = \{1\}$ then since $\#K^*[2] = \#K[2] = \#K[2]^*$ we have $K^*[2]_{rs} = K[2]^*$. That implies that in this case the last expression of (A4.5) is equal, for any χ :

$$\frac{1}{2d(2)} \cdot (d + d(2) \cdot \tau(\chi |_{K[2]}))$$

which equal to $\frac{1}{2}(r(2) + \tau(\chi |_{K[2]}))$. To sum up:

The representation $W_{(n,\chi,\tau)}$ appear with the following multiplicity in $Sym^2(\Gamma(X, L))$:

i) $\frac{1}{2} \cdot r(2)$ *if* $K^*[2]_{tr} \neq 1$,

ii) $\frac{1}{2}(r(2) + \tau(\chi|_{K[2]}))$ *if* $K^*[2]_{tr} = \{1\}$.

EXAMPLE. Consider the special case where $L = M^2$ where M is a symmetric ample line bundle of degree 1.

In this case we have $H(L) = X[2]$ and a maximal level group K is an elementary abelian 2-group of order 2^g. Clearly $r(2) = 1$. Consequently, any representation $W_{(2,\chi,\tau)}$ (which is $\#K^{*2} = 1$ dimensional) appears with multiplicity $\frac{1}{2}(1 + \tau(\chi))$ which is either zero or 1.

Let us call a representation $W_{(2,\chi,\tau)}$ even or odd if $\tau(\chi|_K)$ is equal to 1 or -1 respectively. The dimension of $\Gamma(X, L^2)$ is 2^{2g} and according to (6.2) it is the sum of all $W_{(n,\chi,\tau)}$ each appearing with multiplicity 1, and each is 1-dimensional. Now, it is classical fact (See Mumford / "Tata Lectures on Theta I", Proposition 1.3, p.124 and [Mum2] Proposition 3.2 p.40) that over the complex numbers the global sections of (the pull-back to the universal covering space of) M^2 are given by $\Theta\begin{bmatrix} a \\ 0 \end{bmatrix}(2z, 2\tau)$, $a \in \frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$, and the global sections of M^4 are given both by $\Theta\begin{bmatrix} a \\ 0 \end{bmatrix}(4z, 4\tau)$, $a \in \frac{1}{4}\mathbb{Z}^g/\mathbb{Z}^g$ and by $\Theta\begin{bmatrix} a/2 \\ b/2 \end{bmatrix}(2z, \tau)$ where $a, b \in \mathbb{Z}^g$. It is reasonable to guess that each of the $\Theta\begin{bmatrix} a/2 \\ b/2 \end{bmatrix}(2z, \tau)$ spans a unique $W_{(2,\chi,\tau)}$ and that the new notion of even/odd corresponds to the classical notion of even/odd characteristics. This is indeed the case.

IV. $\Gamma(X, L)$ as a representation of order n of $G(L^{\otimes n})$, via η_n.

It is obvious that this is an irreducible representation. The determination of which (n, χ, τ) belong to this representation is a ' combinatorial problem ' which is of no importance in this paper.

V. $\Gamma(X, L)$ as a representation of order n^2 of $G(L)$, via δ_n.

Choosing a splitting of $G(L)$ as in **II** we can write the homomorphism δ_n as

$$(\alpha, h) \longmapsto (\alpha^{n^2} \cdot s(h), nh)$$

where s is a character of $H(L)$.

The character Φ of the aforementioned representation is given by

$$\text{(A4.6)} \qquad \Phi(\alpha, x, l) \begin{cases} d \cdot \alpha^{n^2} \cdot s(h) & \text{if } nh = 0. \\ 0 & \text{otherwise.} \end{cases}$$

We can easily compute now the product

$$< \Phi, \Phi_{(n,^2\chi,\tau)} > = \frac{1}{d^4} \cdot \sum_{\substack{\mu \in \mu_{d^2} \\ x \in K[n] \\ l \in K^*[n]}} d \cdot s(x + \sigma(l)) \cdot r(n^2)\, \chi^{-1}(x) \tau^{-1}(l)$$

$$= \frac{r(n^2)}{d} \cdot \sum_{\substack{x \in K[n] \\ l \in K^*[n]}} [s \cdot (\chi \times \tau)^{-1}](x + \sigma(l)) \; .$$

