CLASSIFICATION AND EXTENSIONS OF L_∞ ALGEBRAS OF ONE EVEN AND TWO ODD DIMENSION

ALICE FIALOWSKI AND MICHAEL PENKAVA

Abstract. We classify L_∞ algebras of one even and two odd dimension, which are related to $2|1$-dimensional $\mathbb{Z}_2$-graded Lie algebras. What makes this case interesting is that there are many nonequivalent L_∞ examples, in addition to the $\mathbb{Z}_2$-graded Lie algebra (or superalgebra) structures, yet the moduli space is simple enough that we can give a complete classification up to equivalence.

1. Introduction

L_∞ algebras are natural generalizations of Lie algebras and superalgebras if one considers $\mathbb{Z}_2$-graded vector spaces. An ordinary 3-dimensional Lie algebra is the same as an L_∞ algebra structure on a $0|3$ (0 even and 3 odd) dimensional $\mathbb{Z}_2$-graded vector space, while differential graded Lie algebras, also called superalgebras are another examples of L_∞ algebras. L_∞ algebras were first described in [12] and have recently been the focus of much attention both in mathematics ([13]) and in mathematical physics ([8, 1, 2, 9]).

Even though L_∞ algebras appear in many contexts, there are only a few examples known, even in low dimensions. In [6], the authors classified L_∞ algebras of dimension less than or equal to 2. In [7], we have constructed miniversal deformations for all L_∞ structures on a space of dimension $0|3$. Since $0|3$ -dimensional vector spaces in our model of L_∞ algebras, correspond to $3|0$-dimensional spaces in the usual grading, the theory of $0|3$- dimensional L_∞ algebras coincides

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with the theory of Lie algebra structures on non-graded 3-dimensional vector spaces, whose classification is well known.

In [3], L_∞ algebras of dimension $2|1$ were studied, and a classification of the L_∞ algebra structures determined by codifferentials whose leading term has degree less than or equal to two was given. That classification includes the deformation of $\mathbb{Z}_2$-graded Lie algebra structures into more general L_∞ structures. The theory in the $2|1$ dimensional case is more complicated than the $1|2$-dimensional algebras we will study in this paper, because the space of n-cochains on a $1|2$ dimensional space has dimension $6|6$ for $n > 1$, while the space of n-cochains on a $2|1$- dimensional space has dimension $3n + 2|3n + 1$, making it more difficult to classify the nonequivalent structures. Thus the $1|2$ dimensional case which we study here is intermediate in complexity.

Our purpose in this paper is to add to the body of examples of L_∞ algebras by classifying all such structures on a $1|2$-dimensional vector space over a field $\mathfrak{K}$. We hope that these examples will help to build a picture of the techniques necessary to attack the classification problem in greater generality.

In Section 2 we introduce L_∞ algebras, deformations, and give some useful facts on equivalent codifferentials and extensions. Section 3 treats the classification of codifferentials with leading term $d_1 \neq 0$. In Section 4 we classify codifferentials with leading term d_N, where $N > 1$. Sections 5 through 8 deal with cohomology computations for the codifferentials we obtained in the previous section, in order to get a better picture of the moduli space. In Section 9 we determine extensions of each of the codifferentials.

2. Basic Definitions and Facts

2.1. L_∞ **Algebras.** A detailed introduction to L_∞ algebras can be obtained in [9, 10, 11]. Here we summarize the basic notions. If W is a $\mathbb{Z}_2$-graded vector space, then $S(W)$ denotes the symmetric coalgebra of W. If we let $T(W)$ be the reduced tensor algebra $T(W) = \bigoplus_{n=1}^{\infty} W^{\otimes n}$, then the reduced symmetric algebra $S(W)$ is the quotient of the tensor algebra by the graded ideal generated by $u \otimes v - (-1)^{uv} v \otimes u$ for elements u, $v \in W$. The symmetric algebra has a natural coalgebra structure, given by

$$\Delta(w_1 \dots w_n) = \sum_{k=1}^{n=1} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma) w_{\sigma(1)} \dots w_{\sigma(k)} \otimes w_{\sigma(k+1)} \dots w_{\sigma(n)},$$

where we denote the product in $S(W)$ by juxtaposition, $\mathrm{Sh}(k, n-k)$ is the set of *unshuffles* of type $(k, n-k)$, and $\epsilon(\sigma)$ is a sign determined by σ (and $w_1 \dots w_n$) given by

$$w_{\sigma(1)} \dots w_{\sigma(n)} = \epsilon(\sigma) w_1 \dots w_n.$$

A *coderivation* on $S(W)$ is a map $\delta : S(W) \to S(W)$ satisfying

$$\Delta \circ \delta = (\delta \otimes I + I \otimes \delta) \circ \Delta.$$

Let us suppose that the even part of W has basis $e_1 \dots e_m$, and the odd part has basis $f_1 \dots f_n$, so that W is an $m|n$ dimensional space. Then a basis of $S(W)$ is given by all vectors of the form $e_1^{k_1} \dots e_m^{k_m} f_1^{l_1} \dots f_n^{l_n}$, where k_i is any nonnegative integer, and $l_i \in \mathbb{Z}_2$. An L_∞ structure on W is simply an odd codifferential on $S(W)$, that is to say, an odd coderivation whose square is zero. The space $\operatorname{Coder}(W)$ can be naturally identified with $\operatorname{Hom}(S(W), W)$, and the Lie superalgebra structure on $\operatorname{Coder}(W)$ determines a Lie bracket on $\operatorname{Hom}(S(W), W)$ as follows. Denote $L_m = \operatorname{Hom}(S^m(W), W)$ so that $L = \operatorname{Hom}(S(W), W)$ is the direct product of the spaces L_i. If $\alpha \in L_m$ and $\beta \in L_n$, then $[\alpha, \beta]$ is the element in L_{m+n-1} determined by

$$\begin{aligned}(1)\quad [\alpha,\beta](w_1 \dots w_{m+n-1}) &= \\ &\sum_{\sigma \in \operatorname{Sh}(n,m-1)} \epsilon(\sigma) \alpha(\beta(w_{\sigma(1)} \dots w_{\sigma(n)}) w_{\sigma(n+1)} \dots w_{\sigma(m+n-1)}) \\ &- (-1)^{\alpha\beta} \sum_{\sigma \in \operatorname{Sh}(m,n-1)} \epsilon(\sigma) \beta(\alpha(w_{\sigma(1)} \dots w_{\sigma(m)}) w_{\sigma(m+1)} \dots w_{\sigma(m+n-1)}).\end{aligned}$$

Another way to express this bracket is in the form

$$[\alpha, \beta] = \alpha \tilde{\beta} - (-1)^{\alpha\beta} \beta \tilde{\alpha},$$

where for $\phi \in \operatorname{Hom}(S^k(W), W)$, $\tilde{\phi}$ is the associated coderivation, given by

$$\tilde{\phi}(w_1 \dots w_n) = \sum_{\sigma \in \operatorname{Sh}(k,n-k)} \epsilon(\sigma) \phi(w_{\sigma(1)} \dots w_{\sigma(k)}) w_{\sigma(k+1)} \dots w_{\sigma(n)}.$$

If W is completely odd, and $d \in L_2$, then d determines an ordinary Lie algebra on W, or rather on its parity reversion. This is the case we considered in [7]. The symmetric algebra on W looks like the exterior algebra on W if we forget the grading. If we define $[a, b] = d(ab)$ for $a, b \in W$, then the bracket is antisymmetric because $ba = -ab$, and moreover

$$\begin{aligned} 0 = [d,d](abc) =& \frac{1}{2}\sum_{\sigma\in\mathrm{Sh}(2,1)} \epsilon(\sigma)d(d(\sigma(a)\sigma(b))\sigma(c)) \\ =& d((d(ab)c) + d(d(bc)a) - d(d(ac)b) \\ =& [[a,b],c] + [[b,c],a] - [[a,c],b], \end{aligned}$$

which is the Jacobi identity. When $d \in L_2$ and W has a true grading, then the same principle holds, except that one has to take into account a sign arising from the map $S^2(W) \to \bigwedge^2(V)$, where V is the parity reversion of W. Thus $\mathbb{Z}_2$-graded Lie algebras are also examples of L_∞ algebras.

We work in the framework of the parity reversion $W = \Pi V$ of the usual vector space V on which an L_∞ algebra structure is defined, because in the W framework, an L_∞ structure is simply an odd coderivation d of the symmetric coalgebra $S(W)$, satisfying $d^2 = 0$, in other words, it is an odd codifferential in the $\mathbb{Z}_2$-graded Lie algebra of coderivations of $S(W)$. As a consequence, when studying $\mathbb{Z}_2$-graded Lie algebra structures on V, the parity is reversed, so that a $1|2$-dimensional vector space W corresponds to a $2|1$-dimensional $\mathbb{Z}_2$-graded Lie structure on V. Moreover, the anti-symmetry of the Lie bracket becomes $\mathbb{Z}_2$-graded symmetry of the associated coderivation d.

