ON THE COHOMOLOGY RING OF AN ALGEBRA

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March 1, 1996

To Yu. I. Manin on his 60th birthday

Abstract

We define several versions of the cohomology ring of an associative algebra. These ring structures unify some well known operations from homological algebra and differential geometry. They have some formal resemblance with the quantum multiplication on Floer cohomology of free loop spaces. We discuss some examples, as well as applications to index theorems, characteristic classes and deformations.

0. Introduction.

This paper is a continuation and extension of [NT2].

Let X be a smooth manifold or an affine algebraic variety. Let A be the algebra of smooth or regular functions on X. It is well known that most geometric properties of X can be recovered from the algebra A. By (affine) noncommutative geometry one usually means extending corresponding constructions to the case when A is not necessarily commutative.

Among differential-geometric objects, the easiest ones to generalize to the non-commutative setting are vector fields. A noncommutative vector field is just a

*Partially supported by NSF Grant DMS–9307927 .

derivation of an algebra A. As for noncommutative multivector fields and noncommutative differential forms, the answer has been suggested in [HRK]. Consider the standard cochain complex $C^*(A,A)$ computing the Hochschild cohomology

$$H^*(A,A) = Ext^*_{A\otimes A^o}(A,A)$$

(Section 1) and the standard chain complex computing the Hochschild homology

$$HH_*(A) = H_*(A,A) = Tor_*^{A\otimes A^o}(A,A)$$

(Section 2). It was shown in [HRK] that, if A is the algebra of regular functions on an affine algebraic variety X over a field k of characteristic zero, then

$$HH_i(A) \simeq \Omega^i_{X/k} \tag{0.1}$$

$$H^i(A,A) \simeq \Gamma(X, \wedge^i TX) \tag{0.2}$$

The standard cochain complex $C^*(A,A)$ is a differential graded associative algebra (with the cup product which can also be interpreted as the Yoneda product). It is also a differential graded Lie algebra (with the Gerstenhaber bracket [G]). Under the isomorphism (0.1) these operations turn into the wedge product and the Schouten bracket on multivector fields. The cap product from [CE]

$$C^p(A,A)\otimes C_n(A,A) \to C_{n-p}(A,A) \tag{0.3}$$

becomes a noncommutative analogue of contracting a form with a multivector field.

On the other hand, the standard chain complex $C_*(A,A)$ is not a differential graded algebra unless A is commutative. Instead, there is the exterior product [CE]

$$C_*(A,A)\otimes C_*(B,B) \to C_*(A\otimes B, A\otimes B) \tag{0.4}$$

which induces an isomorphism on homology (for algebras over a field).

In [R] Rinehart defined the differential

$$B : C_*(A,A) \to C_{*+1}(A,A)$$

which commutes with the Hochschild differential and becomes the de Rham differential under the isomorphism (0.1). A systematic study of a noncomutative de Rham theory, or cyclic homology, was undertaken in [C], [FT],[L],[T], etc. One defines the periodic cyclic complex $(CC_*^{per}(A), B+b)$ of an algebra A, and proves that when A is the ring of regular functions on a regular affine variety X over a field k of characteristic zero, the homology of this complex is the de Rham cohomology of X. It is worth mentioning that the correct space of "noncommutative differential forms" is not the Hochschild homology $\mathrm{H}_*(A, A)$ but the space of all Hochschild chains $C_*(A, A)$. The Hochschild differential b itself is an odd noncommutative bivector field and the Hochschild homology is "a space of physical states" $\mathrm{Ker}b/\mathrm{Im}b$. We have already mentioned that there seems to be no way of recovering noncommutative differential forms together with their multiplicative structure. It turns out, however, that if one considers all differential operators on $\Omega^*(X)$ (not just zero order multiplication operators) one gets a ring which has a noncommutative generalization. The construction we are going to present rests upon an idea of Yu.I.Manin ([M]).

Let R be a monoidal category. For an object A of R one calls an object $\underline{\mathrm{End}}\ A$ of R the inner object of endomorphisms of A if there are natural morphisms

$$\underline{\mathrm{End}}\ A \times \underline{\mathrm{End}}\ A \to \underline{\mathrm{End}}\ A \quad ; \quad \underline{\mathrm{End}}\ A \times A \to A \tag{0.5}$$

which are universal and associative in a natural way. In the category of associative algebras inner objects of endomorphisms do not exist (for example, the set of endomorphisms of an algebra is not a linear space). In this paper, however, we will show that, if one takes for $\underline{\mathrm{End}}\ A$ the differential graded algebra $C^*(A, A)$ of Hochschild cochains, the maps (0.5), in a sense, still exist. More precisely, they exist if one passes from the category of algebras to the category of complexes by means of some well known homological functors.

These functors put in correspondence to an algebra its Hochschild or periodic cyclic complex. In the case of bar complex the operations analogous to the ones that we will describe were discovered by Getzler - Jones and by Gerstenhaber - Voronov

([GJ], [GV]). Therefore one can say informally that in the homotopy category of (differential graded) algebras the inner endomorphism objects always exist.

The operations arising from the second map (0.5) (in the above sense) play a crucial role in our approach to index theorems (Section 6); they also allow to construct the bivariant JLO cochain and, hopefully, a Chern character with values in bivariant entire cyclic cohomology. Those operations bear a strong formal resemblance to the multiplication in the quantum cohomology of symplectic manifolds. In particular, one can construct "the Fukaya category" of an associative ring; objects of this category are automorphisms of the ring, and the object $\underline{\mathrm{End}}\ A$ from (0.5) is the object of endomorphisms of the identity.

The second author recalls Yu.I. Manin's first reaction upon receiving A.Connes' paper on noncommutative geometry in 1982. Commenting on Connes' formula

$$da = i[F, a]$$

where F is an odd element, Prof. Manin suggested that perhaps the differentials should belong to the structure sheaf of a more general superscheme, not just of the odd affine line. This idea, together with I.M.Gelfand's notion of an $(\mathfrak{a}, \mathcal{D})$-system, is central to this work.

Let us describe the contents of the paper in more detail.

Let A be an associative unital algebra over a commutative unital ground ring k. Consider A as a bimodule over itself; by $\mathcal{E}_A^*$ we denote the differential graded algebra $(C^*(A, A), \delta, \smile)$ which is the standard complex for computing $\mathrm{Ext}^*_{A\otimes A^\circ}(A, A) = H^*(A, A)$ (the Hochschild cohomology) equipped with the Yoneda product. We recall some well-known properties of this algebra in Section 1. In Section 2 we construct the maps of complexes

$$\bullet : C_*(A, A) \otimes C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \to C_*(A, A)$$

$$\bullet : C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \otimes C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \to C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \tag{0.6}$$

Here C_* stands for the Hochschild complex computing $HH_*(A) = \mathrm{Tor}_*^{A\otimes A^\circ}(A, A)$.

In Sect. 3 we construct similar morphisms

$$\bullet : CC_*^{per}(A) \otimes CC_*^{per}(\mathcal{E}_A^*) \to CC_*^{per}(A)$$

$$\bullet : CC_*^{per}(\mathcal{E}_A^*) \otimes CC_*^{per}(\mathcal{E}_A^*) \to CC_*^{per}(\mathcal{E}_A^*) \tag{0.7}$$

where CC_*^{per} is the periodic cyclic complex.

In particular, the center Z_A of A is a (differential) subalgebra of $C^*(A,A)$. One gets the usual products

$$C_*(Z_A) \otimes C_*(A) \to C_*(A)$$

$$CC_*^{\mathrm{per}}(Z_A) \otimes CC_*^{\mathrm{per}}(A) \to CC_*^{\mathrm{per}}(A)$$

([CE], [HJ]).

On the other hand, let D be a Hochschild cochain of A (in particular a derivation). Put

$$i_D(a) = (-1)^{\deg a \cdot \deg D} a \bullet D$$

in the Hochschild complex where D, being an element of $\mathcal{E}_A^*$, is regarded as a zero-chain of $\mathcal{E}_A^*$; let

$$I_D(a) = (-1)^{\deg a \cdot \deg D} a \bullet D$$

but in the periodic cyclic complex; let

$$L_D(a) = (-1)^{\deg a (\deg D - 1)} a \bullet (1 \otimes D)$$

(in this case the formulas are the same for Hochschild or periodic cyclic case). Then L_D is the Lie derivative and I_D is the contraction operator of Rinehart [R]; the fact that $\bullet$ is a morphism of complexes implies the Cartan homotopy formula

$$[B + b, \ \ I_D] = L_D + I_{\delta D} \tag{0.8}$$

(cf. [R]).

There is another product on $C^*(A,A)$, the Gerstenhaber bracket. It makes $C^*(A,A)[1]$ a differential graded Lie algebra which we denote by $(\mathfrak{g}(A), \delta, [\ ,\ ])$ (cf [Ge]). The Lie derivative L_D turns $C_*(A)$ and $CC_*^{\mathrm{per}}(A)$ into differential graded

$\mathfrak{g}(A)$-modules. Let $R_*(\mathfrak{g}(A)) = \wedge^*(\mathfrak{g}(A)) \otimes U(\mathfrak{g}(A))$ be the standard Koszul resolution of the trivial right $\mathfrak{g}(A)$-module k. We construct a morphism of complexes of modules (Sect. 4):

$$J : R_*(\mathfrak{g}(A)) \to \underline{\mathrm{End}}\ CC_*^{\mathrm{per}}(A) \tag{0.9}$$

In our view, the existence of this morphism is one of the main features of noncommutative differential geometry. Compare this with the usual situation: let X be a smooth manifold, and let D_i be vector fields on X. Put

$$J(D_1 \wedge \cdots \wedge D_m) = i_{D_1} \ldots i_{D_m} : \Omega_X^* \to \Omega_X^{*-m}$$

Then

$$\begin{aligned}[d, J(D_1 \wedge \cdots \wedge D_m)] = &\sum_i (-1)^{m-i+1} J(\cdots \wedge \widehat{D}_i \ldots) L_{D_i} + \\ &+ \sum_{i<j} (-1)^{i+j-1} J([D_i D_j] \wedge \cdots \wedge \widehat{D}_i \wedge \cdots \wedge \widehat{D}_j \ldots)\end{aligned} \tag{0.10}$$

The morphism (0.9) satisfies a similar formula in the general context (when, for example, D_i are derivations of A; of course, one has to replace d by $B + b$).

Note that one could simply put $J(D_1 \wedge \cdots \wedge D_m) = I_{D_1} \ldots I_{D_m}$ if the operators I_D (anti)-commuted; but they do not, so the formula is different.

