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NEW SERIES OF RATIONAL MODULI COMPONENTS OF RANK 2 BUNDLES ON PROJECTIVE SPACE

CHARLES ALMEIDA, MARCOS JARDIM, ALEXANDER TIKHOMIROV, AND SERGEY TIKHOMIROV

Abstract. We present a new family of monads whose cohomology is a stable rank two vector bundle on $\mathbb{P}^3$. We also study the irreducibility and smoothness together with a geometrical description of some of these families. These facts are used to construct a new infinite series of rational moduli components of stable rank two vector bundles with trivial determinant and growing second Chern class. We also prove that the moduli space of stable rank two vector bundles with trivial determinant and second Chern class equal to 5 has exactly three irreducible components.
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1. Introduction

In [26] Maruyama proved that the rank r stable vector bundles on a projective variety X with fixed Chern classes $c_1, ..., c_r$ can be parametrized by an algebraic quasi-projective variety, denoted by $\mathcal{B}_X(r, c_1, ..., c_r)$. Although this result has been known for almost 40 years, there are just a few concrete examples and established facts about such varieties, even for cases like $X = \mathbb{P}^3$ and $r = 2$. For instance, $\mathcal{B}_{\mathbb{P}^3}(2, 0, 1)$ was studied by Barth in [2], $\mathcal{B}_{\mathbb{P}^3}(2, 0, 2)$ was described by Harthorne in [14], $\mathcal{B}_{\mathbb{P}^3}(2, -1, 2)$ studied by Harthorne and Sols in [17] and by Manolache in [25], while $\mathcal{B}_{\mathbb{P}^3}(2, -1, 4)$ was described by Banica and Manolache in [1]. This probably happened due to the fact that the questions of irreducibility (solved in [30] and [31]), and smoothness (answered in [22]) of the so-called *instanton component* of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, c_2)$ remained opened until 2014.

In this paper, we continue the study of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, n)$, which we will simply denote by $\mathcal{B}(n)$ from now on, with the goal of providing new examples of families of vector bundles, and understanding their geometry. It is more or less clear from the table in [16, Section 5.3] that $\mathcal{B}(1)$ and $\mathcal{B}(2)$ should be irreducible, while $\mathcal{B}(3)$ and $\mathcal{B}(4)$ should have exactly two irreducible components; see [12] and [8], respectively, for the proof of the statements about $\mathcal{B}(3)$ and $\mathcal{B}(4)$. For $n \geq 5$, two families of irreducible components have been studied, namely the *instanton components*, whose generic point corresponds to an instanton bundle, and the *Ein components*, whose generic point corresponds to a bundle given as cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0$$

where $b \geq a \geq 0$ and $c > a + b$. All of the components of $\mathcal{B}(n)$ for $n \leq 4$ are of either of these types; here we focus on a new family of bundles that appear as soon as $n \geq 5$.

More precisely, we study the family of vector bundles in $\mathcal{B}(a^2+k)$ for each $a \geq 2$ and $k \geq 1$ which arise as cohomologies of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

which will be denoted by $\mathcal{G}(a,k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} \tilde{E} \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

where $\tilde{E}$ is a rank 4 instanton bundle of charge k. When $k = 1$ these facts, are used to prove our first main result. (See Theorem 20 below.)

Main Theorem 1. *For each $a \geq 2$ not equal to 3, $\mathcal{G}(a,1)$ is a nonsingular dense subset of an irreducible rational component of $\mathcal{B}(a^2+1)$ of dimension*

$$4 \cdot \binom{a+3}{3} - a - 1.$$

Our second main result provides a complete description of all the irreducible components of $\mathcal{B}(5)$. (See Theorem 22 below.)

Main Theorem 2. *The moduli space $\mathcal{B}(5)$ has exactly 3 irreducible components, namely:*

(i) *the* instanton component*, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{12} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \text{ and} \tag{1}$$

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{2}$$

(ii) *the* Ein component*, of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{3}$$

(iii) *the closure of the family $\mathcal{G}(2,1)$, of dimension* 37*, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \text{ and} \tag{4}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{5}$$

(iv) *All these components are rational varieties.*

Indeed, Hartshorne and Rao proved in [16] that every stable rank 2 bundle on $\mathbb{P}^3$ with Chern classes $c_1(E) = 0$ and $c_1(E) = 5$ is the cohomology of one of the monads listed above. Rao showed in [29] that bundles given as cohomology of monads of the form (2) lie in the closure of the family of instanton bundles of charge 5, which was shown to be irreducible firstly by Coanda, Tikhomirov and Trautmann in [9]; see also [30]. The irreducibility of the family of bundles which arise as cohomology of monads of the form (3) was established by Ein in [11].

Finally, our first main result yields the third component, and we also show that the family of bundles given by the monads of the form (5) lies in the closure of the family $\mathcal{G}(2,1)$.

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Notation and Conventions.

- In this work, $\mathbf{k}$ is an algebraically closed field of characteristic zero,
- V_n denotes a $\mathbf{k}$-vector space of dimension n.
- $\mathbf{P}(F) := \mathrm{Proj}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_X} F)$, for given scheme X and a coherent $\mathcal{O}_X$-sheaf F,
- $\mathcal{O}_{\mathbf{P}}(F)(1)$ the Grothendieck sheaf on $\mathbf{P}(F)$,
- $\mathbf{V}(F) := \mathrm{Spec}(\mathrm{Sym}^{\bullet}_{\mathcal{O}_X} F)$, for X and F as above,
- $\mathbb{P}(F) := \mathbf{P}(F^{\vee})$,
- $\mathbb{P}^3 := \mathbb{P}(V_4)$ the projective 3-space,
- $\mathbf{Isom}(V_n \otimes \mathcal{O}_X, F) \to X$ the principal $GL(n, \mathbf{k})$-bundle of frames of a rank n locally free $\mathcal{O}_X$-sheaf F,
- $\mathbf{X} := \mathbb{P}^3 \times X$, for a given scheme X,
- $p_X : \mathbf{X} \to X$ the projection onto the second factor, for $\mathbf{X}$ and X as above,
- $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ the morphism induced by the morphism of schemes $f : X \to Y$,
- $F_X := f^*F$, $\mathbf{E_X} := \mathbf{f}^*\mathbf{E}$, for a given $\mathcal{O}_Y$-sheaf F, a given $\mathcal{O}_{\mathbf{Y}}$-sheaf (or, a complex of sheaves) $\mathbf{E}$, and $f : X \to Y$ and $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ as above,
- $\mathbf{E}(a, 0) := \mathbf{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_X$, for X and $\mathbf{E}$ as above, and $a \in \mathbb{Z}$,
- $X \xleftarrow{g_X} X \times_Z Y \xrightarrow{f_Y} Y$ the projections of the fibre product $X \times_Z Y$ induced by the morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$,
- $H^i(F)$ the i-th cohomology group of the sheaf F on $\mathbb{P}^3$,
- $Gr(n, V_k)$ the grassmannian variety of n-dimensional subspaces of V_k.
- Since we are working with rank 2 vector bundles on $\mathbb{P}^3$, and Gieseker stability is equivalent to $\mu-$stability, we will not make any distinction between these two concepts.
- We will not make any distinction between vector bundles and locally free sheaves.
- $[E]$ the isomorphism class of a given rank 2 stable vector bundle E on $\mathbb{P}^3$ considered as a point in the moduli space M of stable rank 2 sheaves on $\mathbb{P}^3$,
- $\Phi_X : X \to M,\ x \mapsto [\mathbf{E}|_{\mathbb{P}^3 \times \{x\}}]$ the morphism defined by the $\mathcal{O}_{\mathbf{X}}$-sheaf $\mathbf{E}$ which is family of stable rank 2 vector bundles on $\mathbb{P}^3$ with base X, for M as above. We call Φ_X the *modular morphism defined by the family* $\mathbf{E}$.

2. Monads

Recall that a monad is a complex of vector bundles of the form:

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{6}$$

such that α is injective, and β is surjective. We call the sheaf $E := \ker\beta/\operatorname{im}\alpha$ the cohomology of the monad (6). When α is locally left invertible, then E is a vector bundle.

The notion of monad is important in the study of vector bundles on $\mathbb{P}^3$ because Horrocks proved in [18] that every vector bundle on $\mathbb{P}^3$ is cohomology of a monad of the form (6) with A, B and C being sums of line bundles.

For completeness, we include in this section some useful results about monads that will be required in this work. The following lemma gives a relation between isomorphism classes of monads and its cohomology vector bundles; a proof can be found in [27, Lemma 4.1.3].

Lemma 1. *Let E and E' be, respectively, cohomology of the following monads:*

$$M: \qquad A \xrightarrow{a} B \xrightarrow{b} C \tag{7}$$

$$M': \qquad A' \xrightarrow{a'} B' \xrightarrow{b'} C' \tag{8}$$

If one has that

$$\begin{gathered}\operatorname{Hom}(B,A') = \operatorname{Hom}(C,B') = \operatorname{Ext}^1(C,A') = \\ = \operatorname{Ext}^1(B,A') = \operatorname{Ext}^1(C,B') = \operatorname{Ext}^2(C,A') = 0,\end{gathered}$$

then there exists a bijection between the set of all morphisms from E to E' and the set of all morphisms of monads from (7) to (8).

The following important corollary will be used several times in what follows, and a proof can also be found in [27, Lemma 4.1.3, Corollary 2].

Corollary 2. *Consider the monad*

$$M: \qquad A \xrightarrow{a} B \xrightarrow{b} C$$

and its dual monad:

$$M^\vee: \qquad C^\vee \xrightarrow{b^\vee} B^\vee \xrightarrow{a^\vee} A^\vee.$$

If these monads satisfy the hypothesis of Lemma 1, and there exists an isomorphism $f : E \to E^\vee$ between its cohomology bundles such that $f^\vee = -f$, then there are isomorphisms $h : C \to A^\vee$, and $q : B \to B^\vee$, such that $q^\vee = -q$, and $h \circ b = a^\vee \circ q$.

Recall that every locally free sheaf E on $\mathbb{P}^3$ is the cohomology of a monad of the form [18]:

$$0 \to \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^s \mathcal{O}_{\mathbb{P}^3}(b_j) \to \oplus_{k=1}^t \mathcal{O}_{\mathbb{P}^3}(c_k) \to 0 \tag{9}$$

In this work we will be interested in rank 2 locally free sheaves with vanishing first Chern class. Under these conditions, we have $E^\vee \simeq E$, thus the monad (9) is self dual, which implies that $t = r$, $s = 2r+2$, and $\{a_i\} = \{-c_k\}$. In addition, the middle entry of the monad is also self dual, so that (9) reduces to

$$0 \to \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(a_i) \to \oplus_{j=1}^{r+1} \left(\mathcal{O}_{\mathbb{P}^3}(b_j) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_j)\right) \to \oplus_{i=1}^r \mathcal{O}_{\mathbb{P}^3}(-a_i) \to 0.$$

Finally, recall also that r coincides with the number of generators of $\mathrm{H}^1_*(E) = \bigoplus_{p\in\mathbb{Z}} \mathrm{H}^1(E(p))$ as a graded module over the ring of homogeneous polynomials in four variables, while a_i are the degrees of these generators, cf. [21, Theorem 2.3].

3. Symplectic instanton bundles

Instanton bundles are a particularly important class of stable rank 2 vector bundles due to their many remarkable properties and applications in mathematical physics. Besides this, instanton bundles form the only known irreducible component of the moduli space $\mathcal{B}(c)$ for every $cin\mathbb{N}$.

We will now present the main results concerning instanton sheaves that will be used below. We start by recalling the definition of instanton sheaves on $\mathbb{P}^3$; see [19, Introduction] for further information on these objects.

Definition 3. *An* instanton sheaf *on* $\mathbb{P}^3$ *is a torsion free coherent sheaf* E *with* $c_1(E) = 0$ *satisfying the following cohomological conditions:*

$$h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0. \tag{10}$$

The integer $n := c_2(E)$ *is called the* charge *of* E*. When* E *is locally free, we say that* E *is an* instanton bundle*.*

We remark that instanton bundles of rank $r > 2$ and non locally free instanton sheaves of rank $r \geq 2$ on $\mathbb{P}^3$ are not μ-semistable in general, and also The vanishing of $h^1(E(-2))$ does not imply the vanishing of $h^2(E(-2))$. The definition above is the right generalization of the usual definition of an instanton vector bundle in the sense that, applying the Beilison spectral sequence [27, Ch. II, Thm. 3.1.4]

$$\mathrm{E}_1^{pq} = H^q(E(-p-1)\otimes\Omega^{-p}_{\mathbb{P}^3})\otimes \ op3(p+1) \Rightarrow \mathrm{E}_\infty^{p+q} = \begin{cases} E, & p+q=0,\\ 0, & p+q\neq 0,\end{cases} \tag{11}$$

to an arbitrary rank r instanton sheaf E of charge k, the vanishing (10) yields that E is the cohomology of a monad of the form

$$0 \to V_k\otimes\mathcal{O}_{\mathbb{P}^3}(-1) \to V_{r+2k}\otimes\mathcal{O}_{\mathbb{P}^3} \to V_k\otimes\mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{12}$$

Conversely, the cohomology of a monad as above is an instanton sheaf as defined in Definition 3, see [19, Theorem 3].

The cokernel N of any monomorphism of sheaves $\mathcal{O}_{\mathbb{P}^3}(-1)\to\Omega^1_{\mathbb{P}^3}(1)$ is called a *null correlation sheaf*:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s} \Omega^1_{\mathbb{P}^3}(1) \to N \to 0. \tag{13}$$

Such sheaves are precisely the rank 2 instanton sheaves of charge 1, and are parametrized by the projective space $\mathbb{P}H^0(\Omega^1_{\mathbb{P}^3}(2)) \simeq \mathbb{P}^5$. If N is locally free, we say that N is a *null correlation bundle*. The set of non locally free null correlation sheaves are parametrized by the Grassmanian of lines in $\mathbb{P}^3$: given a line $l\subset\mathbb{P}^3$ the corresponding null correlation sheaf N_l is defined by the sequence

$$0 \to N_l \to V_2\otimes\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \mathcal{O}_l(1) \to 0. \tag{14}$$

For the purposes of this paper, it is important to study rank 4 instanton bundles of charge 1. Some of the following facts might be well known, but for lack of a reference we include proofs here.

Lemma 4. *Every rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$ fits into an exact sequence:*

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \to E \to N \to 0, \tag{15}$$

where N is a null correlation sheaf. If N is a null correlation bundle, then sequence (15) *splits. In addition,*

$$h^0(E) = 2, \qquad h^i(E) = 0, \quad i \geq 1. \tag{16}$$

Proof. As observed in the paragraph right below Definition 3, E can be obtained as cohomology of a monad (12) for $r = 4$ and $k = 1$:

$$M_E : \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0. \tag{17}$$

Without loss of generality, we can choose homogeneous coordinates $[x : y : z : w]$ in $\mathbb{P}^3$ and a basis in V_6, such that the map β can be written as

$$\beta := \begin{pmatrix} x & y & z & w & 0 & 0 \end{pmatrix}. \tag{18}$$

Hence using the display of the above monad, we have that E fits into the following short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \Omega(1) \to E \to 0. \tag{19}$$

From the above short exact sequence we can build up the following commutative diagramm

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & = & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \Omega(1) & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & \Omega(1) & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The rightmost column is the desired sequence.