A formal power series $d = d_1 + \cdots$, with $d_i \in L_i$ determines an element in $L = \mathrm{Hom}(S(W), W)$, which is naturally identified with $\mathrm{Coder}(S(W))$, the space of coderivations of the symmetric coalgebra $S(W)$. Thus L is a $\mathbb{Z}_2$-graded Lie algebra, and we say that *an odd element d in L is a codifferential* if $d^2 = 0$, which is equivalent to $[d,d] = 0$. Then d is said to be an L_∞ structure on W.

If $g = g_1 + \cdots \in \mathrm{Hom}(S(W), W)$, and $g_1 : W \to W$ is invertible, then it determines a coalgebra automorphism of $S(W)$: $\tilde{g} : S(W) \to S(W')$, that is a map satisfying

$$\Delta' \circ \tilde{g} = (\tilde{g} \otimes \tilde{g}) \circ \Delta.$$

If d and d' are L_∞ algebra structures on W and W', resp., then $\tilde{g}$ is a homomorphism between these structures if $\tilde{g} \circ d = d' \circ \tilde{g}$. Two L_∞ structures d and d' on W are equivalent, and we write $d' \sim d$ when there is a coalgebra automorphism $\tilde{g}$ of $S(W)$ such that $d' = \tilde{g}^*(d) = \tilde{g}^{-1} \circ d \circ \tilde{g}$. Furthermore, if $d = d'$, then $\tilde{g}$ is said to be an automorphism of the L_∞ algebra.

2.2. **Deformations.** An augmented local ring $\mathcal{A}$ with maximal ideal $\mathfrak{m}$ will be called an *infinitesimal base* if $\mathfrak{m}^2 = 0$, and a *formal base* if $\mathcal{A} = \varprojlim_n \mathcal{A}/\mathfrak{m}^n$. A deformation of an L_∞ algebra structure d on

W with base given by a local ring $\mathcal{A}$ with augmentation $\epsilon : \mathcal{A} \to \mathfrak{K}$, where $\mathfrak{K}$ is the field over which W is defined, is an $\mathcal{A}$-L_∞ structure $\tilde{d}$ on $W\hat{\otimes}\mathcal{A}$ such that the morphism of $\mathcal{A}$-L_∞ algebras $\epsilon_* = 1 \otimes \epsilon : L_{\mathcal{A}} = L \otimes \mathcal{A} \to L \otimes \mathfrak{K} = L$ satisfies $\epsilon_*(\tilde{d}) = d$. (Here $W\hat{\otimes}\mathcal{A}$ is an appropriate completion of $W \otimes \mathcal{A}$.) The deformation is called infinitesimal (formal) if $\mathcal{A}$ is an infinitesimal (formal) base.

In general, the cohomology $H(D)$ of d given by the operator $D : L \to L$ with $D(\phi) = [\phi, d]$ may not be finite dimensional. However, L has a natural filtration $L^n = \prod_{i=n}^{\infty} L_i$, which induces a filtration H^n on the cohomology, because D respects the filtration. Then $H(D)$ is of finite type if H^n/H^{n+1} is finite dimensional. Since this is always true when W is finite dimensional, the examples we study here will always be of finite type. A set δ_i will be called a *basis of the cohomology*, if any element δ of the cohomology can be expressed uniquely as a formal sum $\delta = \delta_i a^i$. If we identify $H(D)$ with a subspace of the space of cocycles $Z(D)$, and we choose a *basis* β_i of the coboundary space $B(D)$, then any element $\zeta \in \mathbb{Z}(D)$ can be expressed uniquely as a sum $\zeta = \delta_i a^i + \beta_i b^i$.

For each δ_i, let u^i be a parameter of opposite parity. Then the infinitesimal deformation $d^1 = d + \delta_i u^i$, with base $\mathcal{A} = \mathfrak{K}[u_i]/(u_i u_j)$ is universal in the sense that if $\tilde{d}$ is any infinitesimal deformation with base $\mathcal{B}$, then there is a unique homomorphism $f : \mathcal{A} \to \mathcal{B}$, such that the morphism $f_* = 1 \otimes f : L_{\mathcal{A}} \to L_{\mathcal{B}}$ satisfies $f_*(\tilde{d}) \sim d$.

For formal deformations, there is no universal object in the sense above. A *versal deformation* is a deformation d^∞ with formal base $\mathcal{A}$ such that if $\tilde{d}$ is any formal deformation with base $\mathcal{B}$, then there is some morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f_*(d^\infty) \sim \tilde{d}$. If f is unique whenever $\mathcal{B}$ is infinitesimal, then the versal deformation is called *miniversal*. In [5], we constructed a miniversal deformation for L_∞ algebras with finite type cohomology in general, and in [6, 7] we computed versal deformations of the examples we considered there.

In this paper we do not compute versal deformation, but we use the concept of deformation in describing the moduli space.

2.3. **Equivalent codifferentials and extensions.** We will construct all equivalence classes of L_∞ structures on W. We shall use the following facts, which are established in [5, 4], to aid in the classification.

If d is an L_∞ structure on W, and d_N is the first nonvanishing term in d, then d_N is a codifferential, which we call the *leading term of* d, and we say that d is *an extension* of d_N. Define the cohomology operator D by $D(\phi) = [\phi, d_N]$, for $\phi \in L$. Then the following formula holds for any extension d of d_N as an L_∞ structure, and all $n \geq N$:

$$D(d_{n+1}) = -\frac{1}{2} \sum_{k=N+1}^{n} [d_k, d_{n+N-k+1}]. \tag{2}$$

Note that the terms on the right all have index less than $n+1$. If a coderivation d has been constructed up to terms of degree m, satisfying equation (2) for $n = 1 \dots m-1$, then the right hand side of equation (2) for $n = m$ is automatically a cocycle. Thus d can be extended to the next level precisely when the right hand side is a coboundary. There may be many nonequivalent extensions, because the term d_{n+1} which we add to extend the coderivation is only determined up to a cocycle. An extension d is given by any coderivation which satisfies equation (2) for every $m = N+1\dots$ whose leading term is d_N.

Classifying the extensions of d_N can be quite complicated. However, the following theorem often makes it easy to classify the extensions.

Theorem 2.1. *If the cohomology $H^n(d_N) = 0$, for $n > N$, then any extension of d_N to an L_∞ structure d is equivalent to the structure d_N.*

Before classifying the extensions of a codifferential d_N, we need to classify the codifferentials in L_N up to equivalence, that is, we need to study the *moduli space* of degree N codifferentials. A *linear automorphism* of $S(W)$ is an automorphism determined by an isomorphism $g_1 : W \to W$. If g is an arbitrary automorphism, determined by maps $g_i : S^i(W) \to W$, and W is finite dimensional, then g_1 is an isomorphism, so this term alone induces an automorphism of $S(W)$ which we call the *linear part* of g.

The following theorem simplifies the classification of equivalence classes of codifferentials in L_N.

Theorem 2.2. *If d and d' are two codifferentials in L_N, and g is an equivalence between them, then the linear part of g is also an equivalence between them.*

Thus we can restrict ourself to linear automorphisms when determining the equivalence classes of elements in L_N.

We will also use the following result.

Theorem 2.3. *Suppose that d and d' are equivalent codifferentials. Then their leading terms have the same degree and are equivalent.*

As a consequence of these theorems, we proceed to classify the codifferentials as follows. First, find all equivalence classes of codifferentials of degree N. For each equivalence class, study the equivalence classes of extensions of the codifferential.

Let us first establish some basic notation for the cochains. Suppose $W = \langle w_1, w_2, w_3\rangle$, with w_1 and w_2 being odd elements and w_3 an even element. If $I = \{i_1, i_2, i_3\}$ is a multi-index, with i_1 and i_2 either zero or one, let $w_I = w_1^{i_1} w_2^{i_2} w_3^{i_3}$. For simplicity, we will denote w_I simply by I.

