Recent progress in analytic arithmetic of positive definite quadratic forms

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§ 1. Introduction

By a well–known theorem of Linnik's, [11], the integral points on a two–dimensional sphere $x_1^2 + x_2^2 + x_3^2 = n$ are asymptotically equidistributed as n varies over the infinite sequence of positive rational integers satisfying two conditions: $n \not\equiv 7,4 \pmod 8$ and $\left(\frac{-n}{p}\right) = 1$ for a fixed rational prime p. Unfortunately the second condition $\left(\frac{-n}{p}\right) = 1$, unnatural as it is, could not be removed unless one assumes a weak but still unproved hypothesis about the zero–free region of a certain Dirichlet's L–function; another drawback of the method is a poor (logarithmic) error term in the asymptotic formula for the number of integral points in the chosen region on the sphere. As it has been pointed out by Yu. V. Linnik, [13, p. 56], one can expect to repair this situation only by developing completely new methods that would, in particular, lead to better understanding of the nature of Kloosterman's sums (one should not fail to recall at this point that the far–reaching Linnik's conjecture, [12, p. 277], on possible cancellations in a sum of Kloosterman's sums remains so far unsettled). Recently D.R. Heath–Brown, [7, p. 137–138], has put forward a conjecture to the extent that every sufficiently large integer congruent to 7 modulo 8 can be represented in the form $x_1^2 + x_2^2 + p^3x_3^2$, where p is a fixed rational prime congruent to 5 modulo 8. The goal of this report is to describe new developments in analytic number theory that, in particular, allow to solve each of these problems. To be more precise, let f be an integral positive definite quadratic form of s variables and let $s \geq 3$. We seek an asymptotic formula for the quantity

$$r_f(n;\Omega) = \text{card}\{u \mid u \in \mathbb{Z}^s,\ f(u) = n,\ \frac{u}{\sqrt{n}} \in \Omega\},$$

where $\Omega \subseteq \{u \mid u \in \mathbb{R}^s,\ f(u) = 1\}$, as $n \longrightarrow \infty$. Such a formula should, in particular, allow to conclude that $r_f(n;\Omega) \longrightarrow \infty$ as n varies over an infinite subsequence of positive rational integers satisfying certain natural restrictions. For $s \geq 4$ such an asymptotic formula was already obtained thirty years ago, [15], [16], by Hardy–Littlewood's circle method. General as it is, this method, however, may not lead to the best error term in a specific problem, and indeed the theory of quadratic forms is intimately related to the theory of modular functions that seems to be a natural tool for investigation of the problem in question. By a careful application of this theory O.M. Fomenko, [5], has recently given a new proof of the asymptotic formula for $r_f(n;\Omega)$ with a better error term than the one known previously. On the other hand, H. Iwaniec, [8], has obtained a new estimate for the Fourier coefficients of a holomorphic cusp–form of half–integral weight larger than 2, allowing to deduce an asymptotic formula for $r_f(n;\Omega)$ in the case $s = 3$. Such a formula has been derived by O.M. Fomenko & E.P. Golubeva, [6]. In the case $s \geq 4$ an estimate from below of the main term in the asymptotic formula for $r_f(n;\Omega)$ is a comparatively easy matter, and one could prove, [15], [16], that $r_f(n) >> n^{s/2-1-\epsilon}$ for $\epsilon > 0$ as soon as n satisfies the natural generic conditions (and, in the case $s = 4$, is not divisible by a high power of an "exceptional" prime). Here and in what follows $r_f(n) := \text{card}\{u \mid u \in \mathbb{Z}^s,$ $f(u) = n\}$ is the representation number of n by f. To estimate $r_f(n)$ from below in the case $s = 3$ is a classical unsolved problem, and the efforts of many authors (cf., for instance, [13], [17], [28] and references therein) have been devoted to its solution, starting from the pioneering work by C.L. Siegel, [23], [24], and Yu.V. Linnik, [10], [13]. Recently W. Duke, [4], has obtained an estimate for the Fourier coefficients of a cusp–form of weight 3/2 by an extension of H. Iwaniec's method, [8]. When combined with a theorem of R. Schulze–Pillot's, [26] (cf. also [27] and references therein), this estimate leads to a solution of the long–standing problem of representation of integers by a positive definite ternary form.

Acknowledgement. The author of this report claims no originality and acknowledges his deep intellectual debt to the work cited in this introduction. A preliminary draft of these notes was prepared during my start at the I.H.E.S. (Bures–sur–Yvette) in September 1987; as a visiting professor at the University of Genova (in winter 1988) and at the University of Strasbourg (in summer 1989) I had a possibility to give a few lectures on the analytic theory of quadratic forms. It is my pleasant duty to thank these institutions for their hospitality and financial assistance. Last but not least my thanks are due to Professor W. Kohnen for useful consultations on the theory of modular functions and to Miss M. Grau who has been patiently typing my manuscripts during the last few years.

§ 2. Statement of the main results

For the general background and terminology regarding integral quadratic forms we refer to G.L. Watson's tract, [29]. Let us start by explaining the notation to be used here. The variables n and p range over the positive rational integers and over the rational primes respectively; $\mathbb{Z}$ is the ring of rational integers, $\mathbb{Z}_p$ is the ring of p–adic integers; $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of rational, real and complex numbers respectively. Let $f(x) = \frac{1}{2} x' Ax$ be an integral positive definite s–ary quadratic form, so that $A = (a_{ij})$ is a symmetric matrix, $a_{ij} \in \mathbb{Z}$ and $2 \mid a_{ii}$ for $1 \leq i,\ j \leq s$; $x = \begin{bmatrix} x_1 \\ \dots \\ x_s \end{bmatrix}$ is a column vector and $x' = (x_1,\dots,x_s)$ is a row vector. We write $D = \det A$ and fix a decomposition $A = 2B'B$ with $B \in GL(s,\mathbb{R})$, where $B \longmapsto B'$ denotes the operation of matrix transposition. Let $S_\ell = \{x \mid x \in \mathbb{R}^{\ell+1},\ |x| = 1\}$ be the ℓ–dimensional unit sphere in $\mathbb{R}^{\ell+1}$; departing slightly from the notation used in § 1 we let $r_f(n,w) = \mathrm{card}\{u \mid u \in \mathbb{R}^s, \frac{Bu}{\sqrt{n}} \in w\}$ for $w \subseteq S_{s-1}$, so that $r_f(n) = r_f(n,S_{s-1})$ denotes the representation number of n by f ; here $|x| = (x'x)^{1/2}$ denotes the Euclidean norm in $\mathbb{R}^\ell$ for $\ell \in \mathbb{Z}$, $\ell \geq 1$. Let μ be the Euclidean measure on S_ℓ normalised by the condition $\mu(S_\ell) = 1$. Let K be a class of integral positive definite quadratic forms and suppose that $f \in K$. We write $r(K,n) := r_f(n)$ and $\nu(K) := \mathrm{card}\{U \mid U \in GL(s,\mathbb{Z}), U'AU = A\}$; clearly, $r(K,n)$ and $\nu(K)$ are well–defined (being independent of the choice of f in K). Let $L = \bigcup_{i=1}^{g} K_i$ be the union of g classes K_i, $1 \leq i \leq g$, of positive definite quadratic forms; one defines Siegel's average $r(L,n)$ of $r_f(n)$ over L by letting $r(L,n) = \sum_{1 \leq i \leq g} \frac{r(K_i,n)}{\nu(K_i)} \Big[\sum_{1 \leq i \leq g} 1/\nu(K_i) \Big]^{-1}$. We write gen f for the genus of quadratic forms containing f and spin f for the spinor genus containing f . Finally let us recall that f is said to represent n properly over a ring $\mathfrak{o}$ containing $\mathbb{Z}$ if $f(u) = n$ for some u in $\mathfrak{o}^s$ satisfying the condition g.c.d. $(u_1,\dots,u_s) = 1$ (as usual u_i, $1 \leq i \leq s$, denotes the $i^{\underline{th}}$ component of u). The following theorem results as a consequence of the work of several authors, [23], [24], [15], [16], [20] (cf. also the papers refered to in these articles).