If N is locally free, then $\mathrm{Ext}^1(N, \mathcal{O}_{\mathbb{P}^3}) \simeq H^1(N) = 0$, so the sequence in display (15) splits. The equality (16) follows from (15). □

Note that, substituting N instead of E into the Beilinson spectral sequence (11) yields the monad for N:

$$M_N : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\overline{\alpha}} V_4 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\overline{\beta}} \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad N = \ker \overline{\beta} / \mathrm{im} \overline{\alpha}, \tag{20}$$

fitting together with the monad (17) in the commutative diagram

(21)
$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & V_2\otimes\mathcal{O}_{\mathbb{P}^3} & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\alpha} & V_6\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\beta} & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{\overline{\alpha}} & V_4\otimes\mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\overline{\beta}} & \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0\\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

in which the middle column is obtained from the exact triple $0 \to 6 \to 0V_2 \to V_6 \to V_4 \to 0$ arising as the cohomology sequence of the exact triple $0 \to V_2 \otimes \Omega_{\mathbb{P}^3} \to E \otimes \Omega_{\mathbb{P}^3} \to N \otimes \Omega_{\mathbb{P}^3} \to 0$ induced by the triple (15). In addition, from (21) and (18) we obtain

(22) $$\overline{\beta} = \begin{pmatrix} x & y & z & w \end{pmatrix}.$$

Proposition 5. *Let E be a rank 4 instanton bundle E of charge 1 over $\mathbb{P}^3$, then* $h^0(S^2E) = 3,\ h^1(S^2E) = 5,\ h^2(S^2E) = 0.$

Proof. Taking the symmetric power of the sequence in display (19), we obtain that S^2E fits into the following short exact sequence:

$$0 \longrightarrow V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \Omega \longrightarrow (S^2V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) \oplus (V_2 \otimes \Omega(1)) \oplus S^2\Omega(2) \longrightarrow S^2E \longrightarrow 0.$$

From the long exact sequence of cohomology we have

$$0 \to S^2V_2 \to H^0(S^2E) \to \mathbf{k} \to \Lambda^2W^\vee \to H^1(S^2E) \to 0,$$

where W is the 4-dimensional $\mathbf{k}-$vector space such that $\mathbb{P}^3 = \mathbb{P}(W)$, and

$$0 \to H^2(S^2E) \to 0.$$

From which we conclude that $H^2(S^2E) = 0$. The map $\mathbf{k} \to \Lambda^2W^\vee$ is given by the skew-form corresponding to the morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to \Omega(1)$, in the definition of E, and in particular is non-zero, which implies that $\mathbf{k} \to \Lambda^2W^\vee$ is injective, and therefore

$$H^0(S^2E) \simeq S^2V_2 \text{ and } H^1(S^2E) \simeq \Lambda^2W^\vee/\mathbf{k}$$

from which our result follows. □

In the remaining part of this section we will discuss the existence of a symplectic structure on an arbitrary rank 4 instanton bundle of charge 1. Recall that a locally free sheaf E is said to be *symplectic* if it admits a symplectic structure, that is, there exists an isomorphism $\varphi : E \to E^\vee$, such that $\varphi^\vee = -\varphi$. A *symplectic instanton bundle* is a pair (E, φ) consisting of an instanton bundle E together with a symplectic structure φ on it; two symplectic instanton bundles (E, φ) and (E', φ') are isomorphic if there exists a bundle isomorphism $g : E \xrightarrow{\sim} E'$ such that $\varphi = g^\vee \circ \varphi' \circ g$.

Proposition 6. *Any rank 4 instanton bundle E of charge 1 admits a symplectic structure. In particular, if E splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$ where N is a null correlation bundle, then any symplectic structure φ on E splits as $\varphi = \varphi_1 \oplus \varphi_2$ where φ_1 and φ_2 are symplectic structures on $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and N, respectively.*

Proof. Let E be an instanton rank 4 bundle. If E splits as $E = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N$, where N is a null correlation bundle, then $\det(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = \det N = \mathcal{O}_{\mathbb{P}^3}$, hence both rank 2 bundles $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ and N admit symplectic structures, say,

$$\varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\sim} (V_2 \otimes \mathcal{O}_{\mathbb{P}^3})^\vee, \qquad \varphi_2 : N \xrightarrow{\sim} N^\vee. \tag{23}$$

Then

$$\varphi = \varphi_1 \oplus \varphi_2 : \ E \xrightarrow{\sim} E^\vee \tag{24}$$

is a symplectic structure on E. Since

$$\mathrm{Hom}(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}, N) = \mathrm{Hom}(N, V_2 \otimes \mathcal{O}_{\mathbb{P}^3}) = 0, \tag{25}$$

it follows immediately that any symplectic stucture on E splits as in (24).

Now let E be a non-splitting instanton, i. e. $E/V_2 \otimes \mathcal{O}_{\mathbb{P}^3}$ is a null correlation sheaf N_l which is is not locally free at the points of the line l given by the equations, say, $\{x = y = 0\}$. This means that the morphism $\overline{\alpha}$ in the monad (20) for $N = N_l$ is vanishes at l, so that

$$\overline{\alpha} = A\binom{x}{y}, \qquad A = (\alpha_{ij}), \quad 1 \le i \le 4, \ 1 \le j \le 2, \tag{26}$$

where A is a (4×2)-matrix of rank 2. The condition that $\overline{\beta} \circ \overline{\alpha}$ in (20) is the zero morphism together with (26) and (22) implies that all the coefficients α_{ij} of the matrix A, except α_{12} and α_{21}, vanish and $\alpha_{12} + \alpha_{21} = 0$. Thus, taking without loss of generality $\alpha_{12} = 1$, we obtain

$$\overline{\alpha} = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \end{pmatrix}. \tag{27}$$

Since the cohomology sheaf of the middle monad in (21) is locally free, the morphism α in that diagram is a subbundle morphism. This together with (27) implies, again without loss of generality, that there exists a (2×2)-matrix $C = (c_{ij})$ such that

$$\alpha = \begin{pmatrix} y \\ -x \\ 0 \\ 0 \\ c_{11}x + c_{12}y + z \\ c_{21}x + c_{22}y + w \end{pmatrix} \tag{28}$$

It now follows from (28) and (18) that the skew symmetric (6×6)-matrix J of the following (2×2)-block form

$$J = \begin{pmatrix} Q & \mathbb{O} & -C^t \\ \mathbb{O} & \mathbb{O} & -\mathbb{1} \\ C & \mathbb{1} & \mathbb{O} \end{pmatrix}, \qquad \text{where} \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

satisfies the condition $\alpha = J\beta^t$. This means that, taking $-J$ for the matrix of the symplectic form $q : V_6 \to V_6^\vee$ with respect to the above choice of the basis in V_6, we obtain that α and β as morphisms satisfy the condition $\beta = \alpha^\vee \circ q$. In other words, the monad (17) is symplectic. Then by Corollary 2 its cohomology bundle E also admits a symplectic structure. □

4. Modified instanton monads

We will now study monads of the following form, with $a \geq 2$ and $k \geq 1$:

$$(29)\quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

which we call *modified instanton monads.* The family of isomorphism classes of bundles arising as cohomology of such monads will be denoted by $\mathcal{G}(a,k)$. Note that, so far, $\mathcal{G}(a,k)$ could possibly be empty.

Proposition 7. *For each $a \geq 2$ and $k \geq 1$, the family $\mathcal{G}(a,k)$ is non-empty and contains stable bundles, while every $[\mathcal{E}] \in \mathcal{G}(a,k)$ is μ-semistable. In addition, every $[\mathcal{E}] \in \mathcal{G}(a,1)$ is stable.*

Proof. Let F be an rank 2 instanton bundle of charge k. Let $a \geq 2$ and take $\sigma \in \mathrm{H}^0(F(2a))$ such that its zero locus $X = (\sigma)_0 := \{\sigma = 0\}$ is a curve; such σ always exists if F is a 't Hooft instanton bundle, for instance. Let Y be a complete intersection curve given by the intersection of two surfaces of degree a such that $X \cap Y = \emptyset$. According to [16, Lemma 4.8], there exists a bundle E and a section $\tau \in \mathrm{H}^0(E(a))$ such that $(\tau)_0 = Y \cup X$ which is given as cohomology of a monad of the form (29). In addition, since F is stable, X is not contained in any surface of degree a, hence neither is $Y \cup X$, and $\mathcal{E}$ is also stable.

It is straightforward to check that every $[\mathcal{E}] \in \mathcal{G}(a,k)$ satisfies $h^0(\mathcal{E}(-1)) = 0$, thus $\mathcal{E}$ is μ-semistable.

Now fix $k = 1$, and assume that there is $[\mathcal{E}] \in \mathcal{G}(a,1)$ satisfying $h^0(\mathcal{E}) \neq 0$. Setting $K := \ker \beta$, it follows that $h^0(K) \neq 0$, hence the quotient $K' := K/\mathcal{O}_{\mathbb{P}^3}$ fits into the following exact sequence

$$0 \to K' \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta'} \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0.$$

By [6, Theorem 2.7] K' is μ-stable. However, the monomorphism $\alpha : \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to K$ induces a monomorphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to K'$; by the μ-stability of K', we should have

$$-1 < \mu(K') = -\frac{a+1}{3} \implies a < 2,$$

providing the desired contradiction. □

Next, we provide a cohomological characterization for modified instanton bundles.

Proposition 8. *A vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ is the cohomology of a monad of the form (29) if and only if $\mathrm{H}^1_*(\mathcal{E})$ has one generator in degree $-a$ and k generators in degree -1, and its Chern classes are $c_1(\mathcal{E}) = 0$, and $c_2(\mathcal{E}) = a^2 + k$.*

Proof. The "only if" part is straightforward. If $\mathcal{E}$ is a self dual vector bundle on $\mathbb{P}^3$ with one generator in degree $-a$ and k generators in degree -1, then by [21, Theorem 2.3], $\mathcal{E}$ is cohomology of a monad of the type:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \oplus_{i=1}^{2k+4} \mathcal{O}_{\mathbb{P}^3}(k_i) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(a) \oplus V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$$

Computing the Chern class give us $c_2(\mathcal{E}) = a^2 + k - \sum_{i=1}^6 k_i^2$, since $c_2(\mathcal{E}) = a^2 + k$, we have $k_i = 0$ for all i. □

The modified instanton bundles are also related to usual instanton bundles of higher rank in a very important way. The precise relationship is outlined in the next couple of lemmas, and then summarized in Proposition 12 below.

Lemma 9. *(i) Given a vector bundle $[\mathcal{E}] \in \mathcal{G}(a,k)$, there exists a rank 4 instanton bundle E of charge k, and sections $\sigma \in H^0(E(a))$, $\tau \in H^0(E^\vee(a))$ such that the complex:*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \tag{30}$$

is a monad whose cohomology coincides with $\mathcal{E}$.
(ii) The construction of the monad (30) is functorial in the sense that, if $E \xrightarrow{\sim} E'$, then the induced isomorphism $E \xrightarrow{\sim} E'$ extends to an isomorphism of monads

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & E & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \\ & & \simeq \downarrow f & & \simeq \downarrow g & & \simeq \downarrow h & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma'} & E' & \xrightarrow{\tau'} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0. \end{array} \tag{31}$$

Proof. The monad (29) naturally includes into a diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^3}(a) & \xrightarrow{i} & V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0 \\ \uparrow & & \uparrow \beta & & \uparrow \\ 0 & \longrightarrow & V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & 0 \\ \uparrow & & \uparrow \alpha & & \uparrow \\ 0 & \longrightarrow & V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\varepsilon} & \mathcal{O}_{\mathbb{P}^3}(-a), \end{array} \tag{32}$$

where i, resp., ε, is a canonical monomorphism, resp., epimorphism. In the bounded derived category $D^b(\mathrm{Coh}\mathbb{P}^3)$ of the category $\mathrm{Coh}\mathbb{P}^3$ of coherent sheaves on $\mathbb{P}^3$ this diagram can be considered as a complex of morphisms

$$\mathcal{O}_{\mathbb{P}^3}(a)[-1] \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^3}(-a)[1], \tag{33}$$

and the image $[E]$ of the bundle E under the natural inclusion of $\mathrm{Coh}\mathbb{P}^3$ as a full subcategory in $D^b(\mathrm{Coh}\mathbb{P}^3)$ is the convolution of this complex:

$$[E] = \mathrm{Conv}\Big(\mathcal{O}_{\mathbb{P}^3}(a)[-1] \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^3}(-a)[1]\Big). \tag{34}$$

Note that the complex (33) clearly satisfies the conditions of Lemma 1.5 from [28], hence by this Lemma the convolution (34) is defined uniquely up to an isomorphism.

On the other hand, we may look at the diagram (32) as a double complex $K^{\bullet\bullet}$ in $\mathrm{Coh}\mathbb{P}^3$, and from the definition of convolution it follows that (34) coincides in $D^b(\mathrm{Coh}\mathbb{P}^3)$ with the middle cohomology of the total complex $\mathrm{Tot}^\bullet(K^{\bullet\bullet})$ of the double complex $K^{\bullet\bullet}$, i.e.:

$$[E] = \big[\mathcal{H}^0\big(\mathrm{Tot}^\bullet(K^{\bullet\bullet})\big)\big]. \tag{35}$$

To obtain the monad (30), define the morphisms

$$\tilde{\alpha} : V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \oplus (V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) \overset{\alpha}{\hookrightarrow} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3},$$

where the first morphism is the inclusion, and

$$\tilde{\beta} : V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\beta}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}(-a) \oplus (V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) \twoheadrightarrow V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1),$$

where the last morphism is the natural projection. Consider also

$$\tilde{\sigma} : \mathcal{O}_{\mathbb{P}^3}(-a) \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(-a) \oplus (V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) \overset{\alpha}{\hookrightarrow} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}, \text{ and}$$

$$\tilde{\beta} : V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \overset{\beta}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}(-a) \oplus (V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(a).$$

These morphisms give rise to a double complex $\mathcal{A}^{p,q}$ (we put $p = q = -1$ in the lower left corner):

(36)
$$\begin{array}{ccccc}
0 & \longrightarrow & V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \longrightarrow & 0 \\
\uparrow & & \uparrow \tilde{\beta} & & \uparrow \\
\mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\tilde{\sigma}} & V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\tilde{\tau}} & \mathcal{O}_{\mathbb{P}^3}(a) \\
\uparrow & & \uparrow \tilde{\alpha} & & \uparrow \\
0 & \longrightarrow & V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \longrightarrow & 0
\end{array}$$

Associated to $\mathcal{A}^{p,q}$, there is a spectral sequence ${}^{''}E_d^{p,q}$, whose first page is ${}^{''}E_1^{p,q} =: H^q(A^{\bullet,p})$. More precisely, ${}^{''}E_1^{p,q} = 0$ when $q \neq 0$ and ${}^{''}E_1^{p,1}$ is the complex

(37)
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

where σ and τ are the morphisms induced from $\tilde{\sigma}$ and $\tilde{\tau}$, respectively, and $E = \ker\tilde{\beta}/\mathrm{im}\tilde{\alpha}$; note that σ is a monomorphism, while τ is an epimorphism. It follows that ${}^{''}E_d^{p,q}$ converges in the second page with ${}^{''}E_2^{p,q} = 0$ when $p, q \neq 0$ and ${}^{''}E_2^{1,1} = \ker\tau/\operatorname{im}\sigma$. By general theory, $\ker\tau/\operatorname{im}\sigma$ coincides with the cohomology of the total complex associated to the bicomplex $\mathcal{A}^{p,q}$, which is precisely a monad as in display (29).

Note also that from (32) it follows easily that $\mathcal{H}^0\big(\mathrm{Tot}^\bullet(K^{*,*})\big)$ is isomorphic to the cohomology sheaf of the middle vertical complex in (36), so that, by (35),

(38)
$$[E] = \Big[\mathcal{H}^0\Big(V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1)\Big)\Big].$$

□

Lemma 10. *Given a monad*

(39)
$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{\sigma} E \xrightarrow{\tau} \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

with $\tilde{E}$ being a rank 4 instanton bundle of charge k, there is a monad of the form (29) *whose cohomology coincides with the cohomology of the above monad.*

Proof. Since E is the cohomology of a monad of the form

(40)
$$0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\beta}} V_k \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$$

for some subbundle morphism $\tilde{\alpha}$ and some epimorphism $\tilde{\beta}$, we have exact triples $0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \ker(\tilde{\beta}) \xrightarrow{\delta} \tilde{E} \to 0$, $0 \to \ker(\tilde{\beta}) \xrightarrow{j} V_k \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0.$

They induce the exact sequences

$$
\begin{aligned}
&\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-a),\ker(\tilde\beta)) \xrightarrow{\delta_*} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-a),E)\to 0,\\
&0\to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-a),\ker(\tilde\beta)) \xrightarrow{j_*} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-a),V_{2k+4}\otimes\mathcal{O}_{\mathbb{P}^3}),\\
&\mathrm{Hom}(V_{2k+4}\otimes\mathcal{O}_{\mathbb{P}^3},\mathcal{O}_{\mathbb{P}^3}(a)) \xrightarrow{j^*} \mathrm{Hom}(\ker(\tilde\beta),\mathcal{O}_{\mathbb{P}^3}(a))\to 0,\\
&0\to \mathrm{Hom}(E,\mathcal{O}_{\mathbb{P}^3}(a)) \xrightarrow{\delta^*} \mathrm{Hom}(\ker(\tilde\beta),\mathcal{O}_{\mathbb{P}^3}(a)).
\end{aligned}
\tag{41}
$$

Take any $\sigma'\in\delta_*^{-1}(\sigma)$ and any $\tilde\tau\in(j^*)^{-1}(\delta^*(\tau))$ and set $\tilde\sigma=j_*(\sigma)$. Consider the double complex (36) with these morphisms $\tilde\sigma$, $\tilde\tau$ and the morphisms a, b instead of $\tilde\alpha$, resp., $\tilde\beta$. Proceeding with this complex as in the proof of Lemma (9), we obtain the monad (37) with the cohomology sheaf E. □

Next, we argue that the instanton bundle E obtained in Lemma 9 comes with a natural symplectic structure.

