# A $p$-adic property of Cohen's numbers


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## Introduction

The generalized class numbers $H(r,N)$ were inveted by H. Cohen [1]. They coinside with the usual class numbers of the binary positive defined quadratic forms when $r=1$. Since they are Fourier coefficients of the Eisenstein series of half integral weight $r+1/2$,

$$\mathcal{H}_{r+1/2}(\tau)=\sum_{N\geq 0} H(r,N)q^N \qquad q=\exp(2\pi i\tau),$$

one can prove their nice properties. In particular, generalizations of the Kronecker - Hurwitz class number relation

$$\sum_{s\in\mathbf{Z}} H(1,4N-s^2)+\sum_{\substack{N=\lambda\lambda'\\ \lambda,\lambda'>0}} \min(\lambda,\lambda')=\sigma_1(n) \qquad N>0$$

are investigated in [1]. These generalizations are based on the explicite construction of spaces of modular forms of weight $r+1$. The relations obtained in [1] involve the numbers $H(r,N)$ for $r\leq 5$. The basis problem becomes more complicated when the weight increases. It yields that one can not hope to obtain nice relations of this type when $r$ is considerably large. In the present note we get $p$-adic information about the numbers $H(r,N)$ for arbitrary $r$.

Throughout the paper we fix an odd regular prime $p$.

For positive integers $m$,$n$ we introduce the finite sets of non-negative integers:

$$S(m,n)=\{4mp^n-s^2\geq 0|s\in\mathbf{Z}\}$$
$$S^*(m,n)=\{4mp^n-s^2\geq 0|s\in\mathbf{Z},p\not| s\}$$

**Theorem 1** *Let $r$,$m$ be positive integers such that $m$ is not a perfect square and is not divisible by $p$. Denote by $\chi_{-N}$ the quadratic character associated with $\mathbf{Q}(\sqrt{-N})$.*

**a.** *Suppose that*

$$r\equiv 3,5,7,9,13 \bmod p-1 \qquad (1)$$

*Then the double series*

$$\mathcal{F}(l)=\sum_{n\geq 0}\sum_{N\in S^*(m,n)}(1-\chi_{-N}(p)p^{r-1})H(r,N)N^{(p-1)l}$$

*converges p-adically for every non-negative integer l.*

*Its value at $l = 0$ is*

$$\mathcal{F}(0) = 2\sigma_r(m)\frac{1-p^{2r-1}}{1-p^r}\frac{\zeta(1-2r)}{\zeta(-r)}$$

**b.** *Suppose that $p = (-1)^{r+1} \bmod 4$ and*

$$r + \frac{p-1}{2} \equiv 3, 5, 7, 9, 13 \bmod p-1 \tag{2}$$

*Then the double series*

$$\mathcal{G}(l) = \sum_{n\geq 1} \sum_{N \in S^{*}(m.n)} H(r, pN) N^{(p-1)l}$$

*converges p-adically for every non-negative integer l.*

*Its value at $l = 0$ is*

$$\mathcal{G}(0) = 2\sigma_r(m)\frac{1-p^{2r-1}}{1-p^r}\frac{\zeta(1-2r)}{\zeta(-r)}$$

*If p=3,5,7 then one can omit the conditions (1) and (2).*

## Remark

The condition that $m$ is not a perfect square is technical. The case when $m$ is a square, and in particular $m = 1$ brings nothing essential new but slight modifications of the formulae.

The proof of the Theorem 1 is based on the methods and results of [1] and [3].

The contents of the paper are as follows. In Chapter 1 we recall (slightly modified) propositions from [1] and [3]. These propositions will be used in the proof of Theorem 2. This theorem asserts some $p$-adic properties of Fourier coefficients of modular forms of half integral weight. $p$-adic analytic functions associated with a modular form of half integral weight are constructed in Corollary 1. Theorem 2 and Corollary 1 are formulated and proven in Chapter 2. These constructions might be of independent interest. The proof of Theorem 1 concludes Chapter 2.