Then for $n \geq 0$,

$$S^{n+2}(W) = \langle (1,0,n+1), (0,1,n+1), (0,0,n+2), (1,1,n)\rangle.$$

Note that $(1,0,n+1)$ and $(0,1,n+1)$ are odd, while $(0,0,n+2)$ and $(1,1,n)$ are even, so $\dim(S^{n+2}(W)) = 2|2$. If g is a linear automorphism of $S(W)$, then because automorphisms preserve parity, we must have $g(w_1) = w_1 r + w_2 p$, $g(w_2) = w_1 r + w_2 s$, $g(w_3) = q w_3$, for some coefficients l, p, r, s, q such that $q(ls - pr) \neq 0$. If $q = 1$, then we will express g as a matrix $g = \begin{pmatrix} l & r \\ p & s \end{pmatrix}$. There is a natural extension of g to an automorphism of $S(W)$, given by

$$\begin{aligned}
g(1,0,n+1) &= (1,0,n+1) l q^{n+1} + (0,1,n+1) p q^{n+1} \\
g(0,1,n+1) &= (1,0,n+1) r q^{n+1} + (0,1,n+1) s q^{n+1} \\
g(0,0,n+2) &= (0,0,n+2) q^{n+2} \\
g(1,1,n) &= (1,1,n)(ls - pr) q^n
\end{aligned}$$

Let $L_n := \operatorname{Hom}(S^n(W), W)$. Define

$$\phi_j^I(w_J) = I! \delta_J^I w_j,$$

where $I! = i_1! i_2! i_3!$. If we let $|I| = i_1 + i_2 + i_3$, then $L_n = \langle \phi_j^I, |I| = n\rangle$. If ϕ is odd, we denote it by the symbol ψ to make it easier to distinguish the even and odd elements. Then

$$\begin{aligned}
(L_{n+2})_0 &= \langle \phi_1^{1,0,n+1}, \phi_2^{1,0,n+1}, \phi_1^{0,1,n+1}, \phi_2^{0,1,n+1}, \phi_3^{0,0,n+2}, \phi_3^{1,1,n}\rangle \\
(L_{n+2})_1 &= \langle \psi_3^{1,0,n+1}, \psi_3^{0,1,n+1}, \psi_1^{0,0,n+2}, \psi_2^{0,0,n+2}, \psi_1^{1,1,n}, \psi_2^{1,1,n}\rangle,
\end{aligned}$$

so $\dim L_{n+2} = 6|6$.

For L_1, $n = -1$, and no terms of the form $\{1,1,n\}$ occur, so $\dim L_1 = 5|4$. We study the special case of codifferentials with leading term in L_1 first.

3. Classification of codifferentials with $d_1 \neq 0$

Let us suppose that d is an odd element of L_1. Then

$$d = \psi_3^{1,0,0} a_1 + \psi_3^{0,1,0} a_2 + \psi_1^{0,0,1} a_3 + \psi_2^{0,0,1} a_4.$$

We have

$$\frac{1}{2}[d,d] = \phi_1^{1,0,0}a_1a_3 + \phi_1^{0,1,0}a_3a_2 + \phi_2^{1,0,0}a_4a_1 + \phi_2^{0,1,0}a_2a_4 + \phi_3^{0,0,1}(a_2a_4 + a_1a_3).$$

For this bracket to vanish, it is clear that either $a_1 = a_2 = 0$ or $a_3 = a_4 = 0$. Let us suppose that the former occurs, so that $d = \psi_1^{0,0,1}a_3 + \psi_2^{0,0,1}a_4$. Then we claim that d is equivalent to $d' = \psi_1^{0,0,1}$. For if we let g be a linear automorphism such that $g(w_1) = w_1a_3 + w_2a_4$, then $g^{-1}dg = d'$. Similarly, if $a_3 = a_4 = 0$, then $d = \psi_3^{1,0,0}a_1 + \psi_3^{0,1,0}a_2$, which is equivalent to $\phi_3^{0,1,0}$. Thus there are exactly two equivalence classes of codifferentials of degree 1.

In order to classify their extensions, we need to compute the cohomology. Let $z_n = \dim(Z^n = \ker D : L_n \to L_n)$ be the dimension of the cocycles, and $b_n = \dim(B^n = \operatorname{Im} D : L_n \to L_n)$ be the dimension of the space of coboundaries. Then $h_n = z_n - b_n$. (Note that dimensions are represented as a pair $k|l$ consisting of the dimension of the even, followed by the dimension of the odd part of the space.) Note that for a codifferential of degree 1, $D : L_n \to L_n$. It follows that $h_1 = \dim(H^1)$ cannot be zero. We shall see that for both codifferentials, $h_n = \dim(H^n) = 0$ for $n > 1$. Thus we can apply Theorem (2.1) to conclude that all codifferentials which have a nonzero leading term of degree 1 are equivalent to one of these two codifferentials.

Let $d_0 = \psi_3^{1,0,0}$. The following table of coboundaries allows us to compute H^1 immediately.

$$\begin{array}{ll}
D(\phi_1^{1,0,0}) = 0, & D(\psi_3^{1,0,0}) = 0 \\
D(\phi_2^{1,0,0}) = -\psi_3^{1,0,0}, & D(\psi_3^{0,1,0}) = 0 \\
D(\phi_1^{0,1,0}) = 0, & D(\psi_1^{0,0,1}) = \phi_1^{0,1,0} \\
D(\phi_2^{0,1,0}) = -\psi_3^{0,1,0}, & D(\psi_2^{0,0,1}) = \phi_2^{0,1,0} + \phi_3^{0,0,1} \\
D(\phi_3^{0,0,1}) = \psi_3^{0,1,0} &
\end{array}$$

Thus $z_1 = 3|2$, $b_1 = 2|2$ and so $h_1 = 1|0$. Moreover, we can take $H^1 = \langle\phi_1^{1,0,0}\rangle$. Next, we compute coboundaries in L_{n+2} for $n \geq 0$.

$$
\begin{aligned}
D(\phi_1^{1,0,n+1}) &= -\psi_3^{1,1,n}, & D(\psi_3^{1,0,n+1}) &= -\phi_3^{1,1,n}\\
D(\phi_2^{1,0,n+1}) &= \psi_2^{1,1,n} - \psi_3^{1,0,n+1}, & D(\psi_3^{0,1,n+1}) &= 0\\
D(\phi_1^{0,1,n+1}) &= 0, & D(\psi_1^{0,0,n+2}) &= \phi_1^{0,1,n+1}\\
D(\phi_2^{0,1,n+1}) &= \psi_3^{0,1,n+1}, & D(\psi_2^{0,0,n+2}) &= \phi_2^{0,1,n+1} + \phi_3^{0,0,n+2}\\
D(\phi_3^{0,0,n+2}) &= \psi_3^{0,1,n+1}, & D(\psi_1^{1,1,n}) &= 0\\
D(\phi_3^{1,1,n}) &= 0, & D(\psi_2^{1,1,n}) &= \phi_3^{1,1,n}
\end{aligned}
$$

Notice that if you omit the three terms with negative indices, this table coincides with the previous table when $n = -1$. From the table, we can see that the space of even cocycles has dimension 3, so the space of odd coboundaries must also have dimension 3. Similarly, the odd cocycles span a three dimensional space, so the even coboundaries span a three dimensional space. Thus $z_n = 3|3 = b_n$, so $h_n = 0$ when $n > 1$.

If $d_* = \psi_1^{0,0,1}$, then a similar calculation gives $H^1 = \langle \phi_2^{0,1,0} \rangle$, and $H^n = 0$ if $n > 1$. So far, nothing much interesting has emerged.

4. Codifferentials with leading term d_N, where $N > 1$

Suppose that d is a coddiferential in L_{m+2}, where $m \geq 0$. Then we can express

$$d = \psi_3^{1,0,m+1}a_1 + \psi_3^{0,1,m+1}a_2 + \psi_1^{0,0,m+2}a_3 + \psi_2^{0,0,m+2}a_4 + \psi_1^{1,1,m}a_5 + \psi_2^{1,1,m}a_6.$$

The bracket of d with itself is given by

$$
\begin{aligned}
\tfrac{1}{2}[d,d] = {}& \phi_1^{1,0,2m+2}(-a_4a_5 + a_1a_3(m+2)) + \phi_1^{0,1,2m+2}a_3(a_5 + a_2(m+2))\\
&+ \phi_2^{1,0,2m+2}a_4(-a_6 + a_1(m+2)) + \phi_2^{0,1,2m+2}(a_3a_6 + a_2a_4(m+2))\\
&+ \phi_3^{1,1,2m+1}(a_2a_6 + a_1a_5) + \phi_3^{0,0,2m+3}(a_2a_4 + a_1a_3)
\end{aligned}
$$

A little arithmetic reduces the set of equations which follow from the above to

$$\begin{aligned} a_2a_6+a_1a_5&=0\\ a_1a_3+a_2a_4&=0\\ a_3(a_5+(m+2)a_2)&=0\\ a_4(a_5+(m+2)a_2)&=0\\ a_3(a_6-(m+2)a_1)&=0\\ a_4(a_6-(m+2)a_1)&=0. \end{aligned}$$

It follows that either $a_3=a_4=0$ or both $a_5=-(m+2)a_2$ and $a_6=(m+2)a_1$. Let us consider the case $a_3=a_4=0$ first. Then the equations reduce to $a_2a_6+a_1a_5=0$. If $a_1=a_2=0$ then we obtain no conditions on a_5 or a_6, so we obtain the codifferential $d=\psi_1^{1,1,m}a_5+\psi_2^{1,1,m}a_6$. Let b_1, b_2 be such that $b_2a_5-b_1a_6=1$. Let $d'=\psi_1^{1,1,m}$. Then $d'=g^{-1}dg$ for the automorphism $g=\begin{pmatrix}a_5&b_1\\a_6&b_2\end{pmatrix}$, so d' and d are equivalent.