Lemma 11. *If E is a rank 4 instanton bundle of charge k that fits in a monad of the form (30), such that the cohomology is a vector bundle, then E admits a symplectic structure, and τ is determined by σ.*

Proof. Since E is a rank 2 vector bundle with $c_1(E)=0$, there is a (unique up to scaling) symplectic isomorphism $\varphi:\ E\xrightarrow{\simeq}E^\vee$. By Corollary 2, there is an isomorphism of monads:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\sigma} & E & \xrightarrow{\tau} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0\\
 & & \simeq\downarrow g & & \simeq\downarrow\varphi & & \simeq\downarrow h & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{\tau^\vee} & E^\vee & \xrightarrow{\sigma^\vee} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0
\end{array}
$$

such that $\varphi^\vee=-\varphi$, so (E,φ) is a symplectic instanton bundle, and $\tau=\sigma^\vee\circ\varphi$. □

Putting Lemmas 9, 10 and 11 together, we obtain the following statement.

Proposition 12. *A rank 2 bundle $\mathcal{E}$ belongs to $\mathcal{G}(a,k)$, i.e., $\mathcal{E}$ is the cohomology of a monad of the form (29) if and only if it is also the cohomology $\mathcal{E}=\mathcal{H}^0(A_{E,\varphi,\sigma})$ of a monad of the form:*

$$
A_{E,\varphi,\sigma}:\qquad 0\to\mathcal{O}_{\mathbb{P}^3}(-a)\xrightarrow{\sigma}E\xrightarrow{\sigma^\vee\circ\varphi}\mathcal{O}_{\mathbb{P}^3}(a)\to 0,
\tag{42}
$$

where (E,φ) is a rank 4 symplectic instanton bundle of charge k.

5. Set $\mathcal{G}(a,1)$ and related families of sheaves

We introduce a piece of notation which we will use below. Denote by $\mathcal{I}(k)$ the set of isomorphism classes of symplectic rank 4 instanton bundles with $c_2=k$. as before, let V_k and V_{2k+4} be the fixed vector spaces of dimensions k and $2k+4$, respectively, and let $(\wedge^2V_{2k+4}^\vee)^0$ be an open subset of the vector space $\wedge^2V_{2k+4}^\vee$ consisting of nondegenerate symplectic forms on V_{2k+4}. Next, for a given morphism $\tilde\alpha:V_k\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to V_{2k+4}\otimes\mathcal{O}_{\mathbb{P}^3})$ we denote by a the homomorphism $V_k\otimes U_4\to V_{2k+4}$ corresponding to the morphism $\tilde\alpha$ under the isomorphism $\mathrm{Hom}(V_k\otimes$

$\mathcal{O}_{\mathbb{P}^3}(-1), V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3}) \cong W := \mathrm{Hom}(V_k \otimes U_4, V_{2k+4})$, where $U_4 := H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$. We will call $\tilde{\alpha}$ the *morphism associated to* $a \in W$.

Recall the description of symplectic rank 4 instantons (E, φ) in terms of symplectic monads (43) below. Namely, for a given point

$$m = (a, q) \in W \times (\wedge^2 V_{2k+4}^\vee)^0$$

consider the monad (40) in which $\tilde{\alpha}$ the morphism associated to the homomorphism a, and the morphism $\tilde{\beta}$ is such that $\tilde{\beta} = \tilde{\alpha}^t(q)$, where $\tilde{\alpha}^t(q)$ is the composition $V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{q \otimes \mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}}} V_{2k+4}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^\vee} V_k^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$:

$$A_m : \quad 0 \to V_k \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_{2k+4} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^t(q)} V_k^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{43}$$

We call A_m a *symplectic monad*. We also will denote by $\mathcal{H}^0(A_m)$ the cohomology bundle of the monad A_m.

Consider the set $\mathcal{M}(k)$ of symplectic monads (43):

$$\mathcal{M}(k) = \{(a, q) \in W \times (\wedge^2 V_{2k+4}^\vee)^0 \mid (a, q) \text{ satisfies the conditions (i)-(ii)}\} \tag{44}$$

where:
(i) the morphism $\tilde{\alpha}$ associated to a is a subbundle morphism,
(ii) the composition $\tilde{\alpha}^t(q) \circ \tilde{\alpha}$ is the zero morphism.
Since W is a vector space, and the condition (i), resp., (ii) is an open, resp., closed condition on the point $a \in W$, it follows that $\mathcal{M}(k)$ has a natural structure of a locally closed subscheme of the affine space $W \times \wedge^2 V_{2k+4}^\vee$. In the sequel we will assume that this scheme structure is reduced.

From now on we will restrict to the case $k = 1$. Set $\widetilde{M} := \mathcal{M}(1)$. Note that the condition (i) of the definition of $\mathcal{M}(k)$ is empty in the case $k = 1$, since in this case the the vanishig of $\wedge^2(V_1^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ clearly implies $\alpha^t(q) \circ \alpha = 0$. Hence, $\widetilde{M}$ is a nonempty open (hence dense) subset of the affine space $W \times \wedge^2 V_6^\vee$, where $W = \mathrm{Hom}(V_1 \otimes U_4, V_6) \simeq \mathbf{k}^{24}$. In particular, $\widetilde{M}$ is irreducible and

$$\dim \widetilde{M} = \dim W + \dim \wedge^2 V_6^\vee = 45. \tag{45}$$

Proposition 13. *Any rank 4 instanton of charge 1 appears as a cohomology bundle of a symplectic monad*

$$A_m : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\tilde{\alpha}} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\alpha}^t(q)} \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{46}$$

from $\widetilde{M}$.

Proof. Let E be a rank 4 instanton of charge 1. According to Proposition (6), E admits a symplectic structure $\varphi : E \xrightarrow{\sim} E^\vee$. It then known from [7, Section 3] that, under the condition $h^0(E) = h^1(-2) = 0$ on a symplectic bundle E, this bundle is a cohomology of a symplectic monad from $\widetilde{M}$. However, the proof given therein, works without changes under the slightly weaker conditions (10) used in the Definition 3. □

On $\widetilde{\mathbf{M}} = \mathbb{P}^3 \times \widetilde{M}$ there is the universal symplectic monad

$$\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}} : \quad 0 \to \mathcal{O}_{\widetilde{\mathbf{M}}}(-1, 0) \xrightarrow{\boldsymbol{\alpha}} V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} \xrightarrow{\boldsymbol{\alpha}^t} \mathcal{O}_{\widetilde{\mathbf{M}}}(1, 0) \to 0 \tag{47}$$

with the cohomology sheaf

$$\widetilde{\mathbf{E}} = \ker \boldsymbol{\alpha}^t / \mathrm{im} \boldsymbol{\alpha} \tag{48}$$

Here $\boldsymbol{\alpha}^t = \boldsymbol{\alpha}^\vee \circ \mathbf{q}_{\widetilde{\mathbf{M}}}$ and

$$\mathbf{q}_{\widetilde{\mathbf{M}}} : \ V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} \xrightarrow{\simeq} V_6^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{M}}}$$

is the tautological symplectic structure on $V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}}$. From now on we fix an isomorphism of the monad $\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}}$ with its dual monad $\boldsymbol{\mathcal{A}}^\vee_{\widetilde{\mathbf{M}}}$ by the following diagram:

$$\begin{array}{ccccccccc}
\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}} : 0 & \longrightarrow & \mathcal{O}_{\widetilde{\mathbf{M}}}(-1,0) & \xrightarrow{\boldsymbol{\alpha}} & V_6 \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} & \xrightarrow{\boldsymbol{\alpha}^t} & \mathcal{O}_{\widetilde{\mathbf{M}}}(1,0) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle -\mathrm{id}\ \simeq} & & \Big\downarrow {\scriptstyle \mathbf{q}_{\widetilde{\mathbf{M}}}\ \simeq} & & \Big\downarrow {\scriptstyle \mathrm{id}\ \simeq} & & \\
\boldsymbol{\mathcal{A}}^\vee_{\widetilde{\mathbf{M}}} : \ 0 & \longrightarrow & \mathcal{O}_{\widetilde{\mathbf{M}}}(-1,0) & \xrightarrow{(\boldsymbol{\alpha}^t)^\vee} & V_6^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{M}}} & \xrightarrow{\boldsymbol{\alpha}^\vee} & \mathcal{O}_{\widetilde{\mathbf{M}}}(1,0) & \longrightarrow & 0
\end{array}$$

This isomorphism induces the symplectic structure

$$\boldsymbol{\varphi}_{\widetilde{\mathbf{M}}} : \widetilde{\mathbf{E}} \xrightarrow{\simeq} \widetilde{\mathbf{E}}^\vee, \tag{49}$$

so that, for any $m \in \widetilde{M}$,

$$E_m = \widetilde{\mathbf{E}}|_{\mathbb{P}^3 \times \{m\}}, \qquad \varphi_m = \boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}|_{\mathbb{P}^3 \times \{m\}} : E_m \xrightarrow{\simeq} E_m^\vee, \tag{50}$$

is a symplectic rank 4 instanton on $\mathbb{P}^3$. Note that, by the universality of the space $\widetilde{M}$, for any given symplectic rank 4 instanton (E, φ), there exists a unique point $m \in \widetilde{M}$ such that $(E, \varphi) = (E_m, \varphi_m)$, where E_m and φ_m are given by (50).

Let $p_{\widetilde{M}} : \widetilde{\mathbf{M}} \to \widetilde{M}$ be the projection. It follows from (16) and the Base Change that the $\mathcal{O}_{\widetilde{M}}$-sheaf

$$\widetilde{\mathbf{U}} := p_{\widetilde{M}*} \widetilde{\mathbf{E}} \tag{51}$$

is a rank 2 locally free sheaf and there is an exact triple on $\widetilde{\mathbf{M}}$:

$$0 \to \widetilde{\mathbf{U}}_{\widetilde{\mathbf{M}}} \xrightarrow{\mathbf{ev}} \widetilde{\mathbf{E}} \to \widetilde{\mathbf{N}} \to 0, \qquad \widetilde{\mathbf{N}} := \mathrm{coker}(\mathbf{ev}), \tag{52}$$

and, for any $m \in \widetilde{M}$, the restriction of this triple onto $\mathbb{P}^3 \times \{m\}$ coincides with the triple (15) for $E = E_m$. We thus have a map

$$\Psi : \widetilde{M} \to \mathbb{P}^5 = \mathbb{P}(\wedge^2 V_4^\vee), \qquad m \mapsto [\widetilde{\mathbf{N}}|_{\mathbb{P}^3 \times \{m\}}]. \tag{53}$$

The map Ψ has the following explicit description. Given a point $m = (a, q) \in \widetilde{M}$, consider a homomorphism $f(a, q) : V_4 \xrightarrow{a} V_6 \xrightarrow{q} V_6^\vee \xrightarrow{a^\vee} V_4^\vee$. It is clearly skew-symmetric: $f(a, q) \in \wedge^2 V_4^\vee$. An easy diagram chasing with the display of the monad $\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}}|_{\mathbb{P}^3 \times \{m\}}$ (i. e., equivalently, of the monad (46)) using (52) shows that

$$\Psi(m) = \mathbf{k} f(a, q) \in \mathbb{P}(\wedge^2 V_4^\vee), \tag{54}$$

so that Ψ is a well-defined morphism. By the universality of the monad $\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}}$, Ψ is surjective.

We next consider the set

$$M := \{ m \in \widetilde{M} \mid \widetilde{\mathbf{N}}|_{\mathbb{P}^3 \times \{m\}} \text{ is locally free} \}. \tag{55}$$

From the definition om M it follows that it is a nonempty open subset of $\widetilde{M}$, hence it is irreducible, since $\widetilde{M}$ is irreducible. Denote

$$\mathbf{E} := \widetilde{\mathbf{E}}_{\mathbf{M}}, \qquad \boldsymbol{\varphi}_{\mathbf{M}} := \boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}|_{\mathbf{M}} : \mathbf{E} \xrightarrow{\simeq} \mathbf{E}^\vee, \tag{56}$$

where $\boldsymbol{\varphi}_{\widetilde{\mathbf{M}}}$ is the symplectic structure defined in (49),

$$\mathbf{U} := \widetilde{\mathbf{U}}_M, \qquad \mathbf{N} := \widetilde{\mathbf{N}}_{\mathbf{M}}. \tag{57}$$

Note that, by Lemma 4, for any $m \in M$, the triple (52) restricted onto $\mathbb{P}^3 \times \{m\}$ splits:

$$E_m = \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus N_m, \tag{58}$$

where N_m is a null correlation bundle. Moreover, in view of (25) and the Base Change, these splittings for all $m \in M$ yield the global splitting

$$\mathbf{E} = \mathbf{U}_{\mathbf{M}} \oplus \mathbf{N}. \tag{59}$$

Now, for $a \geq 2$ and any $m \in M$, the triple (15) twisted by $\mathcal{O}_{\mathbb{P}^3}(a)$, in which we set $E = E_m$, yields:

$$h^0(E_m(a)) = 4\binom{a+3}{3} - a - 2, \qquad h^i(E_m(a)) = 0, \quad i > 0. \tag{60}$$

Formulas (50), (60) and the Base Change show that the sheaf

$$F = p_{M*}(\mathbf{E}(a, 0)) \tag{61}$$

is a locally free $\mathcal{O}_M$-sheaf of rank $r = h^0(E_m(a))$. Consider the scheme

$$T = \mathbf{P}(F^\vee) \tag{62}$$

By the above, T is set-theoretically described as

$$T = \{(m, \mathbf{k}\sigma) \mid m \in M, \ 0 \neq \sigma \in H^0(E_m(a))\}, \tag{63}$$

and the natural projection $\rho : T \to M$, $(m, \mathbf{k}\sigma) \mapsto m$ is a locally trivial $\mathbb{P}^{r-1}$-bundle. Note that, since M is an open subset of the affine space W, it follows that T is an irreducible variety, and from (45) and (60) we have

$$\dim T = h^0(E_m(a)) - 1 + \dim M = 4\binom{a+3}{3} - a + 42. \tag{64}$$

On T and $\mathbf{M} = \mathbb{P}^3 \times M$ we have canonical morphisms $F_T^\vee \overset{\mathrm{ev}}{\twoheadrightarrow} L$ and $F_{\mathbf{M}} \xrightarrow{\mathrm{can}} \mathbf{E}(a, 0)$, respectively, where $L = \mathcal{O}_{\mathbf{P}(F^\vee)}(1)$ is the Grothendieck sheaf. Consider the composition of morphisms

$$\boldsymbol{\sigma} : \mathcal{O}_{\mathbb{P}^3} \boxtimes L^\vee \xrightarrow{\mathrm{ev}_{\mathbf{T}}^\vee} F_{\mathbf{T}} \xrightarrow{\mathrm{can}_{\mathbf{T}}} \mathbf{E}_{\mathbf{T}}(a, 0). \tag{65}$$

By definition, for any point $(m, \mathbf{k}\sigma) \in T$ the restriction $\boldsymbol{\sigma}|_{\mathbb{P}^3 \times \{(m, \mathbf{k}\sigma)\}}$ coincides, up to a twist by $\mathcal{O}_{\mathbb{P}^3}(-a)$, with the morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$. In view of (58) we may represent σ as

$$\sigma = (\sigma_1, \sigma_2), \qquad \sigma_1 \in H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a)), \qquad \sigma_2 \in H^0(N_m(a)).$$

For the pair $\sigma = (\sigma_1, \sigma_2) \neq (0, 0)$ we will adopt in the sequel, together with the notation $\mathbf{k}\sigma$ for the set $\{(\lambda\sigma_1, \lambda\sigma_2) | \lambda \in \mathbf{k}^\times\}$, the equivalent notation:

$$[\sigma_1 : \sigma_2] := \{(\lambda\sigma_1, \lambda\sigma_2) | \lambda \in \mathbf{k}^\times\}, \tag{66}$$

and also understand $[\sigma_1 : \sigma_2]$ as a point of the projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a)) \oplus H^0(N_m(a)))$. Under this notation, define an open subset S of T as

$$\begin{aligned} S := \{(m, [\sigma_1 : \sigma_2]) \in T \mid (i)\ \sigma = (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \\ \text{is a subbundle morphism and } (ii)\ \sigma_1 \neq 0, \ \sigma_2 \neq 0.\}. \end{aligned} \tag{67}$$

The subset S is clearly open in T. Moreover, it is nonempty. Indeed, for any point $m \in M$, E_m decomposes as in (58). Take any $a \geq 2$. Since the direct summand N_m is a null correlation bundle, it follows quickly from the triple (13) for $N = N_m$, twisted by $\mathcal{O}_{\mathbb{P}^3}(a)$, that $N_m(a)$ is generated by global sections. From this it follows easily (cf. [14, Proof of Prop. 1.4]) that a general section $\sigma_1 \in H^0(N_m(a))$ has 1-dimensional zero-locus $Z(\sigma_1)$. Next, since a general section $\sigma_2 \in H^0(\mathcal{O}_{\mathbb{P}^3}^{\oplus 2}(a))$ has for its zero locus a complete intersection curve $Z(\sigma_2) = D_1 \cap D_2$ for two surfaces D_1, D_2 of degree a, it follows that for general D_1 and D_2 we have $Z(\sigma_1) \cap Z(\sigma_2) = \emptyset$. Hence, the section $\sigma = (\sigma_1, \sigma_2) \in H^0(E_m(a))$ has no seroes and therefore defines a subbundle morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m$.