In section 5 we construct another multiplication which is crucial for our approach to index theorems. Let A be an associative algebra. By $\overline{C}^{\lambda}_{*-1}(A)$ we denote the reduced cyclic complex of A; one can define this complex also as

$$\overline{C}^{\lambda}_{*-1}(A) = \mathrm{Prim}\ C_*(\mathfrak{gl}(A), \mathfrak{gl}(k); k)$$

(the subcomplex of primitive Lie algebra chains; cf. [L]). We construct the operation

$$\overline{C}^{\lambda}_{*-1}(A) \otimes CC_*^{per}(A) \longrightarrow CC_*^{per}(A)$$

or in other words the morphism of complexes

$$\chi : \ \overline{C}^{\lambda}_{*-1}(A) \to \underline{\mathrm{End}}\ CC_*^{per}(A)$$

To obtain this operation one applies the homomorphism J in the case when A is replaced by its matrix algebra. Let $M_\infty(A)$ be the algebra of matrices over A with finitely many non-zero diagonals; let $M(A)$ be its ideal of matrices with finitely many non-zero entries and let $\mathfrak{gl}$ be the same algebra viewed as a Lie algebra. We will see in Section 5 that one can modify the operation J to get a homomorphism

$$C^*(\mathfrak{gl}(A), \mathfrak{gl}(k)) \otimes CC_*^{per}(M_\infty(A)) \to CC_*^{per}(M(A))$$

Here $\mathfrak{gl}(A)$ is viewed as a subalgebra of the algebra of inner derivations. To get the operation χ one uses the embedding $CC_*^{per}(A) \to CC_*^{per}(M_\infty(A))$ and the trace map $CC_*^{per}(M(A)) \to CC_*^{per}(A)$.

The next four sections of the paper are devoted to examples and applications. We study the product $\bullet$ on the Hochschild complex. One can define the Hochschild chain complex of the algebra of cochains in two different ways:

$$\mathcal{C}^n(A) = \bigoplus_{i+j=n} C_{-i}(\mathcal{E}_A^*, \mathcal{E}_A^*)^j$$
$$\mathcal{C}_\infty^n(A) = \prod_{i+j=n} C_{-i}(\mathcal{E}_A^*, \mathcal{E}_A^*)^j$$

Therefore we get two versions of the cohomology ring, $\mathcal{H}^*(A)$ and $\mathcal{H}_\infty^*(A)$, together with a map $\mathcal{H}^* \to \mathcal{H}_\infty^*$. First let $A = k[X]$, where X is regular affine over a ring of characteristic zero. We show that

$$\mathcal{H}^*(A) = \mathcal{D}(\Omega_X^*)^o; \qquad \mathcal{H}_\infty^*(A) = \mathrm{End}(\Omega_X^*)$$

Here $\mathcal{D}(\Omega_X^*)$ stands for the ring of differential operators on the graded module of differential forms, that is to say, differential operators on the exterior algebra of the cotangent bundle; the symbol ^{o} means the opposite ring).

Next we consider the case of a deformed algebra of functions. Let (M, ω) be a symplectic manifold and $*$ be a star product on $C^\infty(M)$ (cf. [BFFLS]) such that $f * g - g * f = i\hbar\{f, g\} + O(\hbar^2)$. Isomorphism classes of such star products are parametrized by $\mathrm{H}^2(M, \mathbb{C}[[\hbar]])$ ([LDW], [De], [NT]). Let $A = \mathbb{A}^\hbar(M)[\hbar^{-1}] = C^\infty(M)[[\hbar, \hbar^{-1}]$ with the product $*$. Then

$$\mathcal{H}^*(A) \xrightarrow{\sim} \mathrm{H}^*(M^{S^1}, \mathbb{C}[[\hbar, \hbar^{-1}])$$

if M is simply connected;

$$\mathcal{H}^*_\infty(A) \widetilde{\to} \mathrm{H}^*(M, \mathbb{C}[[\hbar, \hbar^{-1}])$$

Note some resemblance with the multiplication on Floer cohomology.

Let us now mention some applications of our constructions.

In Section 6 we outline the proof of the algebraic index theorem from [Fe] and [NT]. Our approach owes very much to the ideas of B.Feigin and I.M.Gelfand. Let us mention some other applications, namely to deformations and characteristic classes.

Let ∇ be an odd element of $\mathfrak{g}(A)$ such that

$$\delta\nabla + \frac{1}{2}[\nabla\nabla] = 0. \tag{0.11}$$

(δ is the Hochschild differential). We show that, formally, *i.e.* without regard for convergence, if

$$e^\nabla = \sum_{n\geq 0} \frac{\wedge^n \nabla}{n!} \otimes 1 \tag{0.12}$$

in $\wedge^* \mathfrak{g} \otimes U\mathfrak{g}$, then for $X(\nabla) = J(e^\nabla \otimes 1)$

$$[B + b, X(\nabla)] = L_\nabla X(\nabla) \tag{0.13}$$

Assume, for example, that A is a $\mathbb{Z}_2$-graded algebra and $\not{D}$ is an odd element of A. Then

$$\nabla = ad(\not{D}) - \not{D}^2$$

is an odd cochain satisfying (0.11). Using (0.9), we construct a morphism

$$ch(\not{D}) : CC^{\mathrm{per},(0)}_*(A) \to CC^{\mathrm{per}}_*(A)$$

where $CC^{\mathrm{per},(0)}_*$ is the subcomplex of finite cochains:

$$CC^{\mathrm{per},(0)}_n(A) = \bigoplus_{i\equiv n(2)} A \otimes \overline{A}^{\otimes n},$$

while, as usually,

$$CC^{\mathrm{per}}_n(A) = \prod_{i\equiv n(2)} A \otimes \overline{A}^{\otimes n}.$$

The component

$$ch_0(\not{D}) : CC_*^{\mathrm{per},(0)}(A) \to A$$

is given by the formula from [JLO], [GS]:

$$ch_0(\not{D}) = \sum_{n\geq 0} \int_{\substack{t_0+\dots+t_n=1 \\ t_i\geq 0}} a_0 e^{-t_0\not{D}^2}[\not{D}, a_0]\dots e^{-t_{n-1}\not{D}^2}[\not{D}, a_n]e^{-t_n\not{D}^2} dt_0 \dots dt_{n-1}$$

Another application: let A be an algebra; assume that another multiplication law on A, $*$, is given. Put

$$\nabla(a, b) = a * b - ab;$$

then ∇ satisfies (0.11). The operator $X(\nabla)$ is a morphism of complexes

$$X(\nabla) : CC_*^{\mathrm{per},(0)}(A, \cdot) \to CC_*^{\mathrm{per}}(A, *)$$

When $ab - a * b \in I$ where I is an ideal of A such that A is I-adically complete, then $X(\nabla)$ gives a well-defined isomorphism of periodic cyclic complexes, an explicit version of the theorem of Goodwillie. If one composes this isomorphism with a trace on the algebra $(A, *)$, one recovers the cocycle of [CFS]. One can expect that, if the multiplications $\cdot$ and $*$ are "close", one would be able to use $X(\nabla)$ to compare Connes' entire cyclic cohomologies.

In Section 10 we try to clarify the analogy between the multiplication $\bullet$ and the quantum multiplication, as well as to advance B.Feigin's idea that Lagrangian intersections might be related to the cohomology of deformed algebras. We construct "the Fukaya category" of an associative algebra A; the objects of this category are automorphisms of A and, for two endomorphisms α and β, the complex $Hom(\alpha, \beta)$ is the twisted cochain complex $C_*(\mathcal{E}^*_{A}, {}_\alpha \mathcal{E}^*_\beta)$. We also construct a functor putting in correspondence to α the twisted chain complex $C_*(A, A_\alpha)$. Conjecturally, these are an A_∞ category and an A_∞ functor. When A is a deformed ring of functions on a symplectic manifold then the cohomology of the above complexes is related to their fixed point sets and to the cohomology of certain path and loop spaces.

Acknowledgements The second author has many fond memories of the time when he was Yu. I. Manin's graduate student in Moscow from 1982 to 1985. Some of the ideas and methods of this paper are due to his collaboration with Yu.L. Daletski, B. Feigin, I.M. Gelfand, as well as to discussions with Yu. I. Manin, A. Beilinson, V. Schechtman, M. Wodzicki and many others back then (and later); some other methods and ideas came from the first author's collaboration with G. Elliot and T. Natsume. We would like to thank P. Bressler, J.L. Brylinski, A.Connes, B.V. Fedosov, M. Flato, E.Getzler, J. Kaminker, C. Kassel, M. Kontsevich, J.L. Loday, A.Radul, N.Reshetikhin, D. Sternheimer, D.Tamarkin and A. Voronov for fruitful discussions. Note that homological operations somewhat similar to ours were first considered by Getzler in [G]; there, too, a homological complex of the differential graded algebra $(C^*(A,A),\delta,\smile)$ appeared. Finally, we would like to thank the referee whose many suggestions greatly improved the presentation.

Section 1. Hochschild cohomological complex.

Let A be a graded algebra with unit over a commutative unital ring k. Let M be a graded bimodule over A. A Hochschild d-cochain is a linear map $A^{\otimes d} \to M$. Put, for $d \geq 0$,

$$C^d(A,M) = \mathrm{Hom}_k(\overline{A}^{\otimes d}, M)$$

where $\overline{A} = A/k\cdot 1$. For a cochain D in $C^*(A,A)$ put

$$\deg\, D = (\text{degree of the linear map } D) + d$$

$$|D| = \deg\, D - 1; \quad |a| = \deg\, a - 1$$

Given a tensor $a_1 \otimes \cdots \otimes a_N$ in $A^{\otimes N}$, we will denote it by $(a_1,\ldots,a_n)$. We will write $\eta_j = \sum_{i=1}^{j} |a_i|$ (as in [G]). Put for cochains D and E in $C^*(A,A)$,

$$(D \smile E)(a_1,\ldots,a_{d+e}) = (-1)^{\deg\, E\cdot \eta_d} D(a_1,\ldots,a_d) \times E(a_{d+1},\ldots,a_{d+e}); \tag{1.1}$$

$$(D \circ E)(a_1,\ldots,a_{d+e-1}) = \sum_{j\geq 0} (-1)^{|E|\eta_j} D(a_1,\ldots,a_j, E(a_{j+1},\ldots,a_{j+e}),\ldots); \tag{1.2}$$

$$[D,\ E] = D\circ E - (-1)^{|D||E|} E\circ D$$

These operations define the graded associative algebra $(C^*(A,A), \deg, \smile)$ and the graded Lie algebra $(C^*(A,A), |\cdot|, [\, ,\,])$ (cf. [CE]; [Ge]). Let

$$m(a_1,a_2) = (-1)^{\deg a_1}\ a_1 a_2;$$

this is a 2-cochain of A (not in C^2, because it is a bilinear map from A to A, not from $\overline{A}$ to A). Put

$$\begin{aligned}
&\delta D = [m, D];\\
&(\delta D)(a_1,\ldots,a_{d+1}) = (-1)^{|a_1||D|+|a_1|+1}\times\\
&\times a_1 D(a_2,\ldots,a_{d+1}) + \\
&+\sum_{j=1}^{d}(-1)^{|D|+\eta_j} D(a_1,\ldots,a_j\ a_{j+1},\ldots,a_{d+1})\\
&+(-1)^{|D|+\eta_d+1} D(a_1,\ldots,a_d)a_{d+1}
\end{aligned}\tag{1.3}$$

The last formula defines a differential of degree $+1$ on $C^*(A,M)$ for any M. For an element x of A, let $\underline{x}$ be the corresponding zero-cochain in $C^*(A,A)$. By definition

$$(\delta\underline{x})(a) = (-1)^{\deg x}[x,a];$$

a one-cochain D is a cocycle iff it is a derivation. One has

$$\delta^2 = 0; \quad \delta(D\smile E) = \delta D \smile E + (-1)^{\deg D} D \smile \delta E$$

$$\delta[D,E] = [\delta D, E] + (-1)^{|D|}\ [D, \delta E]$$

($\delta^2 = 0$ follows from $[m,m]=0$).