<u>Theorem 1</u>. Let $s \geq 3$. Then $r(\mathrm{gen}\, f,n) = n^{s/2-1}\alpha_\infty(f)A_f(n)$, wher $\alpha_\infty(f)$ denotes the Lebesgue measure of the ellipsoid $\{u \mid u \in \mathbb{R}^s, f(u) = 1\}$ and where $A_f(n) = \prod_p \alpha(p,n)$ with

$$\alpha(p,n) := \lim_{a \to \infty} p^{-a(s-1)}\, \mathrm{card}\{u \mid u \in (\mathbb{Z}/p^a\mathbb{Z})^s,\ f(u) = n(\mathrm{mod}\ p^a)\} \ ;$$

moreover, $A_f(n) \ll_{f,\epsilon} n^\epsilon$ for $\epsilon > 0$ (here $\prod_p$ extends over all the primes p in $\mathbb{Z}$).

Furthermore, suppose that f represents n over $\mathbb{Z}_p$ for each p and satisfies one of the following conditions: (i) $s \geq 5$; (ii) $(n,2D) = 1$ and $s \geq 4$; (iii) $(n,2D) = 1$ and f represents n properly over $\mathbb{Z}_p$ for each p . Then $A_f(n) \gg_{f,\epsilon} n^{-\epsilon}$ for $\epsilon > 0$.

The next theorem in this report is a rather direct consequence of the estimates of coefficients of modular forms, [3], [4], discussed in § 3.

<u>Theorem 2</u>. If $s \geq 4$ and $2|s$ then $r_f(n) = r(\text{gen } f,n) + O(n^{s/4-1/2+\epsilon})$ for $\epsilon > 0$; if $s > 4$, $2 \nmid s$ and $(n,2D) = 1$ then $r_f(n) = r(\text{gen } f,n) + O(n^{s/4-2/7+\epsilon})$ for $\epsilon > 0$.

Theorem 3 and Corollary 1 have been alluded to at the beginning of this memoir and constitute one of its main results.

<u>Theorem 3</u>. Let f(x) be an integral positive definite ternary form. If $(n,2D) = 1$ then $r_f(n) = r(\text{gen } f,n) + O(n^{1/2-1/28+\epsilon})$ for $\epsilon > 0$.

<u>Corollary 1</u>. Let p be a rational prime congruent to 5 modulo 8 and let $f(x) = x_1^2 + x_2^2 + p^3x_3^2$. Then $r_f(8n + 7) \xrightarrow[n \to \infty]{} \infty$; in particular, every sufficiently large rational integer congruent to 7 modulo 8 is represented by f .

Finally, following [5], [6], we shall prove Theorem 4 about asymptotic equidistribution of integral points on an ellipsoid.

<u>Theorem 4</u>. Let $s \geq 3$, $w \subseteq S_\ell$ with $\ell = s-1$, and suppose that the (topological) boundary ∂w of the set w is a smooth submanifold of S_ℓ of codimension one. If $(n,2D) = 1$ then $r_f(n,w) = \mu(w)r_f(n) + O(n^{s/2-1-\delta(s)})$ for any $\delta(s)$ such that $\delta(s) < \frac{s-2}{3s+2}$ when $2|s$ and $\delta(s) < \frac{s-3-1/24}{3(s+1)}$ when $2 \nmid s$.

<u>Remark 1</u>. For the standard definition of a spinor genus (going back to [9]) see [19,

p. 298]; it seems to be different from the definition used in [29] (cf. [1, p. 85–86]). Following [26] we use here the standard definition.

Remark 2. Corollary 1 confirms a conjecture of D.R. Heath–Brown's, [7]; when combined with a theorem of Gauß's asserting that a positive rational integer n is either a sum of three squares or it is of the shape $n = 4^{\ell}(7k+8)$ with $\ell \in \mathbb{Z}$ and $k \in \mathbb{Z}$, this corollary implies that every sufficiently large positive integer is a sum of at most three square–full numbers. According to [7, p. 137] this answers a question posed by P. Erdős and A. Ivic (and first answered by D.R. Heath–Brown, [7, Theorem 1]).

Remark 3. Condition $(n,2D) = 1$ in Theorem 4 is redundant (and may be omitted) when $f(x) = x_1^2 + x_2^2 + x_3^2$, as it can observed by analysing the proof of this theorem.

After collecting the necessary results from the theory of modular functions in the next section we prove theorems 1–3 and corollary 1 in § 4. In the last section (§ 6) we make a few final notes on the subject–matter of this report.

§ 3. On coefficients of holomorphic cusp–forms

Let $\Gamma_0(N) = \{\gamma \mid \gamma \in SL_2(\mathbb{Z}),\ \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ c \equiv 0(N)\}$ be a congruence subgroup of $SL_2(\mathbb{Z})$, let $\epsilon_d = \begin{cases} i, & d \equiv -1(4) \\ 1, & d \equiv 1(4) \end{cases}$ be the sign of the Gauß sum, and let

$j(\gamma,z) = \epsilon_d^{-1} \left[\frac{c}{d}\right] (cz+d)^{1/2}$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\left[\frac{c}{d}\right]$ is the generalised Legendre symbol defined as in [25]. For $\nu \in \mathbb{R}$ let $S_\nu(N,\chi)$ denote the finite–dimensional Hilbert space of $\Gamma_0(N)$–cusp–forms for which $f(\gamma z) = j(\gamma,z)^{2\nu}\chi(\gamma)f(z)$ whenever $\gamma \in \Gamma_0(N)$ and

$z \in \mathbb{C}_+$ (here $\mathbb{C}_+ = \{z \mid z \in \mathbb{C},\ \operatorname{Im} z > 0\}$ denotes the upper half–plane), the Petersson's inner product being defined by the equation: $\langle f|g\rangle = \int_{\Gamma_0(N)\backslash \mathbb{C}_+} f(z)\overline{g(z)}y^{\nu-2}dxdy$ for $f,g \in S_\nu(N,\chi)$. We write $\|f\| = \langle f|f\rangle^{1/2}$ for $f \in S_\nu(N,\chi)$. We are interested here in modular forms of integral or half–integral weight only, so that it is assumed in what follows that $2\nu \in \mathbb{Z}$ and $\nu \geq 0$.

Lemma 1. Let $\nu \in \mathbb{Z}$, $\nu > 0$ and let $\varphi(z) \in S_\nu(N,\chi)$ with $\chi(\gamma) = \left[\frac{D}{d}\right]$ for some D in $\mathbb{Z}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\gamma \in \Gamma_0(N)$. On writing $\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ we have $a(n) \ll_{\varphi,\epsilon} n^{(\nu-1)/2+\epsilon}$ for $\epsilon > 0$; to be more precise, if φ is a common eigenfunction of all the Hecke operators T_p with $p \nmid N$ normalised by the condition $a(1) = 1$ then $a(n) \ll_\epsilon n^{(\nu-1)/2+\epsilon}$ for $\epsilon > 0$.

Proof. It is the famous Ramanujan–Petersson's conjecture proved by P. Deligne [3, Théorème (8.2)].

Proposition 1. Suppose that $\nu \geq 2$, $\nu \in \mathbb{Z}$ and let $\varphi(z) \in S_\nu(N,\chi)$ with $\chi(\gamma) = \left[\frac{D}{d}\right]$ for some D in $\mathbb{Z}$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. On writing $\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ we have

$$a(n) \ll_{N,\epsilon} \frac{(4\pi)^{(\nu-1)/2}\nu^{1/2}}{\Gamma(\nu-1)^{1/2}}\, n^{\frac{\nu-1}{2}+\epsilon}\|\varphi\| \text{ for } \epsilon > 0 \text{ and } (n,N) = 1.$$

Proof. We fix an orthogonal basis $\{\varphi_j \mid 1 \leq j \leq g\}$, $g = \dim S_\nu(N,\chi)$, consisting of common eigenfunctions for the set of Hecke operators T_p with $p \nmid N$ and normalise φ_j, $1 \leq j \leq g$, so that the first non–zero Fourier coefficient of φ_j is equal to 1. Let