It follows that S is irreducible and dense in T since T is irreducible. The morphism $\boldsymbol{\sigma}_{\mathbf{S}}$ is included in the monad $\boldsymbol{\mathcal{A}} := (\boldsymbol{\mathcal{A}}_{\widetilde{\mathbf{M}}})_{\mathbf{S}}$ on $\mathbf{S}$:

$$\boldsymbol{\mathcal{A}}:\ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^\vee \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}} \mathbf{E}_{\mathbf{S}} \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L \to 0 \tag{68}$$

where $\boldsymbol{\sigma}_{\mathbf{S}}^t$ is the composition $\mathbf{E}_{\mathbf{S}} \xrightarrow{\boldsymbol{\varphi}_{\mathbf{S}}} \mathbf{E}_{\mathbf{S}}^\vee \xrightarrow{\boldsymbol{\sigma}_{\mathbf{S}}^\vee} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L$. By construction, for any point $(m, \mathbf{k}\sigma) \in S$, the restriction of the monad $\boldsymbol{\mathcal{A}}$ onto $\mathbb{P}^3 \times \{(m, \mathbf{k}\sigma)\}$ is isomorphic to the monad $A_{E_m, \varphi_m, \sigma}$ in (42). Hence,

$$\mathcal{H}^0(\boldsymbol{\mathcal{A}})_{\mathbb{P}^3 \times \{(m,\mathbf{k}\sigma)\}} = \mathcal{H}^0(A_{E_m,\varphi_m,\sigma}), \qquad (m, \mathbf{k}\sigma) \in S. \tag{69}$$

In (73)-(80) below we will extend the constructions (61)-(63), (67)-(69) of the data F, T, S, $\boldsymbol{\mathcal{A}}$ and $\mathcal{H}^0\boldsymbol{\mathcal{A}}$) over M to the constructions of the corresponding data $\widetilde{F}$, $\widetilde{T}$, $\widetilde{S}$, $\widetilde{\boldsymbol{\mathcal{A}}}$ and $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$ over $\widetilde{M}$. As a consequence, it will follow that:

$$F = \widetilde{F}_{\mathbf{M}}, \qquad T = M \times_{\widetilde{M}} \widetilde{T}, \tag{70}$$

$$S \xhookrightarrow{\text{open dense}} \widetilde{S}, \tag{71}$$

$$\boldsymbol{\mathcal{A}} = (\widetilde{\boldsymbol{\mathcal{A}}})_{\mathbf{S}}, \qquad \mathcal{H}^0(\boldsymbol{\mathcal{A}}) = (\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}}))_{\mathbf{S}}. \tag{72}$$

For this, we first remark that formulas (60) are still true for any $m \in \widetilde{M}$, so that the sheaf

$$\widetilde{F} := p_{\widetilde{M}*}(\widetilde{\mathbf{E}}(a, 0)) \tag{73}$$

is a locally free $\mathcal{O}_{\widetilde{M}}$-sheaf of rank $r = h^0(E_m(a))$ given by (60), and the scheme

$$\widetilde{T} := \mathbf{P}(\widetilde{F}^\vee). \tag{74}$$

is set-theoretically described as

$$\widetilde{T} = \{(m, \mathbf{k}\sigma) \mid m \in \widetilde{\mathcal{M}},\ 0 \neq \sigma \in H^0(E_m(a))\}. \tag{75}$$

The natural projection $\widetilde{\rho} : \widetilde{T} \to \widetilde{M},\ (m, \mathbf{k}\sigma) \mapsto m$ is a locally trivial $\mathbb{P}^{r-1}$-bundle, so that, since $\widetilde{M}$ is an open subset of the vector space W, it follows that $\widetilde{T}$ is an irreducible variety of dimension

$$\dim \widetilde{T} = h^0(E_m(a)) - 1 + \dim \widetilde{M} = 4\binom{a+3}{3} - a + 42. \tag{76}$$

Next, we have an open subset $\widetilde{S}$ of $\widetilde{T}$ defined as

$$\widetilde{S} := \{(m, \mathbf{k}\sigma) \in \widetilde{T} \mid \sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \text{ is a subbundle morphism.}\} \tag{77}$$

Since the condition (ii) in (67) is open, comparing (67) and (77) we obtain that S is an open subset of $T \cap \widetilde{S}$, where the intersection is taken in $\widetilde{T}$. Since S is nonempty and $\widetilde{T}$ is irreducible, (71) follows and, moreover,

$$\widetilde{\rho}_S = \rho. \tag{78}$$

Next, we have the extension of the universal monad (68) from $\mathbf{S}$ to $\widetilde{\mathbf{S}}$:

$$\widetilde{\boldsymbol{\mathcal{A}}}:\ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^\vee \xrightarrow{\boldsymbol{\sigma}} \widetilde{\mathbf{E}}_{\widetilde{\mathbf{S}}} \xrightarrow{\boldsymbol{\sigma}^t} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L \to 0, \tag{79}$$

satisfying the relation similar to (69):

$$\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})|_{\mathbb{P}^3 \times \{(m,\mathbf{k}\sigma)\}} = \mathcal{H}^0(A_{E_m,\varphi_m,\sigma}), \qquad (m,\mathbf{k}\sigma) \in \widetilde{S}. \tag{80}$$

Clearly, the relations (70) follow from (56), (73)-(75) and the Base Change; respectively, the relations (72) follow from (79)-(80).

Now (71), (80), and Proposition 12 together with the irreducibility of $\widetilde{S}$ yield

Proposition 14. *(i) For $a \geq 2$, the set $\mathcal{G}(a,1)$ of isomorphism classes of cohomology sheaves of monads (29) for $k=1$ is the image of the modular morphism*

$$\Phi_{\widetilde{S}}:\ \widetilde{S} \to \mathcal{B}(a^2+1), \qquad (m,\mathbf{k}\sigma) \mapsto [\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})|_{\mathbb{P}^3 \times \{(m,\mathbf{k}\sigma)\}}],$$

defined by the family $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$ of sheaves over $\widetilde{S}$. Its closure $\overline{\mathcal{G}(a,1)}$ in $\mathcal{B}(a^2+1)$ is an irreducible variety.
(ii) The set $\mathcal{G}(a,1)_0 := \Phi_S(S)$ is dense in $\overline{\mathcal{G}(a,1)}$.

In the remaining part of this section we will construct a new family of monads $\mathbf{A_Y}$ on $\mathbb{P}^3$, with base Y and cohomology sheaves belonging to $\mathcal{G}(a,1)$, for which the related modular morphism

$$\Phi_Y:\ Y \to \mathcal{B}(a^2+1), \qquad y \mapsto [\mathcal{H}^0(\mathbf{A}_Y)|_{\mathbb{P}^3 \times \{y\}}]$$

has $\mathcal{G}(a,1)_0$ as its image (see Proposition 15 below). This family will be used in the next Section to prove one of the main results of the paper - the rationality of $\overline{\mathcal{G}(a,1)}$.

To construct the variety Y, consider the moduli space of $B := \mathcal{B}(1)$ of locally free null correlation bundles on $\mathbb{P}^3$. This is well known to be isomorphic to $\mathbb{P}^5 \smallsetminus G(2,4)$, where $G(2,4)$ is the Plücker hyperquadric (see, e.g., [27, Thm. 4.3.4]). Moreover, on $\mathbf{B} = \mathbb{P}^3 \times B$ there is the universal family $\boldsymbol{\mathcal{N}}$ of null correlation bundles. Consider the vector bundle

$$\boldsymbol{\mathcal{E}} = V_2 \otimes \mathcal{O}_{\mathbf{B}} \oplus \boldsymbol{\mathcal{N}}. \tag{81}$$

and denote

$$E_b = \boldsymbol{\mathcal{E}}|_{\mathbb{P}^3 \times \{b\}}, \quad N_b = \boldsymbol{\mathcal{N}}|_{\mathbb{P}^3 \times \{b\}}, \qquad b \in B, \tag{82}$$

so that

$$E_b = V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b, \qquad b \in B. \tag{83}$$

Next, consider the varieties

$$B_1 = \mathbf{V}(\wedge^2(V_2 \otimes \mathcal{O}_B)) \smallsetminus \{0-\mathrm{section}\} \to B,$$
$$B_2 = \mathbf{V}(p_{B*}(\wedge^2 \boldsymbol{\mathcal{N}})) \smallsetminus \{0-\mathrm{section}\} \to B.$$

Over these varieties there are tautological symplectic structures

$$\varphi_{\mathbf{B}_1} : V_2 \otimes \mathcal{O}_{\mathbf{B}_1} \xrightarrow{\simeq} (V_2 \otimes \mathcal{O}_{\mathbf{B}_1})^\vee, \qquad \varphi_{\mathbf{B}_2} : \boldsymbol{\mathcal{N}}_{\mathbf{B}_2} \xrightarrow{\simeq} \boldsymbol{\mathcal{N}}^\vee_{\mathbf{B}_2}. \tag{84}$$

Consider the variety

$$\widetilde{B} := B_1 \times_B B_2 \tag{85}$$

with natural projections $\widetilde{\zeta}_i : \widetilde{B} \to B_i,\ i = 1,2$, and $\zeta : \widetilde{B} \to B$. The symplectic structures (23) induce on the vector bundle

$$\boldsymbol{\mathcal{E}}_{\widetilde{\mathbf{B}}} = V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{B}}} \oplus \boldsymbol{\mathcal{N}}_{\widetilde{\mathbf{B}}} \tag{86}$$

the symplectic structure

$$\boldsymbol{\varphi}_{\widetilde{\mathbf{B}}} = \boldsymbol{\varphi}_1 \oplus \boldsymbol{\varphi}_2 :\ \boldsymbol{\mathcal{E}}_{\widetilde{\mathbf{B}}} \to \boldsymbol{\mathcal{E}}_{\widetilde{\mathbf{B}}}^{\vee}, \tag{87}$$

where

$$\boldsymbol{\varphi}_1 := (\varphi_{\mathbf{B}_1})_{\widetilde{\mathbf{B}}} : V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{B}}} \xrightarrow{\simeq} (V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{B}}})^{\vee}, \qquad \boldsymbol{\varphi}_2 := (\varphi_{\mathbf{B}_2})_{\widetilde{\mathbf{B}}} : \boldsymbol{\mathcal{N}}_{\widetilde{\mathbf{B}}} \xrightarrow{\simeq} \boldsymbol{\mathcal{N}}_{\widetilde{\mathbf{B}}}^{\vee}.$$

Under the notation (82), we thus have the following description of the varieties B_1, B_2 and $\widetilde{B}$:

$$\begin{aligned} &B_1 = \{(b,\varphi_1) \mid b \in B,\ V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \underset{\simeq}{\xrightarrow{\varphi_1}} (V_2 \otimes \mathcal{O}_{\mathbb{P}^3})^{\vee} \text{ is a symplectic structure}\}, \\ &B_2 = \{(b,\varphi_2) \mid b \in B,\ N_b \underset{\simeq}{\xrightarrow{\varphi_2}} N_b^{\vee} \text{ is a symplectic structure}\}, \\ &\widetilde{B} = \{(b,\varphi_1,\varphi_2) \mid (b,\varphi_i) \in B_i,\ i = 1,2\}. \end{aligned} \tag{88}$$

The following constructions (see (89)-(100)) are parallel to the constructions (61)-(68). Twisting the equality (83) by $\mathcal{O}_{\mathbb{P}^3}(a)$, we obtain as in (60): $h^0(E_b(a)) = 4\binom{a+3}{3} - a - 2,\ h^i(E_b(a)) = 0,\ i > 0$. Thus, as in (61), the sheaf

$$F_B = p_{B*}(\boldsymbol{\mathcal{E}}(a,0)) \tag{89}$$

is a locally free $\mathcal{O}_B$-sheaf of rank $r = h^0(E_b(a))$. Consider the variety

$$\mathcal{T} := \mathbf{P}(F_B^{\vee}). \tag{90}$$

Similarly to (63) we have

$$\mathcal{T} = \{(b, \mathbf{k}\sigma) \mid b \in B,\ 0 \neq \sigma \in H^0(E_b(a))\}, \tag{91}$$

For any point $(b, \mathbf{k}\sigma) \in \mathcal{T}$ in view of (83) we may represent σ as a pair $\sigma = (\sigma_1, \sigma_2),\ \sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(a)),\ \sigma_2 \in H^0(N_b(a))$, Thus, using the notation (66) we can rewrite (91) as

$$\mathcal{T} = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B,\ [\sigma_1 : \sigma_2] \in \mathbb{P}(H^0(E_b(a)))\}, \tag{92}$$

On the other hand, representing σ as a morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_b$, we see that, when $(b, \mathbf{k}\sigma)$ runs through $\mathcal{T}$, the morphisms σ, as in (65), globalize to the morphism on $\boldsymbol{\mathcal{T}}$:

$$\boldsymbol{\sigma}_{\boldsymbol{\mathcal{T}}} :\ \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L_{\mathcal{T}}^{\vee} \to \boldsymbol{\mathcal{E}}_{\boldsymbol{\mathcal{T}}}, \tag{93}$$

where $L_{\mathcal{T}}$ is the Grothendieck sheaf $\mathcal{O}_{\mathcal{T}/B}(1)$.

Next, similar to (67), we define an open subset $\mathcal{S}$ of $\mathcal{T}$ as

$$\begin{aligned} \mathcal{S} := \{(b, [\sigma_1 : \sigma_2]) \in \mathcal{T} \mid (i)\ (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \\ \text{is a subbundle morphism and } (ii)\ \sigma_1 \neq 0,\ \sigma_2 \neq 0.\}. \end{aligned} \tag{94}$$

Note that $\mathcal{S}$ is a nonempty set. (The proof mimics that of nonemptiness of the subset M of T given in paragraph after (67).)

By the Base Change, the sheaf

$$F_{\widetilde{B}} = p_{\widetilde{B}*}(\boldsymbol{\mathcal{E}}_{\widetilde{\mathbf{B}}}(a,0)) \tag{95}$$

is isomorphic to the sheaf $(F_B)_{\widetilde{B}}$. Thus the variety $\widetilde{Y} := \mathbf{P}(F^\vee_{\widetilde{B}})$ is isomorphic to $\widetilde{B} \times_B \mathcal{T}$:

$$\widetilde{Y} = \widetilde{B} \times_B \mathcal{T}. \tag{96}$$

Thus by (88) and (91) we have

$$\widetilde{Y} = \{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mid (b, \varphi_1, \varphi_2) \in \widetilde{B},\ [\sigma_1 : \sigma_2] \in \mathbb{P}(H^0(E_b(a)))\}, \tag{97}$$

and the natural projection $\widetilde{Y} \to \widetilde{B},\ (\beta, \mathbf{k}\sigma) \mapsto \beta$ is a locally trivial $\mathbb{P}^{r-1}$-bundle.