Thus $C^*(A,A)$ becomes a complex; the cohomology of this complex is $H^*(A,A)$ or the Hochschild cohomology. The $\smile$ product induces the Yoneda product on $H^*(A,A) = \mathrm{Ext}^*_{A\otimes A^0}(A,A)$. The operation $[\, ,\,]$ is the Gerstenhaber bracket [Ge].

If $(A,\ \partial)$ is a differential graded algebra then one can define the differential ∂ acting on $C^*(A,A)$ by

$$\partial D\ =\ [\partial, D]$$

Definition 1.1. *Define the differential graded algebra $\mathcal{E}^*_A$ as the complex $C^*(A,A)$ with the differential δ (or $\delta+\partial$ if (A,∂) is a differential graded algebra), the grading deg and the product $\smile$.*

In Sections 1–9 the only cochains we will be considering will be those from $C^*(A,A)$, *i.e.* when $A = M$. For Hochschild cochains D_i define a new Hochschild cochain $D_0\{D_1, \ldots, D_m\}$ by the following formula of Gerstenhaber ([Ge]) and Getzler ([G]):

$$D_0\{D_1, \ldots, D_m\}(a_1, \ldots, a_n) =$$

$$= \sum (-1)^{\sum_{p=1}^{m} \eta_{i_p} |D_p|} D_0(a_1, \ldots, a_{i_1}, D_1(a_{i_1+1}, \ldots), \ldots, D_m(a_{i_m+1}, \ldots), \ldots) \tag{1.4}$$

The following statements are contained in [Ge], [G], [GV].

Proposition 1.2. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q \le i_p} |E_p||F_q|} \times$$
$$\times D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots, \}, \ldots, E_k\{F_{i_k+1}, \ldots, \}, \ldots, \}$$

Proof. Direct computation ∎

Example 1.3. *One has*

$$D \smile E = (-1)^{\deg D} m\{D, E\}; \quad D \circ E = D\{E\}$$

Corollary 1.4.

$$1). \quad (D \circ E) \circ F - D \circ (E \circ F) = D\{E, F\} + (-1)^{|E||F|} D\{F, E\};$$

$$(-1)^{|D||F|}[[D, E], F] + (-1)^{|E||D|}[[E, F], D] + (-1)^{|F||E|}[[F, D], E] = 0 \ .$$

2). Let M be the cochain defined above, but for the differential graded algebra $\mathcal{E}_A^*$: $M(D_1, D_2) = (-1)^{\deg D_1} D_1 \smile D_2$. For any cochain D in $C^*(A, A)$ define a cochain $\overline{D}$ in $C^*(A, A)$ by $\overline{D}(E_1, \ldots, E_k) = D\{E_1, \ldots, E_k\}$ for all k. Then

i).

$$\overline{(D \smile E)} = \overline{D} \smile \overline{E};$$

ii).

$$\overline{\delta D} = [\delta, \overline{D}] + [M, \overline{D}].$$

Proof. To prove the first identity in 1), one applies Proposition 1.2 in the case when $k = l = 1$. The second identity of 1) follows from the first if one antisymmetrizes the first (with appropriate signs). To prove the identity 2), i) one applies Proposition 1.2 to compute $(m\{D,E\})\{F_1,\dots,F_m\}$ to get the following formula

$$(D\smile E)\{F_1,\dots,F_m\}=\sum_{p=0}^{m}(-1)^{\deg E\sum_{i\le p}|F_i|}D\{F_1,\dots,F_p\}\smile E\{F_{p+1},\dots,F_m\}\ .$$

which is equivalent to 2), i).

To prove 2), ii) one applies Proposition 2.2 to compute $(m\{D\})\{F_1,\dots,F_m\}$, $(D\{m\})\{F_1,\dots,F_m\}$, and $(D\{F_1,\dots,F_m\})\{m\}$; their sum, with appropriate sings, gives the following identity

$$\begin{gathered}\delta D(\{F_1,\dots,F_m\})=(\delta D)\{F_1,\dots,F_m\}-\\ -\sum_{p=1}^{m}(-1)^{\deg D+\sum_{i<p}|F_i|}D\{F_1,\dots,\delta F_p,\dots,F_m\}+\\ +\sum_{p=1}^{m-1}(-1)^{\deg D+\sum_{i\le p}|F_i|}D\{F_1,\dots,F_p\smile F_{p+1}\dots,F_m\}-\\ -(-1)^{\deg D+\sum_{i<m}|F_i|}D\{F_1,\dots,F_{m-1}\}\smile F_m+\\ +(-1)^{\deg D|F_1|}F_1\smile D\{F_2,\dots,F_m\}\end{gathered}$$

which is equivalent to 2), ii).

Section 2. Operations on the Hochschild Complex.

Let M be a differential graded bimodule over a differential graded algebra A. Recall that the Hochschild homological complex $(C_*(A,M),b)$ is the following:

$$C_n(A,M)=M\otimes\overline{A}^{\otimes n}$$

We will always write, as in Section 1,

$$(a_0,\dots,a_n)=a_0\otimes\cdots\otimes a_n$$

For a_0 in M and a_i in A, $i>0$, define

$$b(a_0,\dots,a_n)=\sum_{j=0}^{n-1}(-1)^{\eta_{j+1}+1}(a_0,\dots,a_j\,a_{j+1},\dots,a_n)\\ +(-1)^{(|a_n|+1)(\eta_n+1)+1}(a_na_0,a_1,\dots,a_{n-1}).\quad(2.1)$$

We shall denote $C_*(A,A)$ simply by $C_*(A)$. We introduce a grading on $C_*(A,A)$ by the formula

$$\deg\,(a_0,\ldots,a_n)=\sum \deg a_i+n$$

(A rule for remembering the signs: let $|\epsilon|=1$, $\epsilon^2=0$; map $C_n(A)$ to $A[\epsilon]/[A[\epsilon],A[\epsilon]]$, $a_0\otimes\cdots\otimes a_n\mapsto a_0\epsilon a_1\epsilon\cdots a_n\epsilon$; then b becomes $\frac{\partial}{\partial\epsilon}$).

The homology of $C_*(A,M)$ is the Hochschild homology $HH_*(A,M)=\mathrm{Tor}_*^{A\otimes A^\circ}(A,M)$. We will denote $HH_*(A,A)$ simply by $HH_*(A)$. If A is a differential graded algebra and ∂ the differential in A, one extends ∂ to $C_*(A)$:

$$\partial(a_0,\ldots,a_n)=\sum_{j=0}^{n}(-1)^{\eta_j}(a_0,\ldots,\partial a_j,\ldots,a_n) \tag{2.2}$$

For a in $C_*(A,A)$ and x in $C_*(\mathcal{E}_A^*,\mathcal{E}_A^*)$ define

$$a\bullet x=a\bullet_1 x+a\bullet_2 x \tag{2.3}$$

where

$$\begin{aligned}(a_0,\ldots,a_n)\bullet_1(D_0,\ldots,D_m)&=\sum(-1)^{\sum_{i>0}|a_i|\deg D_0+\sum_{p>0}\sum_{i>i_p}|a_i||D_p|}\times\\ &\times(a_0D_0(a_1,a_2,\ldots),\ldots,a_{i_1},D_1(a_{i_1+1},\ldots),\ldots,D_m(a_{i_m+1},\ldots),\ldots)\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(a_0,\ldots,a_n)\bullet_2(D_0,\ldots,D_m)=\\ &=\sum_{q\le n+1}(-1)^{\sum_{i<q}|a_i|\sum_{i\ge q}|a_i|+\sum_{i_0<i<q}|a_i|\deg D_0+\sum_{p>0}\sum_{i_p<i<q}|a_i||D_p|}\times\\ &\qquad\times(D_m(a_q,\ldots,a_n,a_0,\ldots,a_{i_0})D_0(a_{i_0+1},\ldots),\ldots,a_{i_1},\\ &\qquad\qquad,D_1(a_{i_1+1},\ldots),\ldots,D_{m-1}(a_{i_{m-1}+1},\ldots),\ldots)\end{aligned} \tag{2.5}$$

The sum in (2.5) is taken over all q, $i_0,\ldots,$ i_{m-1} for which a_0 is inside D_m.

Theorem 2.1. *The map*

$$\bullet: C_*(A,A)\otimes C_*(\mathcal{E}_A^*,\mathcal{E}_A^*)\to C_*(A,A) \tag{2.6}$$

is a morphism of complexes, i.e.

$$b(a\bullet x)=(ba)\bullet x+(-1)^{\deg a}a\bullet(b+\delta)x$$

Proof. Let $\alpha = (a_0, \dots , a_n)$ and $D = (D_0, \dots , D_m)$. Compute $b(\alpha \bullet_1 D) - (b\alpha) \bullet_1 D$. It will consist of the following terms.

1). All the terms with $a_i a_{i+1}$ or $a_n a_0$ outside D_j, $i > 0$. They will cancel out (because they appear once in $b(\alpha \bullet_1 D)$ and once in $(b\alpha) \bullet_1 D$).

1a).$(a_0 a_1 D_0(\dots), \dots D_1(\dots), \dots)$

2). All the terms with $a_i a_{i+1}$ or $a_n a_0$ inside D_j, $i > 0$.

3). All the terms with $a_i D_j(\dots)$ or $D_j(\dots)a_i$, $i > 0$.

The terms 1a), 2), 3) will cancel out with the term $\alpha \bullet_1 \delta D$.

4).$(a_0 D_0(\dots) D_1(\dots), \dots , D_k(\dots), \dots)$;

$$(a_0 D_0(\dots), \dots , D_k(\dots) D_{k+1}(\dots), \dots);$$

These terms will cancel out with all the terms in $\alpha \bullet_1 bD$ except the following:

$$(a_0 D_m(\dots) D_0(\dots), \dots , D_1(\dots), \dots) \tag{2.7}$$

5). $(D_m(\dots) a_0 D_0(\dots), \dots , D_1(\dots), \dots)$.

Now compute $b(\alpha \bullet_2 D) - (b\alpha) \bullet_2 D$. It will consist of the following terms.

6). All the terms with $a_i a_{i+1}$ or $a_n a_0$ outside D_j. They will cancel out (because they appear once in $b(\alpha \bullet_2 D)$ and once in $(b\alpha) \bullet_2 D$).

7). All the terms with $a_i a_{i+1}$ or $a_n a_0$ inside D_j.

8). All the terms with $a_i D_j(\dots)$ or $D_j(\dots)a_i$.

The terms 7), 8) will cancel out with some of the terms in $\alpha \bullet_2 \delta D$

9).$D_m(\dots a_0 \dots) D_0(\dots), \dots , D_1(\dots), \dots)$.

9a). $D_m(\dots a_0 \dots) D_0(\dots) D_1(\dots), \dots , D_2(\dots), \dots)$.

9b).$D_m(\dots a_0 \dots) D_0(\dots), \dots , D_k(\dots) D_{k+1}(\dots), \dots)$.

The terms 9) and 9a) cancel out with all the terms in $\alpha \bullet_2 bD$ except the following:

$$((D_{m-1} \smile D_m)(\dots , a_0, \dots) D_0(\dots), \dots , D_1(\dots), \dots) \tag{2.8}$$

and

$$D_{m-1}(\dots , a_0, \dots) D_m(\dots) D_0(\dots), \dots , D_1(\dots), \dots) \tag{2.9}$$

The combination of the terms 9b), (2.8) and (2.9) cancels out. The remaining terms are (2.7) and 5). But they will cancel out with the terms in $\alpha \bullet_2 \delta D$ that do not cancel out with 7), 8).

