Geometry of Determinants of Elliptic Operators

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1. INTRODUCTION

D.B. Ray and I.M. Singer invented zeta-regularized determinants for positive definite elliptic pseudo-differential operators (PDOs) of positive orders acting in the space of smooth sections of a finite-dimensional vector bundle E over a closed finite-dimensional manifold M ([RS1], [RS2]).

Recall that for any such invertible operator A its zeta-function, defined for $\operatorname{Re} s \gg 0$ by the formula

$$\zeta_A(s) = \sum_{\{\lambda_i\} \in \operatorname{Spec} A} \lambda_i^{-s} = \operatorname{Tr} A^{-s} ,$$

has a meromorphic continuation to $\mathbb{C}$ without pole at zero. (Here, the sum includes the algebraic multiplicities.) A zeta-regularized determinant of A is, by definition,

$$\det_\zeta(A) := \exp\left(-\frac{\partial}{\partial s}\zeta_A(s)\Big|_{s=0}\right) .$$

We are interested in this paper in multiplicative properties of these determinants, i.e., we want to compute the ratio

$$F(A,B) := \det_\zeta(AB)/(\det_\zeta(A)\det_\zeta(B)) . \tag{1.1}$$

We call it the multiplicative anomaly. In general, it is not equal to 1. For example for $A = \Delta + \mathrm{Id}$ and $B = \Delta + 2\,\mathrm{Id}$, where Δ is the Laplacian acting on functions on an even-dimensional Riemannian manifold, $F(A,B)$ is defined and it is almost never equal to 1.

The determinant $\det_\zeta(A)$ is defined for an invertible elliptic PDO A, $\operatorname{ord} A > 0$, admitting a spectral cut. Such a cut exists, if A satisfies the Agmon-Nirenberg condition formulated as follows (for closed M). There exists a closed conical sector $V = \{\lambda\colon \theta_1 \le \arg\lambda \le \theta_2\}$, $\theta_1 < \theta_2$, in the spectral plane $\mathbb{C}$ such that all eigenvalues of the principal symbol $\sigma_d(A)(x,\xi)$ do not belong to V for any $(x,\xi) \in T^*M \setminus M$. If such a condition is satisfied for A, then in V there is no more than a finite number of eigenvalues of A including their algebraic multiplicities.

Note that this condition on $\sigma_d(A)(x,\xi)$ cannot be satisfied for any $d = \operatorname{ord} A \in \mathbb{C}\backslash\mathbb{R}$ because for any such d the curve $\mathbb{R}_+ \ni t \to t^d \in \mathbb{C}^\times$ crosses all the rays L_θ infinitely many times. Note also that the Agmon-Nirenberg condition is formulated in terms of the principal symbol of A. So it is a micro-local condition, and it can be checked effectively. It provides us with an information about the spectrum of A which we cannot compute in general.

Let us pick a cut $L_\theta = \{\lambda\colon \arg\lambda = \theta\}$, $\theta_1 < \theta < \theta_2$, such that $\operatorname{Spec} A \cap L_\theta = \emptyset$, and define a zeta-function $\zeta_{A,\theta}(s)$ of A corresponding to this cut. Namely we define $A_{(\theta)}^{-s}$ for $\operatorname{Re} s \in \mathbb{R}_+$ large enough by

$$A_{(\theta)}^{-s} := \frac{i}{2\pi}\int_{\Gamma_{(\theta)}} (A-\lambda)^{-1}\lambda_{(\theta)}^{-s}d\lambda , \tag{1.2}$$

where $\Gamma_{(\theta)}$ is a contour $\Gamma_{1,\theta}(\rho)\cup\Gamma_{0,\theta}(\rho)\cup\Gamma_{2,\theta}(\rho)$, $\Gamma_{1,\theta}(\rho)\colon = \{\lambda = x\exp(i\theta), +\infty > x \ge \rho\}$, $\Gamma_{0,\theta}(\rho)\colon = \{\lambda = \rho\exp(i\varphi), \theta \ge \varphi \ge \theta - 2\pi\}$, $\Gamma_{2,\theta}(\rho)\colon = \{\lambda = x\exp i(\theta-2\pi), \rho \le x < +\infty\}$, and ρ is a positive number such that all the eigenvalues in $\operatorname{Spec}(A)$ are outside of the disk $D_\rho := \{\lambda\colon |\lambda| \le \rho\}$. Here, $\lambda_{(\theta)}^{-s} := \exp\left(-s\log_{(\theta)}\lambda\right)$ with a branch $\log_{(\theta)}\lambda$, $\theta \ge \operatorname{Im}\log_{(\theta)}\lambda \ge \theta - 2\pi$. Then a family $A_{(\theta)}^{-s}$ for any s is defined as $A^k A_{(\theta)}^{-(s+k)}$ for $k \in \mathbb{Z}_+$ large enough (and depending on $\operatorname{Re} s$). This definition of $A_{(\theta)}^{-s}$ is independent of k. Then $\zeta_{A,\theta}(s)$ is defined as $\mathrm{Tr}\left(A^{-s}_{(\theta)}\right)$ for $\mathrm{Re}\, s$ large enough.[1] This zeta-function has a meromorphic continuation to the whole complex plane and it is regular at $s = 0$. This zeta-function depends on an admissible cut L_θ. Nevertheless the corresponding determinant is independent of such a cut for $L_\theta \subset V$. The reason is that if the number m, $m \in \mathbb{Z}_+ \cup 0$, of eigenvalues of A in the sector between L_θ and $L_{\bar\theta}$ is finite, then

$$\partial_s \left(\zeta_{A,\theta}(s) - \zeta_{A,\bar\theta}(s)\right)\Big|_{s=0} = \pm 2\pi i m \ .$$

Note that in general $\det_\zeta(A)$ depends on the homotopy class of an admissible spectral sector V in the Agmon-Nirenberg condition for A.