We now use (96) and the open subset $\mathcal{S}$ of $\mathcal{T}$ to define an open subset Y of $\widetilde{Y}$ as

$$Y := \widetilde{B} \times_B \mathcal{S}. \tag{98}$$

Here Y is a nonempty open in $\widetilde{Y}$ since $\mathcal{S}$ is nonempty. It follows that Y is irreducible and dense in $\widetilde{Y}$ since $\widetilde{Y}$ is irreducible. In addition, using (94) and (97), we obtain the description of Y as:

$$\begin{aligned} Y = \{(b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \in \widetilde{Y} \mid (i)\ (\sigma_1, \sigma_2) : \mathcal{O}_{\mathbb{P}^3}(-a) \to E_m \\ \text{is a subbundle morphism and } (ii)\ \sigma_1 \neq 0,\ \sigma_2 \neq 0.\}. \end{aligned} \tag{99}$$

The morphism $\boldsymbol{\sigma}_{\mathbf{Y}} := (\boldsymbol{\sigma}_{\mathcal{T}})_{\mathbf{Y}}$ is included in the universal monad on $\mathbf{Y}$:

$$\mathbf{A}_{\mathbf{Y}} :\ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L_Y^\vee \xrightarrow{\boldsymbol{\sigma}_{\mathbf{Y}}} \boldsymbol{\mathcal{E}}_{\mathbf{Y}} \xrightarrow{\boldsymbol{\sigma}^t_{\mathbf{Y}}} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L_Y \to 0, \tag{100}$$

where $L_Y = (L_{\mathcal{T}})_Y$ and $\boldsymbol{\sigma}^t_Y$ is the composition $\boldsymbol{\mathcal{E}}_{\mathbf{Y}} \xrightarrow{\boldsymbol{\varphi}_{\mathbf{Y}}} \boldsymbol{\mathcal{E}}^\vee_{\mathbf{Y}} \xrightarrow{\boldsymbol{\sigma}^\vee_{\mathbf{Y}}} \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes L_Y$. By construction, for any point $(\beta, \mathbf{k}\sigma) \in Y$, $\beta = (b, \varphi_1, \varphi_2)$, the restriction of the monad $\mathbf{A}_{\mathbf{Y}}$ onto $\mathbb{P}^3 \times \{(\beta, \mathbf{k}\sigma)\}$ is isomorphic to the monad $A_{E_b, \varphi_1 \oplus \varphi_2, \sigma}$ in (42). Hence,

$$\mathcal{H}^0(\mathbf{A}_Y)|_{\mathbb{P}^3 \times \{(\beta, \mathbf{k}\sigma)\}} = \mathcal{H}^0(A_{E_b, \varphi_1 \oplus \varphi_2, \sigma}), \qquad (\beta, \mathbf{k}\sigma) \in Y, \qquad \beta = (b, \varphi_1, \varphi_2). \tag{101}$$

Now consider the rank 2 the vector bundle $\mathbf{U}$ on M defined in (51)and (57) and its associated principal frame bundle

$$I := \mathbf{Isom}(V_2 \otimes \mathcal{O}_M, \mathbf{U}) \xrightarrow{\xi} M \tag{102}$$

together with the tautological isomorphism on I

$$V_2 \otimes \mathcal{O}_I \xrightarrow{\sim} \mathbf{U}_I. \tag{103}$$

Consider the sheaves $\mathbf{E}_{\mathbf{I}}$ and $\mathbf{N}_{\mathbf{I}}$. Applying to (59) the functor $\boldsymbol{\xi}^*$ and using (103) we obtain the isomorphism

$$\mathbf{E}_{\mathbf{I}} \cong V_2 \otimes \mathcal{O}_{\mathbf{I}} \oplus \mathbf{N}_{\mathbf{I}}. \tag{104}$$

Besides, by (56), we have a symplectic structure on $\mathbf{E}_{\mathbf{I}}$:

$$\boldsymbol{\varphi}_{\mathbf{I}} := (\boldsymbol{\varphi}_{\mathbf{M}})_{\mathbf{I}} : \mathbf{E}_{\mathbf{I}} \xrightarrow{\simeq} \mathbf{E}^\vee_{\mathbf{I}}. \tag{105}$$

This symplectic structure in view of (104) splits into a direct sum of two symplectic structures

$$\begin{aligned} \boldsymbol{\varphi}_{\mathbf{I}} = \boldsymbol{\varphi}_{\mathbf{I},1} \oplus \boldsymbol{\varphi}_{\mathbf{I},2}, \\ \boldsymbol{\varphi}_{\mathbf{I},1} : V_2 \otimes \mathcal{O}_{\mathbf{I}} \xrightarrow{\simeq} (V_2 \otimes \mathcal{O}_{\mathbf{I}})^\vee, \qquad \boldsymbol{\varphi}_{\mathbf{I},2} : \mathbf{N}_{\mathbf{I}} \xrightarrow{\simeq} \mathbf{N}^\vee_{\mathbf{I}}. \end{aligned} \tag{106}$$

Remark that, by the definition of the morphism Ψ given in (53) and (54), we have $\Psi(M) = B$. Now, comparing (86)-(88) with (104)-(106), we obtain a morphism

$$\Gamma : I \to \widetilde{B}, \quad x \mapsto (b, \varphi_1, \varphi_2), \quad b = \Psi(\xi(x)), \quad \varphi_i = \boldsymbol{\varphi}_{\mathbf{I},i}|_{\mathbb{P}^3 \times \{x\}},\ i = 1, 2, \tag{107}$$

such that

$$\mathbf{E_I} \cong (\boldsymbol{\mathcal{E}}_{\widetilde{\mathbf{B}}})_{\mathbf{I}}, \qquad \boldsymbol{\varphi}_{\mathbf{I}} \cong (\varphi\ _{widetilde\mathbf{B}})_{\mathbf{I}}, \tag{108}$$

and these isomorphisms are compatible with the direct sum decompositions (104), (106) and (86), (87). From (107) and the surjectivity of Ψ it follows that Γ is also surjective.

Set

$$X := I \times_M S, \qquad Y \xleftarrow{\Gamma_Y} X \xrightarrow{\xi_S} S, \qquad F_I := p_{I*}(\mathbf{E_I}(a,0)). \tag{109}$$

From (61), (89), (108) and the Base Change we obtain

$$F_I \cong\ (F_{\widetilde{B}})_I,$$

so that, in view of (96) and (62), the variety $\widetilde{X} := \mathbf{P}(F_{\mathcal{X}}^{\vee})$ satisfies the relations

$$I \times_M T = \widetilde{X} = I \times_{\widetilde{B}} \widetilde{Y}. \tag{110}$$

The definition of X (see (109)) and the first relation (110) imply that there exists an open embedding $X \hookrightarrow \widetilde{X}$ such that $X = \widetilde{X} \times_T S$. Therefore, comparing the descriptions (99) and (67) of Y and S and using the second relation (110), we obtain:

$$X = I \times_{\widetilde{B}} Y. \tag{111}$$

This together with (108) implies that

$$\mathbf{E_X} = (\boldsymbol{\mathcal{E}}_{\mathbf{Y}})_{\mathbf{X}}. \tag{112}$$

Moreover, since $X = I \times_M S$, we have

$$\boldsymbol{\mathcal{A}}_{\mathbf{X}} = (\mathbf{A_Y})_{\mathbf{X}}, \tag{113}$$

where the monads $\boldsymbol{\mathcal{A}}$ and $\mathbf{A_Y}$ were defined in (68) and (100), respectively.

Consider the modular morphisms

$$\Phi_X : X \to \mathcal{B}(a^2+1), \qquad \Phi_Y : Y \to \mathcal{B}(a^2+1), \tag{114}$$

$$\Phi_S : S \to \mathcal{B}(a^2+1), \quad \Phi_{\widetilde{S}} : \widetilde{S} \to \mathcal{B}(a^2+1), \tag{115}$$

defined by the (families of) sheaves $\mathcal{H}^0(\boldsymbol{\mathcal{A}}_{\mathbf{X}})$, $\mathcal{H}^0(\mathbf{A_Y})$, $\mathcal{H}^0(\boldsymbol{\mathcal{A}})$, $\mathcal{H}^0(\widetilde{\boldsymbol{\mathcal{A}}})$, respectively. From (113), (111) and (109) it follows that Φ_X factors through Γ_Y and through ξ_X:

$$\Phi_X = \Phi_Y \circ \Gamma_Y = \Phi_S \circ \xi_S. \tag{116}$$

Here ξ_S is surjective by the surjectivity of ξ, and Γ_Y is surjective as Γ is surjective. Hence,

$$\mathcal{G}(a,1)_0 = \Phi_S(S) = \Phi_Y(Y). \tag{117}$$

On the other hand, in view of (71), $\mathcal{G}(a,1)_0$ is dense in $\mathcal{G}(a,1) = \Phi_{\widetilde{S}}(\widetilde{S})$, hence also dense in $\overline{\mathcal{G}(a,1)}$. We thus obtain

Proposition 15. *Let $\Phi_Y : Y \to \mathcal{B}(a^2+1)$ be the modular morphism defined by the family of sheaves $\mathcal{H}^0(\mathbf{A_Y})$, where $\mathbf{A_Y}$ is the monad* (100)*. Then $\mathcal{G}(a,1)_0 = \Phi_Y(Y)$ is dense in $\overline{\mathcal{G}(a,1)}$.*

6. Series of irreducible rational components of the moduli spaces $\mathcal{B}(a^2+1)$

Consider the variety Y defined in (98). We first will relate to Y a new variety $\mathcal{P}_a$, together with a natural projection

$$\pi: \ Y \to \mathcal{P}_a.$$

The morphism π will be later related to the modular morphism $\Phi_Y : Y \to \mathcal{B}(a^2+1)$ (for the precise formulation see Theorem 18).

For this, take any point $y \in Y$. According to (99), y is a collection of data

$$y = (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]),$$

where
(i) $b \in B$,
(ii) $\varphi_1 : V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\simeq} (V_2 \otimes \mathcal{O}_{\mathbb{P}^3})^\vee$ and $\varphi_2 : N_b \xrightarrow{\simeq} N_b^\vee$ are symplectic isomorphisms:

(118) $$\varphi_1 \in H^0(\wedge^2 (V_2 \otimes \mathcal{O}_{\mathbb{P}^3})^\vee) \smallsetminus \{0\} = \wedge^2 V_2^\vee \smallsetminus \{0\} \cong \mathbf{k}^\times,$$

(119) $$\varphi_2 \in H^0(\wedge^2 N_b^\vee) \smallsetminus \{0\} = H^0(\mathcal{O}_{\mathbb{P}^3}) \smallsetminus \{0\} \cong \mathbf{k}^\times,$$

(iii)

(120) $$0 \neq \sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(a)) = \mathrm{Hom}(V_2^\vee, W_a), \qquad W_a := H^0(\mathcal{O}_{\mathbb{P}^3}(a)),$$

(121) $$0 \neq \sigma_2 \in H^0(N_b(a)),$$

(iv) $\sigma = (\sigma_1, \sigma_2)$ considered as a morphism $\sigma : \mathcal{O}_{\mathbb{P}^3}(-a) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus N_b$ is a subbundle morphism.

In $\mathrm{Hom}(V_2^\vee, W_a)$ consider an open subset

$$\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) = \{\sigma_1 \in \mathrm{Hom}(V_2^\vee, W_a) \mid \sigma_1 : V_2^\vee \to W_a \text{ is a monomorphism}\}.$$

One can easily see (use the argument in paragraph after (67)) that

(122) $$\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) = \{\sigma_1 \in \mathrm{Hom}(V_2^\vee, W_a) \mid \dim Z(\sigma_1) = 1\},$$

where by $Z(\sigma_1)$ we denote, as before, the zero-locus of the section $\sigma_1 \in H^0(V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(a))$. Besides, note that the group $GL(V_2)$ naturally acts on $\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a)$ via its action on $V_2^\vee$, and we have an isomorphism

(123) $$\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a)/GL(V_2) \xrightarrow{\simeq} Gr(2, W_a)$$

and the factorization morphism

(124) $$\tau_1 : \mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \to Gr(2, W_a), \quad \sigma_1 \mapsto \mathrm{im}(\sigma_1 : V_2^\vee \hookrightarrow W_a).$$

Next, as it was mentioned in Section 5 (see paragraph after (67)), the set

$$H^0(N_b(a))^* := \{\sigma_2 \in H^0(N_b(a)) \mid \dim Z(\sigma_2) = 1\}$$

is open dense in $H^0(N_b(a))$. Besides, it is clearly invariant under the action of the group

(125) $$\mathrm{Aut}(N_b(a)) = \mathbf{k}^\times.$$

(Recall that the null correlation bundle N_b is stable and therefore simple, i.e., $\mathrm{End}(N_b) = \mathbf{k}$.) Hence,

(126) $$\mathbb{P}(H^0(N_b(a)))^* := H^0(N_b(a))^*/\mathrm{Aut}(N_b(a)) \overset{\text{open}}{\hookrightarrow} \mathbb{P}(H^0(N_b(a))) \simeq \mathbb{P}^r,$$

where $r = 2\binom{a+3}{3} - a - 3$, and we have the factorization morphism

$$\tau_2 : H^0(N_b(a))^* \to \mathbb{P}(H^0(N_b(a)))^*, \qquad \sigma_2 \mapsto \mathbf{k}\sigma_2. \tag{127}$$

Now the condition (iv) imposed on (σ_1, σ_2) can be rewritten in the form:

$$(\sigma_1, \sigma_2) \in H_{b,a} := \{(\sigma_1, \sigma_2) \in \mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \times H^0(N_b(a))^* \mid Z(\sigma_1) \cap Z(\sigma_2) = \emptyset\}. \tag{128}$$

Clearly, $H_{b,a}$ is a dense open subset of $\mathrm{Hom}^{\mathrm{in}}(V_2^\vee, W_a) \times H^0(N_b(a))^*$. This subset is invariant under the action of the group $\mathbf{k}^\times$ by homotheties and, denoting

$$\mathbb{P}(H_{b,a}) := H_{b,a}/\mathbf{k}^\times,$$

and using (124) and (127), we obtain the factorization morphism

$$\tau : \mathbb{P}(H_{b,a}) \to Gr(2, W_a) \times \mathbb{P}(H^0(N_b(a)))^*, \qquad [\sigma_1 : \sigma_2] \mapsto (\tau_1(\sigma_1), \tau_2(\sigma_2)). \tag{129}$$

To globalize the above pointwise (for $b \in B$) constructions over B, let $\mathcal{K} = p_{B*}(\boldsymbol{\mathcal{N}}(a, 0))$. The variety $\mathbf{P}(\mathcal{K}^\vee)$ has the description $\mathbf{P}(\mathcal{K}^\vee) = \{(b, \mathbf{k}\sigma_2) \mid b \in B,\ \mathbf{k}\sigma_2 \in \mathbb{P}(H^0(N_b(a)))\}$. Consider its dense open subset

$$\Pi_a := \{(b, \mathbf{k}\sigma_2) \in \mathbf{P}(\mathcal{K}^\vee) \mid \mathbf{k}\sigma_2 \in \mathbb{P}(H^0(N_b(a)))^*\}$$

and set

$$\mathcal{G}_a := Gr(2, W_a) \times \Pi_a, \qquad \mathcal{G}_a = \{(b, V, \mathbf{k}\sigma_2) \mid V \in Gr(2, W_a), (b, \mathbf{k}\sigma_2) \in \Pi_a\}. \tag{130}$$

By construction, $\mathcal{G}_a$ is a rational variety.

Next, remark that, comparing the definitions (94) (128) of $\mathcal{S}$ and $H_{b,a}$, we obtain

$$\mathcal{S} = \{(b, [\sigma_1 : \sigma_2]) \mid b \in B,\ [\sigma_1 : \sigma_2] \in \mathbb{P}(H_{b,a})\}.$$

Thus, by (129), we have a well-defined morphism

$$\tau :\ \mathcal{S} \to \mathcal{G}_a,\ (b, [\sigma_1 : \sigma_2]) \mapsto (b, \tau_1(\sigma_1), \tau_2(\sigma_2)). \tag{131}$$

Consider the group $\tilde{G} = GL(V_2) \times \mathbf{k}^\times$, its normal subgroup $G' = \{(\rho \mathrm{id}_{V_2}, \rho) \mid \rho \in \mathbf{k}^\times\}$, and let

$$G = \tilde{G}/G' \tag{132}$$

be the factor group. We will use the following notation for elements of G:

$$[g_1 : \lambda] := (g_1, \lambda)H = \{(\rho g_1, \rho\lambda) \mid \rho \in \mathbf{k}^\times\}, \qquad (g_1, \lambda) \in \tilde{G}.$$

The group G naturally acts on $\mathcal{S}$ as:

$$a_{\mathcal{S}} :\ \mathcal{S} \times G \to \mathcal{S}, \qquad ((b, [\sigma_1 : \sigma_2]), [g_1 : \lambda]) \mapsto (b, [g_1 \circ \sigma_1 : \lambda\sigma_2]), \tag{133}$$

and formulas (123)-(131) show that

$$\mathcal{G}_a = \mathcal{S}/G \tag{134}$$

and the morphism

$$\tau :\ \mathcal{S} \to \mathcal{G}_a \tag{135}$$

in (131) is the quotient morphism for this action and it is a principal G-bundle and therefore in view of (60) we have:

$$\dim \mathcal{G}_a = \dim \mathbb{P}(H_{b,a}) + \dim B - \dim G = 4\binom{a+3}{3} - a - 2. \tag{136}$$

The principal G-bundle $\mathcal{S} \xrightarrow{\tau} \mathcal{G}_a$ by construction is locally trivial, hence there exists a local section $U \overset{s}{\hookrightarrow} \mathcal{S}$ of the projection $\tau : \ \mathcal{S} \to \mathcal{G}_a$:

$$\begin{array}{ccc} & & \mathcal{S} \\ & \overset{s}{\nearrow} & \downarrow \tau \\ U & \xhookrightarrow{\text{open}} & \mathcal{G}_a. \end{array} \tag{137}$$

Here U is rational since $\mathcal{G}_a$ is rational as it was mentioned above.