## Notations

Let $X$ denote the group of continious $p$-adic characters of ${\bf Z}_p{}^*$. For $t \in {\bf Z}_p$, $u \in {\bf Z}/(p-1){\bf Z}$ we let $(t,u) \in X$ be the character which sends $z \in {\bf Z}_p$ to $< z >^t \omega(z)^u$, where $\omega$ is the Teichmüller character and $< z >= z/\omega(z) \in 1 + p{\bf Z}_p$. All elements of $X$ are of the form $(t,u)$. For a residue $r$ modulo $p-1$ we write $(t,u) \equiv r \bmod p-1$ iff $u \equiv r \bmod p-1$.

For a formal power series

$$g = \sum_{n\geq 0} b(n)q^n \qquad b(n) \in {\bf Q}_p \tag{3}$$

we define $v_p(g)$ be the minimum $p$-adic ordinal of its Fourier coefficients $b(n)$.

We call the series (3) a $p$-adic modular form of integral (half integral) weight if there exists a sequence of modular forms of even weights $k_i$ on $SL(2,{\bf Z})$ (of half integral weights $r_i + 1/2$ on $\Gamma_0(4)$ ) with rational Fourier coefficients such that $\lim_{i\to\infty} f_i = g$ i.e. $v_p(f_i - g)$ tends to infinity. It is known [3], [2] that in this case the sequence $k_i$ $(r_i)$ converges in $X$.

The symbol lim will denote $p$-adic limit.

We denote by $\zeta^*$ the Kubota - Leoplodt $p$-adic $\zeta$-function. The group $X$ is its area of definition.

## Chapter 1.

For non-negative integers $l, r, s, N$ put

$$P_{2l}^{(r)}(s,N) = \sum_{l\geq\mu\geq 0} (-1)^\mu \frac{(2l)!}{\mu!(2l-2\mu)!} \frac{(r+2l-\mu-1)!}{(r+l-1)!} s^{2l-2\mu} N^\mu.$$

**Proposition 1** *Let $\phi = \sum_{N\geq 0} c(N)q^N$ be a modular form of half integral weight $r + 1/2 \geq 5/2$ on congruence subgroup $\Gamma_0(4)$.*

*Let $D$ be an positive integer such that $D \equiv (-1)^{r-1} \bmod 4$; let $l$ be a positive integer.*

*Then*

$$F = \sum_{N\geq 0} q^N \sum_{s\in{\bf Z}} P_{2l}^{(r)}(s,N)c\left(\frac{4N-s^2}{D}\right)$$

*is a modular form of weight $2l+r+1$ on congruence subgroup $\Gamma_0(D)$ with character $\chi_{(-1)^{r-1}D}$. It is a cusp form if $l>0$.*

This Proposition essentially coinside with Theorem 6.2 from [1]. We change the normalization of the Gegenbauer polinomial $P_{2l}^{(r)}$ and consider arbitrary modular form of half integral weight $\phi$ instead of the Cohen series $\mathcal{H}$. The argument atays the same as in [1] up to the described modifications.

**Proposition 2** *Let $f = \sum_{n\geq 0} a(n)q^n$ be a p-adic modular form of even weight $k\neq 0$. Let m be a positive integer not divisible by p. Suppose that*

$$k \equiv 4,6,8,10,14 \bmod p-1. \tag{4}$$

*Then*

$$2a(0)\sigma_{k-1}(m) = \zeta^*(1-k)\ lim_{n\to\infty}a(mp^n)$$

*If $p=3,5,7$ then one can omit the condition (4).*

**Proof.**

Acting as in [3], proof of Theorem 4, p.209-210, one gets a $p$-adic modular form $f|_kT(m)$ of the same weight $k$:

$$f|_kT(m) = \sum_{n\geq 0} q^n \sum_{d|(m,n)} d^{k-1}a(mn/d^2).$$

Application of Theorem 7 of [3] (see also Remark, p. 216) completes the proof.