If either $a_1\neq 0$ or $a_2\neq 0$, then we have $a_5=ka_2$ and $a_6=-ka_1$ for some k. Thus we obtain the codifferential

$$d=\psi_3^{1,0,m+1}a_1+\psi_3^{0,1,m+1}a_2+\psi_1^{1,1,m}ka_2-\psi_2^{1,1,m}ka_1.$$

Let $g=\begin{pmatrix}a_2&b_1\\-a_1&b_2\end{pmatrix}$, where $a_1b_1+a_2b_2=1$. Then one can easily check that $gd'=dg$, for $d'=\psi_3^{0,1,m+1}+\psi_1^{11m}k$.

Finally, suppose that at least one of a_3 or a_4 does not vanish. Then $a_5=-(m+2)a_2$ and $a_6=(m+2)a_1$. Define $g=\begin{pmatrix}a_3&b_1\\a_4&b_2\end{pmatrix}$, where $a_3b_2-a_4b_1=1$. Let $k=(a_1b_1+a_2b_2)$. Then d is equivalent to the codifferential $d'=\psi_3^{0,1,m+1}k+\psi_1^{0,0,m+2}-\psi_1^{1,1,m}k(m+2)$. If $k\neq 0$ then we can reduce this codifferential to $d=\psi_3^{0,1,m+1}+\psi_1^{0,0,m+2}-\psi_1^{1,1,m}(m+2)$.

Thus we arrive at the following table of codifferentials of degree $m+2$:

$$\begin{aligned} d_\infty&=\psi_1^{1,1,m}\\ d_\lambda&=\psi_3^{0,1,m+1}+\psi_1^{1,1,m}\lambda\\ d_*&=\psi_1^{0,0,m+2}\\ d_\#&=\psi_3^{0,1,m+1}+\psi_1^{0,0,m+2}-\psi_1^{1,1,m}(m+2) \end{aligned}$$

If we wish to emphasize the degree of the codifferential, we will indicate it parenthetically. Thus $d_\lambda(k)$ represents the degree k version of d_λ. Note that for $m=-1$, only d_0 and d_* are well defined, and they coincide with the codifferentials of degree 1 which we have already studied. It is easy to check that none of these four types of codifferentials is equivalent to any of the others. Moreover, $d_\lambda(k)$ is equivalent to $d_\mu(k)$ precisely when $\lambda = \mu$. Thus we have classified all codifferentials in L_{m+2} for $m \geq 0$.

Notice that any neighborhood of d_* contains an element equivalent to $d_\#$ and similarly, any neighborhood of d_λ, for $\lambda = -(m+2)$ also contains an element equivalent to $d_\#$. We say that $d_\#$ is infinitesimally close to these elements. It seems reasonable to characterize the moduli space of codifferentials of degree $N > 1$ as consisting of $\mathfrak{K}\mathbb{P}^1$ plus two special points. In order to get a better picture of our moduli space, we need to look at the cohomology.

5. Cohomology of $d_\lambda = \psi_3^{0,1,m+1} + \psi_1^{1,1,m}\lambda$

First, we compute a table of coboundaries for $D = [\cdot, d_\lambda]$. The coboundaries of the even elements are given by

$$\begin{aligned}
D(\phi_1^{1,0,n+1}) &= -\psi_1^{1,1,m+n+1}(n+1)\\
D(\phi_2^{1,0,n+1}) &= \psi_2^{1,1,m+n+1}(\lambda - n - 1) - \psi_3^{1,0,m+n+2}\\
D(\phi_1^{0,1,n+1}) &= 0\\
D(\phi_2^{0,1,n+1}) &= -\psi_1^{1,1,m+n+1}\lambda - \psi_3^{0,1,m+n+2}\\
D(\phi_3^{0,0,n+2}) &= -\psi_1^{1,1,m+n+1}\lambda m + \psi_3^{0,1,m+n+2}(n+1-m)\\
D(\phi_3^{1,1,n}) &= 0
\end{aligned}$$

The odd coboundaries are

$$\begin{aligned}
D(\psi_3^{1,0,n+1}) &= -\phi_3^{1,1,m+n+1}(-\lambda - m + n)\\
D(\psi_3^{0,1,n+1}) &= 0\\
D(\psi_1^{0,0,n+2}) &= \phi_1^{0,1,m+n+2}(\lambda + n + 2)\\
D(\psi_2^{0,0,n+2}) &= -\phi_1^{1,0,m+n+2}\lambda + \phi_2^{0,1,m+n+2}(n+2) + \phi_3^{0,0,m+n+3}\\
D(\psi_1^{1,1,n}) &= 0\\
D(\psi_2^{1,1,n}) &= \phi_3^{1,1,m+n+1}
\end{aligned}$$

Let us define

$$\begin{aligned}
z_n &= \ker(D : L_n \to L_{n+m+1})\\
b_n &= \operatorname{Im}(D : L_n \to L_{n+m+1})
\end{aligned}$$

Then $h_{n+2} = z_{n+2} - b_{n-m+1}$. Note that if $n < m$, $h_{n+2} = z_{n+2}$, while $h_{m+2} = z_{m+2} - b_1$. Furthermore, $z_{n+2} + b_{n+2} = 6|6$, for $n \geq 0$ and $z_1 + b_1 = 5|4$.

Let us first consider the case $n \neq -1$. Then there are exactly 3 even independent $(n+2)$-cocycles, and therefore three odd independent $(n+2)$-coboundaries. The situation for odd cocycles is a bit more complicated. It is clear that generically, we have 3 odd $(n+2)$-cocycles and therefore 3 even $(n+2)$-coboundaries. When $\lambda = -(n+2)$, we obtain an extra cocycle $\psi_1^{0,0,n+2}$. Thus, when λ is a negative integer, the cohomology is a bit different than in the generic case.

When $n = -1$, there are always three even independent cocycles, and thus 2 odd independent coboundaries. Generically, there is only one odd 1-cocycle, but when $\lambda = -(m+1)$, we pick up an extra odd cocycle $\psi_3^{1,0,0}$. Notice, that in this case, λ is also a negative integer.

5.1. **Cohomology when λ is not a negative integer.** Let us suppose that λ is generic, that is, it is not a negative integer. Then $z_{n+2} = 3|3$ and $b_{n+2} = 3|3$ for $n \geq 0$. Also we have $z_1 = 3|1$, so $b_1 = 3|2$. Therefore we have

$$\begin{aligned} &h_1 = 3|1 \\ &h_{n+2} = 3|3, \quad \text{if } 0 \leq n < m \\ &h_{m+2} = 0|1 \\ &h_{n+2} = 0, \quad \text{if } n > m \end{aligned}$$

We can give a basis for the cohomology in each dimension.

$$\begin{aligned} &H^1 = \langle \psi_3^{0,1,0}, \phi_1^{1,0,0}, \phi_1^{0,1,0}, m\phi_2^{0,1,0} - \phi_3^{0,0,1} \rangle \\ &(H^{n+2})_o = \langle \psi_3^{0,1,n+1}, \psi_1^{1,1,n}, \psi_3^{1,0,n+1} + \psi_2^{1,1,n}(-\lambda - m + n) \rangle \\ &(H^{n+2})_e = \langle \phi_1^{0,1,n+1}, \phi_3^{1,1,n}, -\phi_1^{1,0,n+1}\lambda + \phi_2^{0,1,n+1}(n+1-m) + \phi_3^{0,0,n+2} \rangle \\ &\quad \text{if } 0 \leq n < m \\ &H^{m+2} = \langle \psi_1^{1,1,m} \rangle \\ &H^{n+2} = 0 \quad \text{if } n > m. \end{aligned}$$

The terms of degree $n+2$, with $-1 \leq n < m$, govern deformations of d_λ into codifferentials of lower degree. Looking at H^1, we see that d_λ generically deforms into a codifferential which is equivalent to the codifferential $d_0(1) = \psi_3^{0,1,0}$ of degree 1.

When the leading term of a deformation is of degree $n+2$, where $0 \leq n < m$, our codifferential deforms into three distinct types of codifferentials of degree $n+2$. First, we have $d_0 = \psi_3^{0,1,n+1}$, then $d_\infty = \psi_1^{1,1,n}$, and finally $d_{-\lambda-m+n} = \psi_3^{1,0,n+1} - \psi_2^{1,1,n}(-\lambda - m + n)$. Note that since λ is not a negative integer, $-\lambda - m + n \neq 0$, so we do not obtain two different ways to deform into d_0. Of course, we could deform into a sum of terms of these types as well.

We are especially interested in the deformations of d_λ on the same level, because they tell us how the moduli space of degree $m+2$ codifferentials fits together. Consider the degree $m+2$ part of the universal infinitesimal deformation of d_λ. It is of the form

$$d^1_\lambda = \psi_3^{0,1,m+1} + \psi_1^{1,1,m}(\lambda + t).$$

Thus we see that d_λ deforms into $d_{\lambda+t}$. This is a deformation along the family d_λ.