Now consider the variety $\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbb{P}^3 \times B}) \oplus \wedge^2 \boldsymbol{\mathcal{N}})$ together with the embeddings $\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}})) \hookrightarrow \mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}}) \oplus \wedge^2 \boldsymbol{\mathcal{N}}) \hookleftarrow \mathbf{P}(\wedge^2 \boldsymbol{\mathcal{N}})$ and denote

$$\mathbb{P}\widetilde{B} := \mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}}) \oplus \wedge^2 \boldsymbol{\mathcal{N}}) \smallsetminus \{\mathbf{P}(\wedge^2(V_2 \otimes \mathcal{O}_{\mathbf{B}})) \sqcup \mathbf{P}(\wedge^2 \boldsymbol{\mathcal{N}})\}$$

By construction, the natural projection $\mathbb{P}\widetilde{B} \to B$ is a locally trivial fibration with fiber

$$\mathbf{F} \simeq \mathbb{P}^1 \smallsetminus \{2 \text{ points}\}. \tag{138}$$

Using the description (88) of the varieties B_1, B_2 and the notation (66) in which we put φ_1, φ_2 in place of σ_1, σ_2, we obtain

$$\mathbb{P}\widetilde{B} = \{(b, [\varphi_1 : \varphi_2]) \mid (b, \varphi_i) \in B_i, \ i = 1, 2\}. \tag{139}$$

Remark that the group $\mathbf{k}^\times$ naturally acts on $\widetilde{B}$ as

$$\widetilde{B} \times \mathbf{k}^\times \to \widetilde{B}, \qquad ((b, \varphi_1, \varphi_2), \lambda) \mapsto (b, \lambda\varphi_1, \lambda\varphi_2), \tag{140}$$

(here we use the description (88) of $\widetilde{B}$), so that

$$\mathbb{P}\widetilde{B} = \widetilde{B}/\mathbf{k}^\times, \tag{141}$$

and we have the factorization morphism

$$\pi_{\widetilde{B}} : \ \widetilde{B} \to \mathbb{P}\widetilde{B}, \qquad (b, \varphi_1, \varphi_2) \mapsto (b, [\varphi_1 : \varphi_2]). \tag{142}$$

Consider the varieties

$$\mathbb{P}Y := \mathbb{P}\widetilde{B} \times_B \mathcal{S} = \{(b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]) \mid (b, [\varphi_1 : \varphi_2]) \in \mathbb{P}\widetilde{B}, \ (b, [\sigma_1 : \sigma_2]) \in \mathcal{S}\}, \tag{143}$$

and

$$\mathcal{P}_a := \mathbb{P}\widetilde{B} \times_B \mathcal{G}_a = \{(b, [\varphi_1 : \varphi_2], V, \mathbf{k}\sigma_2) \mid (b, [\varphi_1 : \varphi_2]) \in \mathbb{P}\widetilde{B}, \ (b, V, \mathbf{k}\sigma_2) \in \mathcal{G}_a\}, \tag{144}$$

where $\mathcal{G}_a$ was defined in (130). From (136) and (138) we have

$$\dim \mathcal{P}_a = \dim \mathcal{G}_a + \dim \mathbf{F} = 4\binom{a+3}{3} - a - 1. \tag{145}$$

Note that the local triviality of the fibration $\mathbb{P}\mathcal{B} \to B$ yields that the natural projection

$$pr_{\mathcal{Y}} : \ \mathbb{P}Y \to \mathcal{S} \tag{146}$$

is a locally trivial fibration with fiber $\mathbf{F}$ given in(138).

The morphism $\pi_{\widetilde{B}}$ in (142) induces the morphism

$$\pi_Y : \ Y \to \mathbb{P}Y, \ (b, \varphi_1, \varphi_2, [\sigma_1 : \sigma_2]) \mapsto (b, [\varphi_1 : \varphi_2], [\sigma_1 : \sigma_2]), \tag{147}$$

and from (140)-(142) it follows that π_Y is a factorization morphism of the following $\mathbf{k}^\times$-action on Y:

(148) $$a_Y:\ Y\times\mathbf{k}^\times\to Y,\qquad ((b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2]),\lambda)\mapsto(b,\lambda\varphi_1,\lambda\varphi_2,[\sigma_1:\sigma_2]).$$

Respectively, the morphism $\tau: Y_a\to\mathcal{G}_a$ defined in (131) induces a morphism

(149) $$\tau_Y:\ \mathbb{P}Y\to\mathcal{P}_a,\ (b,[\varphi_1:\varphi_2],[\sigma_1:\sigma_2])\mapsto(b,[\varphi_1:\varphi_2],\tau_1(\sigma_1),\tau_2(\sigma_2)).$$

We now define the morphism $\pi: Y\to\mathcal{P}_a$ as the composition

(150) $$\pi=\tau_Y\circ\pi_Y:\ Y\to\mathcal{P}_a,\qquad(b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2])\mapsto(b,[\varphi_1:\varphi_2],\tau_1(\sigma_1),\tau_2(\sigma_2)).$$

We will now proceed to the study of the fibers of the morphism π.

Definition 16. *Introduce on Y the following equivalence relation:*

(151) $$y=(b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2])\ \sim\ (\tilde b,\tilde\varphi_1,\tilde\varphi_2,[\tilde\sigma_1:\tilde\sigma_2])=\tilde y$$

if there exists an isomorphism of symplectic monads

$A_y:\ 0\to\mathcal{O}_{\mathbb{P}^3}(-a)\xrightarrow{(\sigma_1,\sigma_2)}V_2\otimes\mathcal{O}_{\mathbb{P}^3}\oplus N_b\xrightarrow{(\sigma_1^\vee\circ\varphi_1,\sigma_2^\vee\circ\varphi_2)}\mathcal{O}_{\mathbb{P}^3}(a)\to 0$

and

$A_{\tilde y}:\ 0\to\mathcal{O}_{\mathbb{P}^3}(-a)\xrightarrow{(\tilde\sigma_1,\tilde\sigma_2)}V_2\otimes\mathcal{O}_{\mathbb{P}^3}\oplus N_{\tilde b}\xrightarrow{(\tilde\sigma_1^\vee\circ\tilde\varphi_1,\tilde\sigma_2^\vee\circ\tilde\varphi_2)}\mathcal{O}_{\mathbb{P}^3}(a)\to 0,$

i. e., a commutative diagram with vertical isomorphisms

(152) $$\begin{array}{ccccccccc} A_y:\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{(\sigma_1,\sigma_2)} & V_2\otimes\mathcal{O}_{\mathbb{P}^3}\oplus N_b & \xrightarrow{(\sigma_1^\vee\circ\varphi_1,\sigma_2^\vee\circ\varphi_2)} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0\\ & & h_-\downarrow\simeq & & (g_1,g_2)\downarrow\simeq & & h_+\downarrow\simeq & & \\ A_{\tilde y}:\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-a) & \xrightarrow{(\tilde\sigma_1,\tilde\sigma_2)} & V_2\otimes\mathcal{O}_{\mathbb{P}^3}\oplus N_{\tilde b} & \xrightarrow{(\tilde\sigma_1^\vee\circ\tilde\varphi_1,\tilde\sigma_2^\vee\circ\tilde\varphi_2)} & \mathcal{O}_{\mathbb{P}^3}(a) & \longrightarrow & 0.\end{array}$$

We denote by

(153) $$[y]=[b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2]]$$

the equivalence class of a point $y=(b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2])\in Y$ under the equivalence relation (151).

Note that, in diagram (152),

(154) $$g_1\in\mathrm{Isom}(V_2\otimes\mathcal{O}_{\mathbb{P}^3},V_2\otimes\mathcal{O}_{\mathbb{P}^3})\cong GL(V_2);$$

and $g_2\in\mathrm{Isom}(N_b,N_{\tilde b})$ which in view of the stability of N_b implies that

(155) $$b=\tilde b,\qquad g_2=\lambda\cdot\mathrm{id}_{N_b},\qquad\lambda\in\mathbf{k}^\times;$$

besides, the isomorphisms $h_-,\ h_+$ are multiplications by some constants $\mu,\ \nu\in\mathbf{k}^\times$, respectively:

(156) $$h_-=\mu\cdot\mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}(-a)},\qquad h_+=\nu\cdot\mathrm{id}_{\mathcal{O}_{\mathbb{P}^3}(a)}.$$

Furthermore, in view of (118) and (118) we have

(157) $$\tilde\varphi_1=\lambda_1\varphi_1,\qquad\tilde\varphi_2=\lambda_2\varphi_2,\qquad\lambda_1,\lambda_2\in\mathbf{k}^\times,$$

and, in view of the symplecticity of $\varphi_1,\ \varphi_2$, we obtain using (154) and (155):

(158) $$g_1^\vee\circ\varphi_1\circ g_1=\det(g_1)\varphi_1,\qquad g_2^\vee\circ\varphi_2\circ g_2=\lambda^2\varphi_2.$$

The leftmost square of diagram (152) together with (156) yields:

(159) $$\tilde\sigma_1=\frac{1}{\mu}g_1\circ\sigma_1,\qquad\tilde\sigma_2=\frac{\lambda}{\mu}\sigma_2,$$

Respectively, the rightmost square of diagram (152) yields

$$\nu\sigma_1^\vee \circ \varphi_1 = \tilde{\sigma}_1^\vee \circ \tilde{\varphi}_1 \circ g_1, \qquad \nu\sigma_2^\vee \circ \varphi_2 = \lambda\tilde{\sigma}_2^\vee \circ \tilde{\varphi}_2.$$

Substituting here (156)-(159) we obtain the relations

$$\nu = \frac{\lambda_1 \det(g_1)}{\mu} \quad \text{and} \quad \nu = \frac{\lambda_2\lambda^2}{\mu},$$

respectively. Whence

$$\lambda_1 \det(g_1) = \lambda_2\lambda^2. \tag{160}$$

This relation shows that the G-action (133) on $\mathcal{S}$ lifts to the following G-action on $\mathbb{P}Y$:

$$\begin{aligned} &a_{\mathbb{P}Y}: \ \mathbb{P}Y \times G \to \mathbb{P}Y, \\ &((b,[\varphi_1:\varphi_2],[\sigma_1:\sigma_2]),[g_1:\lambda]) \mapsto (b,[\frac{\varphi_1}{\det(g_1)}:\frac{\varphi_2}{\lambda^2}],[g_1\circ\sigma_1:\lambda\sigma_2]). \end{aligned} \tag{161}$$

$$\mathcal{P}_a = \mathbb{P}Y/G \tag{162}$$

and the morphism

$$\tau_Y: \ \mathbb{P}Y \to \mathcal{P}_a \tag{163}$$

in (149) is the quotient morphism for this action and it is a locally trivial principal G-bundle. We thus have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}Y & \xrightarrow{\tau_Y} & \mathcal{P}_a \\ \big\downarrow{\scriptstyle pr_{\mathcal{Y}}} & & \big\downarrow{\scriptstyle pr_{\mathcal{G}}} \\ \mathcal{S} & \xrightarrow{\tau} & \mathcal{G}_a, \end{array}$$

where $pr_{\mathcal{G}}$ is a natural projection. Since by (146) the morphism $pr_{\mathcal{Y}}: \ \mathbb{P}Y \to \mathcal{S}$ is a locally trivial fibration with fibre $\mathbf{F}$ the open section $U \overset{s}{\hookrightarrow} \mathcal{S}$ in the diagram (137), after possible shrinking U, can be lifted to an open section $\mathbf{F} \times U \overset{\tilde{s}}{\hookrightarrow} \mathbb{P}Y$ of the projection $\tau_Y: \mathbb{P}Y \to \mathcal{P}_a$:

$$\begin{array}{ccc} & & \mathbb{P}Y \\ & \overset{s}{\nearrow} & \big\downarrow{\scriptstyle \tau_Y} \\ \mathbf{F} \times U & \xhookrightarrow{\text{open}} & \mathcal{P}_a. \end{array}$$

Since $\mathbf{F}$ is rational by (138) and U is rational, it follows that

$$\mathcal{P}_a \ \text{ is rational.} \tag{164}$$

Next, from (147)-(148), (161) and (163) it follows that the morphism $\pi: Y \to \mathcal{P}_a$ in (150) is the quotient morphism of the following action of the group

$$\widetilde{G} = \mathbf{k}^\times \times G \tag{165}$$

on Y:

$$\begin{aligned} &a_Y: \ Y \times \widetilde{G} \to Y, \\ &((b,\varphi_1,\varphi_2,[\sigma_1:\sigma_2]),(\mu,[g_1:\lambda])) \mapsto (b,\frac{\mu\varphi_1}{\det(g_1)},\frac{\mu\varphi_2}{\lambda^2},[g_1\circ\sigma_1:\lambda\sigma_2]). \end{aligned} \tag{166}$$

Moreover,

$$\pi : Y \to \mathcal{P}_a = Y/\widetilde{G} \quad \text{is a principal } \widetilde{G}\text{-bundle,} \tag{167}$$

and computations (154)-(161) show that the equivalence class $[y]$ of any point $y \in Y$ coincides with the $\widetilde{G}$-orbit of y:

$$[y] = a_Y(\{y\} \times \widetilde{G}) = \pi^{-1}(\pi(y)), \qquad y \in Y. \tag{168}$$

In other words, $\mathcal{P}_a$ is the set of equivalence classes of points of Y:

$$\mathcal{P}_a = \{[y] \mid y \in Y\}. \tag{169}$$

Remark that, by Corollary 2, the equality $[y] = [\tilde{y}]$, i.e. the isomorphism of symplectic monads A_y and $A_{\tilde{y}}$ in (152) is equivalent to the isomorphism of their cohomology rank 2 bundles as symplectic bundles $(\mathcal{H}^0(A_y), \psi_y)$ and $(\mathcal{H}^0(A_{\tilde{y}}), \psi_{\tilde{y}})$, i.e., to the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}^0(A_y) & \xrightarrow[\simeq]{\psi_y} & \mathcal{H}^0(A_y)^\vee \\ f \downarrow \simeq & & f^\vee \uparrow \simeq \\ \mathcal{H}^0(A_{\tilde{y}}) & \xrightarrow[\simeq]{\psi_{\tilde{y}}} & \mathcal{H}^0(A_{\tilde{y}})^\vee. \end{array} \tag{170}$$

Here ψ_y, respectively, $\psi_{\tilde{y}}$, is a symplectic isomorphism induced by the symplectic isomorphism of the monad A_y with its dual $A_y^\vee$, respectively, of $A_{\tilde{y}}$ with $A_{\tilde{y}}^\vee$. Thus, denoting by $[\mathcal{H}^0(A_y), \psi_y]$ the isomorphism class of the pair $(\mathcal{H}^0(A_y), \psi_y)$, we have:

$$[y] = [\mathcal{H}^0(A_y), \psi_y] = [\mathcal{H}^0(A_y)]. \tag{171}$$

This together with (167)-(169) shows that the modular morphism

$$\Phi_Y : \ Y \to \mathcal{B}(a^2+1), \qquad y \mapsto [\mathcal{H}^0(A_y)]$$

factors through an injective map $\Theta : \mathcal{P}_a \to \mathcal{B}(a^2+1)$, i.e.

$$\Phi_Y = \Theta \circ \pi. \tag{172}$$

Since Y is clearly smooth, the map Θ is actually a morphism. This outcomes from the following well known general result. (For the convenience of the reader we give its proof here.)

Lemma 17. *Let X, Y, Z be quasiprojective varieties with Y smooth, and let $a : X \to Y$ and $b : X \to Z$ be morphisms such that a is surjective and b is constant on the fibers of a. Then there exists a morphism $f : Y \to Z$ such that $b = f \circ a$.*

Proof. Consider the morphism $g : X \to Y \times Z$, $x \mapsto (a(x), b(x))$, and let $Y \xleftarrow{a'} Y \times Z \xrightarrow{b'} Z$ be the projections onto factors so that $a = a' \circ g$ and $b = b' \circ g$. Since b is constant on the fibers of p, it follows that $\tilde{a} := a'|_{g(X)} : g(X) \to Y$ is a bijection. Therefore, as Y is smooth, $\tilde{a}$ is an isomorphism (see, e.g., [S, Ch.2, Section 4.4, Thm. 2.16]). The desired morphism f is now the composition $f = b' \circ \tilde{a}^{-1}$. □

Now Proposition 15 together with (145), (164), (167) and (172) yields

Theorem 18. *There exists an injective morphism $\Theta : \mathcal{P}_a \hookrightarrow \mathcal{B}(a^2+1)$ such that the modular morphism $\Phi_Y : Y \to \mathcal{B}(a^2+1)$ factorizes as*

$$\Phi_Y : \ Y \xrightarrow{\pi} \mathcal{P}_a \overset{\Theta}{\hookrightarrow} \mathcal{B}(a^2+1), \tag{173}$$

where $\pi : Y \to \mathcal{P}_a$ *is a principal* $\widetilde{G}$*-bundle with the group* $\widetilde{G}$ *defined in* (165) *and* (132)*. The irreducible variety* $\overline{\mathcal{G}(a,1)}$ *containing a rational variety* $\mathcal{G}(a,1)_0 = \Theta(\mathcal{P}_a)$ *as a dense subset is rational of dimension* $4\binom{a+3}{3} - a - 1$.