## Chapter 2.

**Theorem 2** *Let $f = \sum_{n\geq 0}$ be a p-adic modular form of half integral weight $r+1/2$. Let l,m be positive integers, m is not divisible by p.*

**a.** *Suppose that*

$$r+2l \equiv 3,5,7,9,13 \bmod p-1. \tag{5}$$

*Then for $2l+r+1\neq 0$ in $X$, one has*

$$\textbf{1a.} \quad \lim_{n\to\infty} \sum_{N\in S^*(m,n)} c(N)N^l = 0$$

$$\textbf{2a.} \quad \lim_{n\to\infty} \sum_{N\in S(m,n)} c(N) = 2\frac{\sigma_r(m)c(0)}{\zeta^*(-r)}$$

**b.** *Suppose that*

$$r + 2l + \frac{p-1}{2} \equiv 3, 5, 7, 9, 13 \bmod p - 1. \qquad (6)$$

*Then for* $p = (-1)^{r+1} \bmod 4$, *one has*

**1b** $$\lim_{n\to\infty} \sum_{N \in S^*(m,n)} c(pN)N^l = 0$$

**2b.** $$\lim_{n\to\infty} \sum_{N \in S(m,n)} c(N) = 2\frac{\sigma_r(m)c(0)}{\zeta^*(-r, -r-(p-1)/2)}$$

*If* $p = 3, 5, 7$ *then one can omit the conditions (5), (6).*

**Corollary 1** *Let* $f = \sum_{N\geq 0} c(N)q^N$ *be a p-adic modular form of weight* $r + 1/2$. *Let* $m$ *be an integer not divisible by* $p$. *Denote by* $l$ *the element* $(s, l_0)$ *of* $X$, *where* $l_0$ *is a fixed residue modulo* $p-1$ *and* $s$ *is a p-adic integer.*

**a.** *Let* $r$ *be odd and* $2l + r + 1 \neq 0$. *Suppose that*

$$r + 2l_0 \equiv 3, 5, 7, 9, 13 \bmod p - 1. \qquad (7)$$

*Then the series*

$$\Phi_{f,m,l_0}(s) = \sum_{n\geq 0} \sum_{N \in S^*(m,n)} c(N)N^l$$

*converge p-adically for every* $s \in \mathbf{Z}_p$ *and the function* $\Phi_{f,m,l_0}(s)$ *is analytic in variable* $s$.

**b.** *Suppose that* $p = (-1)^{r+1} \bmod 4$ *and*

$$r + 2l_0 + \frac{p-1}{2} \equiv 3, 5, 7, 9, 13 \bmod p - 1. \qquad (8)$$

*Then the series*

$$\Psi_{f,m,l_0}(s) = \sum_{n\geq 0} \sum_{N \in S^*(m,n)} c(pN)N^l$$

*converge p-adically for every* $s \in \mathbf{Z}_p$ *and the function* $\Psi_{f,m,l_0}(s)$ *is analytic in variable* $s$.

*If* $p = 3, 5, 7$ *then one can omit the conditions (7) and (8).*

**Proof of the Corollary 1**

Let us assume that part **a** of Theorem 2 is valid and prove part **a** of the Corollary. Part **b** is similar.

It follows from **1a** of Theorem 2 that the series in question converges for $s \in \mathbf{Z}_p$. The finite sum $\sum_{0\leq n\leq n_0} \sum_{N\in S^*(m,n)} c(N)N^l$ is an analytic function. It follows that the function $\Phi_{f,m,l_0}$ is the limit of the sequence of analytic functions. The application of [3], Lemma 12 completes the proof.

**Proof of the Theorem 2**

Consider the sequence $f_i = \sum_{n\geq 0} c_i(n)q^n$ of modular forms of half integral weights $r_i + 1/2$ which defines the $p$-adic modular form $f$.

Since $\lim_{i\to\infty} c_i(N) = c(N)$ uniformly in $N$, it is enough to prove the assertion of the Theorem for the forms $f_i$.

**a.** Applying to these forms Proposition 1 with $D = 1$ we obtain the modular forms

$$F_{i,l} = \sum_{N\geq 0} q^N \sum_{s\in\mathbf{Z}} P_{2l}^{(r_i)}(s,N)c_i(4N-s^2)$$

of weights $2l + r_i + 1$ on $SL_2(\mathbf{Z})$. The constant term of the $q$ expansion of $F_{i,l}$ is equal to $c_i(0)$ if $l = 0$ and vanishes if $l > 0$. Let us denote this number by $a_{i,l}$ and apply Proposition 2 to modular forms $F_{i,l}$:

$$2a_{i,l}\sigma_{2l+r_i}(m) = \zeta^*(-2l-r_i) \lim_{n\to\infty} \sum_{s\in\mathbf{Z}} P_{2l}^{(r_i)}(s,4mp^n)c_i(4mp^n - s^2). \qquad (9)$$