The last expression is zero unless $s = \chi \times \tau$ on $K[n] \times K^*[n]$. If this happens then this expression is equal to $r(n^2) \cdot d(n)^2 / d = d(n)^2 / d(n^2)$. Fix some χ, τ such that $s = \chi \times \tau$ then $\Gamma(X, L)$ as a representation of order n^2 of $G(L)$ decomposes as follows :

$$\text{(A4.7)} \qquad \Gamma(X, L) = \bigoplus_{\substack{\chi_1 \in K^{*n} / K^{*n^2} \\ \tau_1 \in K^*[n^2]^{*n}}} \frac{d(n)^2}{d(n^2)} W_{(n^2, \chi \cdot \chi_1, \tau \cdot \tau_1)} \; .$$

With regard to adelic Heisenberg groups we have the following assetions :

(i) $\hat{\Gamma}\;(L^{\otimes n})$ is the unique irreducible representation of order n of $\hat{G}\;(L)$ acting via ε_n ;

(ii) $\hat{\Gamma}\;(L)$ is the unique irreducible representation of order n of $\hat{G}\;(L^{\otimes n})$ acting via η_n ;

(iii) $\hat{\Gamma}\;(L)$ is the unique irreducible representation of order n^2 of $\hat{G}\;(L)$ acting via δ_n ;

(iv) $\hat{\Gamma}\;(L)^{\otimes n}$ is the unique irreducible representation of order n of $\hat{G}\;(L)$ with infinite countable multiplicity. The same is true for $Sym^n(\hat{\Gamma}\;(L))$.

The irreducibility of these representations follows immediately from the fact that the homo - morphisms through which they are obtained are all surjective and from the fundamental result ([Mum2] Proposition 5.3) stating that for every ample invertible sheaf M, $\hat{\Gamma}\;(M)$ is the unique irreducible continuous representation of order 1 of $\hat{G}\;(M)$. Their uniquness follows from Theorem A3.1.

As an immediate consequence of (i) we get the following corollary (compare[Mum2] Theorem 7.1) : Choose an non-zero section $s \in \hat{\Gamma}\;(L)_1$, where the decomposition is with respect to a maximal level subgroup $\boldsymbol{Z}$ of $\hat{G}\;(L)$. Then, denoting by U the action of $\hat{G}\;(L)$ on $\hat{\Gamma}\;(L)$, we have that $\{\, (U_z s)^n \mid z \in \boldsymbol{Z} \,\}$ span $\hat{\Gamma}\;(L^{\otimes n})$. Indeed, these generators are permuted - up to scalar factors - by the ε_n action of $\hat{G}\;(L)$.

REFERENCES

[CG] L. CORWIN AND F. P. GREENLEAF, "Representations of nilpotent Lie groups and their applications; Part 1: Basic theory and examples", Cambridge studies in advanced mathematics 18, Cambridge University Press 1989.

[Kem] G. R. KEMPF, Linear systems on abelian varieties, *American Journal of Mathematics* 111 (1989), 65 - 94.

[LB] H. LANGE AND CH. BIRKENHAKE , "Complex Abelian Varieties", *in* A Series of Comprehensive Studies in Mathematics 302, Springer Verlag 1992.

[Mum1] D. MUMFORD, On the equations defining abelian varieties I, *Invent. Math.* 1, 287-354 (1966).

[Mum2] D. MUMFORD, On the equations defining abelian varieties II, *Invent. Math.* 3, 75-135 (1967).

[Mum3] D. MUMFORD WITH M. NORI AND P. NORMAN, "Tata Lectures on Theta III", *in* "Progress in Mathematics", Vol. 97, Birkhauser, 1991.

[Ser] J. P. SERRE, " Algebraic Groups and Class fields ", *in* Graduate Texts in Mathematics 117, Springer - Verlag 1988.