The rest of the paper is devoted to the study of properties of the multiplicative anomaly and related algebraic and geometric objects. Using Fredholm determinants we introduce a central $\mathbb{C}^\times$-extension $\widetilde{G}$ of the group G of elliptic symbols and a partially defined section d_0 of it. All properties of multiplicative anomaly are encoded in these objects.

One of our results is an extension of the notion of the zeta-regularized determinant to a larger class of operators (including operators of nonreal orders). The modified definition of $\det(A)$ does not use the existence of any holomorphic family A^{-s} for a given A and does not use any analytic continuations. The main tool is a new trace class functional TR defined for classical PDOs of noninteger orders. We discovered a simple Lie-algebraic description of $\widetilde{G}$ and of d_0 in a neighborhood of the identity $\mathrm{Id} \in \widetilde{G}$ purely in terms of symbols. There is an interesting interplay between invariant quadratic forms and 2-cocycles on Lie algebras. We also describe an analogue of the determinant Lie group $\widetilde{G}$ for a certain natural class of PDOs on odd-dimensional manifolds. We prove, in particular, that for positive self-adjoint elliptic differential operators on such manifolds the multiplicative property holds.

This paper is essentially a compressed version of our previous paper [KV]. The aim of the current paper is to give a short and clear exposition of our present understanding of the subject. In comparison with [KV] we change the general structure of the text and present some new proofs. Here we drop minor details of the proofs but try to give main ideas. Our notaitions differ a little from the notations of [KV].

1.1. Formula for multiplicative anomaly. Let A and B be invertible elliptic PDOs of real nonzero orders α and β such that $\alpha + \beta \neq 0$ and such that their principal symbols $\sigma_\alpha(A)$, $\sigma_\beta(B)$, and $\sigma_{\alpha+\beta}(AB)$ obey the Agmon-Nirenberg condition (with appropriate spectral cuts). Let A_t be a smooth deformation of the elliptic PDO $A = A_0$ such that $\mathrm{ord}\, A_t \equiv \mathrm{ord}\, A$. Hence A_t and $A_t B$ satisfy the Agmon-Nirenberg condition for small t. The complex powers of A_t, B, and of $A_t B$ are defined for such

[1] The trace $\mathrm{Tr}\left(A^{-s}_{(\theta)}\right)$ for $\mathrm{Re}(s\, \mathrm{ord}\, A) > \dim M$ is equal to the sum $\sum \lambda^{-s}_{i,(\theta)}$ (including algebraic multiplicities) as it follows from the Lidskii Theorem [Li], [Re], XI.

t by (1.2) with appropriate spectral cuts. Thus the determinants of these operators are defined.

Proposition 1.1. *Under the conditions above, the variation formula for the multiplicative anomaly (*1.1*) holds (for small t)*

$$\frac{\partial}{\partial t}\log F\left(A_t, B\right) = -\operatorname{res}\left(\sigma\left(\frac{\partial}{\partial t}A_t \cdot A_t^{-1}\right)\cdot\sigma\left(\frac{\log\left(A_t B\right)}{\alpha+\beta} - \frac{\log A}{\alpha}\right)\right). \tag{1.3}$$

This formula is proved in [KV], Section 2.

The logarithms in (1.3) are defined as the derivatives at $s=0$ of complex powers $(A_tB)^s$ and B^s. Note that $\log\left(A_tB\right)/(\alpha+\beta) - \log A/\alpha \in CL^0$ (i.e., it is a classical PDO of order zero).

To remind, the noncommutative residue of a classical PDO-symbol a of an integer order is equal to the integral over M, $\dim M = n$, of a density defined by

$$\operatorname{res}_x(a) := \int_{S_x^*M} \operatorname{tr}\left(a_{-n}(x,\xi)\right) d'\xi \ . \tag{1.4}$$

This density on M is independent of a choice of local coordinates on M. The integral (1.4) is taken over the unit sphere $S_x^*M = \{\xi \in T_x^*M : |\xi| = 1\}$.

Remark 1.1. Using formula (1.3) one can obtain an explicit local expression in terms of symbols for the multiplicative anomaly (1.1) if A and B are sufficiently close to positive definite self-adjoint PDOs. Namely, in this case, one can connect A with $A_1 := B^{\alpha/\beta}$ by a smooth path in the space of elliptic PDOs of order α admitting a spectral cut close to $\mathbb{R}_- \subset \mathbb{C}$.

Remark 1.2. A formula for $F(A,B)$ for commuting self-adjoint positive elliptic DOs was obtained by M. Wodzicki, see [Kas]. For noncommuting positive self-adjoint elliptic PDOs a variation formula for $F(A,B)$ in a form different from (1.3) was obtained by L. Friedlander, [Fr].

2. Determinant Lie group

From now on all elliptic PDOs are supposed to be invertible. Let A, B, and AB admit spectral cuts and let their orders be nonzero real numbers. Then the multiplicative anomaly $F(A,B)$ depends on symbols $\sigma(A)$ and $\sigma(B)$ only (for fixed admissible spectral cuts). This statement is an immediate consequence of the following lemma.

Lemma 2.1. *For an elliptic operator A, $\operatorname{ord} A \in \mathbb{R}^\times$, admitting a spectral cut L_θ and for any invertible operator Q of the form $Q = \mathrm{Id} + S$, where the Schwartz kernel of S is C^∞ on $M \times M$ (i.e., S is smoothing), the equality holds*

$$\det{}_\zeta(AQ) = \det{}_\zeta(A)\det{}_{Fr}(Q) \ . \tag{2.1}$$

Here, $\det_\zeta$ for A, AQ are taken with respect to any admissible spectral cuts close to L_θ. The Fredholm determinant $\det_{Fr}$ is defined by

$$\det_{Fr}(\mathrm{Id} + S) = 1 + \operatorname{Tr} S + \operatorname{Tr} \wedge^2 S + \dots \quad . \tag{2.2}$$

This series is absolutely convergent for any trace class operator S. (Smoothing operators are of trace class.) Formula (2.2) is valid in a finite-dimensional case also.

