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The twisted Spin^c Dirac operator on Kähler submanifolds of the complex projective space

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Abstract

In this paper, we estimate the eigenvalues of the twisted Dirac operator on Kähler submanifolds of the complex projective space $\mathbb{C}P^m$ and we discuss the sharpness of this estimate for the embedding $\mathbb{C}P^d \to \mathbb{C}P^m$.

1 Introduction

In his Ph.D. thesis [4], N. Ginoux gave an upper bound for the eigenvalues of the twisted Dirac operator for a Kähler spin submanifold M^{2d} of a Kähler spin manifold $\widetilde{M}^{2m}$ carrying Kählerian Killing spinors (see Eq.(3)). More precisely, he showed that there are at least μ eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_\mu$ of the square of the twisted Dirac operator satisfying

$$\lambda \leqslant \begin{cases} (d+1)^2 & \text{if } d \text{ is odd}, \\ d(d+2) & \text{if } d \text{ is even}. \end{cases} \tag{1}$$

Here μ denotes the dimension of the space of Kählerian Killing spinors on $\widetilde{M}^{2m}$. Recall that the normal bundle is endowed with the induced spin

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structure coming from both manifolds M and $\widetilde{M}$. The idea consists in computing the so-called Rayleigh-quotient applied to the Kählerian Killing spinor restricted to the submanifold M. The upper bound is then deduced by using the min-max principle. This technique was used by C. Bär in [1] for submanifolds in $\mathbb{R}^{n+1}, \mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$.

The complex projective space $\mathbb{C}P^m$ is a spin manifold if and only m is odd. In this case, the sharpness of the upper bound (1) was studied in [5] for the canonical embedding $\mathbb{C}P^d \to \mathbb{C}P^m$, where d is also odd. In fact, it is shown that for $d = 1$, the upper estimate is optimal for $m = 3, 5, 7$ while it is not for $m \geq 9$.

Kähler manifolds are not necessary spin but every Kähler manifold has a canonical Spin^c structure (see Section 2) and any other Spin^c structure can be expressed in terms of the canonical one. Moreover, O. Hijazi, S. Montiel and F. Urbano [7] constructed on Kähler-Einstein manifolds with positive scalar curvature, Spin^c structures carrying Kählerian Killing spinors. Thus one can consider the result of N. Ginoux for Spin^c manifolds.

Section 2 is devoted to recall some basic facts on Spin^c structures on Kähler manifolds. In Section 3, we extend the estimate (1) to the eigenvalues of the twisted Dirac operator for a Kähler submanifold of the complex projective space (see Theorem 3.1). Finally, we discuss the sharpness for the embedding $\mathbb{C}P^d \to \mathbb{C}P^m$ with different values of m.

2 Kähler Submanifolds of Kähler manifolds

Let (M^{2m}, g, J) be a Kähler manifold of complex dimension m. Recall that the complexified tangent bundle splits into the orthogonal sum $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$ where $T_{1,0}M$ (resp. $T_{0,1}M$) denotes the eigenbundle of $T^{\mathbb{C}}M$ corresponding to the eigenvalue i (resp. $-i$) of the extension of J. Using this decomposition, we define $\Lambda^{0,r}M := \Lambda^r(T^*_{0,1}M)$ (resp. $\Lambda^{r,0}M$) as being the bundle of complex r-forms of type $(0, r)$ (resp. of type $(r, 0)$). Recall also that every Kähler manifold has a *canonical* Spin^c structure whose complex spinorial bundle is given by $\Sigma M = \Lambda^{0,*}M = \oplus_{r=0}^m \Lambda^{0,r}M$, where the auxiliary bundle of this Spin^c structure is given by K_M^{-1}. Here K_M is the canonical bundle of M defined by $K_M = \Lambda^{m,0}M$ [3, 10]. On the other hand, the spinor bundle of any other Spin^c structure can be written as [3, 7]:

$$\Sigma M = \Lambda^{0,*}M \otimes \mathfrak{L},$$

where $\mathfrak{L}^2 = K_M \otimes L$ and L is the auxiliary bundle associated with this Spin^c structure. Moreover, the action of the Kähler form Ω of M splits the spinor bundle into [3, 9, 8]:

$$\Sigma M = \oplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ denotes the eigensubbundle corresponding to the eigenvalue $i(2r-m)$ of Ω with complex rank $\binom{m}{k}$. For any vector field $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma_r M)$, we have the following property $p_\pm(X)\cdot\psi \in \Gamma(\Sigma_{r\pm 1}M)$, where $p_\pm(X) = \frac{1}{2}(X \mp iJX)$.