■

Consider some examples of the product $\bullet$. When all D_i are zero-cochains then one gets the shuffle product from [CE]. On the other hand, put, for a d-cochain D and for a in $C_*(A,A)$, $i_D(a) = (-1)^{\deg a \deg D}(a \bullet D)$;

$$i_D(a_0, \dots, a_n) = (-1)^{\deg a_0 \cdot \deg D_0}(a_0 D(a_1, \dots, a_d),\ a_{d+1}, \dots, a_n) \tag{2.7}$$

One gets the cap product from [CE]:

$$C^d(A,A) \otimes C_n(A,A) \to C_{n-d}(A,A)$$

Now, for a d-cochain D and for a in $C_*(A,A)$, let $L_D(a) = (-1)^{|D| \cdot \deg a} a \bullet (1, D)$; one has

$$\begin{aligned} L_D(a_0, \dots, a_n) = & \sum_{q=0}^{n-d+2} (-1)^{(\eta_{n+1} - \eta_q)\eta_q + |D|}(D(a_q, \dots, a_0, \dots), \dots, a_{q-1}) + \\ & + \sum_{k=1}^{n-d} (-1)^{(\eta_{k+1}+1)|D|}(a_0, \dots, a_k, D(a_{k+1}, \dots), \dots,) \end{aligned} \tag{2.10}$$

One has

$$[b, L_D] = -L_{\delta D}$$

Now construct the product

$$\bullet : C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \otimes C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \to C_*(\mathcal{E}_A^*, \mathcal{E}_A^*) \tag{2.11}$$

as follows. For a cochain D let $D^{(k)}$ be the following k-cochain of $\mathcal{E}_A^*$:

$$D^{(k)}(D_1, \dots, D_k) = D\{D_1, \dots, D_k\}$$

Proposition 2.2. *The map*

$$D \mapsto \sum_{k\geq 0} D^{(k)}$$

is a morphism of differential graded algebras $\mathcal{E}_A^* \to \mathcal{E}_{\mathcal{E}_A^*}^*$.

Proof. Follows from Corollary 1.4, 2), 3). ■

Now we construct the product $\bullet$ by composing the one for the algebra $\mathcal{E}_A^*$ with the above map.

The product (2.11) is given by formulas (2.3-2.5) where a_i are now viewed as Hochschild cochains and $D(a_1, \ldots, a_d)$ is replaced by $D\{a_1, \ldots, a_d\}$.

Proposition 2.3. *The product (2.11) is homotopically associative.*

We omit the proof.

Section 3. Operations on the periodic cyclic complex.

In the situation of Sect. 2 define

$$B(a_0, \ldots, a_n) = \sum_{j=0}^{n} (-1)^{(\eta_{n+1}-\eta_j)\eta_j} (1, a_j, \ldots, a_n, a_0, \ldots, a_{j-1}) \tag{3.1}$$

Then $B^2 = b^2 = Bb + bB = 0$; denote

$$CC_n^{\mathrm{per}}(A) = \prod_{i\equiv n(2)} A \otimes \overline{A}^{\otimes n} \tag{3.2}$$

with the differential $b + B + \partial$ (for the differential graded algebra (A, ∂)). Put

$$\begin{aligned}(a_0, \ldots, a_n) \bullet_3 (D_1, \ldots, D_m) &= \\ &= \sum_{0\leq p\leq m} (-1)^{\sum\limits_{r<p}(|D_r|+1)\sum\limits_{r\geq p}(|D_r|+1)+\sum\limits_{r}|D_r|} \times \\ \times \sum_{0\leq j\leq n} (-1)^{(\eta_{n+1}-\eta_j)\eta_j} &\lambda'(D_p, \ldots, D_m, D_0, \ldots, D_{p-1})(1, a_j, \ldots, a_n,\ a_0, \ldots, a_{j-1})\end{aligned} \tag{3.3}$$

Here

$$\lambda(D_1, \ldots, D_m)(a_0, \ldots, a_n) =$$
$$= \sum_{j_1>0} (-1)^{|D_1|(\eta_{n+1}-\eta_{j_1})+\cdots+|D_m|(\eta_{n+1}-\eta_{j_m})} \times$$
$$\times (a_0, \ldots, D_1(a_{j_1} \ldots), \ldots, D_m(a_{j_m}, \ldots), \ldots); \tag{3.4}$$

λ' is the sum of all those terms in λ which contain $D_0(\ldots)$ after a_0.

Theorem 3.1. *The map*

$$\bullet : CC_*^{per}(A) \otimes CC_*^{per}(\mathcal{E}_A^*) \to CC_*^{per}(A) \tag{3.5}$$

is a morphism of complexes.

Proof. Let $\alpha = (a_0, \ldots, a_n)$ and $D = (D_0, \ldots, D_m)$. Compute $b(\alpha \bullet_3 D) - (b\alpha) \bullet_3 D - \alpha \bullet_3 bD$. It will consist of the following terms.

1). All the terms with $a_i a_{i+1}$ or $a_n a_0$ outside D_j. They will cancel out (because they appear once in $b(\alpha \bullet_3 D)$ and once in $(b\alpha) \bullet_3 D$).

2). All the terms with $a_i a_{i+1}$ or $a_n a_0$ inside D_j.

3). All the terms with $a_i D_j(\ldots)$ or $D_j(\ldots)a_i$.

The terms 2), 3) except for the following one (formula (3.6) will cancel out with the term $\alpha \bullet_3 \delta D$.

$$(1, \ldots, a_0 D_0(\ldots), \ldots, D_1(\ldots), \ldots) \tag{3.6}$$

4). All the terms with $D_i(\ldots)D_{i+1}(\ldots)$ or $D_m(\ldots)D_0(\ldots)$ but for the following (3.7) will cancel out.

$$(1, \ldots, D_m(\ldots, a_0, \ldots)(D_0(\ldots), \ldots, D_1(\ldots), \ldots) \tag{3.7}$$

5).$(D_k(\ldots, a_{0,l}\, dots), \ldots, D_0(\ldots), \ldots)$.

6).$(a_0, \ldots, D_0(\ldots), \ldots)$.

7). $D_0(\ldots), \ldots, a_0, \ldots, D_m(\ldots), \ldots)$.

(Here a_0 may be inside or outside D_i).

8). $(a_k, \dots, a_0, \dots, D_0(\dots], \dots)$

where a_0 may be inside or outside D_i. These terms will cancel out because they will enter twice (namely, in the first and in the last summands of $b(\alpha \bullet_3 D)$.

The term 5) will cancel out with $\alpha \bullet_2 BD$, the term 6) with $\alpha \bullet_1 BD$, and the term 7) with $B\alpha \bullet_1 D$. The term (3.6) cancels out with $B(\alpha \bullet_1 D)$ and the term (3.7) with $B(\alpha \bullet_2 D)$. Now, $B\alpha \bullet_2 D = 0$ and $B\alpha \bullet_3 D = \alpha \bullet_3 BD = B(\alpha \bullet_3 D) = 0$, which finishes the proof. ■

For a d-cochain D and for a in $CC_*^{per}(A)$, let $I_D(a) = (-1)^{\deg D \cdot \deg a}(a \bullet D)$; $L_D(a) = (-1)^{|D| \cdot \deg a} a \bullet (1, D)$; one has

$$[B + b,\ I_D] - I_{\delta D} = L_D.$$

■

This is the homotopy formula of Rinehart [R].

One can define the product

$$\bullet : CC_*^{per}(\mathcal{E}_A^*) \otimes CC_*^{per}(\mathcal{E}_A^*) \to CC_*^{per}(\mathcal{E}_A^*) \tag{3.8}$$

exactly in the same way as in the end of Section 2.

Remark 3.2. *Considering $CC_*^{per}(A) = CC_*^{per}(\mathcal{E}_A^0)$ as a part of $CC_*^{per}(\mathcal{E}_A^*)$ and restricting the product $\bullet$ to it, we get the Hood-Jones product on $CC_*^{per}(A)$. When A is commutative then $CC_*^{per}(A)$ is a subcomplex of $CC_*^{per}(\mathcal{E}_A^*)$ and one gets a product on the complex $CC_*^{per}(A)$.*

Section 4. The Lie algebra complex.

Let $(\mathfrak{g}, \partial)$ be a differential graded Lie algebra. Put $\wedge^* \mathfrak{g} = T(\mathfrak{g}) / < D \otimes E -$

$(-1)^{(|D|-1)(|E|-1)}E\otimes D>$;

$$R_*(\mathfrak{g})=\wedge^*\mathfrak{g}\otimes U(\mathfrak{g});$$

$$\partial^{\mathrm{Lie}}(D_1\wedge\cdots\wedge D_m\otimes f)=\sum_i(-1)^{\theta_i}D_1\wedge\dots$$

$$\wedge\widehat{D}_i\wedge\cdots\wedge D_m\otimes D_i f+$$

$$+\sum_{i<j}(-1)^{\theta_{ij}}D_1\wedge\cdots\wedge\widehat{D}_i\wedge\cdots\wedge[D_iD_j]\wedge\cdots\otimes f$$

where

$$\theta_i=\sum_{r<i}(|D_r|+1)+|D_i|\sum_{r>i}(|D_r|+1);$$
$$\theta_{ij}=\sum_{r<i}(|D_r|+1)+|D_i|(\sum_{i<r<j}(|D_r|+1)+1);$$

Put also

$$\partial(D_1\wedge\cdots\wedge D_m)=\sum(-1)^{\mu_j}(D_1,\dots,\delta D_j,\dots,D_m)$$

where

$$\mu_j=1+\sum_{r<j}(|D_r|+1).$$

A rule for remembering the signs: Let $|\epsilon|=1$, $\epsilon^2=0$; map $R_*(\mathfrak{g})$ to $U(\mathfrak{g}[\epsilon])$ by $(D_1\wedge\cdots\wedge D_m)\otimes f\mapsto\epsilon D_1\dots\epsilon D_m\cdot f$; then ∂^{Lie} becomes $\frac{\partial}{\partial\epsilon}$; ∂ is the induced derivation on U but with the opposite sign.

Recall that in Section 2 we defined Lie derivatives $L_D:CC_*^{per}(A)\mapsto CC_*^{per}(A)$.

Lemma 4.1.

$$[L_D,L_E]=L_{[D,E]}$$

Proof. Straightforward. ∎

Put $\mathfrak{g}(A)=(C^*(A,A);\ |\cdot|;\ [\ ,\ ];\ \delta)$. One sees that $\mathfrak{g}(A)$ is a differential graded Lie algebra and that $U(\mathfrak{g}(A))$ acts on the complex of endomorphisms End $CC_*^{\mathrm{per}}(A)$ from the right: $X\mapsto X\cdot L_D$ for $D\in\mathfrak{g}(A)$. For $f\in U(\mathfrak{g}(A))$, we will denote the corresponding operator by L_f.

The purpose of this Section is to show that the complex $R_*(\mathfrak{g}(A))$ acts $\mathfrak{g}(A)$ - equivariantly on the complex $CC_*^{per}(A)$, thus generalizing the standard calculus from the commutative case.