$\varphi_j(z) = \sum_{n=1}^{\infty} a_j(n)e^{2\pi inz}$ and suppose, without loss of generality, that $a_j(1) = 1$ for $1 \le j \le g_0$ and $a_j(1) = 0$ for $j > g_0$. Then (see, for instance, [22, p. 319]) $a_j(n) = 0$ if $(n,N) = 1$ and $j > g_0$. Write $\varphi = \sum_{j=1}^{g} \beta_j\varphi_j$, so that $a(n) = \sum_{j=1}^{g} \beta_j a_j(n)$ and therefore $|a(n)|^2 \le \sum_{j=1}^{g_0} |\beta_j|^2 \sum_{j=1}^{g_0} |a_j(n)|^2$ for $(n,N) = 1$. Since $g_0 \le g << \nu$ (cf., for instance, Theorem 4.2.1 in [22, p. 102]) it follows from lemma 1 that

$|a(n)|^2 << n^{\nu-1+\epsilon} \nu \sum_{j=1}^{g_0} |\beta_j|^2$ for $(n,N) = 1$. On the other hand,

$$\|\varphi\|^2 = \sum_{j=1}^{g} |\beta_j|^2\|\varphi_j\|^2 \ge \sum_{j=1}^{g_0} |\beta_j|^2\|\varphi_j\|^2 \text{ and}$$

$$\|\varphi_j\|^2 = \int_{\Gamma_0(N)\backslash \mathbb{C}_+} |\varphi_j(z)|^2 y^{\nu-2}dxdy \ge \int_1^{\infty} dy \int_0^1 dx\,|\varphi_j(z)|^2 y^{\nu-2} =$$

$$= \int_1^{\infty} y^{\nu-2}dy \sum_{n=1}^{\infty} |a_j(n)|^2 e^{-4\pi ny} \text{ so that if } 1 \le j \le g_0 \text{ then}$$

$\|\varphi_j\|^2 \ge \int_1^{\infty} y^{\nu-2}e^{-4\pi y}\,dy >> \frac{\Gamma(\nu-1)}{(4\pi)^{\nu-1}}$. Thus $\|\varphi\|^2 >> \frac{\Gamma(\nu-1)}{(4\pi)^{\nu-1}} \sum_{j=1}^{g_0} |\beta_j|^2$, and

therefore $|a(n)|^2 << n^{\nu-1+\epsilon}\nu\|\varphi\|^2 \frac{(4\pi)^{\nu-1}}{\Gamma(\nu-1)}$, as claimed.

To obtain a non–trivial (uniform in the weight) estimate for the coefficients of a cusp–form of half–integral weight one follows a different path. We start with a general lemma.

Lemma 2. Let $\nu \in \mathbb{R}$, $\nu > 2$. Denoting by $\hat{S}_\nu(N,v)$ the Hilbert space of $\Gamma_0(N)$–cusp–forms φ which transform according to the equation $\varphi(\gamma z) = v(\gamma)(cz+d)^\nu \varphi(z)$ for $\gamma \in \Gamma_0(N)$, $z \in \mathbb{C}_+$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we choose an orthonormal basis $\{\varphi_j \mid 1 \le j \le g\}$ of $\hat{S}_\nu(N,v)$ and write $\varphi_j(z) = \sum_{n=1}^{\infty} a_j(n) e^{2\pi i n z}$ for $1 \le j \le g$. The following identity holds:

$$\sum_{j=1}^{g} |a_j(n)|^2 = \frac{(4\pi n)^{\nu-1}}{\Gamma(\nu-1)} \left[1 + 2\pi i^{-\nu} \sum_{\substack{c=0\,(N) \\ c>0}} \frac{K_v(n,c)}{c} J_{\nu-1}\left[\frac{4\pi n}{c}\right]\right],$$

where $K_v(n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \overline{v(\gamma)} \exp(\frac{2\pi i n}{c}(d+d^{-1}))$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is a Kloosterman sum and where $J_{\nu-1}$ denotes a Bessel function. Here it is tacitly assumed that $|v(\gamma)| = 1$.

Proof. We follow [22]. One defines a Poincaré series

$$G_\nu(z,m) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \exp(2\pi i m \gamma z)(cz+d)^{-\nu} v(\gamma)^{-1}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $\Gamma_\infty = \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z}\}$, and proves (see Theorem 5.1.2 in [22, p. 136]) that

$G_\nu(z,m) \in \hat{S}_\nu(N,v)$ for $m > 0$ and that $G_\nu(z,m) = \sum_{n=1}^{\infty} A_m(n) e^{2\pi inz}$ with

$$A_m(n) = \delta_{mn} + 2\pi e^{-i\nu\pi/2} \left[\frac{n}{m}\right]^{\nu-1} \sum_{\substack{c \equiv 0\,(N) \\ c>0}} \frac{W(n,m;c)}{c} J_{\nu-1}\left[\frac{4\pi\sqrt{mn}}{c}\right], \text{ where}$$

$$W(n,m;c) = \sum_{\gamma \in \Gamma_0(N)} \overline{v(\gamma)} \exp\left[2\pi i \frac{ma + nd}{c}\right], \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ (see (5.3.32) in [22, p. 136]).

Furthermore, let $\varphi(z) \in \hat{S}_\nu(N,v)$ and let $\varphi(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi inz}$. By Theorem 5.1.2 in [22, p. 136] (note that due to a different normalization of the inner product $< \cdot \mid \cdot >$, which is defined here by the same equation as in $S_\nu(N,\chi)$, one has to omit the factor μ in this theorem), $<\varphi \mid G(\cdot,m)> = a(m) \frac{\Gamma(\nu-1)}{(4\pi m)^{\nu-1}}$, so that

$$G_\nu(\cdot,m) = \sum_{j=1}^{g} \overline{<\varphi_j \mid G(\cdot,m)>} \varphi_j = \frac{\Gamma(\nu-1)}{(4\pi m)^{\nu-1}} \sum_{j=1}^{g} \overline{a_j(m)}\, \varphi_j \text{ and, in particular,}$$

$$A_m(n) = \frac{\Gamma(\nu-1)}{(4\pi m)^{\nu-1}} \sum_{j=1}^{g} \overline{a_j(m)}\, a_j(n).$$ On letting $m = n$ one obtains the required identity.

<u>Corollary 2</u>. Let $\{\varphi_j \mid 1 \le j \le g\}$ be an orthonormal basis of $S_\nu(N,\chi)$ and let $\varphi_j(z) = \sum_{n=1}^{\infty} a_j(n) e^{2\pi inz}$. Then the following identity holds:

$$\sum_{j=1}^{n} |a_j(n)|^2 = \frac{(4\pi n)^{\nu-1}}{\Gamma(\nu-1)} \left[1 + 2\pi i^{-\nu} \sum_{\substack{c \equiv 0\,(N) \\ c>0}} \frac{K(n,c)}{c} J_{\nu-1}\left[\frac{4\pi n}{c}\right]\right], \text{ where}$$

$$K(n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \overline{\chi(\gamma)} \left[\frac{c}{d}\right]^{2\nu} \epsilon_d^{2\nu} \exp(2\pi i \frac{n(d+d^{-1})}{c}) .$$

Proof. By definition $S_\nu(N,\chi) = \hat{S}_\nu(N,v_\chi)$ with $v_\chi(\gamma) = \chi(\gamma)\epsilon_d^{-2\nu} \left[\frac{c}{d}\right]^{2\nu}$, therefore Corollary 2 is a direct consequence of lemma 2.

Lemma 3. If ψ is a character of $(\mathbb{Z}/_{c\mathbb{Z}})^*$ and $\chi(\gamma) = \psi(d)$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$|K(n,c)| \leq \tau(e)\,(n,c)^{1/2}\, c^{1/2} \text{ with } \tau(c) := \sum_{\alpha | c} 1 .$$

Proof. It is a well–known theorem of A. Weil's, [30].

Definition 1. We let $\mathcal{N} = \{pN \,|\, P < p \leq 2P,\ p \nmid 2n\}$, where p ranges over all the rational primes, $n \in \mathbb{Z}$, $n > 0$, $P > 0$.

The following lemma, due to H. Iwaniec, takes account of cancellations in a sum of Kloosterman sums on average. Before stating it we define three sums.