Finally, note that because $h_{n+2} = 0$ for $n > m$, there are no nonequivalent extensions of d_λ.

5.2. **Cohomology for $\lambda = -p$, if $p \neq m+1$.** Suppose that $\lambda = -p$ is a negative integer, not equal to $-(m+1)$. Then, depending on the value of p, we have several distinct cases.

5.2.1. *Case* $p = 1$. Now $h_1 = 3|2$, and we have an extra even $(m+2)$-cohomology class, $\phi_3^{1,1,m}$, so $h_{m+2} = 1|1$. We obtain a deformation in the $d_*(1) = \psi_1^{0,0,1}$ direction. Thus the table of cohomology for d_λ is modified by

$$\begin{aligned} H^1 &= \langle \psi_3^{0,1,0}, \psi_1^{0,0,1}, \phi_1^{1,0,0}, \phi_1^{0,1,0}, m\phi_2^{0,1,0} - \phi_3^{0,0,1} \rangle \\ H^{m+2} &= \langle \psi_1^{1,1,m}, \phi_3^{1,1,m} \rangle \end{aligned}$$

5.2.2. *Case* $2 \leq p < m+2$. In this case we obtain an extra odd p-cohomology class $\psi_1^{0,0,p}$, which means we again obtain an extra deformation in the $d_*(p)$ direction. Of course, there is a corresponding extra even $(m+p+1)$-cohomology class $\phi_1^{0,1,m+p}$. Notice that this value of λ coincides with the value of λ at which we already noticed something peculiar happening in the moduli space. The table of cohomology for d_λ becomes modified by

$$\begin{aligned} (H^p)_o &= \langle \psi_3^{0,1,p-1}, \psi_1^{1,1,p-2}, \psi_3^{1,0,p-1} - \psi_2^{1,1,p-2}(2p-m), \psi_1^{0,0,p} \rangle \\ H^{m+p+1} &= \langle \phi_1^{0,1,m+p} \rangle. \end{aligned}$$

5.2.3. *Case* $p = m+2$. In this case we get a deformation of d_λ in the $d_* = \psi_1^{0,0,m+2}$ direction on the same level, and an extra even $(2m+3)$-cohomology class $\phi_1^{0,1,2m+2}$. Thus we have an extra direction of deformation in the moduli space of degree $m+2$ codifferentials. The table of cohomology is modified by

$$\begin{aligned} H^{m+2} &= \langle \psi_1^{1,1,m}, \psi_1^{0,0,m+2} \rangle \\ H^{2m+3} &= \langle \phi_1^{0,1,2m+2} \rangle. \end{aligned}$$

5.2.4. *Case $p > m + 2$.* In this final case, we obtain something really interesting, because there is a nontrivial extension $d_{\lambda,e} = d_\lambda + \psi_1^{0,0,n+2}$ of d_λ. We will discuss this extension later. The table of cohomology is altered in degrees p and $m + p + 1$ as follows:

$$\begin{aligned} H^p &= \langle \psi_1^{0,0,p} \rangle \\ H^{m+p+1} &= \langle \phi_1^{0,1,m+p} \rangle. \end{aligned}$$

5.3. **Cohomology for $\lambda = -(m + 1)$.** For $n = -1$, there is an extra odd 1-cohomology class, $\psi_3^{1,0,0}$, so there is an extra even $(m + 2)$-cohomology class $\phi_3^{1,1,m}$. Then $h_1 = 3|2$ and $h_{m+2} = 1|1$. We also obtain an extra odd $(m + 1)$-cohomology class $\psi_1^{0,0,m+1}$ and an extra even $(2m+2)$-cohomology class $\phi_1^{0,1,2m+1}$. Thus the table of cohomology becomes modified as follows:

$$\begin{aligned} H^1 &= \langle \psi_3^{0,1,0}, \psi_3^{1,0,0}, \phi_1^{1,0,0}, \phi_1^{0,1,0}, m\phi_2^{0,1,0} - \phi_3^{0,0,1} \rangle \\ (H^{m+1})_o &= \langle \psi_3^{0,1,m}, \psi_1^{1,1,m-1}, \psi_3^{1,0,m} - \psi_2^{1,1,m-1}(2m) \rangle \\ H^{m+2} &= \langle \psi_1^{1,1,m}, \phi_3^{1,1,m} \rangle \\ H^{2m+2} &= \langle \phi_1^{0,1,2m+2} \rangle. \end{aligned}$$

6. Cohomology of $d_\infty = \psi_1^{1,1,m}$

The coboundaries are

$$\begin{aligned} D(\phi_1^{1,0,n+1}) &= 0, & D(\psi_3^{1,0,n+1}) &= \phi_3^{1,1,m+n+1} \\ D(\phi_2^{1,0,n+1}) &= \psi_2^{1,1,m+n+1}, & D(\psi_3^{0,1,n+1}) &= 0 \\ D(\phi_1^{0,1,n+1}) &= 0, & D(\psi_1^{0,0,n+2}) &= \phi_1^{0,1,m+n+2} \\ D(\phi_2^{0,1,n+1}) &= -\psi_1^{1,1,m+n+1}, & D(\psi_2^{0,0,n+2}) &= -\phi_1^{1,0,m+n+2} \\ D(\phi_3^{0,0,n+2}) &= -\psi_1^{1,1,m+n+1}m, & D(\psi_1^{1,1,n}) &= 0 \\ D(\phi_3^{1,1,n}) &= 0, & D(\psi_2^{1,1,n}) &= 0 \end{aligned}$$

It is easy to see that $z_1 = 3|1$ and $z_{n+2} = 4|3$ for $n > -1$. Thus $b_{n+2} = 3|2$ for $n \geq -1$. From this we get

$$\begin{aligned} h_1 &= 3|1 \\ h_{n+2} &= 4|3, \quad \text{if } 0 \leq n < m \\ h_{n+2} &= 1|1, \quad \text{if } n \geq m. \end{aligned}$$

Consequently we always have nontrivial extensions of d_∞. If we look at H^1, we see that the odd cohomology class $d_0 = \psi_3^{0,1,0}$ will deform into a codifferential of type d_0 in degree 1. For $0 \leq n < m$, we have three independent odd cohomology classes, $d_0 = \psi_3^{0,1,n+1}$, $\tilde{d}\infty = \psi_1^{1,1,n}$ and $\psi_2^{1,1,n}$. The last cohomology class also is of type d_∞. For $n \geq m$, there is only one odd cohomology class, $\psi_3^{0,1,n+1}$. When $n = m$, this cohomology class governs deformations along the family. For $n > m$, the existence of this cohomology class implies that there is a nontrivial extension of d_∞. In fact, there are many nonequivalent extensions, which we will discuss later. A table of the basis of the cohomology for d_∞ is

$$\begin{aligned} &H^1 = \langle \psi_3^{0,1,0}, \phi_1^{1,0,0}, \phi_1^{0,1,0}, \phi_2^{0,1,0} m - \phi_3^{0,0,1} \rangle \\ &(H^{n+2})_o = \langle \psi_3^{0,1,n+1}, \psi_1^{1,1,n}, \psi_2^{1,1,n} \rangle \\ &(H^{n+2})_e = \langle \phi_1^{0,1,n+1}, \phi_3^{1,1,n}, \phi_1^{1,0,n+1}, \phi_2^{0,1,n+1} m - \phi_3^{0,0,n+2} \rangle \\ &\quad \text{if } 0 \leq n < m \\ &H^{n+2} = \langle \psi_3^{0,1,n+1}, \phi_2^{1,0,n+1} m - \phi_3^{0,0,n+2} \rangle \quad \text{if } n \geq m. \end{aligned}$$

7. Cohomology of $d_* = \psi_1^{0,0,m+2}$

First, we compute a table of coboundaries for $D = [\cdot, d_\lambda]$. The boundaries of the elements are given by

$$\begin{aligned} &D(\phi_1^{1,0,n+1}) = \psi_1^{0,0,m+n+3}, && D(\psi_3^{1,0,n+1}) = \phi_1^{1,0,m+n+2}(m+2) + \phi_3^{0,0,m+n+3} \\ &D(\phi_2^{1,0,n+1}) = \psi_2^{0,0,m+n+3}, && D(\psi_3^{0,1,n+1}) = \phi_1^{0,1,m+n+2}(m+2) \\ &D(\phi_1^{0,1,n+1}) = 0, && D(\psi_1^{0,0,n+2}) = 0 \\ &D(\phi_2^{0,1,n+1}) = 0, && D(\psi_2^{0,0,n+2}) = 0 \\ &D(\phi_3^{0,0,n+2}) = -\psi_1^{0,0,m+n+3}(m+2), && -D(\psi_1^{1,1,n}) = \phi_1^{0,1,m+n+2} \\ &D(\phi_3^{1,1,n}) = -\psi_1^{1,1,m+n+1}(m+2) + \psi_3^{0,1,m+n+2}, && D(\psi_2^{1,1,n}) = \phi_2^{0,1,m+n+2} \end{aligned}$$