We next obtain the following important formula.

Lemma 19. *For every* $[\mathcal{E}] \in \mathcal{G}(a,1)_0$ *with* $a \geq 2$*, it holds*

$$h^1(\mathcal{E}nd(\mathcal{E})) = 4 \cdot \binom{a+3}{3} - a - 1 + \varepsilon_a,$$

where $\varepsilon(a) = 1$ *when* $a = 3$*, and* $\varepsilon(a) = 0$ *when* $a \neq 3$.

Proof. Since $\mathcal{E}$ is a self dual rank 2 bundle, we have $\mathcal{E}nd(\mathcal{E}) \simeq S^2\mathcal{E} \oplus \Lambda^2\mathcal{E} = S^2\mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^3}$, thus $h^1(\mathcal{E}nd(\mathcal{E})) = h^1(S^2\mathcal{E})$. We will compute the latter.

For $[\mathcal{E}] \in \mathcal{G}(a,1)_0$, consider a monad of the form (30) whose cohomology sheaf is isomorphic to E as a complex $M^\bullet$ with terms $M^{-1} = \mathcal{O}_{\mathbb{P}^3}(-a)$, $M^0 = E$, $M^1 = \mathcal{O}_{\mathbb{P}^3}(a)$. Proceed to the double complex $M^\bullet \otimes M^\bullet$, and to its total complex $T^\bullet$. The last complex naturally decomposes into its symmetric and antisymmetric parts; the symmetric part is the complex

$$0 \to E(-a) \to S^2E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0, \tag{174}$$

whose middle cohomology sheaf is isomorphic to S^2E. Therefore the monad (174) can be broken into two short exact sequences

$$0 \to K \to S^2E \oplus \mathcal{O}_{\mathbb{P}^3} \to E(a) \to 0 \text{ and } 0 \to E(-a) \to K \to S^2E \to 0.$$

Since $h^0(E(-a)) = h^0(S^2E) = 0$, it follows that $h^0(K) = 0$; in addition, $h^1(E(a)) = h^2(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 0$ (use Proposition 5) implies that $h^2(K) = 0$. It then follows that

$$h^1(S^2\mathcal{E}) = h^1(K) + h^2(E(-a)) = h^1(K) + \varepsilon(a), \tag{175}$$

since $h^1(E(-a)) = 0$ for $a \geq 2$.

To complete our calculation, consider the exact sequence

$$0 \to H^0(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) \to H^0(E(a)) \to H^1(K) \to H^1(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Since $h^0(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 4$ and $h^1(S^2E \oplus \mathcal{O}_{\mathbb{P}^3}) = 5$ by Corollary 5, we conclude that

$$h^1(K) = h^0(E(a)) + 1 = h^0(N(a)) + V_2 \otimes h^0(\mathcal{O}_{\mathbb{P}^3}(a)) + 1,$$

which, together with the equality in equation (175), yields the desired formula. □

It is interesting to observe that the right hand side of the formula in Lemma 19 yields the expected value when $a = 2$ and $a = 3$, respectively 37 and 77; when $a \geq 4$, one can check that $4 \cdot \binom{a+3}{3} - a - 1 > 8(a^2+1) - 3$.

Noting that, in view of Theorem (18), the dimension of $\overline{\mathcal{G}(a,1)}$ matches $h^1(\mathcal{E}nd(E))$ for $a = 2$ and $a \geq 4$, as calculated in Lemma 19, and using Proposition 14, we have therefore completed the proof of the first main result of this paper.

Theorem 20. *For* $a = 2$ *and* $a \geq 4$*, the rank 2 bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0$$

fill out a dense subset of an irreducible rational component of $\mathcal{B}(a^2+1)$ *of dimension*

$$4 \cdot \binom{a+3}{3} - a - 1.$$

In particular, for the case $a = 2$, we conclude that rank 2 bundles given as cohomology of monads of the form (4) yield an open subset of an irreducible component of $\mathcal{B}(5)$ with expected dimension 37.

7. Monads of the form (5)

We finally consider the set

$$\mathcal{H} = \{[E] \in \mathcal{B}(5) \mid E \text{ is cohomology of a monad of the form (5)}\}.$$

We prove:

Proposition 21. *The set $\mathcal{H}$ satisfies the condition*

$$\dim(\mathcal{H} \setminus (\mathcal{G}(a,1) \cap \mathcal{H})) \leq 36. \tag{176}$$

Proof. Let E be the cohomology bundle of the following monad:

(177)

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\alpha^\vee} V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0.$$

Since the bundle $V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined subbundle of the bundle $\mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)$ (respectively, $\mathcal{O}_{\mathbb{P}^3}(-1)$ is a uniquely defined quotient bundle of $\mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$), there is a well-defined morphism

(178)

$$\tilde{\alpha}: V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1).$$

Consider first the case

$$\tilde{\alpha} \neq 0. \tag{179}$$

It follows that $\tilde{\alpha}$ is a surjection, hence the kernel of the composition map is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-1)$. In this case we obtain a morphism $\alpha_1 = \alpha|_{\ker \tilde{\alpha}} : \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. Thus similar to (178) there is a well-defined morphism

$$\alpha': \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_1} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow V_6 \otimes \mathcal{O}_{\mathbb{P}^3},$$

together with its dual morphism β' as in (40) with $k = 1$, so that, eventually, we obtain the anti self dual monads (40) with $k = 1$ and (30) with E being a rank 4 instanton bundle of charge 1, which implies that $E \in \mathcal{G}(2,1)$. This means that the condition (179) is equivalent to $[\mathcal{E}] \in \mathcal{H} \cap \mathcal{G}(2,1)$, that is:

$$[\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)) \iff \tilde{\alpha} = 0.$$

We therefore proceed to the case

$$\tilde{\alpha} = 0.$$

This condition implies that $\mathrm{im}(\alpha_0) \subset V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$, where $\alpha_0 := \alpha|_{V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1)}$. Moreover, since α is a subbundle morphism, it follows that $\mathrm{im}(\alpha_0) \not\subset \mathcal{O}_{\mathbb{P}^3}(1)$, so that there is a well-defined injective morphism

$$\bar{\alpha}: V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \twoheadrightarrow V_6 \otimes \mathcal{O}_{\mathbb{P}^3}.$$

Again similar to the anti self dual monads (40) and (30) we obtain the anti-self-dual monads

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha_0} \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\alpha_0^\vee} V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0,$$

$$E_1 := \ker \alpha_0^\vee / \mathrm{im}\alpha_0,$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\gamma} E_1 \xrightarrow{\gamma^\vee} \mathcal{O}_{\mathbb{P}^3}(2) \to 0, \quad E = \ker \gamma^\vee / \mathrm{im}\gamma, \tag{180}$$

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\bar{\alpha}} V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\bar{\alpha}^\vee} V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \quad E_2 := \ker \bar{\alpha}^\vee / \mathrm{im}\bar{\alpha}, \tag{181}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta} E_1 \xrightarrow{\delta^\vee} \mathcal{O}_{\mathbb{P}^3}(-1) \to 0, \quad E_2 \simeq \ker \delta^\vee / \mathrm{im}\delta, \tag{182}$$

where γ and δ are the induced morphisms and E_2 is a rank 2 bundle with $c_1(E_2) = 0$ and $c_2(E_2) = 2$.

The monad (180) induces an exact triple

$$0 \to E \to G \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{183}$$

where $G := \mathrm{coker}\, \gamma$ and ε is the induced morphism. Consider the composite morphisms

$$\delta' : \ \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta} E_1 \twoheadrightarrow G, \quad E' := \mathrm{coker}\, \delta',$$

and

$$\delta'' : \ \mathcal{O}_{\mathbb{P}^3}(1) \xrightarrow{\delta'} G \overset{\varepsilon}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}(2).$$

Here $\delta'' \neq 0$, since otherwise by (183) $h^0(E(-1)) \neq 0$, contrary to the stability of E. Hence,

$$\mathrm{coker}\, \delta'' = \mathcal{O}_{\mathbb{P}^2_a}(2)$$

for some projective plane $\mathbb{P}^2_a$ in $\mathbb{P}^3$, and we have an induced exact triple:

$$0 \to E \to E' \to \mathcal{O}_{\mathbb{P}^2_a}(2) \to 0. \tag{184}$$

Besides, (180) and (182) yield exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\gamma'} E_3 \xrightarrow{\lambda} E' \ to 0, \tag{185}$$

$$0 \to E_2 \xrightarrow{\mu} E_3 \xrightarrow{\nu} \mathcal{O}_{\mathbb{P}^3}(-1) \to 0.$$

where $E_3 := \mathrm{coker}\, \delta$ and $\gamma', \lambda, \mu, \nu$ are the induced morphisms. Note that (182) implies that $h^0(E_2(-2)) = 0$, hence by (185) the composition $\lambda \circ \mu$ is a nonzero morphism. Moreover, one easily sees that this morphism is injective. Therefore, since E' is a rank 2 sheaf, it follows that the composition $\nu \circ \gamma' : \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3}(-1)$ is a nonzero morphism and $\mathrm{coker}(\nu \circ \gamma') = \mathcal{O}_{\mathbb{P}^2_b}(-1)$ for some projective plane $\mathbb{P}^2_b$ in $\mathbb{P}^3$. We thus obtain an exact triple

$$0 \to E_2 \xrightarrow{\lambda \circ \mu} E' \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^2_b}(-1) \to 0, \tag{186}$$

where θ is the induced morphism. Now remark that the triple (184) does not split, since otherwise, as E_2 is locally free, the composition $\mathcal{O}_{\mathbb{P}^2_a}(2) \hookrightarrow E' \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^2_b}(-1)$ is nonzero which is impossible. Thus $\mathbb{P}^2_a = \mathbb{P}^2_b =: \mathbb{P}^2$ and the triple (184) as an extension is given by a nonzero element

$$s \in \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(2), E) \simeq H^0(\mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(2), E)) \simeq H^0(E|_{\mathbb{P}^2}(-1)).$$

Remind that, since E is cohomology of (177) by [16, Table 5.3, page 804] it has spectrum $(-1, 0, 0, 0, 1)$ and then follows that

$$h^1(E(-3)) = 0, \quad h^1(E(-2)) = 1. \tag{187}$$

The zero-scheme $Z = (s)_0$ of this section s is 0-dimensional. Indeed, otherwise $h^0(E|_{\mathbb{P}^2}(-2)) \neq 0$, which contradicts to the cohomology sequence of the exact triple $0 \to E(-3) \to E(-2) \to E|_{\mathbb{P}^2}(-2) \to 0$ as $h^0(E(-2)) = 0$ by the stability of E and the first equalty in (187). Besides, the cohomology sequence of the last triple twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ in view of the stability of E and the second equalty in (187) yields:

$$h^0(E|_{\mathbb{P}^2}(-1)) = 1. \tag{188}$$

Furthermore, applying the functor $-\otimes\mathcal{O}_{\mathbb{P}^2}$ to the triple (184) we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{s} E|_{\mathbb{P}^2} \to E'|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0.$$

By (188), the leftmost morphism s here is the above section of $E|_{\mathbb{P}^2}(-1)$, so that $\operatorname{coker}(s) \simeq \mathcal{I}_{Z,\mathbb{P}^2}(-1)$, and the last sequence yields an exact triple

$$0 \to \mathcal{I}_{Z,\mathbb{P}^2}(-1) \to E'|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0.$$

Apply to this sequence the functor $\mathcal{H}om(-,\mathcal{O}_{\mathbb{P}^2}(2))$. Since $\dim Z = 0$, it follows that $\mathcal{H}om(\mathcal{I}_{Z,\mathbb{P}^2}(-1),\mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathcal{O}_{\mathbb{P}^2}(3)$, and we obtain an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{H}om(E'|_{\mathbb{P}^2},\mathcal{O}_{\mathbb{P}^2}(2)) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0.$$

Hence, $\dim \operatorname{Hom}(E',\mathcal{O}_{\mathbb{P}^2}(2)) = h^0(\mathcal{H}om(E'|_{\mathbb{P}^2},\mathcal{O}_{\mathbb{P}^2}(2))) = 11$ and therefore

$$\mathbb{P}(\operatorname{Hom}(E',\mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}.$$

This equality will be used below.

We now proceed to the study of the sheaf E_2 defined in (181). The results obtained here will complete the proof of Proposition 21.

Consider the monad (181), and suppose that the homomorphism

$$h^0(\bar{\alpha}^\vee):\ H^0(V_6\otimes\mathcal{O}_{\mathbb{P}^3}) \to H^0(V_2\otimes\mathcal{O}_{\mathbb{P}^3}(1))$$

has rank at most 4. We will show that this leads to a contradiction. Indeed, the assumption means that the morphism $\bar{\alpha}^\vee$ factors through a morphism $\varphi : V_4\otimes\mathcal{O}_{\mathbb{P}^3} \to V_2\otimes\mathcal{O}_{\mathbb{P}^3}(1)$. Let $\tilde{\varphi}:\ H := \mathcal{H}om(V_4\otimes\mathcal{O}_{\mathbb{P}^3}, V_2\otimes\mathcal{O}_{\mathbb{P}^3}) \to \mathcal{O}_{\mathbb{P}^3}(1)$ be the morphism induced by φ. Since $\bar{\alpha}^\vee$ is an epimorphism, $\tilde{\varphi}$ is also an epimorphism. Consider the scheme $Y = \mathbf{P}(H)$. Since H is a trivial sheaf of rank 8, we have an isomorphism $Y \simeq \mathbb{P}^3\times\mathbb{P}^7$, with projections onto the factors $\mathbb{P}^3 \xleftarrow{p_1} Y \xrightarrow{p_2} \mathbb{P}^7$ and the isomorphism $\mathcal{O}_{Y/\mathbb{P}^3}(1) = p_2^*\mathcal{O}_{\mathbb{P}^7}(1)$, where $\mathcal{O}_{Y/\mathbb{P}^3}(1)$ is the Grothendieck sheaf, and the canonical epimorphism $\varepsilon : p_1^*H \twoheadrightarrow \mathcal{O}_{Y/\mathbb{P}^3}(1)$. Now by the universal property of the Proj-scheme Y (see, e. g. [13, II, Prop. 7.12]), there exists a section $s : \mathbb{P}^3 \to Y$ of the projection $p_1 : Y \to \mathbb{P}^3$ such that $\tilde{\varphi} : H \to \mathcal{O}_{\mathbb{P}^3}(1)$ coincides with the morphism $s^*\varepsilon$; in particular, $\mathcal{O}_{\mathbb{P}^3}(1) = s^*p_2^*\mathcal{O}_{\mathbb{P}^7}(1)$. This means that $f = p_2\circ s : \mathbb{P}^3 \to \mathbb{P}^7$ is a nonconstant morphism and thus $\dim f(\mathbb{P}^3) = 3$. On the other hand, in $\mathbb{P}^7 = P(\operatorname{Hom}(\mathbf{k}^2,\mathbf{k}^4))$ the determinantal locus $\Delta = \{\mathbf{k}\varphi \in \mathbb{P}^7 \mid \varphi : \mathbf{k}^2 \to \mathbf{k}^4$ is not injective $\}$ has codimension 3, hence $M = f(\mathbb{P}^3)\cap\Delta \neq \emptyset$ and, by construction, $f^{-1}(M)$ is a subset of points in $\mathbb{P}^3$ at which $\tilde{\varphi}$ is not surjective, contrary to the surjectivity of $\bar{\alpha}^\vee$.

Hence, $h^0(\bar{\alpha}^\vee)$ has rank at least 5, and the monad (181) implies that

$$h^0(E_2) \leq 1.$$

We now analyze both cases, namely: (i) $h^0(E_2) = 1$; (ii) $h^0(E_2) = 0$.