Let (M^{2d}, g, J) be an immersed Kähler submanifold in a Kähler manifold $(\widetilde{M}^{2m}, g, J)$ with the induced complex structure J (i.e. $J(TM) = TM$) and denote respectively by $\Omega_{\widetilde{M}}$, Ω and Ω_N the Kähler form of $\widetilde{M}$, M and of the normal bundle $NM \longrightarrow M$ of the immersion. Since the manifolds M and $\widetilde{M}^{2n}$ are Kähler, they carry Spin^c structures with corresponding auxiliary line bundles L_M and $L_{\widetilde{M}}$. This induces a Spin^c structure on the bundle NM such that the restricted complex spinor bundle $\Sigma\widetilde{M}_{|_M}$ of $\widetilde{M}$ can be identified with $\Sigma M \otimes \Sigma N$, where ΣM and ΣN are the spinor bundles of M and NM respectively ([1], [6]). Moreover, the auxiliary line bundle L_N of this Spin^c structure on NM is given by $L_N := (L_M)^{-1} \otimes (L_{\widetilde{M}})_{|_M}$. Given connections 1-form on L_M and $L_{\widetilde{M}}$, they induce a connection $\nabla := \nabla^{\Sigma M \otimes \Sigma N}$ on $\Sigma M \otimes \Sigma N$. Thus one can state a Gauss-type formula for the spinorial Levi-Civita connections $\widetilde{\nabla}$ and ∇ on $\Sigma\widetilde{M}$ and $\Sigma M \otimes \Sigma N$ respectively [12]. That is, for all $X \in TM$ and $\varphi \in \Gamma(\Sigma\widetilde{M}_{|_M})$, we have

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2}\sum_{j=1}^{2d} e_j \cdot II(X, e_j)\cdot\varphi, \tag{2}$$

where $(e_j)_{1\leq j\leq 2d}$ is any local orthonormal basis of TM and II is the second fundamental form of the immersion. As a consequence of the Gauss formula, the square of the auxiliary Dirac-type operator $\widehat{D} := \sum_{j=1}^{2d} e_j \cdot \widetilde{\nabla}_{e_j}$ is related to the square of the twisted Dirac operator $D_M^{\Sigma N} := \sum_{j=1}^{2d} e_j \cdot \nabla_{e_j}$ by [4, Lemme 4.1]:

$$\widehat{D}^2\varphi = (D_M^{\Sigma N})^2\varphi - d^2|H|^2\varphi - d\sum_{j=1}^{2d} e_j \cdot \nabla^N_{e_j} H \cdot \varphi,$$

where $H := \frac{1}{2d}\mathrm{tr}(II)$ is the mean curvature vector field of the immersion. But in our case $H = 0$, then $\widehat{D}^2$ and $(D_M^{\Sigma N})^2$ coincide.

In the sequel, take the manifold $\widetilde{M}$ as the complex projective space $\mathbb{C}P^m$ endowed with its Fubini-Study metric of constant holomorphic sectional curvature 4. In [7], the authors proved that for every $q \in \mathbb{Z}$, such that $q+m+1 \in 2\mathbb{Z}$, there exists a Spin^c structure on $\mathbb{C}P^m$ whose auxiliary line bundle is given by $\mathcal{L}_m^q$. Here $\mathcal{L}_m$ denotes the tautological bundle of $\mathbb{C}P^m$. In particular for $q=-m-1$ (resp. $q=m+1$), the Spin^c structure is the canonical one (resp. anti-canonical) [11] and for $q=0$ it corresponds to the unique spin structure. Let us denote by $\Sigma^q\mathbb{C}P^m$ the spinor bundle of the corresponding Spin^c structure with $\mathcal{L}^q$ as auxiliary line bundle. Take an integer r in $\{0, \cdots, m+1\}$ and define $q := 2r-(m+1)$. For such a q, the bundle $\Sigma^q\mathbb{C}P^m$ carries a Kählerian Killing spinor field $\psi = \psi_{r-1} + \psi_r$ satisfying, for all $X \in \Gamma(T\mathbb{C}P^m)$ [7]

$$\begin{aligned} \widetilde{\nabla}_X \psi_r &= -p_+(X)\cdot\psi_{r-1}, \\ \widetilde{\nabla}_X \psi_{r-1} &= -p_-(X)\cdot\psi_r, \end{aligned} \tag{3}$$

The space of Kählerian Killing spinors is of rank $\begin{pmatrix} m+1 \\ r \end{pmatrix}$. We point out that for $r=0$ (resp. $r=m+1$) the Kählerian Killing spinor is a parallel spinor which is carried by the canonical structure (resp. anti-canonical). Moreover, for $r=\frac{m+1}{2}$, i.e. m is odd, the Kählerian Killing spinor is the usual one lying in $\Sigma^0_{\frac{m-1}{2}}\mathbb{C}P^m \oplus \Sigma^0_{\frac{m+1}{2}}\mathbb{C}P^m$ defined in [8, 9].