Let us consider the complex $CC_*^{per}(\mathcal{E}_A^*)$ with the Hood-Jones product (Remark 3.2). In other words, we view $\mathcal{E}_A^*$ as $C_0(\mathcal{E}_A^*)$ and consider the product $\bullet$ from section 3. We denote this product by $\star$.

Definition 4.2. *Put*

$$J(D_1 \wedge \cdots \wedge D_m)\alpha = (-1)^{\deg \alpha \sum_{i=1}^m \deg D_i} \frac{1}{m!} \sum_\sigma \epsilon_\sigma \alpha \bullet (D_{\sigma_1} \star (D_{\sigma_2} \star (\ldots \star D_{\sigma_m})) \ldots)$$

where the sign ϵ_σ is taken with respect to the rule

$$\epsilon_{(i,i+1)} = (-1)^{(|D_i|-1)(|D_{i+1}|-1)}$$

If f is in $U(\mathfrak{g}(A))$ put

$$J(D_1 \wedge \cdots \wedge D_m \otimes f) = J(D_1 \wedge \cdots \wedge D_m) L_f.$$

Theorem 4.3.

i) *The map $J : R_*(\mathfrak{g}(A)) \to \underline{End}\ (CC_*^{per}(A))$ is a homomorphism of right $\mathfrak{g}(A)$-modules;*

ii)

$$[B + b,\ J(\gamma)] = J((\partial^{\text{Lie}} + \delta)\gamma)$$

for γ in $R(\mathfrak{g}(A)$;

iii) *If $E_1, \ldots, E_n, D$ are in $\mathcal{E}^0$ or in $\mathcal{E}^1$ then*

$$[L_D, J(E_1 \wedge \ldots \wedge E_n)] = \sum_{j=1}^n (-1)^{|D| \sum_{i<j}(|E_i|+1)} J(E_1 \wedge \ldots [D, E_j] \wedge \ldots E_n)$$

Proof. i) is obvious. To prove ii) and iii), one needs the following partial associativity properties of the product $\bullet$.

Lemma 4.4.. *Let $\alpha \in CC_*^{per}(A)$, $\beta \in CC_*^{per}(\mathcal{E}_A^*)$ and $\gamma \in \mathcal{E}_A^*$. Then*

i) $\alpha \bullet ((1, D) \bullet \beta) = (\alpha \bullet (1, D)) \bullet \beta$;

ii) $(\alpha \bullet \beta) \bullet (1, D) = \alpha \bullet (\beta \bullet (1, D))$ *if D is in $\mathcal{E}^0$ or in $\mathcal{E}^1$*;

iii)

$$\alpha \bullet (1, D) \bullet \beta - (-1)^{|\beta||D|} \alpha \bullet \beta \bullet (1, D) = \alpha \bullet [D, \beta]$$

where D is in $\mathcal{E}^0$ or in $\mathcal{E}^1$ and $[D,\beta]$ stands for the usual action of $\mathfrak{g}(A)$ on its tensor powers;

iv)
$$(1,D)\bullet(E_0,\dots,E_n)=(1,D)\star(E_0,\dots,E_n)+$$
$$+\sum_{j=0}^{n}(-1)^{|D|(\sum_{i\le j}|E_i|+1)}(E_0,\dots,E_j\circ D,\dots,E_n)$$

v) $D\star((1,E)\star\gamma)=(-1)^{(|D|-1)|E|}(1,E)\star(D\star\gamma)$

<u>*Proof.*</u>. Direct computation. ■

Now let us prove the theorem. We shall omit the signs for simplicity. One has

$$[B+b,J(D_1\wedge\dots\wedge D_m)]\alpha=\alpha\bullet\mathrm{Alt}(B+b+\delta)((D_1\star(D_2\star(\dots\star D_m))\dots))=$$
$$=\alpha\bullet(\mathrm{Alt}\sum_j\pm(D_1\star\dots((1,D_j)\star(\dots\star D_m))\dots)+$$
$$\sum_j\pm(D_1\star\dots(\mathrm{ad}(D_j)\star(\dots\star D_m))\dots)+$$
$$+\sum_j\pm(D_1\star\dots(\delta(D_j)\star(\dots\star D_m))\dots))$$

The second sum contributes terms $D_i\smile D_j-(-1)^{(|D_i|-1)(|D_j|-1)}D_j\smile D_i$ and therefore vanishes under antisymmetrization. The third sum contributes the term $J(\delta\gamma)\alpha$. The first sum, because of (v), is equal to

$$\alpha\bullet\mathrm{Alt}\sum_j(1,D_j)\star(\pm(D_1\star\dots(\widehat{D_j}\star(\dots\star D_m))\dots)=$$
$$=\alpha\bullet\mathrm{Alt}\sum_j\pm(1,D_j)\bullet(D_1\star\dots(\widehat{D_j}\star(\dots\star D_m))\dots)+$$
$$+\alpha\bullet\mathrm{Alt}\sum_j\pm(D_1\star\dots(D_i\circ D_j\star(\dots(\widehat{D_j}\star(\dots\star D_m))\dots)+$$
$$=\sum_j\pm J(D_1\wedge\dots\wedge\widehat{D_j}\wedge\dots\wedge D_m)(L_D\alpha)+$$
$$+\sum_{i<j}\pm J([D_i,D_j]\wedge\dots\wedge\widehat{D_i}\wedge\dots\widehat{D_j}\wedge\dots\wedge D_m)(\alpha)$$

■

<u>Section 5.</u> <u>The characteristic map χ</u>

In this section we construct a natural morphism of complexes

$$\overline{C}^{\lambda}_{*-1}(A) \otimes CC^{per}_{*}(A) \to CC^{per}_{*}(A)$$

where $\overline{C}^{\lambda}_{*-1}(A)$ is the reduced cyclic complex equipped with the differential b: $\overline{C}^{\lambda}_{n}(A) = \overline{A}^{\otimes(n+1)}/\mathrm{Im}(1-\tau)$ where $\overline{A} = A/k\cdot 1$ and

$$\tau(a_0, \dots, a_n) = (-1)^{\eta_n(\eta_n - \eta_{n-1})}(a_n, a_0, \dots, a_{n-1}).$$

Note that this complex is well defined ([L]).

Let $M(A)$ be the algebra of matrices $(a_{ij})_{1\le i,j\le\infty}$ for which $a_{ij}\in A$ and all but finitely many of a_{ij} are zero. The same space considered as a Lie algebra is $\mathfrak{gl}(A)$. The formula

$$(a_0, \dots a_n) \mapsto (-1)^{\sum \deg a_i} E^{a_0}_{01} \wedge E^{a_1}_{12} \wedge \dots \wedge E^{a_n}_{n0}$$

defines a morphism of complexes

$$\overline{C}^{\lambda}_{*-1}(A) \ \to \ C_*(\mathfrak{gl}(A), \mathfrak{gl}(k); k)$$

(with the relative Lie algebra chains in the right hand side). This map is an isomorphism of $\overline{C}^{\lambda}_{*}(A)$ with the subcomplex of primitive elements; cf. [T],[FT], [LQ].

Let $M_\infty(A)$ be the associative algebra of matrices $(a_{ij})_{1\le i,j\le\infty}$ such that $a_{ij}\in A$ and $a_{ij} = 0$ on all but finitely many diagonals. Consider the differential graded Lie subalgebra $C^{\le 1}(M(A), M(A))$. There is a morphism to this algebra from the differential graded Lie algebra $(\mathfrak{gl}(A[\eta])$ where η is a formal parameter of degree -1 such that $\eta^2 = 0$; the differential on $A[\eta]$ is $\partial/\partial\eta$. This morphism sends ηa to a in C^0 and a to $\mathrm{ad}(a)$ in C^1. From Theorem 4.3 one gets the pairing

$$R_*(\mathfrak{gl}(A[\eta])) \otimes CC^{per}_{*}(M_\infty(A)) \longrightarrow CC^{per}_{*}(M(A)) \tag{5.1}$$

Consider the following morphisms:

$$CC^{per}_{*}(A) \xrightarrow{i} CC^{per}_{*}(M_\infty(A));$$

$$CC^{per}_{*}(M(A)) \xrightarrow{tr} CC^{per}_{*}(A)$$

The map i is induced by the inclusion $a \mapsto a \cdot 1$; the second map tr acts as follows:

$$(a_0 m_0, \dots, a_n m_n) \longmapsto tr(m_0 \dots m_n)\ (a_0, \dots, a_n)$$

for $a_i \in A$ and $m_i \in M(k)$. Composing (5.1) with these maps we obtain

$$R_*(\mathfrak{gl}(A[\eta])) \otimes CC_*^{per}(M_\infty(A)) \longrightarrow CC_*^{per}(A)$$

or

$$U(\mathfrak{gl}(A[\epsilon, \eta])) \longrightarrow \mathrm{End}(CC_*^{per}(A)) \tag{5.2}$$

where $[\epsilon, \eta] = [\epsilon, \epsilon] = [\eta, \eta] = 0$, $\deg \epsilon = \deg \eta = -1$ and the differential in the L.H.S. is $\dfrac{\partial}{\partial \epsilon} + \dfrac{\partial}{\partial \eta}$. For modules over any Lie algebra $\mathfrak{g}$ we shall write $\otimes_{\mathfrak{g}}$ instead of $\otimes_{U(\mathfrak{g})}$.

Lemma 5.1. *For any Lie Algebra $\mathfrak{g}$ and any right (left) module M the map (5.2) descends to a morphism*

$$k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta]))_{\mathfrak{gl}(k[\epsilon])} \otimes k \longrightarrow \underline{End}\ (CC_*^{per}(A))$$

Proof First it is easy to see that the map descends to $k \otimes_{\mathfrak{gl}(k[\eta])} U$. In fact, this subalgebra acts by the operators L_m and $L_{\mathrm{ad}(m)}$ where $m \in \mathfrak{gl}(k)$, both are zero when followed by the trace map tr. To see that one can pass to $k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta]))_{\mathfrak{gl}(k[\epsilon])} \otimes k$ note that $L_{\mathrm{ad}(m)}$ is zero on the image of i and that, because of the explicit formula in Definition 4.2, the operator $J(\epsilon D_1 \cdot \dots \cdot \epsilon D_m)$ is zero on the image of i if at least one of the D_i is of the form $\mathrm{ad}(m), m \in \mathfrak{gl}(k)$. Indeed, this formula involves terms $D_i(a_j)$ for any D_i which is a one-cochain. ■

Finally there is an explicit map

$$C_*(\mathfrak{gl}(A[\eta]), \mathfrak{gl}(k); k) \longrightarrow k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta]))_{\mathfrak{gl}(k[\epsilon])} \otimes k \tag{5.3}$$

Let us construct this map. Let $\mathfrak{g} = \mathfrak{gl}(A)$ and $\mathfrak{h} = \mathfrak{gl}(k)$. To describe the image of a chain $D_1 \wedge \dots \wedge D_n \wedge \eta E_1 \wedge \dots \wedge \eta E_m$, write the expression

$$D_1(\epsilon - \eta) \cdot \dots \cdot D_n(\epsilon - \eta) \cdot E_1 \epsilon \eta \cdot \dots \cdot E_m \epsilon \eta$$

and then represent it as a sum

$$\sum \pm D_{i_1}\eta \cdot \ldots \cdot D_{i_k}\eta \cdot E_1\epsilon\eta \cdot \ldots \cdot E_n\epsilon\eta \cdot D_{j_1}\epsilon \cdot \ldots \cdot D_{j_m}\epsilon$$

in the symmetric algebra of the graded space $\mathfrak{g}[\epsilon, \eta]$. For example $D \mapsto D\epsilon - D\eta$;

$$D_1 \wedge D_2 \longmapsto D_1\eta \cdot D_2\eta + D_1\eta \cdot D_2\epsilon + (-1)^{(|D_1|+1)(|D_2|+1)} D_2\eta \cdot D_1\epsilon + D_1\epsilon \cdot D_2\epsilon$$

$$D_1 \wedge D_2\eta \longmapsto -D_1\eta \cdot D_2\epsilon\eta + (-1)^{(|D_1|+1)(|D_2|+1)} D_2\epsilon\eta \cdot D_1\eta$$

etc.