(i) $h^0(E_2) = 1$. Since E_2 is a rank 2 bundle with $c_1(E_2) = 0$ and $c_2(E_2) = 2$ (see (181)), it follows that the zero scheme of the section $0 \neq s \in H^0(E_2)$ is a projective line, say, l in $\mathbb{P}^3$ with some locally complete intersection (shortly: l.c.i.) double structure $l^{(2)}$ on it satisfying the triple

$$0 \to \mathcal{O}_l(2) \to \mathcal{O}_{l^{(2)}} \to \mathcal{O}_l \to 0. \tag{189}$$

We thus obtain an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} E_2 \to \mathcal{I}_{l^{(2)}} \to 0. \tag{190}$$

Note that the set of l.c.i. double structures on a given line $l \in G(2,4)$ is the set of epimorphisms $\psi: \ N^\vee_{l,\mathbb{P}^3} \simeq 2\mathcal{O}_l(-1) \twoheadrightarrow \mathcal{O}_l(2)$ (here, $N_{l,\mathbb{P}^3}$ denotes the normal bundle of l), understood up to scalar multiple, i.e. an open dense subset U_l of the projective space $\mathbb{P}(\mathrm{Hom}(V_2 \otimes \mathcal{O}_l(-1), \mathcal{O}_l(2))) \simeq \mathbb{P}^7$, hence $\dim U_l = 7$. Thus space D of all possible l.c.i. double structures $l^{(2)}$ on lines in $\mathbb{P}^3$ has a projection $\rho: \ D \to G(2,4), \ l^{(2)} \mapsto l$ with fibre $\rho^{-1}(l) = U_l$, so that

$$\dim D = \dim G(2,4) + \dim U_l = 11.$$

Next, for a given $l^{(2)} \in D$ the set of isomorphim classes of locally free sheaves E_2 defined as extensions (190) constitutes an open dense subset $V_{l^{(2)}}$ of the projective space $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3})) \simeq \mathbb{P}^3$. To compute this space, apply to the triple

$$0 \to \mathcal{I}_{l^{(2)}} \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{l^{(2)}} \to 0 \tag{191}$$

the functor $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^3})$. We obtain $\mathcal{E}xt^1(\mathcal{I}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{E}xt^2(\mathcal{O}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3})$, and therefore

$$\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3}) \simeq H^0(\mathcal{E}xt^1(\mathcal{I}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3})) \simeq H^0(\mathcal{E}xt^2(\mathcal{O}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3})). \tag{192}$$

Applying the same functor to (189) and using the isomorphisms $\mathcal{E}xt^2(\mathcal{O}_l, \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l(2)$, and $\mathcal{E}xt^2(\mathcal{O}_l(2), \mathcal{O}_{\mathbb{P}^3}) \simeq \mathcal{O}_l$, we obtain an exact triple

$$0 \to \mathcal{O}_l(2) \to \mathcal{E}xt^2(\mathcal{O}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3}) \to \mathcal{O}_l \to 0$$

which together with (192) yields $\mathbb{P}(\mathrm{Ext}^1(\mathcal{I}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^3})) \simeq \mathbb{P}^3$, hence $\dim V_{l^{(2)}} = 3$. Now, denoting by B the space of isomorphism classes of locally free sheaves E_2 defined as extensions (190), we obtain a well defined morphism $\tau: \ B \to D, \ [E_2] \mapsto l^{(2)} = (s)_0$ for $0 \neq s \in H^0(E_2)$ with fibre $\tau^{-1}(l^{(2)}) = V_{l^{(2)}}$. Hence,

$$\dim B = \dim D + \dim V_{l^{(2)}} = 3 + 11 = 14. \tag{193}$$

Now, for any pair $([E_2], \mathbb{P}^2) \in B \times \check{\mathbb{P}}^3$, consider the space $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)$ of extensions (186):

$$0 \to E_2 \to E' \to \mathcal{O}_{\mathbb{P}^2}(-1) \to 0. \tag{194}$$

Since E_2 is locally free, one has

$$\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2) \simeq H^0(\mathcal{E}xt^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)) \simeq H^0(E_2|_{\mathbb{P}^2}(2)). \tag{195}$$

For $l = (\rho \circ \tau)([E_2])$ denote $\check{l} = \{\mathbb{P}^2 \in \check{\mathbb{P}}^3 \mid \mathbb{P}^2 \ni l\}$. Consider the two cases: **(a)** $\mathbb{P}^2 \in \check{l}$; and **(b)** $\mathbb{P}^2 \notin \check{l}$.

(a) $\mathbb{P}^2 \in \check{l}$. In this case one sees using (189) that $\mathcal{T}or_1(\mathcal{O}_{l^{(2)}}, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathcal{O}_l(3)$ and the scheme $\bar{l} = l^{(2)} \cap \mathbb{P}^2$ is described by the triple $0 \to \mathcal{O}_Y \to \mathcal{O}_{\bar{l}} \to \mathcal{O}_l \to 0$, where Y is a 0-dimensional scheme of length 3 supported on l. Thus, after applying the functor $- \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ to the exact sequence (191), we obtain an exact triple

$$0 \to \mathcal{O}_l(3) \to \mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{I}_{Y,\mathbb{P}^2}(1) \to 0.$$

Since $Y \subset l$, it follows that $h^0(\mathcal{I}_{Y,\mathbb{P}^2}(1)) = 1$, hence the last triple yields $h^0(\mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2)) = 5$. Therefore, the triple

$$0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{l^{(2)}} \otimes \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \tag{196}$$

obtained by applying the functor $-\otimes\mathcal{O}_{\mathbb{P}^2}(2)$ to (191), yields

$$h^0(E_2|_{\mathbb{P}^2}(2)) = 11. \tag{197}$$

(b) $\mathbb{P}^2 \notin \check{l}$. In this case $W = l^{(2)} \cap \mathbb{P}^2$ is a 0-dimensional scheme of length 2 supported at the point $l \cap \mathbb{P}^2$, and the triple (196) becomes: $0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{W,\mathbb{P}^2}(2) \to 0$. From this triple we obtain

$$h^0(E_2|_{\mathbb{P}^2}(2)) = 10. \tag{198}$$

Consider the space Σ_1 of isomorphism classes of sheaves E' obtained as extensions (194). One has a natural projection $\pi_1 : \Sigma_1 \to B \times \check{\mathbb{P}}^3$ with fibre described as $\pi_1^{-1}([E_2], \mathbb{P}^2) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2))$. Now by (195), (197) and (198) this fibre has dimension 10, respectively, 9 in case (a), respectively, (b) above. Hence in view of (193) we have

$$\dim \Sigma_1 = 26. \tag{199}$$

Now return to the triple (184). Consider the space W_1 parametrising the surjections $e_1 :\ E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)$ (up to a scalar multiple) for $[E'] \in \Sigma_1$ and $\mathbb{P}^2 = pr_2(\pi([E']))$, where $pr_2 : B \times \check{\mathbb{P}}^3 \to \check{\mathbb{P}}^3$ is the projection. We thus obtain a surjective morphism $p_1 : W_1 \to \Sigma_1$ with fibre $p_1^{-1}(E)$ being an open dense subset in $\mathbb{P}(\mathrm{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$, where $\mathbb{P}^2 = pr_2(\pi_1([E']))$. Thus by (199)

$$\dim W_1 = 36. \tag{200}$$

On the other hand, the triple (184) means that there is a morphism

$$q : W_1 \to \mathcal{H} \setminus (\mathcal{H} \cap \mathcal{G}(2,1)),\ \mathbf{k}e_1 \mapsto \ker(e_1 : E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)). \tag{201}$$

(ii) $h^0(E_2) = 0$. This means that E_2 is stable, i.e. $[E_2] \in \mathcal{B}(2)$. It is well-known (see [14, §9, Lemma 9.5]) that each bundle $[E_2] \in \mathcal{B}(2)$ fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2 \to \mathcal{I}_Y(1) \to 0,$$

where Y is a divisor of the type (3,0) on some smooth quadric surface in $\mathbb{P}^3$. Moreover, for given E_2, this divisor is not unique, but varies in a 1-dimensional linear series without fixed points. Therefore, for any pair $([E_2], \mathbb{P}^2) \in \mathcal{B}(2) \times \check{\mathbb{P}}^3$ one can choose a nontrivial section $s \in E_2|_{\mathbb{P}^2}(1)$ such that its zero scheme $Z = (s)_0$ is a 0-dimensional scheme of length 3, and therefore $h^0(\mathcal{I}_{Z,\mathbb{P}^2}(3)) = 7$. This together with the exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^2}(1) \xrightarrow{s} E_2|_{\mathbb{P}^2}(2) \to \mathcal{I}_{Z,\mathbb{P}^2}(3) \to 0$$

yields $h^0(E_2|_{\mathbb{P}^2}(2)) = 10$, hence in view of (195) we obtain

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)) \simeq \mathbb{P}^9. \tag{202}$$

Now, as above, consider the space Σ_0 of isomorphism classes of sheaves E' obtained as extensions (194) with $[E_2] \in \mathcal{B}(2)$. One has a natural projection $\pi_0 : \Sigma_0 \to \mathcal{B}(2) \times \check{\mathbb{P}}^3$ with fibre described as $\pi_0^{-1}([E_2], \mathbb{P}^2) = \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2))$. Now by (202) this fibre has dimension 9, and we obtain

$$\dim \Sigma_0 = \dim \mathcal{B}(2) + \dim \check{\mathbb{P}}^3 + \dim \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(-1), E_2)) = 13 + 3 + 9 = 25. \tag{203}$$

Again return to the triple (184). Consider the space W_0 parametrising the surjections $e_0 :\ E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)$ (up to a scalar multiple) for $[E'] \in \Sigma_0$ and $\mathbb{P}^2 = pr_2(\pi([E']))$, where $pr_2 : \mathcal{B}(2) \times \check{\mathbb{P}}^3 \to \check{\mathbb{P}}^3$ is the projection. We thus obtain a surjective morphism $p_0 : W_0 \to \Sigma_0$ with fibre $p_0^{-1}(E)$ being an open dense subset in $\mathbb{P}(\mathrm{Hom}(E', \mathcal{O}_{\mathbb{P}^2}(2))) \simeq \mathbb{P}^{10}$, where $\mathbb{P}^2 = pr_2(\pi_0([E']))$. Thus by (203)

$$\dim W_0 = 35. \tag{204}$$

On the other hand, the triple (184) means that there is a morphism

$$q : W_0 \to (\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})), \ \mathbf{k}e_0 \mapsto \ker(e_0 : E' \twoheadrightarrow \mathcal{O}_{\mathbb{P}^2}(2)). \tag{205}$$

Note that, for any E_2 in (184) we have either $h^0(E_2) = 1$ or $h^0(E_2) = 0$. This means that the morphism

$$q : W_1 \cup W_0 \to (\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H}))$$

defined in (201) and (205) is surjective. Hence (176) follows from (200) and (204). □

8. Components of $\mathcal{B}(5)$

We finally have at hand all the ingredients needed to complete the proof of our second main result, namely the characterization of the irreducible components of $\mathcal{B}(5)$. We will proof the following result.

Theorem 22. *The moduli space $\mathcal{B}(5)$ has exactly 3 irreducible components, namely:*

(i) *the* instanton component*, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to V_{12} \otimes \mathcal{O}_{\mathbb{P}^3} \to V_5 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \text{ and} \tag{206}$$

$$0 \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \to 0; \tag{207}$$

(ii) *the* Ein component*, of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \tag{208}$$

iii) *the closure of the family $\mathcal{G}(2,1)$, of dimension 37, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \text{ and} \tag{209}$$

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \to V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0. \tag{210}$$

The first ingredient of the proof is the fact, proved by Hartshorne and Rao, that every bundle in $\mathcal{B}(5)$ is cohomology of one of the above monads, cf. [16, Table 5.3, page 804].

Recall that for each stable rank 2 bundle E on $\mathbb{P}^3$ with vanishing first Chern class, the number $\alpha(E) := h^1(E(-2)) \bmod 2$ is called the Atiyah–Rees α-invariant of E, see [14, Definition in page 237]. Hartshorne showed [14, Corollary 2.4] that this number is invariant on the components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the form (206) and (207) have α-invariant equal to 0, while the cohomologies of the other three types of monads have α-invariant equal to 1.

Rao showed in [29] that the family of bundles obtained as cohomology of monads of the form (207) is irreducible, of dimension 36, and it lies in a unique component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, i.e. the cohomologies of monads of the form (206), yield an irreducible family of dimension 37, it follows that the set

$$\mathcal{I} := \{[E] \in \mathcal{B}(5) \mid \alpha(E) = 0\}$$

forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $\mathrm{H}^1(\mathcal{E}nd(E)) = 37$; this was originaly proved by Katsylo and Ottaviani for instanton bundles [24], and by Rao for the cohomologies of monads of the form (207) [29, Section 3]. Therefore, we also conclude that $\mathcal{I}$ is nonsingular. This completes the proof of the first part of the Main Theorem.

Our next step is to analyse those bundles with Atiyah–Rees invariant equal to 1.

Hartshorne proved in [15, Theorem 9.9] that the family of stable rank 2 bundles E with $c_1(E) = 0$ and $c_2(E) = 5$ whose spectrum is $(-2, -1, 0, 1, 2)$ form an irreducible, nonsingular family of dimension 40. Such bundles are precisely those given as cohomologies of monads of the form (208), cf. [16, Table 5.3, page 804], which is a particular case of a class of monads studied by Ein in [11]. From these references, we conclude that the closure of the family of vector bundle arising as cohomology of monads of the form (208) is an oversized irreducible component of $\mathcal{B}(5)$ of dimension 40.

We proved above that the bundles arising as cohomology of monads of the form (209) form a third irreducible component of dimension 37, while those bundles arising as cohomology of monads of the form (210), denoted by $\mathcal{H}$, form an irreducible family of dimension 36. It follows that latter must lie either in $\overline{\mathcal{G}(2,1)}$ or in $\overline{\mathcal{E}}$, the closures $\mathcal{G}(2,1)$ and $\mathcal{E}$, respectively, within $\mathcal{B}(5)$.

Proposition 23. *$\mathcal{H} \subset \overline{\mathcal{G}(2,1)}$.*

Proof. Suppose by contradiction that there exists a vector bundle $E \in \mathcal{H} \cap \overline{\mathcal{E}}$. By the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(E(-2)) \geq 3$. However, one can check from the display of the monad (210) that $\dim \mathrm{H}^1(E(-2)) = 1 < 3$. It follows that the family $\mathcal{H}$ must lie in $\overline{\mathcal{G}(2,1)}$. □

This last proposition finally concludes the proof of Main Theorem 2. We summarize all the information in the theorem, and the discrete invariants of stable rank 2 bundles with $c_1 = 0$ and $c_2 = 5$ in the following table.

TABLE 1. Irreducible components of $\mathcal{B}(5)$

<table>
<tr><th>Component</th><th>Dimension</th><th>Monads</th><th>Spectra</th><th>α-invariant</th></tr>
<tr><td rowspan="2">Instanton</td><td rowspan="2">37</td><td>(1)</td><td>(0,0,0,0,0)</td><td rowspan="2">0</td></tr>
<tr><td>(2)</td><td>(-1,-1,0,1,1)</td></tr>
<tr><td>Ein</td><td>40</td><td>(3)</td><td>(-2,-1,0,1,2)</td><td>1</td></tr>
<tr><td rowspan="2">Modified Instanton</td><td rowspan="2">37</td><td>(4)</td><td rowspan="2">(-1,0,0,0,1)</td><td rowspan="2">1</td></tr>
<tr><td>(5)</td></tr>
</table>

In order to give a complete description of the vector bundles $E \in \overline{\mathcal{G}(2,1)}$, we include here its cohomology table. Knowing the spectrum of an arbitrary $E \in \overline{\mathcal{G}(2,1)}$ (given in the table above) allows us to conclude that $h^1(E(k)) = 0$ for

$k \leq -3$, and to compute $h^1(E(-2)) = 1$ and $h^1(E(-1)) = 5$. Serre duality tells us that $h^2(E(k)) = 0$ for $k \geq -1$, while stability implies that $h^0(E(k)) = 0$ for $k \leq 0$, and $h^3(E(k)) = 0$ for $k \geq -4$; it follows that $h^1(E) = -\chi(E) = 8$.

TABLE 2. $h^i(E(l))$ for $E \in \overline{\mathcal{G}(2,1)}$

i\l	**-4**	**-3**	**-2**	**-1**	**0**
3	0	0	0	0	0
2	8	5	1	0	0
1	0	0	1	5	8
0	0	0	0	0	0

Remark. Inspired by the techniques introduced in the present paper, the authors of [30] construct another infinite series of irreducible components of $\mathcal{B}(0,n)$ whose special point corresponds to a bundle obtained as the cohomology of a monad similar to the one in display (24), just substituting a direct sum of two rank 2 instantons bundles for the rank 4 instanton bundle of charge 1 in middle term.

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