3 Main result

In this section, we establish the estimates for the eigenvalues of the twisted Dirac operator of complex submanifolds of the complex projective space. We have

Theorem 3.1 *Let (M^{2d}, g, J) be a closed Kähler submanifold of the complex projective space $\mathbb{C}P^m$. For $r \in \{0, \cdots, m+1\}$ let $q = 2r-(m+1)$. There are at least $\begin{pmatrix} m+1 \\ r \end{pmatrix}$-eigenvalues λ of $(D_M^{\Sigma N})^2$ satisfying*

$$\lambda \leqslant \begin{cases} -(q^2-(d+1)^2) + 2|q|(m-d) - 1 & \text{if } m-d \text{ is odd} \\ -(q^2-(d+1)^2) + 2|q|(m-d) & \text{if } m-d \text{ is even.} \end{cases} \tag{4}$$

Proof. The proof relies on computing the Rayleigh-quotient

$$\frac{\int_M \mathrm{Re}\langle (D_M^{\Sigma N})^2\psi, \psi\rangle v_g}{\int_M |\psi|^2 v_g}$$

applied to any non-zero Kählerian Killing spinor $\psi = \psi_{r-1} + \psi_r$ on $\mathbb{C}P^m$. A straightforward computation of the auxiliary Dirac operator leads to

$$\widehat{D}\psi_{r-1} = (q+d+1)\psi_r + i\Omega_N \cdot \psi_r.$$

$$\widehat{D}\psi_r = -(q-d-1)\psi_{r-1} - i\Omega_N \cdot \psi_{r-1}.$$

Summing up the above two equations, we deduce after using the fact that the auxiliary Dirac operator commutes with the normal Kähler form [5], that

$$\widehat{D}^2\psi = -(q^2 - (d+1)^2)\psi - 2iq\Omega^N \cdot \psi + \Omega^N \cdot \Omega^N \cdot \psi.$$

Taking the hermitian product with ψ and using the fact that the seond term can be bounded from above by $2|q|(m-d)$, we get our estimates after using $|\Omega^N \cdot \psi| \geq |\psi|$ if $m-d$ is odd and 0 otherwise. $\square$

In the following, we will test the sharpness of Inequality (4) for the canonical embedding $\mathbb{C}P^d \to \mathbb{C}P^m$ as in [5]. Recall first that the complex projective space $\mathbb{C}P^d$ can be seen as the symmetric space $\mathrm{SU}_{d+1}/\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$ where $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) := \{\begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \mid B \in \mathrm{U}_d\}$. The tangent bundle of $\mathbb{C}P^d$ can be described as a homogeneous bundle which is associated with the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1} \to \mathbb{C}P^d$ via the isotropy representation

$$\begin{array}{rccl} \alpha : & \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_d \\ & \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B)B. \end{array}$$

The normal bundle $T^\perp\mathbb{C}P^d$ of the embedding is isomorphic to $\mathcal{L}_d^* \otimes \mathbb{C}^{m-d}$ where $\mathcal{L}_d$ is the tautological bundle of $\mathbb{C}P^d$. The bundle $\mathcal{L}_d$ is isomorphic to the homogeneous bundle which is associated with the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle SU_{d+1} via the representation

$$\begin{array}{rccl} \rho : & \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_1 \\ & \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & (\det(B))^{-1}. \end{array}$$

Thus the normal bundle is associated with the $\mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1)$-principal bundle $\mathrm{SU}_{d+1} \to \mathbb{C}P^d$ via the representation

$$\begin{array}{rccl} \rho : & \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_{m-d} \\ & \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & \det(B)\mathrm{I}_{m-d}. \end{array}$$