It turns out that this yields the morphism of complexes

$$C_*(\mathfrak{g}[\eta], \mathfrak{h}; k) \longrightarrow k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta]))_{\mathfrak{gl}(k[\epsilon])} \otimes k \tag{5.4}$$

or, which is the same,

$$k \otimes_{\mathfrak{gl}(A[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta])_{\mathfrak{gl}(k[\epsilon])} \otimes k \longrightarrow k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\epsilon, \eta]))_{\mathfrak{gl}(k[\epsilon])} \otimes k$$

<u>*Remark*</u>. These maps are quasi-isomorphisms; one can easily show that the cohomology of both complexes is

$$C_*(\mathfrak{g}[\eta], \mathfrak{h}; k) = S(\mathfrak{h})_{\mathfrak{h}} = H_*(\mathrm{BH}, k)$$

where H is the Lie group of the Lie algebra $\mathfrak{h}$ and BH is the classifying space of H.

At the level of cohomology the inclusion

$$C_*(\mathfrak{g}, \mathfrak{h}; k) \longrightarrow C_*(\mathfrak{g}[\eta], \mathfrak{h}; k) \tag{5.5}$$

is the dual map to the Chern-Weil map

$$C^*(\mathfrak{g}, \mathfrak{h}; k) \longleftarrow W^*(\mathfrak{h})_{basic};$$

$W^*(\mathfrak{h}) = C_*(\mathfrak{h}[\eta], k)$ is the Weil algebra of $\mathfrak{h}$. Another realization of the map which is induced by 5.5) on homology is

$$k \otimes_{\mathfrak{h}[\epsilon]} U(\mathfrak{g}[\eta]) \otimes_{U(\mathfrak{g})} U(\mathfrak{g}[\epsilon])_{\mathfrak{h}[\epsilon]} \otimes k \longrightarrow k \otimes_{\mathfrak{h}[\eta]} U(\mathfrak{g}[\epsilon, \eta])_{\mathfrak{h}[\epsilon]} \otimes k \tag{5.6}$$

Section 6. Index theorems

To illustrate our methods, we will sketch the proof of the algebraic index theorem from [Fe], [NT]. This proof, unlike the original ones, is designed to work in a much more general situation (families, foliations, manifolds with boundaries or with corners, complex varieties, etc.).

Let (M, ω) be a $2n$-dimensional symplectic manifold with a star product $*$.

$$f * g = \sum (i\hbar)^k \varphi_k(f, g).$$

Let $\mathbb{A}^\hbar(M) = C^\infty(M)[[\hbar]]$ with the product $*$. Put also $\mathbb{A}_c^\hbar(M) = C_c^\infty(M)[[\hbar]]$. The following is proved in [Fe], [NT]. Consider the canonical trace

$$\mathrm{Tr} : \mathbb{A}_c^\hbar(M) \to \mathbb{C}[\hbar^{-1}, \hbar]]$$

constructed in [Fe], [NT]. One has

$$\mathrm{Tr}(f) = \frac{1}{(i\hbar)^n n!}[\int_M f\omega^n + \sum_{k=1}^{\infty} (i\hbar)^k D_k(f)\omega^n]$$

where D_k are certain differential operators. All traces on $\mathbb{A}_c^\hbar(M)$ are proportional.

Let $P^2 = P$, $Q^2 = Q$ be two elements of the matrix algebra $M_N(\mathbb{A}^\hbar(M))$ such that $P - Q$ is in $M_N(\mathbb{A}_c^\hbar(M))$. Let $P_0 = P \mathrm{mod} \hbar$, $Q_0 = Q \mathrm{mod} \hbar$, $\mathrm{ch}(P_0) = \sum_k \frac{1}{k!} \mathrm{tr} P_0 (dP_0)^{2k}$ (the Chern character of the connection $dP_0 d$ in the vector bundle $P_0 \mathbb{C}^N$). We regard TM as a $U(n)$-bundle (since $U(n)$ is a maximal compact subgroup of $Sp(2n)$).

Recall that for each deformation quantization of M a characteristic class θ is defined which is an element of $H^2(M)[[\hbar]]$: the class $\theta / i\hbar$ is the curvature of a Fedosov connection defining the deformation (cf. [Fe], [NT]).

Theorem 6.1.

$$Tr(P - Q) = \int_M (ch(P_0) - ch(Q_0)) \widehat{A}(TM) e^{\theta / i\hbar}$$

where θ is the characteristic class of $*$.

Sketch of the proof. Consider the characteristic map

$$\chi : \overline{C}^\lambda_{*-1}(\mathbb{A}^\hbar(M))[\hbar^{-1}] \to \underline{\mathrm{End}}\ CC_*^{per}(\mathbb{A}_c^\hbar(M))[\hbar^{-1}]$$

Note that the right hand side is the space of global sections of a fine sheaf (the left hand side is not). Therefore one can extend χ to the morphism to the Chech - cyclic double complex:

$$\chi : \check{C}^{-*}(\overline{C}^{\lambda}_{*-1}(\mathbb{A}^{\hbar}(M)))[\hbar^{-1}] \to \underline{\text{End}}\ CC^{per}_{*}(\mathbb{A}^{\hbar}_{c}(M))[\hbar^{-1}] \tag{6.1}$$

Next step is to construct the fundamental class Ω in the left hand side of (6.1). Note that for any contractible Darboux chart U_0

$$\overline{HC}_{2n-1}(\mathbb{A}^{\hbar}(U_0))[\hbar^{-1}] \stackrel{\sim}{\to} \mathbb{C}[\hbar^{-1}, \hbar]] \tag{6.2}$$

(the canonical generator is represented by the cycle

$$\Omega_0 = \frac{1}{2n(i\hbar)^n}\text{Alt}(\xi_1 \otimes x_1 \ldots \xi_n \otimes x_n) \tag{6.2.1}$$

where alternation is over the group Σ_{2n} and $[\xi_i, x_j] = i\hbar\delta_i^j$);

$$\overline{HC}_i(\mathbb{A}^{\hbar}(U_0))[\hbar^{-1}] = 0 \tag{6.3}$$

for $i > 2n-1$.

Using the spectral sequence $E^2_{ij} = H^{-i}(M, \overline{HC}_j(\mathbb{A}^{\hbar}))$, we construct a cycle of the complex $\check{C}^{-*}(\overline{C}^{\lambda}_{*-1}(\mathbb{A}^{\hbar}(M))[\hbar^{-1}]$ whose homology class, being restricted to any contractible Darboux chart, gives the canonical generator of $\overline{HC}_{2n-1}$. Let α be a periodic cyclic chain of $\mathbb{A}^{\hbar}_{c}(M)$; $\alpha_0 = \alpha \text{mod}\hbar$ is a periodic cyclic chain of $C^{\infty}_{c}(M)$. (For non-unital algebras the periodic cyclic complex is defined as the kernel $\text{Ker}(CC^{per}(\tilde{A}) \to CC^{per}(k))$ were $\tilde{A}$ is the algebra A with adjoined unit). One proves that

$$\text{Tr}(\chi(\Omega))(\alpha) = \int_M \mu(\alpha_0)\text{mod}\hbar \tag{6.4}$$

where $\mu : CC^{per}_{*}(C^{\infty}_{c}(M)) \to (\Omega^{*}(M), d)$ is the morphism of complexes given by

$$\mu(a_0 \otimes \ldots \otimes a_k) = \frac{(-1)^k}{k!}a_0 da_1 \ldots da_k$$

([L]). Formula (6.4) follows easily from (6.2.1) and from the fact that $[f,g] = i\hbar\{f,g\}\text{mod}\hbar^2$.

Now consider the cycles $(k-1)!\eta^{\otimes k}$ in $\check{C}^{-*}(\overline{C}^{\lambda}_{*-1}(\mathbb{A}^{\hbar}(M)[\eta]))[\hbar^{-1}]$. One has

$$\chi((k-1)!\eta^{\otimes k}) = S^k \tag{6.5}$$

where S is the Bott isomorphism. We know from (6.4) the composition of the canonical trace with $\chi(\Omega)$. To compute the canonical trace itself we have to compare the map $\chi(\Omega)$ to the identity map. In view of (6.5), it is enough to express Ω in terms of the classes $(k-1)!\eta^{\otimes k}$.

<u>Theorem 6.2</u>. *In* $\check{C}^{-*}(\overline{C}^{\lambda}_{*-1}(\mathbb{A}^{\hbar}(M)[\eta])[\hbar^{-1}]$ *one has*

$$\Omega \simeq \sum_k (-1)^k (\widehat{A}(TM)e^{\theta/i\hbar})^{-1}_{2k} \cdot (n+k-1)!\eta^{\otimes(n+k)}$$

Note that, since <u>End</u> $(CC^{per}_*(\mathbb{A}^{\hbar}_c(M)))$ is the space of global sections of a fine sheaf, its cohomology is a module over the algebra $H^*(M,\mathbb{C})$: if $(c_{U_0U_1\ldots U_p})$ is in $\check{C}^p(M,\mathbb{C})$ then <u>End</u> $(CC^{per}_*(\mathbb{A}^{\hbar}_c(M)))$

$$c\cdot\alpha = \sum c_{U_0U_1\ldots U_p}\rho_{U_0}[B+b,\rho_{U_1}]\ldots[B+b,\rho_{U_p}]$$

where $\{\rho_U\}$ is a partition of unity.

We see that for any periodic cyclic cycle α of $CC^{per}_*(\mathbb{A}^{\hbar}_c(M))$

$$\mathrm{Tr}(e^{-\theta/i\hbar}\cdot\alpha) = \int_M \mu(\alpha_0)\widehat{A}(TM)\mathrm{mod}(\hbar) \tag{6.6}$$

Indeed,

$$\mathrm{Tr}(e^{-\theta/i\hbar}\cdot\alpha) = \mathrm{Tr}\sum_k (-1)^k(\widehat{A}(TM)e^{\theta/i\hbar})^{2k}\cdot S^{n+k}\chi(\Omega)(e^{-\theta/i\hbar}\cdot\alpha)$$

by Theorem 6.2; now we use (6.4).

We apply (6.6) to $\alpha = \mathrm{ch}(P) - \mathrm{ch}(Q)$ where ch is the Connes-Karoubi Chern character cycle in the periodic cyclic complex:

$$\mathrm{ch}(P) = \mathrm{tr}(P + \sum_{k\geq 1}(-1)^k\frac{(2k)!}{k!}(P-\frac{1}{2})\otimes P^{\otimes 2k}) \tag{6.7}$$

([L]). Note that $\mu(\mathrm{ch}(P)) = \mathrm{ch}(P_0)$. The left hand side of the last formula stands for the Chern character of the vector bundle $\mathrm{Im}P_0$.