It is easy to compute that

$$\begin{aligned}
&h_1 = 3|2\\
&h_{n+2} = 3|3, \quad \text{if } 0 \le n < m\\
&h_{m+2} = 1|1\\
&h_{n+2} = 0, \quad \text{if } n > m.
\end{aligned}$$

A table of the cohomology is given by

$$\begin{aligned}
&H^1 = \langle \psi_1^{0,0,1}, \psi_2^{0,0,1}, \phi_1^{0,1,0}, \phi_2^{0,1,0}, \phi_2^{1,0,0}(m+2) + \phi_3^{0,0,1} \rangle\\
&(H^{n+2})_o = \langle \psi_1^{0,0,n+2}, \psi_2^{0,0,n+2}, \psi_3^{0,1,n+1} - \psi_1^{1,1,n}(m+2) \rangle\\
&(H^{n+2})_e = \langle \phi_1^{0,1,n+1}, \phi_2^{0,1,n+1}, \phi_2^{1,0,n+1}(m+2) + \phi_3^{0,0,n+2} \rangle\\
&\quad \text{if } 0 \le n < m\\
&H^{m+2} = \langle \psi_3^{0,1,m+1} - \psi_1^{1,1,m}(m+2), \phi_2^{0,1,m+1} \rangle\\
&H^{n+2} = 0, \quad \text{if } n > m.
\end{aligned}$$

In degree 1, our codifferential deforms into d_* in two different ways. For $0 \le n < m$, we have two deformations in the d_* direction, but we also get a deformation into the special element $d_{-(m+2)}$ of the family. This direction of deformation is the only one that remains in degree $m+2$, and there are no higher order deformations, thus no nontrivial extensions of d_*.

8. Cohomology of $d_\# = \psi_3^{0,1,m+1} + \psi_1^{0,0,m+2} - \psi_1^{1,1,m}(m+2)$

The even coboundaries of the elements are

$$\begin{aligned}
D(\phi_1^{1,0,n+1}) &= -\psi_1^{1,1,m+n+1}(n+1) + \psi_1^{0,0,m+n+3}\\
D(\phi_2^{1,0,n+1}) &= -\psi_2^{1,1,m+n+1}(n+m+3) + \psi_2^{0,0,m+n+3} - \psi_3^{1,0,m+n+2}\\
D(\phi_1^{0,1,n+1}) &= 0\\
D(\phi_2^{0,1,n+1}) &= \psi_1^{1,1,m+n+1}(m+2) - \psi_3^{0,1,m+n+2}\\
D(\phi_3^{0,0,n+2}) &= -\psi_1^{1,1,m+n+1}m(m+2) - \psi_1^{0,0,m+n+3}(m+2)\\
&\quad + \psi_3^{0,1,m+n+2}(n+1-m)\\
D(\phi_3^{1,1,n}) &= -\psi_1^{1,1,m+n+1}(m+2) + \psi_3^{0,1,m+n+2}
\end{aligned}$$

while the odd ones are

$$\begin{aligned}
D(\psi_3^{1,0,n+1}) &= \phi_1^{1,0,m+n+2}(m+2) - \phi_3^{1,1,m+n+1}(n+2) + \phi_3^{0,0,n+m+3}\\
D(\psi_3^{0,1,n+1}) &= \phi_1^{0,1,m+n+2}(m+2)\\
D(\psi_1^{0,0,n+2}) &= \phi_1^{0,1,m+n+2}(n-m)\\
D(\psi_2^{0,0,n+2}) &= \phi_1^{1,0,m+n+2}(m+2) + \phi_2^{0,1,m+n+2}(n+2) + \phi_3^{0,0,m+n+3}\\
D(\psi_1^{1,1,n}) &= \phi_1^{0,1,m+n+2}\\
D(\psi_2^{1,1,n}) &= \phi_2^{0,1,n+m+2} + \phi_3^{1,1,m+n+1}.
\end{aligned}$$

A table of the cohomology is given by

$$\begin{aligned}
(H^1)_o &= \langle \psi_3^{0,1,0}(m+1) + \psi_1^{0,0,1}(m+2)\rangle\\
(H^1)_e &= \langle \phi_1^{0,1,0}, \phi_2^{0,1,0}(m+2) - \phi_2^{0,1,0}m + \phi_3^{0,0,0}\rangle\\
(H^{n+2})_o &= \langle \psi_3^{0,1,n+1} - \psi_1^{1,1,n}(m+2), \psi_3^{0,1,n+1} + \psi_1^{0,0,n+2} - \psi_1^{1,1,n}(n+2),\\
&\quad - \psi_3^{1,0,n+1} + \psi_2^{0,0,n+2} - \psi_2^{1,1,n}(n+2)\rangle\\
(H^{n+2})_e &= \langle \phi_1^{0,1,n+1}, \phi_2^{0,1,n+1} + \phi_3^{1,1,n},\\
&\quad \phi_2^{0,1,n+1}(m+2) + \phi_2^{0,1,n+1}(n+1-m) + \phi_3^{0,0,n+2}\rangle\\
&\quad \text{if } 0 \le n < m\\
H^{n+2} &= 0, \quad \text{if } n \ge m.
\end{aligned}$$

Let us analyze the cohomology for $0 \le n < m$ first. The first odd cohomology class in the table is of type $d_{-(n+2)}$ and the last two are of type $d_\#$.

Since $h_{n+2} = 0$, if $n \ge m$, there are no deformations of $d_\#$ on the same level. Thus $d_\#$ is rigid in the moduli space of degree $m+2$ codifferentials.

However, note that we have been so careful up to now to give a basis for H_n consisting of cohomology classes which correspond to actual codifferentials, for $n < m$. But this time, the cohomology class representing the odd part of H^1 doesn't come from any codifferential of degree 1, and there is no way to fix this problem, because there is only one cohomology class in degree 1. If one computes the versal deformation of $d_\#$, then one can show that the parameter t which the cocycle by will have to satisfy $t^2 = 0$. This means that in some sense, we do not get a true deformation of $d_\#$ in this direction, because we do not obtain any nonzero values of the parameter for which the deformation determines an actual codifferential.

9. Extensions of Codifferentials

So far, we have not discussed in any detail how to extend a codifferential d_N which is of fixed degree N. Let G be the subgroup of automorphisms of $S(W)$ fixing d_N, and G_1 be the subgroup G consisting of linear automorphisms. The groups G and G_1 act on the set of nonzero cohomology classes. We shall say that two cohomology classes δ_1 and δ_2 are equivalent if there is an element f in G such that $f_*(\delta_1) = \delta_2$, and linearly equivalent if f lies in G_1.

Suppose that $d = d_N + d_{N+k} + d_{N+k+1} + \cdots$ is an extension of d_N. The following theorem will help to classify such extensions.

Theorem 9.1. *Suppose that $d = d_N+d_{N+k}+d_{N+k+1}+\cdots$ is a codifferential. Then d_{N+k} is a cocycle with respect to the coboundary operator $D = [\bullet, d_N]$. Moreover,*

(1) *d is equivalent to a codifferential whose first nonzero term after d_N is of higher order than $N + k$ iff d_{N+k} is a coboundary,.*
(2) *If $d' = d_N + d'_{N+k} + d'_{N+k+1} + \cdots$ is another codifferential and the cohomology class of d'_{N+k} is not linearly equivalent to the cohomology class of d_{N+k}, then d' and d are not equivalent.*

Let us say that a codifferential d is standard if it is of the form $d_N + d_{N+k} + d_{N+k+1} + \cdots$, where d_{N+k} is a nontrivial cocycle for d_N. By the first part of this theorem, every nontrivial extension of d_N is equivalent to a standard codifferential. For a codifferential in standard form, let us refer to the cohomology class of d_{N+k} as the secondary term of d.

In general, we don't expect $d_N + d_{N+k}$ to be a codifferential for an extension $d = d_N + d_{N+k} + d_{N+k+1} + \cdots$ of d_N. However, for the examples d_∞ and d_λ which arise in this paper, it turns out to be true. We state a theorem which is useful in characterizing the extensions of a codifferential of the form $d = d_N + d_{N+k}$.