Definition 2. Let $K_0(n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \epsilon_d^{-2\nu}\left[\frac{c}{d}\right]\exp\left[\frac{2\pi i n(d+d^{-1})}{c}\right]$ and let

$$K(n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \epsilon_d^{-2\nu}\left[\frac{c}{d}\right]\left[\frac{2D}{d}\right]\exp\left[\frac{2\pi i n(d+d^{-1})}{c}\right]. \text{ Finally let}$$

$$A(n,x;P) = (xP^{-1/2} + xn^{-1/2} + (x+n)^{5/6}(x^{1/4}P^{3/8} + n^{1/8}x^{1/8}P^{1/4}))\cdot\tau(n)(\log n\, x)^2 ,$$

where $\tau(n) = \sum_{a|n} 1$.

Lemma 4. Suppose that $\nu = \frac{1}{2} + \ell$, $\ell \geq 2$, $\ell \in \mathbb{Z}$. If n is square–free and $N = 0(8)$

then $\sum_{Q \in \mathcal{N}} \left| \sum_{\substack{c=0(Q) \\ 1 \leq c \leq x}} c^{-1/2} K_0(n,c) \right| << A(n,x;P)$.

Proof. It is the theorem 3 in [8, p. 399].

Lemma 5. Suppose that $\nu = \frac{1}{2} + \ell$, $\ell \in \mathbb{Z}$, $\ell \geq 2$. If n is square–free, $8 | N$, $2D | N$,

and $(n,N) = 1$ then $\sum_{Q \in \mathcal{N}} \left| \sum_{\substack{c=0(Q) \\ 1 \leq c \leq x}} c^{-1/2} K(n,c) \right| \underset{N}{<<} A(n,x,P)$.

Proof. Since $\epsilon_d^2 = \left[\frac{-1}{d}\right]$ we may write

$$K(n,c) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \epsilon_d^{-2\nu} \left[\frac{c}{d}\right] \left[\frac{-2D}{d}\right] e^{2\pi i \frac{n(d+d^{-1})}{c}}$$. If $2D | c$ we have

$$K(n,c) = \frac{1}{2D} \sum_{\substack{d \bmod (2Dc) \\ (d,2Dc)=1}} \epsilon_d^{-2(\nu+1)} \left[\frac{-2D}{d}\right] \exp\left[\frac{2\pi i(2Dn)(d+d^{-1})}{2Dc}\right]$$, so that

$K(n,c) = \frac{1}{2D} K_0(2Dn,2Dc)$. Without loss of generality it may be assumed that 2D is square–free; conditions $(n,N) = 1$ and $2D | N$ imply then that 2Dn is square–free. Thus lemma 5 may be deduced from lemma 4.

Lemma 5 is our main tool in estimating the coefficients of a cusp–form of half–integral weight larger than 2.

Definition 3. We let $\chi(d) = \left[\frac{2D}{d}\right]$ and fix an orthonormal basis $\{\varphi_j \mid 1 \le j \le g\}$ of $S_\nu(N,\chi)$; let $\varphi_j(z) = \sum_{n=1}^{\infty} a_j(n)e^{2\pi inz}$ for $1 \le j \le g$. We write, for brevity,

$$x_\nu(n,c) := c^{-1}K(n,c)J_{\nu-1}\left[\frac{4\pi n}{c}\right] .$$

Lemma 6. For $P > (4 \log 2n)^2$ we have

$$\frac{\Gamma(\nu-1)}{(4\pi n)^{\nu-1}} \sum_{j=1}^{g} |a_j(n)|^2 << P + (\log P) \sum_{Q\in \mathcal{N}} \Big| \sum_{\substack{c=0\,(Q)\\ c>0}} x_\nu(n,c) \Big| .$$

Proof. Let $Q = pN$ and let $b(p) = [\Gamma_0(Q) : \Gamma_0(N)]$. For $f,g \in S_\nu(N,\chi)$ one has $<f|g>_{S_\nu(Q,\chi)} = <f|g>\, b(p)$. In particular, let $\{\psi_j \mid 1 \le j \le g(p)\}$ be an orthonormal basis of $S_\nu(Q,\chi)$ such that $\psi_j = b(p)^{-1/2}\varphi_j$ for $1 \le j \le g$ and let $\psi_j(z) = \sum_{n=1}^{\infty} \tilde{a}_j(n)e^{2\pi inz}$. Then $\tilde{a}_j(n) = b(p)^{-1/2}a_j(n)$ for $1 \le j \le g$, and clearly $g \le g(p)$. Since $b(p) \le p+1$ it follows that $\sum_{j=1}^{g} |a_j(n)|^2 \le (p+1) \sum_{j=1}^{g(p)} |\tilde{a}_j(n)|^2$.

Therefore corollary 2 gives: $\frac{1}{p+1} \cdot \frac{\Gamma(\nu-1)}{(4\pi n)^{\nu-1}} \sum_{j=1}^{g} |a_j(n)|^2 \le 1 + 2\pi \Big| \sum_{\substack{c=0\,(Q)\\ c>0}} x_\nu(n,c) \Big|$.

Since $\sum_{P<p\leq 2P} \frac{1}{p+1} >> \frac{1}{\log P}$ and $\sum_{P<p\leq 2P} 1 << \frac{P}{\log P}$, on summing the above inequality over $\mathcal{N}$ we complete the proof.

Lemma 7. In the conditions of lemma 5 the following estimate holds:

$$\frac{\Gamma(\nu-1)}{(4\pi n)^{\nu-1}} \sum_{j=1}^{g} |a_j(n)|^2 \ll_{N,\epsilon} \nu^{3/2} n^{1/2-1/48+\epsilon} \quad \text{for } \epsilon > 0 .$$

Proof. Consider two sums:

$$S_1 = \sum_{Q\in\mathcal{N}} \Big| \sum_{\substack{c=0(N)\\ c>n^{1-\gamma}}} x_\nu(n,c) \Big| \text{ and } S_2 = \sum_{Q\in\mathcal{N}} \Big| \sum_{\substack{c=0(Q)\\ 0<c\leq n^{1-\gamma}}} x_\nu(n,c) \Big| .$$

Let $P = n^{1/8}$ and choose $\gamma = \frac{1}{48}$, so that $n^\gamma < P < n$. By lemma 6, $\frac{\Gamma(\nu-1)}{(4\pi n)^{\nu-1}} \sum_{j=1}^{g} |a_j(n)|^2 << P + (\log P)(S_1 + S_2)$. To estimate S_1 we use the following identity:

$$\sum_{\substack{c=0(Q)\\ c>y}} a(c)f(c) = -A_Q(y)f(y) - \int_y^\infty f'(x)A_Q(x)dx , \tag{1}$$

where $A_Q(x) := \sum_{\substack{0<c<x\\ c=0(Q)}} a(c)$ and it is assumed that $f(y)A_Q(y) \xrightarrow[y\to\infty]{} 0$. Let $a(c) = c^{-1/2}K(n,c)$ and let $f(x) = x^{-1/2}J_{\nu-1}\left[\frac{4\pi n}{x}\right]$, so that $x_\nu(n,c) = a(c)f(c)$ and it

follows from (1) that $|S_1| \le |f(y)| \sum_{Q\in\mathcal{N}} |A_Q(y)| + \int_y^\infty |f'(x)| \sum_{Q\in\mathcal{N}} |A_Q(x)|dx$ with $y = n^{1-\gamma}$. In view of the relation $\max_{-1\le x\le 1} |C_m^{3/2}(x)| = \frac{(m+2)(m+1)}{2}$, [14, p. 225], one obtains from the integral representation of the Bessel functions, [14, p. 80] a relation

$$J_{\nu-1}(z) = (2\pi)^{-1/2} z^{3/2} \int_0^\pi e^{iz\cos t} b_m(\cos t)dt \quad \text{with} \quad \max_{-1\le t\le 1} |b_m(t)| \le \tfrac{1}{2}$$

(here C_m^λ denotes a Gegenbauer polynomial), provided $\nu-1 = \frac{3}{2} + m$, $m \in \mathbb{Z}$, and $m \ge 0$. It follows therefore that $|f(y)| << n^{-1/2+2\gamma}$ and $|f'(x)| << n^{3/2}(1 + nx^{-1})x^{-3}$. Combining these estimates with the estimate $\sum_{Q\in\mathcal{N}} |A_Q(x)| << A(n,x,P)$ given in lemma 5 one obtains: $S_1 \lessapprox_{\epsilon}< n^{1/2-1/48+\epsilon}$ for $\epsilon > 0$. To estimate S_2 one notes (cf. [6, p. 61]) that $J_{\nu-1}(z) << \nu^{3/2} z^{-1/2}$ for $z \ge 1$, $\nu = \frac{3}{2} + \ell$, $\ell \in \mathbb{Z}$, $\ell \ge 0$ and therefore, in view of lemma 3, $x_\nu(n,c) << \tau(c)(n,c)^{1/2}\nu^{3/2}$. This results in the estimate $S_2 \lessapprox_{\epsilon,N} n^{1/2-\gamma+\epsilon}$ and completes the proof of lemma 7.