It remains to show that for a periodic cyclic cycle α of $\mathbb{A}^\hbar(M)$ *over* $\mathbb{C}$ (i.e when all the tensor products are taken over $\mathbb{C}$, not $\mathbb{C}[[\hbar]]$) one has

$$\frac{\partial}{\partial\hbar}\mathrm{Tr}(e^{-\theta/i\hbar}\cdot\alpha)=0 \tag{6.8}$$

This can be done by passing to a one-parameter family of deformations

$$f *_\lambda g=\sum(i\hbar\lambda)^k\varphi_k(f,g).$$

One gets the deformed algebra $\mathbb{A}^\hbar(M\times\mathbb{R}^1)$.

It turns out that for any deformed family $M\to B$ of symplectic manifolds one can construct the canonical morphism of complexes ([NT])

$$\mathrm{Tr}_\theta : CC_*^{per}(\mathbb{A}_c^\hbar(M))\to\Omega_c^*(B)[\hbar^{-1},\hbar]]$$

such that the Ω^0 component is, for $b\in B$,

$$\mathrm{Tr}_\theta(\alpha)(b)=\mathrm{Tr}(\alpha(b)\cdot e^{-\theta_b/i\hbar})$$

where $\alpha(b)$ is the restriction of α to $\mathbb{A}_c^\hbar(M_b)$ and θ_b is the characteristic class of $\mathbb{A}^\hbar(M_b)$.

It remains to say a few words about the proof of Theorem 6.2. To prove it one has to construct and to compare various cocycles (cochains) of the complex $\check{C}^*(M,\mathcal{L}^*)$ where $\mathcal{L}^*$ is the sheaf of complexes $\overline{C}^\lambda_{-*}(\mathbb{A}^\hbar(M))[\hbar^{-1}]$. If $\mathcal{L}^*$ were a constant sheaf then one would be able to construct such cochains, for example the ones representing characteristic classes of the tangent bundle. One observes that $\mathcal{L}^*$ is *constant up to homotopy* in the following sense. If $\mathfrak{g}$ is the Lie algebra of infinitesimal coordinate changes (in our case $\mathfrak{g}=\mathrm{Der}(\mathbb{A}^\hbar)$) then for any open contractible chart U the action of $\mathfrak{g}(U)$ on $\mathcal{L}^*(U)$ is homotopically trivial; moreover, for X from $\mathfrak{g}$, let $L_X:\mathcal{L}^*\to\mathcal{L}^*$ be the action of $\mathfrak{g}$ on $\mathcal{L}^*$; one can also construct operators $i_X:\mathcal{L}^*\to\mathcal{L}^{*-1}$ satisfying the usual formulas:

$$[L_X,i_Y]=i_{[X,Y]};[i_X,i_Y]=0;[d,i_X]=L_X$$

In our case $X=\frac{1}{i\hbar}\mathrm{ad}(a)$ and $i_X=L_{\underline{a}}$.

Let $\widehat{\mathbb{A}^\hbar}(\mathbb{R}^{2n})$ be the Weyl algebra $\mathbb{C}[[\xi_1, \ldots, \xi_n, x_1, \ldots, x_n, \hbar]]$ with the Moyal product. Let $\widehat{\mathfrak{g}} = \mathrm{Der}(\widehat{\mathbb{A}^\hbar}(\mathbb{R}^{2n}))$; then $\mathfrak{u}(n)$ is a Lie subalgebra of $\widehat{\mathfrak{g}}$. Put

$$\mathbb{L}^* = \overline{C}^\lambda_{-*}(\widehat{\mathbb{A}^\hbar}(\mathbb{R}^{2n}))[\hbar^{-1}]$$

Let $W^*(\mathfrak{u}(n))$ be the Weil algebra. As a partial case of a general construction one gets a cochain map

$$[W^*(\mathfrak{u}(n)) \otimes \mathbb{L}^*]_{basic} \to \check{C}(M, \mathcal{L}^*) \tag{6.9}$$

The subscript *basic* stands for $\{\alpha : L_X\alpha = i_X\alpha = 0, X \in \mathfrak{u}(n)\}$.

To finish the proof of the theorem one constructs the fundamental class Ω in $[W^*(\mathfrak{u}(n)) \otimes \mathbb{L}^*]_{basic}$ and proves an analogue of Theorem 6.2 by an explicit calculation. ∎

Let us finish by saying a few words about how to carry out all the above computations explicitly at the level of cochains. Recall that one can realize any deformation $\mathbb{A}^\hbar(M)$ as the space of horizontal sections of a Fedosov connection $D : \Omega^0(M, \mathbb{W}) \to \Omega^1(M, \mathbb{W})$ where $\mathbb{W}$ is the Weyl bundle ([Fe]). Then, instead of the Cech complex $\check{C}^*(M, \mathcal{L}^*)$, one considers the de Rham complex $\Omega^*(M, \mathbb{L}^*)$; all the cochains which participate in the proof have an easy explicit Chern - Weil style realization in this complex.

Section 7. Examples

For any differential graded algebra B there is a natural gradind on the Hochschild chain complex $C_*(B, B)$. We denote by $C_*(B, B)^j$ the space of homogeneous elements of degree j in $C_*(B, B)$.Let

$$\mathcal{C}^n(A) = \bigoplus_{i+j=n} C_{-i}(\mathcal{E}^*_A, \mathcal{E}^*_A)^j$$
$$\mathcal{C}^n_\infty(A) = \prod_{i+j=n} C_{-i}(\mathcal{E}^*_A, \mathcal{E}^*_A)^j$$

We have defined a homotopically associative $\bullet$ product on these complexes; their homology is denoted by $\mathcal{H}^*(A)$, resp. by $\mathcal{H}^*_\infty(A)$.

Theorem 7.1. *Let $A = k[X]$ be the algebra of regular functions on an affine algebraic variety X over a field k of characteristic zero. Then*

$$\mathcal{H}^*(A) \tilde{\to} \mathcal{D}(\Omega^*_{X/k})$$

(the ring of all differential operators on Ω);

$$\mathcal{H}^*_\infty(A) \tilde{\to} End(\Omega^*_{X/k})$$

Note that $HH_*(A) \simeq \Omega^*_{X/k}$; the $\bullet$ action of $\mathcal{H}^*$ on HH^* is the obvious one.

Proof of 7.1.

Lemma 7.2. *Let $\mathcal{E}^*$, $\mathcal{F}^*$ be two differential graded algebras together with a quasi-isomorphism $f : \mathcal{E}^* \to \mathcal{F}^*$. Then f induces an quasi-isomorphism*

$$\bigoplus_{i+j=n} C_{-i}(\mathcal{E}^*, \mathcal{E}^*)^j \tilde{\to} \bigoplus_{i+j=n} C_{-i}(\mathcal{F}^*, \mathcal{F}^*)^j$$

Proof. The map induced by f is an isomorphism at the level of E^1 terms of the spectral sequences with $E^1 = \bigoplus_{i+j=n} C_{-i}(H^*(\mathcal{E}^*), H^*(\mathcal{E}^*))^j$, resp.

$\bigoplus_{i+j=n} C_{-i}(H^*(\mathcal{F}^*), H^*(\mathcal{F}^*))^j$, the Hochschild complex of the graded algebra of cohomology of $\mathcal{E}^*$, resp. $\mathcal{F}^*$. Note that we are able to apply this argument to those converging spectral sequences (this is not true if one replaces $\bigoplus$ by $\prod$). ■

Note that by [HRK] $\mathcal{E}^*_A$ is quasi-isomorphic to $\mathcal{F}^* = \Gamma(X, \bigwedge^* TX)$ as differential graded algebras. Locally, $\mathcal{F}^*_U = \mathcal{O}_U \otimes \bigwedge^*(\partial_{x_1}, \ldots, \partial_{x_n})$; $HH_*(\mathcal{F}^*_U) = \Omega^*_{U/k} \otimes \bigwedge^*(\partial_{x_1}, \ldots, \partial_{x_n}) \otimes S^*(d\partial_{x_1}, \ldots, d\partial_{x_n})$ (classes $d\partial_{x_i}$ are represented by cycles $(1, d\partial_{x_i})$). Now, looking at the $\bullet$ action on $HH^*(\mathcal{O}_\mathcal{U}) \simeq \Omega^*_{U/k}$, one sees that $\Omega^*_{U/k}$ acts by multiplication, ∂_{x_i} acts by contraction $i_{\partial_{x_i}}$ and $d\partial_{x_i}$ acts by Lie derivative $L_{\partial_{x_i}}$. Whence the local isomorphism

$$\mathcal{H}^*(\mathcal{O}_U) \tilde{\to} \mathcal{D}(\Omega^*_{U/k});$$

this statement may be globalized exactly as in [HRK] Now let us compute $\mathcal{H}^*_\infty(A)$. We have to use the other spectral sequence whose E_1 term is $HH_*(\mathcal{E}^*_A)$, the

Hochschild homology of the graded anlebra $\mathcal{E}_A^*$ (if one forgets about the differential δ).

For the vector spaces V and W we will denote by $T(V')\widehat{\otimes}W$ the space of multilinear maps from V to W. We will use similar notation for certain subfactors of $T(V')$.

As a graded algebra, $\mathcal{E}_A^*$ is just $T^*(A')\widehat{\otimes}A$ by which we mean the space of all multilinear maps from A to A. One can show that

$$E_2^{-p,n} = HC^{n-1}(A)\widehat{\otimes}\Omega^p + HC^{n-2}(A)\widehat{\otimes}\Omega^{p+1}$$

$$E_3^{-p,n} = (\mathrm{Ker}S)^{n-1}\widehat{\otimes}\Omega^p + (\mathrm{Coker}S)^{n-2}\widehat{\otimes}\Omega^{p+1}$$

where $S : HC^n \to HC^{n+2}$ is the Bott operator ([L]). Recall the Gysin exact sequence ([C], [T], [L]):

$$0 \longrightarrow (\mathrm{Coker}S)^n \longrightarrow HH^n(A) \longrightarrow (\mathrm{ker}S)^{n-1} \longrightarrow 0$$

It is not hard to construct for any element of $HH^*(A)\widehat{\otimes}\Omega^*_{X/k}$ a corresponding cycle of the complex $\mathcal{C}^*$. This shows that the spectral sequence degenerates at E_3 term and that $\mathcal{H}^*_\infty(A) \widetilde{\to} HH^*(A)\widehat{\otimes}\Omega^*_{X/k}$, ∎

Section 8. Characteristic cochain of a flat element.