Theorem 9.2. *Suppose $d = d_N + d_{N+k}$ is a codifferential. Let D_1 be the cohomology given by d_N, and D_2 be the cohomology given by d_{N+k}. Let $d' = d_N + d_{N+k} + d_{N+l} + \cdots$ be an extension of d, where $l > k$. Let $\delta = d_{N+l}$ Then*

(1) *δ is a D_1-cocycle. Moreover $D_2(\delta)$ is a D_1-coboundary.*
(2) *If δ is a D_1-coboundary, then d' is equivalent to an extension whose third term has degree larger than $N + l$.*
(3) *If $H^n(D_1) = 0$ for $n > N + k$, then any extension d' of d is equivalent to d.*
(4) *If $\delta = D_2(\eta)$ for some D_1-cocycle η, then d' is equivalent to an extension whose third term has degree larger than $N + l$.*

(5) *If every cocycle δ whose order is larger than k for which $D_2(\delta)$ is a D_1-coboundary is of the form $\delta = D_2(\eta)$ for some D_1-cocycle η, then any extension of d is equivalent to d.*

The proofs of these theorems are in [4].

9.1. **Extensions of d_λ, for $\lambda = -(n+2)$, $n > m$.** The only nontrivial cocycles of degree larger than $m+2$ are of the form $\psi_1^{0,0,n+2}a$, and for $a \neq 0$, this cocycle is linearly equivalent to $\psi_1^{0,0,n+2}$. Thus any nontrivial extension of d_λ is of the form

$$d = \psi_3^{0,1,m+1} - \psi_1^{1,1,m}(n+2) + \psi_1^{0,0,n+2} + \delta + \cdots,$$

where δ is a cocycle of degree greater than $n+2$, so in fact, δ is a coboundary. But then applying Theorem (9.2), we see that this term can be eliminated. Thus we see that d is equivalent to the extension

$$d_{\lambda,e} = \psi_3^{0,1,m+1} - \psi_1^{1,1,m}(n+2) + \psi_1^{0,0,n+2}.$$

Thus there is only one nontrivial extension of d_λ, up to equivalence. Let us calculate its cohomology. Because there are terms of mixed degree in $d_{\lambda,e}$, any cocycle will also have mixed terms. Let D_λ, D_*, D_e be the coboundary operators determined by d_λ, $d_* = \psi_1^{0,0,n+2}$, and $d_{\lambda,e}$, resp. If $\delta = \delta_{k+2} + \cdots$ is a D_e-cocycle of order $k+2$ (order means the least nonzero degree term), then δ_{k+2} is a D_λ cocycle. Moreover, $D_*(\delta_{k+2}) = -D_\lambda(\delta_{n-m+k+2}$. We claim that if $k > -1$, then the second equality is a consequence of the first equality. In fact, we claim that there is an $(n-m+k+2)$-cochain η such that $\gamma = \delta_{k+2} + \eta$ is a D_e cocycle. Thus $\delta - \gamma$ is a cocycle of higher order than δ.

To see this, we compute D_* on a basis of $Z_{k+2}(D_\lambda)$, the space of D_λ $(k+2)$-cocycles. Let us suppose that k is not -1 or m.

$$D_*(\psi_3^{0,1,k+1} - \psi_1^{1,1,k}(n+2)) = 0$$

$$D_*(\psi_1^{1,1,k}) = \phi_1^{0,1,n+k+2} = D_\lambda(\psi_1^{0,0,n-m+k+2}\tfrac{1}{k-m})$$

$$\begin{aligned}D_*(\psi_3^{1,0,k+1} + \psi_2^{1,1,k}(n-m+k+2)) = \\ \phi_1^{1,0,n+k+2}(n+2) + \phi_3^{0,0,n+k+3} + \phi_2^{0,1,n+k+2}(n-m+k+2) \\ = D_\lambda(\psi_2^{0,0,l+2})\end{aligned}$$

$$D_*(\phi_1^{0,1,k+1}) = 0$$

$$D_*(\phi_3^{1,1,k}) = -\psi_1^{1,1,n+k+1}(n+2) + \psi_3^{0,1,n+k+2} = D_\lambda(-\phi_2^{0,1,n-m+k+1})$$

$$D_*(\phi_1^{1,0,k+1}(n+2) + \phi_2^{1,0,k+1}(k+1-m) + \phi_3^{0,0,k+2}) = 0$$

Now, it is immediate to compute that $\psi_1^{0,0,n-m+k+2}$, $\psi_2^{0,0,n-m+k+2}$, and $\phi^{0,1,n-m+k+1}$ are all cocycles with respect to D_*. It follows that every basis element of $Z_{k+2}(D_\lambda)$ extends to a D_e cocycle.

While it is certainly true that $Z(D_e)$ cannot be decomposed as a direct product of spaces Z^n which are given by elements of degree n, nevertheless, $Z(D_e)$ is a filtered space and we can identify Z^n/Z^{n+1} with a subspace of Z in a natural manner. If k is not -1, m, or n then $z_{k+2}(D_\lambda) = 3|3$ and the elements below give a corresponding basis for $Z_{k+2}(D_e)$.

$$\begin{aligned}
\eta_1(k) &= \psi_3^{0,1,k+1} - \psi_1^{1,1,k}(n+2))\\
\eta_2(k) &= \psi_3^{1,0,k+1} + \psi_2^{1,1,k}(n-m+k+2) - \psi_2^{0,0,n-m+k+2}\\
\eta_3(k) &= \psi_1^{1,1,k} - \psi_1^{0,0,n-m+k+2}\tfrac{1}{k-m}\\
\tau_1(k) &= \phi_1^{0,1,k+1}\\
\tau_2(k) &= \phi_3^{1,1,k} + \phi_2^{0,1,n-m+k+1}\\
\tau_3(k) &= \phi_1^{1,0,k+1}(n+2) + \phi_2^{1,0,k+1}(k+1-m) + \phi_3^{0,0,k+2}
\end{aligned}$$

One can compute that the $(m+k+3)$-cocycles satisfy

$$\begin{aligned}
\eta_1(m+k+1) &= -D_e(\phi_2^{0,1,k+1})\\
\eta_2(m+k+1) &= -D_e(\phi_2^{1,0,k+1,})\\
\eta_3(m+k+1) &= \tfrac{-1}{k+1}D_e(\phi_1^{1,0,k+1})\\
\tau_1(m+k+1) &= D_e(\phi_3^{0,1,k+1})\\
\tau_2(m+k+1) &= \tfrac{1}{n-m+k+2}D_e(-\psi_3^{1,0,k+1} + \psi_2^{0,0,n-m+k+2})\\
\tau_3(m+k+1) &= D_e(\psi_2^{0,0,k+2})
\end{aligned}$$

Thus $H^{k+2} = 0$, if $k > m$, except possibly for $k = n$. The reason we have to single this case out is because then there is an extra D_λ $(n+2)$-cocycle $\eta_4(n) = \psi_1^{0,0,n+2}$, which remains a cocycle for D_e. But $\eta_4(n) = D_e(\phi_1^{1,0,0})$, so it is a D_e-coboundary, so it does not determine a cohomology class for D_e.

For $k = -1$, one can check that one of the basis elements of $Z_1(D_\lambda)$ does not extend to a cocycle for D_e. We have

$$Z_1(D_e) = \langle \phi_1^{0,1,0}, \phi^{1,0,0}(n+2) - \phi_2^{0,1,0}m + \phi_3^{0,0,1}, \psi_3^{0,1,0} - \psi_1^{0,0,n-m+1}\tfrac{n+2}{m+1}\rangle.$$

Note that when $k = m$, η_3 is not defined. The D_λ-cocycle $\phi_1^{1,1,m}$, which generates the cohomology for D_λ in dimension $m+2$ does not extend to a cocycle. Thus the extension has killed the deformation along the family. The formula expressing the cocycles of degree $m+k+3$ applies for $k=-1$, except for the η_3 term, which we don't need, so all $(m+2)$-cocycles are coboundaries as well.

Thus, we finally arrive at the table of cohomology for $d_{\lambda,e}$.

$$\begin{aligned}
H^1 &= \langle \psi_3^{0,1,0}(m+1)+\psi_1^{0,0,n-m+1}(n+2),\\
&\quad \phi_1^{0,1,0}, -\phi_1^{1,0,0}(n+2)+\phi_2^{0,1,0}m-\phi_3^{0,0,1}\rangle\\
H^k &= \langle \eta_1,\eta_2,\eta_3,\tau_1,\tau_2,\tau_3\rangle \qquad \text{if } 0\le k<m\\
H^k &= 0 \qquad \text{if } k\ge m
\end{aligned}$$

Notice that the dimension of the cohomology of the extended codifferential has dropped by one in four different dimensions. This makes sense, because we have killed nontrivial cocycles in dimensions 1 and $m+2$, which create new coboundaries in dimensions $n+2$ and $m+n+2$, thus killing off two more cohomology classes.

It is always true that the dimension of the cohomology of an extended codifferential cannot exceed the dimension of the cohomology of the original codifferential. This is because, as in our example, cocycles may not extend to cocycles, but coboundaries always extend, since the extended coboundary is simply the extended codifferential applied to the same object that gave the original coboundary.