Since $p$ is regular, $\zeta^*(-2l-r_i) \neq 0$. Computation of the limit in the right hand side of (9) yields:

$$2a_{i,l}\sigma_{2l+r_i}(m) = \zeta^*(-2l-r_i) \lim_{n\to\infty} \frac{(r_i+2l-1)!}{(r_i+l-1)!} \sum_{s\in\mathbf{Z}} c_i(4mp^n - s^2)s^{2l}. \quad (10)$$

It follows that the assertion **2a** of the Theorem holds for half the integral weight form $f_i$.

When $l > 0$ (10) yields

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sum_{s\in\mathbf{Z}} c_i(4mp^n - s^2)s^{2l} \\ &= \lim_{n\to\infty} \left( \sum_{N\in S^*(m,n)} c_i(N)N^l + p^{2l} \sum_{N\in S^*(m,n-2)} c_i(p^2N)N^l \right). \end{aligned} \qquad (11)$$

Consider a sequence of rational integers $l_j \to \infty$ such that $\lim_{j\to\infty} l_j = l$ in $X$. To be more specific one can pick $l_j = l + p^j(p-1)$, $j = 1, 2, 3, \ldots$. Since (11) holds for $l = l_j$ for arbitrary $j$, the denominators of the Fourier coefficients $c_i(N)$ of modular forms $f_i$ are bounded, and $N^s$ is $p$-adically continious function on $s$ when $N$ is not divisible by $p$, the assertion **1a** of Theorem 2 for $f = f_i$ follows.

**b.** Applying to the modular forms $f_i$ Proposition 1 with $D = p$ we obtain modular forms

$$F_{i,l} = \sum_{N\geq 0} q^N \sum_{s\in \mathbf{Z}} P_{2l}^{(r_i)}(s,N) c_i\left(\frac{4N - s^2}{p}\right)$$

of weights $2l + r_i + 1$ on congruence subgroup $\Gamma_0(p)$ with character $\chi_{(-1)^{r_i+1}p}$. It follows from [3], Theorem 12 that $F_{i,l}$ is a $p$-adic modular form of weight $\left(l + r_i + 1, l + r_i + \frac{p+1}{2}\right) \in X$. The rest of the proof is essentially the same as of part **a**.

**Proof of Theorem 1**

It is known ([2], Theorem 4) that for any sequence of positive integers $r_i \to \infty$ converging to $r \in X$, the sequence of Eisenstein sries $\mathcal{H}_{r_i+1/2}$ converges $p$-adically to a limit $\mathcal{H}^*_{r+1/2} = \sum_{N\geq 0} H^*(r,N)q^N$. Moreover, the $p$-adic Eisenstein series $\mathcal{H}^*_{r+1/2}$ is invariant under $U_p^2$ operator. In other words, $H^*(r,p^2N) = H^*(r,N)$. It follows that

$$\lim_{n\to\infty} \sum_{N\in S(m,n)} H^*(r,N) =$$

$$\lim_{n\to\infty}\left(\sum_{N\in S^*(m,n)} H^*(r,N) + \sum_{N\in S^*(m,n-2)} H^*(r,p^2N) + \ldots\right) = \Phi_{\mathcal{H}^*_{r+1/2},m,0}(0).$$

In the case under consideration Theorem 2, **a** yields that

$$\Phi_{\mathcal{H}^*_{r+1/2},m,0}(0) = 2\sigma_r(m)\frac{\zeta^*(1-2r)}{\zeta^*(-r)}. \tag{12}$$

It means that we succeeded to calculate the value at $s = 0$ of the $p$-adic analytic function on $\mathbf{Z}_p$ $\Phi_{\mathcal{H}^*_{r+1/2},m,0}(s)$. Let $r = (s, 0) \in X$ for a positive integer $s$. Using the identities ([2], Remark 3, p. 207)

$$H^*(r,0) = \zeta(1-2s)(1-p^{2s-1}),$$

$$H^*(r,N) = (1 - \chi_{(-1)^r N}(p)p^{s-1})H(s,N)$$

and taking in account (12) we derive the assertion of Theorem 1 from Corollary 1, a.

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# References

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