The proof of (2.1) is based on applying a variation formula for an arbitrary smooth 1-parameter family A_t of elliptic PDOs with $\sigma(A_t) = \sigma(A)$, $A_0 = A$, $A_1 = AQ$.

The multiplicative anomaly $F(A, B)$ possesses a cocycle condition

$$F(A, BC)F(B, C) = F(A, B)F(AB, C)$$

(for any fixed spectral cuts for A, B, C, AB, BC, ABC). We consider $F(A, B)$ as a "partially defined and multi-valued 2-cocycle" with the values in $\mathbb{C}^\times$ on the group $\mathrm{SEll} = G$ of elliptic symbols of index zero. However, we can directly construct the corresponding central $\mathbb{C}^\times$-extension $\tilde{G}$ of G. (Hence we do not work with a formalism of partially defined cocycles.) The determinant Lie group $\tilde{G}$ is defined by formula

$$\tilde{G} = \tilde{G}(M, E) := \mathrm{Ell}^\times / H^{(1)}, \tag{2.3}$$

where $H^{(1)}$ is the normal subgroup of the group $\mathrm{Ell}^\times$ of invertible elliptic PDOs of all complex orders,

$$H^{(1)} = \{Q = \mathrm{Id} + S,\ S \text{ are smoothing},\ \det_{Fr} Q = 1\}.$$

Note that the group G of elliptic symbols takes the analogous form,

$$G = \mathrm{Ell}^\times / H, \quad H = \{Q = \mathrm{Id} + S,\ \det_{Fr} Q \neq 0\}. \tag{2.4}$$

There is a natural exact sequence

$$1 \to \mathbb{C}^\times \underset{j}{\to} \tilde{G} \underset{p}{\to} G \to 1 \quad . \tag{2.5}$$

Here, the identification $H/H^{(1)} \xrightarrow[\sim]{} \mathbb{C}^\times$ is given by the Fredholm determinant (2.2).

For any $A, B \in \mathrm{Ell}^\times$ we have

$$d_1(A)d_1(B) = d_1(AB) \ ,$$

where $d_1 \colon \mathrm{Ell}^\times \to \tilde{G}$ is the natural projection. For a symbol $a \in G$, $\operatorname{ord} a = \alpha \in \mathbb{R}^\times$, such that the principal symbol a_α satisfies the Agmon-Nirenberg condition with a sector V, we define a canonical element $d_0(a) \in \tilde{G}$, $p(d_0(a)) = a$, by

$$d_0(a) = d_1(A) \cdot j\left((\det_\zeta(A))^{-1}\right) \ . \tag{2.6}$$

Here, $A \in \mathrm{Ell}^\times$ is an arbitrary invertible elliptic PDO with the symbol a, $\det_\zeta(A)$ is taken with respect to V. Note that $j\left((\det_\zeta(A))^{-1}\right)$ belongs to the central subgroup

$\mathbb{C}^\times$ in $\tilde{G}$. The independence $d_0(a)$ of A (with $\sigma(A) = a$) follows immediately from (2.1).

Thus the multiplicative anomaly is enclosed in the central $\mathbb{C}^\times$-extension $\tilde{G}$ of G with its partially defined multi-valued section d_0 (over elliptic symbols of orders from $\mathbb{R}^\times \subset \mathbb{C}$). Indeed,

$$j(F(A,B)) = d_0(\sigma(A))d_0(\sigma(B))d_0(\sigma(AB))^{-1} .$$

Later on we use Lie algebras $\mathfrak{ell}(M,E)$, $\mathfrak{g} = S_{\log}(M,E)$, $\tilde{\mathfrak{g}}$, $\mathfrak{h}$, $\mathfrak{h}^{(1)}$ of all the Lie groups from above. The Lie algebra $\mathfrak{ell}(M,E)$ of the group $\mathrm{Ell}^\times$ consists of logarithms of invertible elliptic PDOs and any element $l \in \mathfrak{ell}(M,E)$ takes the form $(q/2)\log(\Delta + \mathrm{Id}) + B$, where $q \in \mathbb{C}$ and $B \in CL^0$. (Here, Δ is the Laplacian for a Riemannian metric on M and a unitary connection on E.) The Lie algebra $\mathfrak{g}$ consists of the symbols of elements from $\mathfrak{ell}(M,E)$. These symbols are not classical. In local coordinates on M such a symbol takes the form $q \log|\xi| \cdot \mathrm{Id} + b$, where $q \in \mathbb{C}$ and b is a zero order symbol.

Elements l of $\mathfrak{g}$ are generators of one-parameter subgroups $\exp(sl)$ of $\mathrm{SEll} = G$; $(\partial_s \exp(sl))\exp(-sl) = l$ in $\mathfrak{g}$. Analogously, there are exponential maps from $\mathfrak{ell}(M,E)$, $\mathfrak{g}$, $\tilde{\mathfrak{g}}$, $\mathfrak{h}$, and $\mathfrak{h}^{(1)}$ to $\mathrm{Ell}^\times$, G, $\tilde{G}$, H, and $H^{(1)}$.