9.2. **Extensions of d_∞.** The second part of Theorem (9.1) guarantees that there are an infinite number of nonequivalent extensions of d_∞. In fact, the codifferentials $d_{\infty,n}=\psi_1^{1,1,m}+\psi_3^{0,1,n+1}$, for $n>m$ give an infinite family of nonequivalent codifferentials extending d_∞. Moreover, $\psi_3^{0,1,n+1}a$ is linearly equivalent to $\psi_3^{0,1,n+1}$ when $a\ne 0$, if we can solve the equation $q^{m-n}=a$. In particular, when $\mathfrak{K}$ is algebraically closed, any extension of the form $d=\psi_1^{1,1,m}+\psi_3^{0,1,n+1}a$ is equivalent to the extension $d_{\infty,n}$. Let us assume that $\mathfrak{K}$ is algebraically closed.

Let D_∞ be the coboundary operator given by $\psi_1^{1,1,m}$ and D_0 be the one given by $\psi_3^{0,1,n+1}$. We want to determine how $d_{\infty,n}$ can be extended. Since the next term we add must be a D_∞-cocycle, which may be taken to be a D_λ-cohomology class, it may be taken to be of the form $\psi_3^{0,1,k+1}a$, where $k>n$. Let $\zeta(l)=\phi_2^{0,1,l}m-\phi_3^{0,0,l+1}$. We know that $\zeta(l)$ is a D_λ-cocycle, and moreover $D_0(\zeta(k-n))=\psi_3^{0,1,k+l}(-m-k+2*n)$, which means that we can eliminate this term, by Theorem (9.1), unless $k=2n-m$. However, for this special value of k, we cannot eliminate the term by adding a D_0-coboundary of a D_∞-cocycle, because there is no D_∞-cocycle whose D_0 coboundary coincides with $\psi_3^{0,1,2n-m+1}$. One can examine the linear equivalences which fix $d_{\infty,n}$, and we can check that a linear equivalence which fixes $d_{\infty,n}$ also fixes $\psi_3^{0,1,2n-m+1}a$. Thus we cannot eliminate the term by a linear equivalence, by adding a D_∞-coboundary, or by adding a D_0-coboundary. This means we cannot eliminate the term.

Thus we obtain a nontrivial extension of $d_{\infty,n}$ given by

$$d_{\infty,n,e}=\psi_1^{1,1,m}+\psi_3^{0,1,n+1}+\psi_3^{0,1,2n-m+1}a.$$

any extension of $d_{\infty,n}$ The fact that this is as far as we need to go is easy to explain. First, if we add another term, it can be taken to be of the form $\psi_3^{0,1,k+1}b$, for some $k>2n-m$. But then the term can be eliminated, because it is a D_0-coboundary. Notice that the presence of the extra term $\psi_3^{0,1,2n-m+1}a$ does not interfere with the elimination of the term by the same process as before.

In fact, it is easy to compute the cocycles for the coboundary operator D_e induced by $d_{\infty,n,a}$. For Z_1, we note that the D_∞-cocycles ϕ_1^{100}, $\phi_1^{0,1,0}$ and $\psi_3^{0,1,0}$ remain cocycles for D_a, while the cocycle $\phi_2^{0,1,0}m-\phi_3^{0,0,1}$ does not extend. This is because if we let $D_0=[\bullet,\psi_3^{0,1,n+1}+\psi_3^{0,1,k+1}a]$, we note that

$$D_0(\phi_2^{0,1,0}m-\phi_3^{0,0,1})=\psi_3^{0,1,n+1}(n-m).$$

In order for the original cocycle to extend, this must be D_∞ of something, which it isn't. It turns out that it will be more convenient to take a different basis of the cocycles, which will be more compatible with the basis we will select for arbitrary degrees.

We obtain the following extensions of D_∞ $(k+2)$-cocycles.

$$\begin{aligned}
\eta_1(k)&=\psi_3^{0,1,k+1}+\psi_3^{0,1,n-m+k+1}\tfrac{a(3n-2m-k)}{2n-m-k},\quad k\neq 2n-m\\
\eta_2(k)&=\psi_1^{1,1,k}+\psi_3^{0,1,n-m+k+1}+\psi_3^{0,1,2(n-m)+k+1}a\\
\eta_3(k)&=\psi_2^{1,1,k}-\psi_3^{1,0,n-m+k+1}+\psi_2^{11,n-m+k}(m-k)\\
&\quad-\psi_3^{1,0,2(n-m)+k+1}+\psi_2^{11,2(n-m)+k}a(m-k)\\
\tau_1(k)&=\phi_1^{1,0,k+1}+\phi_2^{0,1,n-m+k+1}(-1+m-k)+\phi_3^{0,0,n-m+k+2}\\
&\quad+\phi_2^{0,1,2(n-m)+k+1}a(-1+m-k)+\phi_3^{0,0,2(n-m)+k+2}a\\
\tau_2(k)&=\phi_1^{0,1,k+1}+\psi_1^{0,1,n-m+k+1}(k-m+1)\\
&\quad+\psi_1^{0,1,2(n-m)+k+1}a(k-m+1)\\
\tau_3(k)&=\phi_3^{1,1,k}+\phi_3^{1,1,n-m+k}(n+1-k+m)\\
&\quad+\phi_3^{1,1,2(n-m)+k}a(n+1-k+m)
\end{aligned}$$

When $k = 2n - m$ we can use $\psi_3^{0,1,2n-m+1}$ as a basis element for the $2n - m + 2$-cocycles. What is interesting is that in this case, it is not a coboundary. The fourth D_∞ even cocycle, $\zeta(l)$ does not extend to a D_e cocycle, except when $l = n - m - 1$. To see this note that

$$D_e(\zeta(l)) = \psi_3^{0,1,n+l+2}((n-m-l-1) + \psi_3^{0,1,2n-m+l+2}a(2(n-m)-l-1).$$

In general, we can add terms of type ζ to this term in order to push the degree of the second term up as high as desired, but the presense of the first term prevents us from eliminating the term altogether. But when $l = n - m - 1$, the first term is zero, so we can add terms to $\zeta(n - m - 1)$ in order to create a cocycle τ_4 of degree $(n - m - 1) + 2$. Note that $0 \leq n - m - 1 \leq n - 1$, so we know something about where that extra even cocycle occurs. The reason for choosing such a strange basis for the cocycles of degree $k+2$ is that we can immediately express them as coboundaries, when the degree is large enough. In fact,

$$\begin{aligned}
\eta_1(k) &= D_e(\zeta(k-n-1)), && \text{if } k \geq n,\ k \neq 2n-m \\
\eta_2(k) &= D_e(-\phi_2^{0,1,k-m}), && \text{if } k \geq m \\
\eta_3(k) &= D_e(\phi_2^{1,0,k-m}), && \text{if } k \geq m \\
\tau_1(k) &= D_e(-\psi_2^{0,0,k-m+1}), && \text{if } k \geq m \\
\tau_2(k) &= D_e(\psi_1^{0,0,k-m+1}), && \text{if } k \geq m \\
\tau_3(k) &= D_e(-\psi_3^{1,0,k-m}), && \text{if } k \geq m
\end{aligned}$$

Notice that it is just the corresponding place where $\zeta(k-n-1)$ extends to a cocycle that we have an extra odd cohomology class, $\eta(2n-m)$. In fact, it makes sense that we should be able to deform in this direction, as we have a family of extended codifferentials, and this cocycle points in the direction of the family. Notice that we can deform along the original family, unlike the previous case where the addition of the extension term killed the deformation in the direction of the family. Of course, in the previous case, we had that the extended term was not a cocycle for a deformed element in the family, but here the extended term is a cocycle for members of the family, but a trivial one except for d_∞. Thus as soon as we deform in the original family direction, the higher order terms which we added become irrelevant.

Finally, note that if $m < k < n$, then we get a deformation in that direction, which, according to the moral that things deform to elements that are earlier, means that we can deform an extension given by adding a term of degree $n+2$ to an extension given by adding a term of smaller degree. The cohomology for the extended codifferential, as long as $a \neq 0$, is given by

$$
\begin{aligned}
&H^1 = \langle \eta_1, \tau_1, \tau_2 \rangle \\
&H^k = \langle \eta_1, \eta_2, \eta_3, \tau_1, \tau_2, \tau_3 \rangle \qquad \text{if } 0 \leq k < m,\ k \neq n-m-1 \\
&H^k = \langle \eta_1 \rangle \qquad \text{if } m \leq k < n \\
&H^k = 0 \qquad \text{if } n \leq k < 2n-m \\
&H^k = \langle \eta_1 \rangle \qquad \text{if } k = 2n-m \\
&H^k = 0 \qquad \text{if } k > 2n-m
\end{aligned}
$$

This completes the description of the extensions of the codifferentials.

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Eötvös Loránd University, Department of Applied Analysis, H-1117 Budapest, Pázmány P. sétány. 1/C, Hungary
E-mail address: fialowsk@cs.elte.hu

University of Wisconsin, Department of Mathematics, Eau Claire, WI 54702-4004
E-mail address: penkavmr@uwec.edu