<u>Proposition 2</u>. Let $\varphi \in S_\nu(N,\chi)$, $\nu = \frac{1}{2} + \lambda$, $\lambda \in \mathbb{Z}$, $\lambda \ge 2$ and suppose that $8|N$, $2D|N$ and $(n,N) = 1$; let $\varphi(z) = \sum_{n=1}^\infty a(n)e^{2\pi inz}$ for $z \in \mathbb{C}_+$. Then

$$a(n) \lessapprox_{\epsilon,N} (4\pi)^{(\nu-1)/2}\Gamma(\nu-1)^{-1/2} \cdot \nu^{3/4} n^{\nu/2-1/4-1/96+\epsilon}\|\varphi\| .$$

<u>Proof</u>. Let $\{\varphi_j | 1 \le j \le g\}$ be an orthonormal basis of $S_\nu(N,\chi)$ consisting of common eigenfunctions for the set of Hecke operators $\{T(p^2) | p \nmid N\}$ and let Φ_t^j, $1 \le j \le g$, be

the Shimura's lift of φ_j, [25] (here, as usual, p ranges over all the rational primes). On writing $\Phi_t^j(z) = \sum_{n=1}^{\infty} A_t^j(n)e^{2\pi inz}$ one obtains $a_j(tn^2) = \sum_{d|n} \mu(d)\chi_t(d)d^{\lambda-1}A_t^j\left[\frac{n}{d}\right]$, since by construction $\sum_{n=1}^{\infty} A_t^j(n)n^{-s} = L(s-\lambda+1,\chi_t) \sum_{n=1}^{\infty} a_j(tn^2)n^{-s}$ with

$\chi_t(m) = \left[\frac{t}{m}\right]\left[\frac{-1}{m}\right]^{\lambda}\chi(m)$, where $L(s,\psi) := \sum_{m=1}^{\infty} \frac{\psi(m)}{m^s}$ and where

$\varphi_j(z) = \sum_{j=1}^{\infty} a_j(n)e^{2\pi inz}$. By [2, Theorem 4.3], we have $\Phi_t^j \in S_{2\lambda}(N,\chi^2)$; moreover, by [2, Proposition 5.1], Φ_t^j is a common eigen–function for the set of Hecke operators $\{T_p \mid p \nmid N\}$. Therefore it follows from lemma 1 that

$$|a_j(tn^2)| \le \sum_{d|n} d^{\lambda-1}\left|A_t^j\left[\frac{n}{d}\right]\right| \le |A_t^j(1)| \sum_{d|n} d^{\lambda-1}\tau(n)\left[\frac{n}{d}\right]^{\lambda-\frac{1}{2}} \ll_{\epsilon} |A_t^j(1)|\, n^{\nu-1+\epsilon}$$

for $\epsilon > 0$. But $A_t^j(1) = a_j(t)$, therefore we have: $a_j(tn^2) \ll_{\epsilon} |a_j(t)|(n^2)^{(\nu-1)/2+\epsilon}$ for $\epsilon > 0$. Let $\varphi = \sum_{j=1}^{g} \beta_j\varphi_j$, so that $\|\varphi\|^2 = \sum_{j=1}^{g} |\beta_j|^2$ and $a(n) = \sum_{j=1}^{g} \beta_j a_j(n)$, and in particular $|a(n)|^2 \le \sum_{j=1}^{g} |\beta_j|^2 \sum_{j=1}^{g} |a_j(n)|^2 = \|\varphi\|^2 \sum_{j=1}^{g} |a_j(n)|^2$. Let $n = tm^2$ with a square–free t, then $\sum_{j=1}^{g} |a_j(n)|^2 \ll (m^2)^{\nu-1+\epsilon} \sum_{j=1}^{g} |a_j(t)|^2$. Combining these estimates with lemma 7 we conclude the proof.

To derive an estimate for coefficients of a cusp–form of weight $3/2$ one argues as follows, [4].(*

Lemma 8. Let $\varphi \in S_{3/2}(N,\chi)$, let $\|\varphi\| = 1$ and suppose that $\varphi(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$. We have $a(n) \ll_{N,\epsilon} n^{1/2-1/28+\epsilon}$ for a square–free n assuming, as always, that $(n,N) = 1$ and $2D|N$, $8|N$, $\epsilon > 0$.

Proof. On choosing in the Kuznetsov's sum formula [4, p. 80]

$$\sum_{\substack{c>0 \\ c\equiv 0\,(N)}} c^{-1} K(N,c) u\left[\frac{4\pi n}{c}\right] = \sum_{\ell=1}^{3} V_\ell(n),$$

where $V_1(n) = 4n \sum_{\lambda_j>0} |\rho_j(n)|^2 \hat{u}(t_j)(\mathrm{ch}\,\pi t_j)^{-1}$, $t_j := \sqrt{\lambda_j - 1/4}$,

$$V_2(n) = \sum_{j=1}^{h} \int_{-\infty}^{\infty} \frac{|\varphi_{jn}(1/2+it)|^2 \hat{u}(t)}{(\mathrm{ch}\,\pi t)|\Gamma(5/4+it)|^2} dt,$$

and $V_3(n) = 4 \sum_{j=1}^{\infty} \frac{\Gamma(3/2+2j)\tilde{u}(3/2+2j)}{(4\pi)^{3/2+2j} n^{1/2+2j}} \sum_{i=1}^{g_j} |a_{ij}(n)|^2 (-1)^j \exp(3\pi i/4)$,

the test function $u(x) = x^{-3/2} J_{13/2}(x) e^{-3\pi i/4}$ one observes that $V_2(n) \geq 0$ and that all

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(* The argument suggested in [4] requires a modification. We are indebted to Professor W. Duke for indicating how it can be best done.

the terms in the sum $V_1(n)$ are positive, moreover, all the terms in the sum $V_3(n)$ with $j \geq 4$ are positive as well. By definition (see [4, p. 78–79]), $a(n) = (4\pi n)^{3/4}\rho_j(n)$ for $\lambda_j = 3/16$; therefore it follows that

$$n^{-1/2}|a(n)|^2 << \Big| \sum_{\substack{c>0 \\ c=0\,(N)}} c^{-1}K(n,c)J_{13/2}\left[\frac{4\pi n}{c}\right]\left[\frac{4\pi n}{c}\right]^{-3/2} \Big| +$$

$$\sum_{1\leq j\leq 3} \frac{\Gamma(1/2+2j)}{(4\pi n)^{1/2+2j}} \sum_{1\leq i\leq g_j} |a_{ij}(n)|^2 .$$ Here $\{\varphi_{ij} | 1 \leq i \leq g_j\}$ is an orthonormal basis for $S_{3/2+2j}(N,\chi)$ and $\varphi_{ij}(z) = \sum_{n=1}^{\infty} a_{ij}(n)e^{2\pi inz}$. Applying corollary 2 and summing over $\mathscr{N}$ one obtains

$$n^{-1/2}|a(n)|^2 \sum_{Q\in\mathscr{N}} \frac{1}{[\Gamma_0(N):\Gamma_0(Q)]} << \sum_{Q\in\mathscr{N}} \Big| \sum_{\substack{c>0 \\ c=0\,(Q)}} c^{-1}K(n,c)J_{13/2}\left[\frac{4\pi n}{c}\right]\left[\frac{4\pi n}{c}\right]^{-3/2} \Big|$$

$$+ \sum_{1\leq j\leq 3} \sum_{Q\in\mathscr{N}} \Big[1 + \Big| \sum_{\substack{c=0\,(Q) \\ c>0}} x_{3/2+2j}(n,c) \Big| \Big] .$$ To apply the argument used in [8, § 8] one requires lemma 5 for the sums

$$\sum_{Q\in\mathscr{N}} \Big| \sum_{\substack{c=0\,(Q) \\ c>0}} c^{-1/2}K(n,c)\exp\left[\frac{4\pi i\nu n}{c}\right] \Big| , \quad \nu \in \{-1,0,1\} , \tag{2}$$

and the estimate $(x\, J_{13/2}\left[\frac{4\pi n}{x}\right])' n^{-3/2} << nx^{-5/2}$ for $n < x$. Both are straightforward and require no further comments. This completes the proof.