Remark 2.1. The extension by $\log\xi$ (not by $\log|\xi|$) of the Lie algebra of scalar Loran PDO-symbols of integer orders in the case of $M = S^1$ was considered in [KrKh]. The authors of this paper also formally constructed a central extension of the Lie algebra of such logarithmic symbols with the help of the Adler-Manin-Lebedev residue. This cocycle is analogous to one appearing on the right in formula (6.4), with $x = \log\xi$. A multi-dimensional analog of this extension was considered in [R]. A connection of a formal Lie algebraic construction of such a type with determinants of elliptic PDOs investigated in [KV] and here (Section 6), is a new fact.

3. New trace type functional

Let $A \in CL^\alpha$ be a classical PDO of a noninteger order $\alpha \in \mathbb{C}\setminus\mathbb{Z}$ acting on sections of a vector bundle E on M, $\dim M = n$. We introduce a canonical density $t(A)$ on M with the values in $\mathrm{End}(E)$ as follows. It is defined in any local coordinate chart U on M together with a trivialization of E over U. The density $t_U(A)$ is given by the restriction to the diagonal $U \hookrightarrow U \times U$ of the difference

$$A(x,y) - \sum_{j=0}^{N} K_{-n-\alpha+j}(x, y-x) \tag{3.1}$$

of the Schwartz kernel $A(x,y)$ of A (restricted to $U \times U$) and the Fourier transforms of the first $N+1$, $N \gg 1$, homogeneous terms $a_\alpha, a_{\alpha-1}, \ldots, a_{\alpha-N}$ of the symbol

$a = \sigma(A)$ with respect to given coordinates in U. Namely

$$K_{-n-\alpha+j}(x, y-x) := (2\pi)^{-n} \int_{\mathbb{R}^n_\xi} \exp(i(x-y, \xi)) a_{\alpha-j}(x, \xi) d\xi \ .$$

This distribution is positive homogeneous in $y - x \in \mathbb{R}^n$ of order $-n - \alpha + j$ for $\alpha \notin \mathbb{Z}$.

Note that any positive homogeneous distribution from $\mathcal{D}'(\mathbb{R}^n \setminus 0)$ of order $\beta \notin \{m \in \mathbb{Z}, m \leq -n\}$ has a unique prolongation to a positive homogeneous distribution from $\mathcal{D}'(\mathbb{R}^n)$ (see [Hö], Theorem 3.2.3). Hence, if we restrict $K_{-n-\alpha+j}(x, y-x)$ to $y \neq x$, we'll not lose any information.

Lemma 3.1. *The difference (3.1) is a continuous on $U \times U$ function for N large enough. Hence its restriction $t_U(A)$ to the diagonal U makes sense.*

Lemma 3.2. *The density $t_U(A)$ with the values in* End E *is independent of large N, of local coordinates on M, and of a local trivialization of E.*

The statement of Lemma 3.1 follows directly from the structure of singularities of PDO-kernels. The independence $t_U(A)$ of the change N by $N+1$ (if N is large enough) follows from the positive homogeneity of $K_{-n-\alpha+N+1}(x, y-x)$ in $y-x$ and from the fact $\mathrm{Re}(-n-\alpha+N+1) > 0$. The invariance of $t_U(A)$ under changes of local coordinates and of trivializations follow from the Taylor's formula, from the non-integrality of α, and from ordinary properties of derivatives of homogeneous functions.

Theorem 3.1. *The linear functional*

$$\mathrm{TR}(A) = \int_M \mathrm{tr}\, t(A)$$

on classical PDOs of orders from $\alpha_0 + \mathbb{Z}$, $\alpha_0 \in \mathbb{C} \setminus \mathbb{Z}$, in the case of a closed M has the following properties.

1. It coincides with the usual trace $\mathrm{Tr}\, A$ *in $L_2(M, E)$ for* $\mathrm{Re}\, \mathrm{ord}\, A < -n$.

2. It is a trace type functional, i.e., $\mathrm{TR}([B, C]) = 0$ *for* $\mathrm{ord}\, B + \mathrm{ord}\, C \in \alpha_0 + \mathbb{Z}$.

3. For any holomorphic family $A(z)$ of classical PDOs on M, $z \in U \subset \mathbb{C}$, $\mathrm{ord}\, A(z) = z$, *the function* $\mathrm{TR}(A(z))$ *is meromorphic with no more than simple poles at $z = m \in U \cap \mathbb{Z}$ and with residues*

$$\mathrm{Res}_{z=m} \mathrm{TR}(A(z)) = -\,\mathrm{res}\, \sigma(A(m)) \ . \tag{3.2}$$

(Here, res *is the noncommutative residue of the symbol of $A(m)$, $m \in \mathbb{Z}$,* [Wo1], [Kas], [Wo2].*)*

The part 2. follows from the parts 1. and 3. applied to arbitrary holomorphic families $B(z)$, $C(z)$, $z \in U$, such that $B(0) = B$, $C(0) = C$, and $B(z)$, $C(z)$ are of trace class in some subdomain $U_1 \subset U$ (i.e., $\operatorname{Re}\operatorname{ord} B(z), \operatorname{Re}\operatorname{ord} C(z) < -n$ for $z \in U_1$).

In the part 3., (3.2), we use that the singularities of densities $t(A(z))$ are the same as of the restriction to the diagonal of the integral

$$\sum_{j=0}^{N} \int (\rho(|\xi|) - 1)|\xi|^{z-j} a_{z-j}(z, x, \xi/|\xi|) \exp(i(x-y,\xi)) d\xi \ . \tag{3.3}$$

Here, $\rho(|\xi|)$ is a smooth cutting function, $\rho(|\xi|) = 1$ for $|\xi| \gg 1$, $\rho(|\xi|) = 0$ for $|\xi| \ll 1$. The integral (3.3) for $x = y$ has an explicit analytic continuation produced with the help of the the equality $\int_0^1 x^\lambda dx = 1/(\lambda + 1)$, $\operatorname{Re}\lambda > -1$.