Let ∇ be an element of $C^*(A,A)$ such that $|\nabla|$ is odd (and therefore deg ∇ is even). Assume that

$$\delta\nabla + \frac{1}{2}[\nabla,\nabla] = 0 \tag{8.1}$$

Consider an element

$$e^{\nabla} = \sum \frac{1}{n!} \wedge^n \nabla \in \wedge^* \mathfrak{g}(A)$$

in the completed space $\prod R_k(\mathfrak{g}(A))$. Put

$$X(\nabla) = J(e^{\nabla} \otimes 1) \tag{8.2}$$

This is a well-defined operator

$$X(\nabla) : CC_*^{\mathrm{per},(0)}(A) \to CC_*^{\mathrm{per}}(A)$$

where $CC_*^{\mathrm{per},(0)}$ is the subcomplex of finite cochains:

$$CC_n^{\mathrm{per},(0)}(A) = \bigoplus_{i \equiv n(2)} A \otimes \overline{A}^{\otimes n},$$

while, as usually,

$$CC_n^{\mathrm{per}}(A) = \prod_{i \equiv n(2)} A \otimes \overline{A}^{\otimes n}.$$

Theorem 8.1.

$$[B + b,\ X(\nabla)] = X(\nabla) \cdot L_{\nabla}.$$

Proof. Indeed, in the completion of $\wedge^* \mathfrak{g}(A) \otimes U(\mathfrak{g}(A))$:

$$(\delta + \partial^{\mathrm{Lie}})(e^{\nabla} \otimes 1) = e^{\nabla}(\delta\nabla + \frac{1}{2}[\nabla, \nabla]) \otimes$$
$$\otimes 1 + e^{\nabla} \otimes L_{\nabla}. \qquad \blacksquare$$

In some cases there is a natural topology on A such that $X(\nabla)$ converges and defines an operator on a complex larger than $CC^{per,0}$.

The formula for $X(\nabla)$ is obtained as follows. In the periodic cyclic complex of the algebra generated by the even elements ∇ and $e^{-t\nabla}$, $t \geq 0$, put

$$e^{I_{\underline{\nabla}}} = \sum_{n \geq 0} \int\limits_{\substack{t_0 + \cdots + t_n = 1 \\ t_i \geq 0}} e^{t_o i_{\nabla}} S_{\underline{\nabla}} e^{t_1 i_{\nabla}} \ldots S_{\underline{\nabla}} e^{t_n i_{\nabla}} \, dt \ldots dt_{n-1}$$

$$e^{t i_{\underline{\nabla}}}(D_0, \ldots, D_m) = (D_0 \smile e^{t\nabla}, D_1, \ldots, D_m);$$

$$S_{\underline{\nabla}}(D_0, \ldots, D_m) = \sum_{j=0}^{n} \sum_{i=0}^{j-1} (-1)^{jn+n+j+i}(1, D_j, \ldots, D_m,$$
$$D_0, \ldots, D_i,\ \nabla, \ldots, D_{j-1})$$

Then

$$X(\nabla)(\alpha) = \alpha \bullet (e^{I_{\underline{\nabla}}} 1)$$

where $\bullet$ is the product from Theorem 3.1. One has

$$e^{I_{\underline{\nabla}}} 1 = e^{\nabla} - \int (e^{t_0 \nabla}, e^{t_1 \nabla}, \nabla) +$$
$$+ \int [(2(e^{t_0 \nabla}, e^{t_1 \nabla}, \nabla, e^{t_2 \nabla}, \nabla) -$$
$$-(e^{t_0 \nabla}, \nabla, e^{t_1 \nabla}, \nabla, e^{t_2 \nabla}) + (e^{t_0 \nabla}, \nabla, e^{t_1 \nabla}, e^{t_2 \nabla}, \nabla)] + \ldots$$

Section 9. Bivariant JLO cochain.

In [JLO] and [GS] a Chern character of a θ-summable Fredholm module over a Banach algebra A was defined. It is given by a cocycle of the complex dual to $CC^{per,0}(A)$; this cocycle satisfies a special growth condition which makes it an entire cyclic cocycle in the sense of Connes. Here we will construct its bivariant version, which means, a homomorphism from $CC^{per,0}(A)$ to $CC^{per}(A)$ whose composition with a trace is given by the formula from [JLO] and [GS].

Given a $\mathbb{Z}_2$-graded algebra A and an odd element $\not{D}$ of A, put

$$\nabla = ad(\not{D}) - \not{D}^2; \tag{9.1}$$

then ∇ satisfies (8.1).

Theorem 9.1. *Put*

$$ch(\not{D}) = X(\nabla) \cdot e^{L_{\not{D}}};$$

then

$$[B + b, ch(\not{D})] = 0 \tag{9.2}$$

Proof. We have seen that

$$[B + b,\ X(\nabla)] = X(\nabla) L_{\nabla}; \tag{9.3}$$

it suffices to show that

$$[B + b, e^{L_{\underline{\not{D}}}}] = -L_{\nabla} \cdot e^{L_{\underline{\not{D}}}}. \tag{9.4}$$

Indeed,

$$[B + b, e^{L_{\underline{\not{D}}}}] = -\int_0^1 e^{tL_{\underline{\not{D}}}} L_{ad\not{D}} e^{(1-t)L_{\not{D}}} dt =$$
$$= -L_{ad\not{D}} \cdot e^{L_{\not{D}}} + \int t\, L_{[\underline{\not{D}}\underline{\not{D}}]} e^{L_{\not{D}}} dt =$$
$$= -L_{\nabla} e^{L_{\underline{\not{D}}}}, \qquad \blacksquare$$

Section 10. "The Fukaya category".

In this Section we extend the results of Section 2 by "moving away from the diagonal". For any automorphism α of an algebra A we define the twisted Hochschild complex $C_*(A, A_\alpha)$ (related to noncommutative geometry of fixed points of α) and construct in these terms a homotopically associative category for which $\mathcal{C}^*(A)$ or $\mathcal{C}^*_\infty(A)$ is the ring of endomorphisms of the identity.

Let α, β be two automorphisms of A. Define the new bimodule ${}_\alpha A_\beta$ over A as follows: ${}_\alpha A_\beta = A$ as k-modules and $a \cdot m \cdot b = \alpha(a)m\beta(b)$ for a, m, b in A. Consider the chain complex $C_*(A,\ {}_\alpha A_\beta)$ as in Section 2 and the cochain complex ${}_\alpha\mathcal{E}^*_\beta = C^*(A,\ {}_\alpha A_\beta)$ as in Section 1. We shall write $A_\alpha =_{\mathrm{id}} A_\alpha$. The cup product on Hochschild cochains is well defined as a morphism ${}_\alpha\mathcal{E}^*_\beta \otimes_\beta \mathcal{E}^*_\gamma \to_\alpha \mathcal{E}^*_\gamma$.

Given an automorphism α of A, one can define its action on a Hochschild cochain from $\mathcal{E}^*_A$ in two ways:

$$(\alpha D)(a_1, \dots, a_n) = \alpha(D(a_1, \dots, a_n));$$

$$(D\alpha)(a_1, \dots, a_n) = D(\alpha a_1, \dots, \alpha a_n)$$

Both yield morphisms of complexes $\mathcal{E}^*_A \to_\alpha \mathcal{E}_\alpha$. We define an $\mathcal{E}^*_A$-bimodule structure on ${}_\alpha\mathcal{E}^*_\beta$ as follows:

$$D \cdot M \cdot E = D\alpha \smile M \smile \beta E$$

for $D, E \in \mathcal{E}^*_A$ and $M \in_\alpha \mathcal{E}^*_\beta$.

Theorem 10.1. *The pairings (0.6) can be extended to the homotopically associative natural morphisms of complexes*

$$\bullet : C_*(A, A_\alpha) \otimes C_*(\mathcal{E}^*_A,_\alpha \mathcal{E}^*_\beta) \to C_*(A, A_\beta) \tag{10.1}$$

$$\bullet : C_*(\mathcal{E}^*_A,\ {}_\alpha\mathcal{E}^*_\beta) \otimes C_*(\mathcal{E}^*_A,\ {}_\beta\mathcal{E}^*_\gamma) \to C_*(\mathcal{E}^*_A,\ {}_\alpha\mathcal{E}^*_\gamma) \tag{10.2}$$

Proof As in Section 2, put for $a \in C_*(A, A_\alpha)$ and $x \in C_*(\mathcal{E}^*_A,\ {}_\alpha\mathcal{E}_\beta)$

$$a \bullet x = a \bullet_1 x + a \bullet_2 x$$

where

$$(a_0, \ldots, a_n) \bullet_1 (D_0, \ldots, D_m) =$$
$$\sum \pm (a_0 \cdot D_0(a_1, \ldots), \ldots, a_{i_1}, D_1(a_{i_1+1}, \ldots), \ldots, D_m(a_{i_m+1}, \ldots), \ldots) \quad (10.3)$$

$$(a_0, \ldots, a_n) \bullet_2 (D_0, \ldots, D_m) =$$
$$= \sum_{q \leq n+1} \pm (D_m(a_q, \ldots, a_n, a_0, \alpha a_1, \ldots, \alpha a_{i_0}) \cdot D_0(a_{i_0+1}, \ldots), \ldots, A_{i_1},$$
$$, D_1(a_{i_1+1}, \ldots), \ldots, D_{m-1}(a_{i_{m-1}+1}, \ldots), \ldots) \quad (10.4)$$

For $A \in C_*(\mathcal{E}_A^*, \ {}_\alpha\mathcal{E}_\beta)$ and $x \in C_*(\mathcal{E}_A^*, \ {}_\beta\mathcal{E}_\gamma)$

$$A \bullet x = A \bullet_1 x + A \bullet_2 x \quad (10.5)$$

where

$$(A_0, \ldots, A_n) \bullet_1 (D_0, \ldots, D_m) =$$
$$\sum \pm (A_0 \smile D_0\{A_1, A_2, \ldots\}, \ldots, A_{i_1}, D_1\{A_{i_1+1}, \ldots\}, \ldots, D_m\{A_{i_m+1}, \ldots\}, \ldots) \quad (10.6)$$

$$(A_0, \ldots, A_n) \bullet_2 (D_0, \ldots, D_m) =$$
$$= \sum_{q \leq n+1} \pm (D_m\{\{A_q, \ldots, A_n, A_0, A_1, \ldots, A_{i_0}\}\} \smile D_0\{A_{i_0+1}, \ldots\}, \ldots, A_{i_1},$$
$$, D_1\{A_{i_1+1}, \ldots\}, \ldots, D_{m-1}\{A_{i_{m-1}+1}, \ldots\}, \ldots) \quad (10.7)$$

and

$$D_m\{\{A_q, \ldots, A_n, A_0, A_1, \ldots, A_{i_0}\}\}(a_1, \ldots, a_n) =$$
$$= \sum \pm D_m(\alpha a_1, \ldots, A_q(\alpha a_j, \ldots,), \ldots,$$
$$, \alpha a_t, A_0(a_{t+1}, \ldots,), \beta a_p, \ldots, \beta A_1(a_r, \ldots), \beta a_s, \ldots)$$

One checks that these maps are homotopically associative morphisms of complexes. ■

One gets the cohomology groups $\mathcal{H}^*(\alpha, \beta)$ and $\mathcal{H}^*_\infty(\alpha, \beta)$ which form a category. When A is the ring of functions on X then, in simple cases, the groups $\mathcal{H}^*(\alpha, \beta)$

are results of some standard D-module constructions on $T^*(X)$. When A is a deformed algebra of functions on a symplectic manifold M then , when M is simply connected, all the automorphisms of A are of the form $exp(\mathrm{ad}\frac{1}{i\hbar}f)$; $\mathcal{H}^*(\alpha,\beta)$ and $\mathcal{H}^*_\infty(\alpha,\beta)$ reflect the geometry of fixed point sets of α and β and of related loop and path spaces. It looks as if there was, along with $\mathcal{H}^*$ and $\mathcal{H}^*_\infty$, the intermediate semi-infinite cohomology which is more closely related to Floer cohomology of loop spaces.

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