Proposition 3. Let $\varphi \in S_{3/2}(N,\chi)$ and suppose that $8|N$ and $2D|N$; let

$\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$. Then $a(n) \underset{\varphi,\epsilon}{<<} n^{1/2-1/28+\epsilon}$ for $(n,N) = 1$, $\epsilon > 0$.

Proof. Since $\Phi_t(z) = \sum_{n=1}^{\infty} A_t(n)e^{2\pi i n z} \in S_2(N,\chi^2)$ for $t \nmid N$ (see [2, Corollary 4.8]), where $\sum_{n=1}^{\infty} A_t(n)n^{-s} = L(s,\chi_t) \sum_{n=1}^{\infty} a(tn^2)n^{-s}$ with $\chi_t(m) = \left[\frac{t}{m}\right]\left[\frac{-1}{m}\right]\chi(m)$, we have

$a(tn^2) = \sum_{d|n} \mu(d)\chi_t(d)A_t\left[\frac{n}{d}\right]$. By lemma 1, $|A_t\left[\frac{n}{d}\right]| \le |A_t(1)| \left[\frac{n}{d}\right]^{1/2+\epsilon}$; on the other

hand, $|A_t(1)| = |a(t)| << t^{1/2-1/28+\epsilon}$ by lemma 8 (since t is assumed to be square–free). Thus $|a(tn^2)| << t^{1/2-1/28+\epsilon}n^{1/2+\epsilon} << (tn^2)^{1/2-1/28+\epsilon}$, as required.

§ 4. On representation of integers by positive definite quadratic forms

We prove here theorems 1–3 and corollary 1. Let us remark first that the identity $r(\text{gen } f,n) = n^{s/2-1}\alpha_\infty(f)A_f(n)$ in theorem 1 is due to C.L. Siegel, [23]; the estimate $A_f(n) \underset{f,\epsilon}{<<} n^{\epsilon}$ for $\epsilon > 0$ is elementary (cf. also [23]); the estimate $A_f(n) \underset{f,\epsilon}{>>} n^{-\epsilon}$ for $\epsilon > 0$ can be found in [15], [16] for $s \geq 4$ and in [20, Satz (3.1)] for $s = 3$ (under the conditions stated in theorem 1). This completes the proof of theorem 1. The estimates given in theorem 2 can be deduced as follows. One remarks first that on defining $a(n) = r_f(n) - r(\text{gen } f,n)$ and $\varphi(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ we get $\varphi(z) \in S_\nu(2N,\chi)$, where $\nu = s/2$, $s \geq 4$, and where N denotes the level of f (cf. [26, p. 283] and [24, p. 376]).

If $2\,|\,s$ theorem 2 follows now from lemma 1. If $2 \nmid s$ we need a variant of proposition 2 asserting that $a(n) \underset{f,\epsilon}{<<} n^{\nu/2-2/7+\epsilon}$ for $\epsilon > 0$ as soon as the conditions of this proposition are satisfied; such a statement can be proved along the lines of [8, § 8] as soon as one has an analogue of lemma 5 for the sums (2) alluded to in § 3 (at the end of the proof of lemma 8). This completes the proof of theorem 2. Finally we note that theorem 3 follows from proposition 3 in view of [26, Korollar 2 and Korollar 3].

Proposition 4. Let $f(x) = x_1^2 + x_2^2 + p^3 x_3^2$ and suppose that $p = 5(\text{mod } 8)$. Then $r_f(n) \underset{\epsilon}{>>} n^{1/2-\epsilon}$ for $\epsilon > 0$ as soon as $n = 7(8)$.

Proof. Let $n = p^{\ell} n_1$, $p \nmid n_1$ and suppose that $n = 7(8)$. If $\ell \geq 3$ the integer $n_2 = p^{-3} n$ is congruent to 3 modulo 8 and therefore $\#\{y \mid y \in \mathbb{Z}^3,\ n_2 = y_1^2 + y_2^2 + y_3^2\} >> n_2^{1/2-\epsilon}$ for $\epsilon > 0$. If $\ell < 3$ it follows from theorem 1 and theorem 2 that $r_g(n_1) >> n_1^{1/2-\epsilon}$ for $\epsilon > 0$, $g_1(x) = x_1^2 + x_2^2 + p^{3-\ell} x_3$. Since $p = 5(8)$ equation $p = z_1^2 + z_2^2$ is solvable in $\mathbb{Z}^2$. The required estimate follows from these observations when one writes $x_1^2 + x_2^2 = p^3(n_2 - y_3^2)$ when $\ell \geq 3$ and $x_1^2 + x_2^2 = p^{\ell}(n_1 - p^{3-\ell} y_3^2)$ when $\ell < 3$ thereby noting that to each solution of the equations $n_2 = y_1^2 + y_2^2 + y_3^2$, $p^3 = z_1^2 + z_2^2$ when $\ell \geq 3$ and $n_1 = y_1^2 + y_2^2 + p^{3-\ell} y_3^2$, $p^{\ell} = z_1^2 + z_2^2$ when $\ell < 3$ corresponds a unique solution of the equation $n = f(x)$ (assuming $x \in \mathbb{Z}^3$, $y \in \mathbb{Z}^3$, $z \in \mathbb{Z}^2$).

Corollary 1 is an immediate consequence of proposition 4.

§ 5. On equidistribution of integral points on an ellipsoid

In this section we prove theorem 4. Let us recall the spectral decomposition theorem for the Laplace operator Δ on S_ℓ, $\ell \geq 2$ (cf., for instance, [18]). One may write

$$L^2(S_\ell) = \sum_{m=0}^{\infty} \oplus \mathscr{H}_m \text{ , where } \Delta_{|\mathscr{H}_m} = m(m+\ell-1)I, \ h_m = \frac{2m+\ell-1}{m}\begin{bmatrix} m+\ell-2 \\ m-1 \end{bmatrix} \text{ and}$$

where I denotes the identical operator, $h_m := \dim \mathscr{H}_m$ for $m \geq 1$, $\dim \mathscr{H}_0 = 1$. On choosing an orthonormal basis $\{\sigma_m^j \mid 1 \leq j \leq h_m\}$ of $\mathscr{H}_m$ one obtains the Gegenbauer polynomials, or ultraspherical harmonics C_m^λ given as follows:

$$\sum_{1\leq j\leq m} \sigma_m^j(y_1)\overline{\sigma_m^j(y_2)} = C_m^{(\ell-1)/2}(y_1'y_2)\, \frac{2m+\ell-1}{\ell-1}, \text{ where } y_i \in S_\ell,\ i = 1,2 .$$

Here are a few basic properties of the polynomials $C_m^\lambda(t)$ (cf. [14. § 5.3]): $C_m^\lambda(t)$ is a polynomial of

degree m and $C_m^\lambda(-t) = (-1)^m C_m^\lambda(t)$; $C_m^\lambda(t) \in \mathbb{R}[t]$;

$$\max_{-1\leq t\leq 1} |C_m^\lambda(t)| = \begin{bmatrix} m+2\lambda-1 \\ m \end{bmatrix} \text{ , and } C_m^\lambda(\cos\theta) << \theta^{-\lambda} m^{\lambda-1} \text{ for } 0 \leq \theta \leq \tfrac{\pi}{2} .$$