Remark 3.1. Theorem 3.1 implies that $\operatorname{res}([b, c]) = 0$ for $\operatorname{ord} b + \operatorname{ord} c \in \mathbb{Z}$ (i.e., res is a trace type functional). This assertion is well known, [Kas], [Wo2], but its usual proof is not so elementary because it uses the spectral interpretation of the noncommutative residue.

4. Applications to zeta-functions

The trace type functional TR introduced in the previous section gives us a tool to define zeta-functions for one-parameter subgroups of $\mathrm{Ell}^\times$ denerated by elements $x \in \mathfrak{ell}(M, E)$ with $\operatorname{ord}(\exp x) \neq 0$. From now on we denote $\operatorname{ord}(\exp x)$ by $\operatorname{ord} x$ for any $x \in \mathfrak{ell}(M, E)$. We define

$$\zeta_x^{\mathrm{TR}}(s) := \mathrm{TR} \exp(-sx)$$

for $s \operatorname{ord} x \notin \mathbb{Z}$. By Theorem 3.1 we conclude the following.

Proposition 4.1. *1. The zeta-function $\zeta_x^{\mathrm{TR}}(s)$ is a meromorphic function on $\mathbb{C} \ni s$ with at most simple poles at $s_k = k/\operatorname{ord} x$, $k \in \mathbb{Z}$, $k \leq n := \dim M$. This function is regular at $s = 0$ by (3.2).*

2. The residue of $\zeta_x^{\mathrm{TR}}(s)$ at s_k is

$$\operatorname{Re}_{s=s_k} \zeta_x^{\mathrm{TR}}(s) = \operatorname{res} \sigma(\exp(-s_k x)) / \operatorname{ord} x \ .$$

3. Let $\operatorname{ord} x \in \mathbb{R}^\times$ and let $\exp x$ possess a spectral cut L_θ such that the $\log \exp x$ defined with respect to L_θ is equal to x. Then we have

$$\zeta_x^{\mathrm{TR}}(s) = \zeta_{\exp x, \theta}(s) \ ,$$

i.e., in this case, $\zeta_x^{\mathrm{TR}}(s)$ coincides with the classical zeta-function.

Note that the functional TR gives us a tool to define zeta-functions without an analytic continuation. For example, our definition has an immediate consequence, which is out of reach of previous methods.

Corollary 4.1. *Let A^s and B^s be two holomorphic families of complex powers such that $A^{s_0} = B^{t_0}$ and let* $\operatorname{ord} A \cdot s_0 \notin \mathbb{Z}$. *Then*

$$\zeta_A(s_0) = \zeta_B(t_0) \quad ,$$

where zeta-functions are defined by the meromorphic continuation from the domains of convergence.

Theorem 3.1 provides us with a general information on the structure of derivatives of zeta-functions at zero.

Theorem 4.1. *There are homogeneous polynomials $T_{k+1}(x)$ of order $k+1 \geq 1$ in $x \in \mathfrak{ell}(M,E)$ such that*

$$\operatorname{ord} x \cdot \partial_s^k \zeta_x^{\mathrm{TR}}(s)|_{s=0} = T_{k+1}(x) \ .$$

These polynomials are invariant with respect to the adjoint action of $\mathrm{Ell}^\times$ *on* $\mathfrak{ell}(M,E)$. *The restriction of $T_{k+1}(x)$ to the Lie ideal (of codimension one) $CL^0 = \{x\colon \operatorname{ord} x = 0\}$ is*

$$T_{k+1}(x)\Big|_{\operatorname{ord} x=0} = \frac{(-1)^{k+1}}{(k+1)} \operatorname{res}\left(\sigma(x)^{k+1}\right)\Big|_{\operatorname{ord} x=0} \ . \tag{4.1}$$

The statement 3. of Theorem 3.1 applied to a holomorphic family $\exp(sx+b)$, $\operatorname{ord} x = 1$, $b \in CL^0$, near $s = 0$ implies that the function $\operatorname{ord} x \cdot \mathrm{TR}\exp(x)$ is holomorphic at $\operatorname{ord} x = 0$ (on $\mathfrak{ell} \ni x$). The polynomials $T_k(x)$ are (up to standard factors) the Taylor coefficients of this function at $x = 0$.

Corollary 4.2. *The function* $\log\det(\exp x)$ *is the ratio* $-T_2(x)/\operatorname{ord} x$ *of a quadratic function and a linear function. But it is not a linear function (by (4.1)). There is no linear function "Tr" on* $\mathfrak{ell}$ *such that* "Tr"$\log A = \log\det(A)$.

A statement analogous to Proposition 4.1 holds also for a generalized zeta-function $\mathrm{TR}\,(B_1 A_1^{s_1} \ldots B_k A_k^{s_k}) =: \zeta_{(A_j),(B_j)}(s_1, \ldots, s_k)$. Here, $A_j^{s_j}$ are holomorphic families of powers of elliptic PDOs A_j (not all of $\alpha_j := \operatorname{ord} A_j$ are equal to zero), $s_j \in \mathbb{C}$, and B_j are classical PDOs of orders β_j.

Proposition 4.2. *The zeta-function $\zeta_{(A_j),(B_j)}(s_1, \ldots, s_k)$ is meromorphic in* $\mathbf{s} := (s_1, \ldots, s_k) \in \mathbb{C}^k$ *with at most simple poles on the hyperplanes* $z(\mathbf{s}) := \sum_j (\beta_j + s_j\alpha_j) = m \in \mathbb{Z}$, $m \geq -n$. *Its residue is equal to* $-\operatorname{res}\sigma\,(B_1 A_1^{s_1} \ldots B_k A_k^{s_k})\,|_{z(\mathbf{s})=m}$ *and thus it is computable in terms of symbols of B_j and of* $\log A_j$.