Consider now the case where d is odd and $\mathbb{C}P^d$ is endowed with its canonical spin structure. The normal bundle of the embedding carries a Spin^c structure with auxiliary line bundle given by $\mathcal{L}_m^q|_{\mathbb{C}P^d}$ which is isomorphic to the q^{th}-power of the tautological bundle $\mathcal{L}_d$ of $\mathbb{C}P^d$. Therefore the Lie-group homomorphism

$$\begin{array}{cccc} \rho: & \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{U}_{m-d} \times \mathrm{U}_1 \\ & \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & (\det(B)\mathrm{I}_{m-d}, \det(B)^{-q}) \end{array}$$

can be lifted through the non-trivial two-fold covering map $\mathrm{Spin}^c_{2(m-d)} \longrightarrow \mathrm{SO}_{2(m-d)} \times \mathrm{U}_1$ to the homomorphism

$$\begin{array}{cccc} \widetilde{\rho}: & \mathrm{S}(\mathrm{U}_d \times \mathrm{U}_1) & \longrightarrow & \mathrm{Spin}^c_{2(m-d)} \\ & \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} & \longmapsto & (\det(B))^{-\frac{q+m-d}{2}} j(\det(B)\mathrm{I}_{m-d}), \end{array}$$

where for any positive integer k, we recall that $j: \mathrm{U}_k \longrightarrow \mathrm{Spin}^c_{2k}$ is given on elements of diagonal form of U_k as

$$j(\mathrm{diag}(e^{i\lambda_1}, \cdots, e^{i\lambda_k})) = e^{\frac{i}{2}(\sum_{j=1}^k \lambda_j)} \widetilde{R}_{e_1, Je_1}(\frac{\lambda_1}{2}) \cdots \widetilde{R}_{e_k, Je_k}(\frac{\lambda_k}{2}).$$

Here J is the canonical complex structure on $\mathbb{C}^k$ and $\widetilde{R}_{v,w}(\lambda) = \cos(\lambda) + \sin(\lambda) v \cdot w \in \mathrm{Spin}_{2k}$ is defined for any orthonormal system $\{v, w\} \in \mathbb{R}^{2k}$. We point out that the integer $q+m-d = 2r-d-1$ is even. Following the same proof as in [5, Cor. 4.4], the complex spinor bundle of $T^\perp\mathbb{C}P^d$ splits into the orthogonal sum

$$\Sigma(T^\perp\mathbb{C}P^d) \cong \bigoplus_{s=0}^{m-d} \begin{pmatrix} m-d \\ s \end{pmatrix} \mathcal{L}_d^{\frac{q+m-d}{2}-s}.$$

Thus one should replace m in Theorem 4.5 of [5] (see also [2]) by $\frac{q+m-d}{2} - s$. In this case, we get the following families of eigenvalues for the square of the twisted Dirac operator:

1. $2(v+l)(1+2l-q-m+2d+2s-2\varepsilon)$ where $v \in \{1, \cdots, d-1\}$, $\varepsilon \in \{0, 1\}$ and $l \geq \max(\varepsilon, \frac{q+m+1}{2} - v - s)$.
2. $2l(2l+2d-1-q-m+2s)$ where $l \geq \max(0, \frac{q+m+1}{2} - s)$.
3. $2(d+l)(2d+1+2l-q-m+2s)$ where $l \geq \max(0, \frac{q+m-2d-1}{2} - s)$.

We will now treat the simplest case where $d = 1$ and $q > 0$ (the same can be done for $q < 0$). That means we are considering the last two families of eigenvalues. By a straightforward computation, the first eigenvalue is 0 with multiplicity equal to

$$\sum_{s=0}^{\frac{q+m-3}{2}} \begin{pmatrix} m-1 \\ s \end{pmatrix} (\frac{q+m-1}{2} - s) + \sum_{s=\frac{q+m+1}{2}}^{m-1} \begin{pmatrix} m-1 \\ s \end{pmatrix} (-\frac{q+m-1}{2} + s)$$

and the second eigenvalue is 4 with multiplicity equal to $4 \begin{pmatrix} m-1 \\ \frac{q+m-1}{2} \end{pmatrix}$. Consider the particular case where $m = 2$ and $q = 1$ (i.e. $r = 2$). By Inequality (4), there are at least 3 eigenvalues satisfying the estimate $\lambda \leq 4$. The multiplicity of 0 is equal to 1 and the multiplicity of the eigenvalue 4 is 4 which means that the estimate is optimal. For $m = 3$ and $q = 2$, the estimate $\lambda \leq 8$ is satisfied for at least 4 eigenvalues. But the multiplicity of 0 is equal to 4 which means that the upper bound is not achieved. For $m = 4$ and $q = 1$, the estimate $\lambda \leq 8$ is satisfied for at least 10 eigenvalues. The multiplicity of 0 is equal to 6 and of 4 is equal to 12. For $q = 3$, we have $\lambda \leq 12$ with 5 eigenvalues. The multiplicity of 0 is 12 which means that the estimate is not optimal.

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