Lemma 9. Let $\ell \geq 2$ and $w \subseteq S_\ell$. Suppose that w satisfies conditions of theorem 4. Given a sufficiently small positive δ in $\mathbb{R}$ there is a function $\chi_\delta : S_\ell \longrightarrow [0,1]$ satisfying the following conditions: $\chi_\delta(y) = \begin{cases} 1 & \text{for } y \in w \\ 0 & \text{for } y \notin w \end{cases}$ when $|y-\partial w| > \delta$;

$$\chi_\delta = \sum_{m=0}^{\infty} H_m \text{ with } H_m \in \mathscr{H}_m,\ \sup_{x\in S_\ell} |H_m(x)| <<_{\alpha, w} \frac{m^{(\ell-3)/2}}{(m\delta)^{2\alpha-1}} \text{ for } \alpha \geq 1,\ \alpha \in \mathbb{Z} .$$

Proof. Choose a function $\varphi_\delta : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi_\delta \in C^\infty(\mathbb{R})$, $\varphi_\delta(t) \geq 0$ and $\varphi_\delta'(t) \leq 0$ for $t \in \mathbb{R}$, $\varphi_\delta(t) = 0$ for $t > \delta$, and $\int_{S_\ell} \varphi_\delta(|x-y|)d\mu(y) = 1$. Assuming $0 < \delta < \frac{1}{2}$ let

$\chi_\delta(y) = \int_{S_\ell} \chi(x)\varphi_\delta(|x-y|)d\mu(x)$ for $y \in S_\ell$, where $\chi(x) = \begin{cases} 0, & x \notin w \\ 1, & x \in w \end{cases}$ is the characteristic function of w . Clearly, $0 \leq \chi_\delta(y) \leq 1$ for $y \in S_\ell$ and $\chi_\delta(y) = \chi(y)$ when $|y-\partial w| > \delta$. Write $\chi_\delta = \sum_{m=0}^{\infty} H_m$ with $H_m = \sum_{j=1}^{h_m} a(j,m)\sigma_m^j$, where a(j,m) is given by the equation $a(j,m) = \int_{S_\ell} \chi_\delta(y)\, \overline{\sigma_m^j(y)}\, d\mu(y)$, or

$$a(j,m) = \frac{1}{[m(m+\ell-1)]^{\alpha}} \int_{S_\ell} \chi_\delta(y)\, (\overline{\Delta^\alpha \sigma_m^j})\, (y)d\mu(y)$$ for $\alpha \geq 1$, $\alpha \in \mathbb{Z}$. Since Δ is self–adjoint in $L^2(S_\ell)$, it follows that

$$H_m(y) = \frac{2m+\ell-1}{(m(m+\ell-1))^{\alpha}(\ell-1)} \int_{S_\ell} (\Delta^\alpha \chi_\delta)(y_1) C_m^{(\ell-1)/2}(y'y_1)d\mu(y_1).$$ By construction,

$$\sup_{y \in S_\ell} |(\Delta^\alpha \chi_\delta)(y)| \ll_{\alpha,w} \delta^{-2\alpha}$$ and $(\Delta^\alpha \chi_\delta)(y) = 0$ for $|y-\partial w| > \delta$, $\alpha \geq 1$. Therefore

$$H_m(y) \ll_{\alpha,w} \frac{m^{(\ell-1)/2}}{(m\delta)^{2\alpha}} \int_{|y_1-\partial w| \leq \delta} \theta^{-(\ell-1)/2} d\mu(y_1),$$ where $y'y_1 = \cos\theta$, $|\theta| \leq \pi/2$,

since $C_m^{(\ell-1)/2} \ll \theta^{-(\ell-1)/2} m^{(\ell-3)/2}$. Denoting by μ_1 the measure on ∂w induced by μ we obtain for a sufficiently small positive δ:

$$\int_{|y_1-\partial w| \leq \delta} \theta^{-(\ell-1)/2} d\mu(y_1) \ll \delta \int_{\partial w} \theta^{-(\ell-1)/2} d\mu_1(y_1) \ll \delta,$$ since ∂w is a smooth submanifold of S_ℓ of dimension $\ell-1$ and $\ell \geq 2$. This gives the required estimate for H_m and concludes the proof.

<u>Definition 4</u>. Given a sufficiently small positive δ let $\mathcal{O}_\delta(A) = \{y \mid y \in S_\ell, |y-A| < \delta\}$ for $A \subseteq S_\ell$, and let $w_+^\delta = w \cup \mathcal{O}_\delta(\partial w)$,

$w_-^\delta = w \setminus \mathcal{O}_\delta(\partial w)$. We denote temporarily by $\hat{H}_m(\delta,w) := H_m$ the functions appearing in lemma 1 and let $H_m^\pm = \hat{H}_m(\delta, w_\pm^\delta)$; we write $\chi_\delta^\pm = \sum_{m=0}^{\infty} H_m^\pm$.

Lemma 10. We have $r_f(n,w) = \mu(w) r_f(n) + O_{w,\epsilon}(\delta r_f(n)) + O(I)$, where

$I = \max_{e \in \{-,+\}} \left| \sum_{\substack{1 \le m \le \delta^{-1-\epsilon} \\ f(x)=n ,\ x \in \mathbb{Z}^s}} H_m^e \left[\frac{Bx}{\sqrt{n}}\right] \right|$, $\epsilon > 0$, assuming $\ell \ge 2$ and w satisfies conditions of theorem 4 (here $\ell = s-1$).

Proof. Clearly,

$$\sum_{\substack{f(x)=n \\ x \in \mathbb{Z}^s}} \chi_\delta^- \left[\frac{Bx}{\sqrt{n}}\right] \le r_f(n,w) \le \sum_{\substack{f(x)=n \\ x \in \mathbb{Z}^s}} \chi_\delta^+ \left[\frac{Bx}{\sqrt{n}}\right] ,$$

and $H_0^\pm = \int_{S_\ell} \chi_\delta^\pm \, d\mu = \mu(w) + O(\delta)$. By lemma 9, $\sum_{\substack{f(x)=n \\ x \in \mathbb{Z}^s}} \chi_\delta^- \left[\frac{Bx}{\sqrt{n}}\right] = I_<^\pm + I_>^\pm$, where

$I_<^\pm = \sum_{0 \le m \le \delta^{-1-\epsilon}} C_m^\pm$ and $I_>^\pm = \sum_{m \le \delta^{-1-\epsilon}} C_m^\pm$, $C_m^\pm := \sum_{\substack{f(x)=n \\ x \in \mathbb{Z}^s}} H_m^\pm \left[\frac{Bx}{\sqrt{n}}\right]$. In particular,

$I_<^\pm = \mu(w) r_f(n) + O(\delta r_f(n)) + O(I)$. It follows from lemma 9 that

$I_>^\pm \ll_{\alpha, w} r_f(n) \sum_{m \le \delta^{-1-\epsilon}} m^{(\ell-3)/2} (m\delta)^{1-2\alpha}$, and therefore

$I_{>}^{\pm} \underset{\alpha, w}{<<} r_f(n)\delta^{2\alpha\epsilon-(\ell-1)(1+\epsilon)/2} \underset{\epsilon, w}{<<} \delta r_f(n)$ for sufficiently large α since $\epsilon > 0$. This completes the proof.

Definition 5. We let $c_m^{\pm}(n) = n^{m/2} \sum_{\substack{f(x)=n \\ x\in\mathbb{Z}^s}} H_m^{\pm}\left[\frac{Bx}{\sqrt{n}}\right]$ and $\theta_m^{\pm}(z) = \sum_{n=1}^{\infty} c_m^{\pm}(n)e^{2\pi inz}$ for $z \in \mathbb{C}_+$; write, for brevity, $k = s/2$, $\ell = s-1$.

Lemma 11. Let $\chi(\gamma) = \left[\frac{D}{d}\right]\left[\frac{2}{d}\right]^s$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\gamma \in SL_2(\mathbb{Z})$ and suppose that $m \geq 1$.

Then $\theta_m^{\pm} \in S_{k+m}(N,\chi)$ and $\|\theta_m^{\pm}\|^2 \underset{f,N, w,\epsilon}{<<} \frac{\Gamma(k+m-1)\, m^{k+\epsilon}}{(4\pi)^{k+m-1}} \left[\frac{m^{k-2}}{(m\delta)^{2\alpha-1}}\right]$ for $\epsilon > 0$, as soon as $2D \mid N$.