5. Canonical determinant

In this section we return to the determinant Lie group G. Above we have constructed, (2.6), the multi-valued section d_0 of the $\mathbb{C}^\times$-bundle $\widetilde{G} \to G$. Here we extend d_0 to its maximal natural domain of definition and introduce (with the help of extended d_0) the canonical determinant of elliptic PDOs.

Let $a \in G :=$ SEll be an elliptic symbol of a nonzero order and let $a = \exp x$ for some $x \in \mathfrak{g} = S_{\log}$. Pick any $b \in \mathfrak{ell}(M, E)$ such that its symbol $\sigma(b)$ is x. Then we define $d_0(a, x)$ as

$$d_0(a,x) := d_1(\exp b) j \left(\exp\left(\partial_s \zeta_b^{\mathrm{TR}}(s)|_{s=0}\right)\right) \in \widetilde{G} \; . \tag{5.1}$$

Lemma 5.1. *The element $d_0(a, x)$ is independent of $b \in \mathfrak{ell}(M, E)$ with $\sigma(b) = x$.*

This lemma together with its proof is analogous to Lemma 2.1.

Remark 5.1. The element $d_0(a, x)$, (5.1), depends on $x \in S_{\log}$ only, because $a = \exp x$. Also $d_0(a, x)$ is analytic in x, $\operatorname{ord} x \neq 0$. Elements $d_0(a, x)$ for $\operatorname{ord} a \in \mathbb{C}^\times$ form the image under the exponential map of a $\mathbb{C}^\times$-cone in the Lie algebra $\widetilde{\mathfrak{g}}$. In the next section we prove that this cone is a quadratic one and give an explicit description of it in terms of symbols.

Lemma 5.2. *For a, $\operatorname{ord} a \in \mathbb{R}^\times$, possessing a spectral cut L_θ and such that the logarithm of a (with respect to L_θ) is equal to x, the definitions (5.1) and (2.6) coincide.*

Definition. Let $A \in \mathrm{Ell}^\times$ be of any nonzero complex order. Let its symbol $a := \sigma(A)$ have a logarithm $x \in S_{\log}$. Then the *canonical determinant* of A is defined as

$$\det(A, x) := j^{-1}\left(d_1(A) d_0(a,x)^{-1}\right) \in \mathbb{C}^\times \; . \tag{5.2}$$

(Here, $j \colon \mathbb{C}^\times \hookrightarrow \widetilde{\mathfrak{g}}$ is the natual inclusion of the central subgroup from (2.5).)

Remark 5.2. This definition does not use any family A^s of complex powers of A. It uses families of powers $\exp(sb) = (\exp(b))^s$ of b with $\sigma(b) = \log\sigma(A)$ constructed elementary by any such b. The necessity of such a construction with powers of other operators follows from the fact that the existence of a logarithm of a generic invertible elliptic PDO cannot be described in terms of his symbol. Also the image of the exponential map $\exp\colon S_{\log} \to \mathrm{SEll} = G$ has much more simple structure than the image of $\exp\colon \mathfrak{ell} \to \mathrm{Ell}^\times$. (See more detailed discussion of this problem in [KV], Remarks 6.3, 6.4, 6.8, 6.9.) In (5.2) we use only the existence of $\log\sigma(A)$.

5.1. Microlocal Agmon-Nirenberg condition. Here we introduce a sufficient condition of the existence of $\log \sigma(A)$ generalizing the Agmon-Nirenberg condition. Let $a_\alpha(x,\xi)$ be the principal elliptic symbol of A, $\alpha = \operatorname{ord} A \in \mathbb{C}^\times$. Let $\theta := \theta(x,\xi)\colon T^*M \setminus M \to \mathbb{R}$ be a continuous map such that $L_{\theta(x,\xi)} \cap \operatorname{Spec} a_\alpha(x,\xi) = \emptyset$ for all $(x,\xi) \in T^*M \setminus M$.

Lemma 5.3. *Under this condition,* $\log \sigma(A)$ *exists. It is explicitly defined by the formula analogous to (1.2) on the level of complete symbols. Namely* $\log_{(\theta)} \sigma(A)(x,\xi)$ *is the derivative at* $s = 0$ *of the family of symbols* $\sigma(A)^s_\theta$. *Here,* $\sigma(A)^{-s}_\theta$ *is defined for* $\operatorname{Re} s > 0$ *by the integral*

$$\frac{i}{2\pi}\int_{\Gamma_{\theta(x,\xi)}} (\sigma(A)-\lambda)^{-1}(x,\xi)\lambda^{-s}_{(\theta(x,\xi))}d\lambda \ , \tag{5.3}$$

and $(\sigma(A)-\lambda)^{-1}$ *is an inverse element in the algebra of symbols (with a parameter* λ *of homogeneity degree* α*). For* $-k < \operatorname{Re} s \le 0$, $k \in \mathbb{Z}_+$, $\sigma(A)^{-s}_\theta := \sigma(A)^k \cdot \sigma(A)^{-s-k}_\theta$.

Remark 5.3. The definition (5.3) of $\sigma(A)^s_\theta$ is invariant under homotopies of a field $\theta(x,\xi)$ of admissible for a_α cuts. By the homotopy equivalence $S^*M \sim T^*M \setminus M$ and positive homogeneity of $a_\alpha(x,\xi)$ it is enough to define θ only over any global smooth section of the $\mathbb{R}^\times_+$-bundle $T^*M \setminus M \to S^*M$. The existence of a field of admissible for $a_\alpha(x,\xi)$ cuts is in a sense nonsensitive to an order $\alpha \in \mathbb{C}^\times$. It is applicable to elliptic symbols of complex orders.

Remark 5.4. The microlocal Agmon-Nirenberg condition of Lemma 5.3 is a rather weak restriction on $\sigma(A)$. Nevertheless there are simple topological obstructions to the existence of $\log \sigma(A)$. For instance, for any (M,E) with $\dim M \ge 2$, $\operatorname{rk} E \ge 2$, there are nonempty open subsets in $\operatorname{Ell}^\times(M,E)$ admitting no continuous logarithms of principal elliptic symbols. For example, let the principal symbol $a_\alpha(x,\xi)$ have at (x_0,ξ_0) a Jordan block $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$. Let the corresponding to λ eigenvalues over a closed curve $S^1 \to S^*M$ be $\lambda_i(\varphi)$, $\lambda_i(\varphi_0) = \lambda$, $i = 1,2$, and let the winding numbers of $\lambda_i(\varphi)$ be $+m$, $-m$, where $m \in \mathbb{Z} \setminus 0$. Then there is no continuous $\log a_\alpha(x,\xi)$. This condition is an open condition on a principal symbol.

6. Determinant Lie algebra and quadratic cone

We know by Theorem 4.1 that the first derivative of the zeta-function at zero is given by

$$\partial_s \zeta^{\mathrm{TR}}_x(s)|_{s=0} = T_2(x)/\operatorname{ord} x \ .$$

Here, T_2 is a quadratic form on $\mathfrak{ell}(M,E) \ni x$. The associated symmetric bilinear form B on $\mathfrak{ell}$,

$$B(x,y) := T_2(x+y) - T_2(x) - T_2(y) \ ,$$

has the following properties.

Lemma 6.1. *1. $B(x,y)$ is invariant under the adjoint action,*

$$B([x,z],y)+B(x,[y,z])=0$$

for $x,y,z\in \mathfrak{ell}$.

2. For $y\in\mathfrak{h}\subset\mathfrak{ell}$ we have

$$B(x,y)=-\operatorname{Tr} y\cdot \operatorname{ord} x\ .$$

(operator $y\in\mathfrak{h}$ is smoothing and hence is of trace class.)

3. For $x,y\in CL^0\subset\mathfrak{ell}$ we have

$$B(x,y)=(\sigma(x),\sigma(y))_{\mathrm{res}}:=\operatorname{res}(\sigma(x)\sigma(y))\ .$$

The properties 1. and 3. of $B(x,y)$ follow immediately from Theorem 4.1, (4.1). The property 2. is a consequence of the equality

$$\partial_t\left(\partial_s\zeta^{\mathrm{TR}}_{x_t}(s)|_{s=0}\right)=-\operatorname{Tr}\left(\int_0^1 ds\cdot \mathrm{Ad}_{\exp(sx_t)}\,\partial_t x_t\right)=-\operatorname{Tr}\left(\partial_t x_t\right)\ . \qquad (6.1)$$

Here, $x_t:=x+ty$, $\operatorname{ord}x\neq 0$. (In (6.1) we use that $\partial_t x_t=y$ is a smoothing operator.)

The property 2. implies that $\mathfrak{h}^{(1)}\subset\operatorname{Ker}B$. Hence B induces an invariant symmetric bilinear form on $\tilde{\mathfrak{g}}:=\mathfrak{ell}/\mathfrak{h}^{(1)}$. We denote this form by the same letter B.

Proposition 6.1. *1. For any $x\in\mathfrak{g}=S_{\log}$, $\operatorname{ord}x\neq 0$, there exists a unique $\tilde{x}\in\tilde{\mathfrak{g}}$ such that $p\tilde{x}=x$ ($p\colon\tilde{\mathfrak{g}}\to\mathfrak{g}$ is the natural projection) and such that $B(\tilde{x},\tilde{x})=0$ (i.e., $\tilde{x}$ is an isotropic vector).*

2. The element $d_0(\exp x,x)$ defined by (5.1) (for $\operatorname{ord}x\neq 0$) is given by

$$d_0(\exp x,x)=\exp(\tilde{x})\ .$$

The part 1. follows from the condition $\operatorname{ord}x\neq 0$ because $B(x,j(1))=-\operatorname{ord}x$ (see Lemma 6.1, *2.*). Here, $1\in\mathfrak{h}/\mathfrak{h}^{(1)}\simeq\mathbb{C}$ is represented by any smoothing operator with the trace equal to 1 and $j\colon\mathfrak{h}/\mathfrak{h}^{(1)}\hookrightarrow\tilde{\mathfrak{g}}$ is the natural inclusion.

The part 2. follows from the equality

$$\partial_s\zeta_b^{\mathrm{TR}}(s)|_{s=0}=\frac{B(\tilde{x},\tilde{x})}{2\operatorname{ord}x}=0$$

for any $b\in\mathfrak{ell}$ such that $\tilde{x}=b\left(\mod \mathfrak{h}^{(1)}\right)$ in $\tilde{\mathfrak{g}}=\mathfrak{ell}/\mathfrak{h}^{(1)}$.

Now we describe a general algebraic construction which in our case gives the description of the determinant Lie algebra $\tilde{\mathfrak{g}}$ (and of the bilinear form B on $\tilde{\mathfrak{g}}$) in terms of symbols.