Proof. By definition, $H_m^{\pm}(y) = \frac{2m+\ell-1}{\ell-1} \int_{S_\ell} C_m^{(\ell-1)/2}(y'y_1)\chi_\delta^{\pm}(y_1)d\mu(y_1)$. On writing $\frac{2m+\ell-1}{\ell-1} C_m^{(\ell-1)/2}(t) = \sum_{0\leq j\leq m} a_j t^j$ one remarks that

$$H_m^{\pm}\left[\frac{Bx}{\sqrt{n}}\right] n^{m/2} = \sum_{\substack{0\leq j\leq m \\ j\equiv m(2)}} a_j \int_{S_\ell} \chi_\delta^{\pm}(y_1)(y_1'Bx)^j |Bx|^{m-j} d\mu(y_1)$$. Therefore, by

definition 5, $\theta_m^{\pm}(z) = \sum_{x\in\mathbb{Z}^s} P_m(x)e^{2\pi if(x)z}$, where $P_m(x)$ is a homogeneous polynomial in $\mathbb{C}[x]$ of degree m . Thus $\theta_m^{\pm} \in S_{k+m}(N,\chi)$ for $m \geq 1$, [21] (or Proposition 2.1 in [25, p. 456]). Now it follows that $\|\theta_m^{\pm}\|^2 = \int_{\Gamma_0(N)\backslash\mathbb{C}_+} |\theta_m^{\pm}(z)|^2 y^{k+m-2} dxdy$. We cover a

$\Gamma_0(N)$–fundamental domain by $\bigcup_b V_b$, where b varies over a complete set of $\Gamma_0(N)$–inequivalent cusps of $\Gamma_0(N)$ and where V_b denotes a neighbourhood of b that can be transformed to a subset of $\{z \mid z = x+iy,\ 0 \le x \le 1,\ y \ge 1/2\}$ by an $SL_2(\mathbb{Z})$–transformation sending b to $i\infty$. Since under $SL_2(\mathbb{Z})$–transformation the series $\theta_m^{\pm}$ is turned to a linear combination of "partial θ–series" of the shape

$$\theta_{m,a}^{\pm}(z) = \sum_{n=1}^{\infty} c_{m,a}^{\pm}(n)\ell^{2\pi i n z}\ , \text{ where } c_{m,a}^{\pm} = n^{m/2} \sum_{\substack{f(x)=n \\ x=a(N)}} H_m^{\pm}\left[\frac{Bx}{\sqrt{n}}\right],\ a \in \mathbb{Z}^s\ ,\ [21], \text{it}$$

follows that

$$\|\theta_m^{\pm}\|^2 \underset{N}{<<} \sum_{\substack{a \bmod N \\ a\ \in\ \mathbb{Z}^s}} \int_0^1 dx \int_{1/2}^{\infty} y^{m+k-2} dy \sum_{1 \le n_1, n_2 < \infty} c_{m,a}^{\pm}(n_1) c_{m,a}^{\pm}(n_2) e^{2\pi i(n_1 z - n_2 \bar{z})}$$

$$= \sum_{\substack{a \bmod N \\ a\ \in\ \mathbb{Z}^s}} \int_{1/2}^{\infty} y^{m+k-2} \sum_{n=1}^{\infty} |c_{m,a}^{\pm}(n)|^2 e^{-4\pi n y} dy\ .$$ By lemma 9,

$$c_{m,a}^{\pm} \underset{w,\alpha}{<<} r_f(n) n^{m/2} \frac{m^{k-2}}{(m\delta)^{2\alpha-1}}\ .$$ On the other hand, $r_f(n) \underset{f,\epsilon}{<<} n^{k-1+\epsilon}$ for $\epsilon > 0$ and

$$\int_{1/2}^{\infty} y^{m+k-2} e^{-4\pi n y} dy = \frac{1}{(4\pi n)^{m+k-1}} \int_{2\pi n}^{\infty} y^{m+k-2} e^{-y} dy\ .$$ Thus

$$\|\theta_m^{\pm}\|^2 << \frac{1}{(4\pi n)^{m+k-1}} \left[\frac{m^{k-2}}{(m\delta)^{2\alpha-1}}\right]^2 S_m\ , \text{ where } S_m = \sum_{n=1}^{\infty} n^{k-1+\epsilon} I_n \text{ and}$$

$$I_n := \int_{2\pi n}^{\infty} y^{m+k-2} e^{-y} dy\ .$$ Since $I_n \le \Gamma(m+k-1)$ it follows that

$$S_m = \sum_{1\le n\le m+k-2} n^{k-1+\epsilon} I_n + S_m^{>} << m^{k+\epsilon}\Gamma(m+k-1) \text{ because}$$

$$S_m^{>} = \sum_{n>m+k-2} n^{k-1+\epsilon} I_n$$ is easily seen to be small enough. Thereby we obtain the required estimate.

Lemma 12. Suppose that $2D\,|\,N$, $8\,|\,N$ and $(n,N) = 1$ and let $s \ge 3$. Then
$$c_m^{\pm}(n) << \frac{m^{3/2(k-1)+\epsilon}}{(m\delta)^{2\alpha-1}} n^{(m+k-1)/2+\epsilon}\beta_s(m,n) \text{ for } m \ge 1 \text{, where}$$

$$\beta_s(m,n) = \begin{cases} 1 & \text{when } 2\,|\,s \\ m^{1/4}n^{23/96} & \text{when } 2 \nmid s \end{cases}, \quad \epsilon > 0 .$$

Proof. If $2 \nmid s$ the required estimate is an immediate consequence of lemma 11 and proposition 2. Suppose that $2\,|\,s$, then we can apply proposition 1 and lemma 11 thereby completing the proof.

Proof of theorem 4. It follows from lemma 10 and definition 5 that
$$r_f(n,w) = \mu(w)r_f(n) + O(\delta r_f(n)) + O(I) \text{, where } I = \max_{e\in\{-,+\}} \Big| \sum_{1\le m\le \delta^{-1-\epsilon}} c_m^e(n)n^{-m/2} \Big|$$
for $\epsilon > 0$. By lemma 12 with $\alpha = 1$, we have
$$I << n^{(k-1)2+\nu_s+\epsilon}\delta^{-1} \sum_{1\le m\le\delta^{-1-\epsilon}} m^{3k/2-5/2+\mu_s}$$, where $\nu_s = \mu_s = 0$ when $2\,|\,s$ and

$\nu_s = 1/4-1/96$, $\mu_s = 1/4$ when $2 \nmid s$. Thus $I << n^{(k-1)/2+\nu_s+\epsilon}\,\delta^{3k/2+1/2-\mu_s+\epsilon}$.

On choosing $\delta = n^{-\gamma(s)}$ with $\gamma(s) = \frac{s-2}{3s+2}$ when $2\,|\,s$ and $\gamma(s) = \frac{s-3-1/24}{3(s+1)}$ when

$2 \nmid s$ and recalling that $k = s/2$, $r_f(n) \ll_\epsilon n^{k-1+\epsilon}$ for $\epsilon > 0$, one obtains the required estimate for $r_f(n,w)$.

§ 6. Concluding remarks

The results described here lead to a few questions for further investigation:

(i) how can one weaken (or get rid of) the condition $(n,2D) = 1$ in the theorems 2–4 ?

(ii) to what extent can the error terms be improved?

(iii) can one treat a more general problem of estimating the number $r_f(n;a,m,w) = \{x \mid x \in \mathbb{Z}^s, \frac{Bx}{\sqrt{n}} \in w, x \equiv a \pmod m\}$, where $a \in \mathbb{Z}^s$, $m \in \mathbb{Z}$, $m > 1$, $w \subseteq S_\ell$?

(iv) can the corresponding problems for an indefinite quadratic form be studied by similar methods?

We abstain from any further comments on these problems and refer the reader to the literature cited in this report. It should be noted here, however, that the estimates for the Fourier coefficients of Maass forms obtained in [4] contribute to the solution of the problem (iv) (cf. especially [4, § 4, § 6]), while the work on the exceptional integers of ternary quadratic forms (cf. [26–28] and references therein) is pertinent to the question (i).

After this report had been written we came across a very interesting article: W. Duke, Lattice points on ellipsoids, Seminaire de Théorie des Nombres de Bordeaux le 20 mai 1988, Année 1987–88, Exposé n°37 (7 pages). It throws further light on our topic.

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