Let us consider a central extension

$$0 \to \mathbb{C} \underset{j}{\to} \tilde{\mathfrak{g}} \underset{p}{\to} \mathfrak{g} \to 0 \tag{6.2}$$

of an abstract Lie algebra $\mathfrak{g}$ with an invariant symmetric bilinear form B on $\tilde{\mathfrak{g}}$ such that $B(j(1), j(1)) = 0$ and $B(j(1), x) \not\equiv 0$ (i.e., $\operatorname{Im} j \not\subset \operatorname{Ker} B$). We associate with (6.2) an exact sequence of Lie algebras

$$0 \to \mathfrak{g}_0 \underset{r}{\to} \mathfrak{g} \underset{q}{\to} \mathbb{C} \to 0 \ . \tag{6.3}$$

Here, $qx := -B(j(1), x_1)$, $x \in \mathfrak{g}$, for any $x_1 \in p^{-1}(x)$, and $\mathfrak{g}_0 := \operatorname{Ker} q$ is a codimension one Lie ideal in $\mathfrak{g}$. The form B on $\tilde{\mathfrak{g}}$ induces a symmetric bilinear form B_0 on $\mathfrak{g}_0$. Namely, $B_0(x, y) := B(x_1, y_1)$ for any $x_1 \in p^{-1}(r(x))$, $y_1 \in p^{-1}(r(y))$. The form B_0 is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_0$.

In our concrete situation $\mathfrak{g}_0$ is CL^0, $\mathfrak{g}$ is $S_{\log}$, $q(x) = \operatorname{ord} x$, r is the natural inclusion, and $B_0(x, y) = (x, y)_{\mathrm{res}}$.

Theorem 6.1. *The exact sequence (6.2) and the symmetric bilinear form B on $\tilde{\mathfrak{g}}$ can be canonically reconstructed from (6.3) and from the form B_0 on $\mathfrak{g}_0$.*

Proof. 1. Suppose we have both sequences, (6.2) and (6.3), related one with another as described above. Then for any $x \in \mathfrak{g}$ with $q(x) = 1$ we have a unique $\tilde{x} \in \tilde{\mathfrak{g}}$, $B(\tilde{x}, \tilde{x}) = 0$, $p(\tilde{x}) = x$. The hyperplane $\{y \in \tilde{\mathfrak{g}} : B(\tilde{x}, y) = 0\} \subset \tilde{\mathfrak{g}}$ defines the splitting of (6.2) (as of the exact sequence of vector spaces) $\Pi_{\tilde{x}} \colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$. One can check that the Lie bracket and the form B on $\tilde{\mathfrak{g}} \simeq \Pi_{\tilde{x}}(\mathfrak{g}) \oplus j(\mathbb{C})$ are as follows.

$$\begin{aligned}&[\Pi_{\tilde{x}}(r(a_1) + t_1 x) + j(c_1), \Pi_{\tilde{x}}(r(a_2) + t_2 x) + j(c_2)] = \\ &\qquad = \Pi_{\tilde{x}}([r(a_1) + t_1 x, r(a_2) + t_2 x]) - j\left(B_0\left(r^{-1}[x, r(a_1)], a_2\right)\right),\end{aligned} \tag{6.4}$$

$$\begin{aligned}&B(\Pi_{\tilde{x}}(r(a_1) + t_1 x) + j(c_1), \Pi_{\tilde{x}}(r(a_2) + t_2 x) + j(c_2)) = \\ &\qquad\qquad = B_0(a_1, a_2) - c_1 t_2 - c_2 t_1 \ .\end{aligned} \tag{6.5}$$

Here we use the parametrization $\Pi_{\tilde{x}}(r(a) + tx) + j(c)$, $a \in \mathfrak{g}_0$, $t, c \in \mathbb{C}$, of $\tilde{\mathfrak{g}}$ (and the parametrization $r(a) + tx$ of $\mathfrak{g}$).

Let we have two elements x, x' in $\mathfrak{g}$ such that $q(x) = q(x') = 1$. Then we have

$$(\Pi_{\tilde{x}} - \Pi_{\tilde{x}'})(y) = B_0\left(r^{-1}(y - q(y)(x + x')/2), r^{-1}(x - x')\right) \cdot j(1) \ . \tag{6.6}$$

2. Formulas (6.4), (6.5) can be interpreted as a construction of the extension (6.2) and of the symmetric bilinear form B on $\tilde{\mathfrak{g}}$ in terms of (6.3) and B_0. This construction of $(\tilde{\mathfrak{g}}, B) \simeq (\tilde{\mathfrak{g}}_x, B_x)$ depends on a choice of $x \in q^{-1}(1)$. Formula (6.6) provides us with an associative system of identifications of these Lie algebras $\tilde{\mathfrak{g}}_x$ (together with the bilinear forms B_x on them) for different x. □

Conclusions. 1. We obtain a description (6.4)–(6.6) of the determinant Lie algebra $\tilde{\mathfrak{g}}$ in terms of $\mathfrak{g} = S_{\log}$, i.e., in terms of symbols (without using Fredholm determinants and so on). Namely, $\tilde{\mathfrak{g}}$ is generated by symbols $\Pi_x y$ for $x \in q^{-1}(1) \subset \mathfrak{g}$, $y \in \mathfrak{g}$, and by $j(c)$, $c \in \mathbb{C}$. Symbols $\Pi_x y$ are linear in y, and $j(c) = cj(1)$. The relations between these symbols are given by (6.4), (6.6), where $\Pi_{\tilde{x}}$ is replaced by Π_x and where y is represented by $r(a) + tx$. Formula (6.5) defines an invariant bilinear form on $\tilde{\mathfrak{g}}$.

2. Let A be an elliptic PDO, $\operatorname{ord} A \neq 0$, with a given $\log \sigma(A) =: x \in S_{\log} = \mathfrak{g}$. Then we have an explicit formula for $\log d_0(\sigma(A), x)$ in $\tilde{\mathfrak{g}}$. Namely

$$\log d_0(\sigma(A), x) = \Pi_{\widetilde{(x/\operatorname{ord} x)}}(x) \ .$$

Remark 6.1. The central extension $\tilde{\mathfrak{g}}$ is not a trivial extension of the topological Lie algebra $\mathfrak{g}$. This fact can be proved with the help of the Atiyah-Singer Index theorem for families of elliptic PDOs. See Lemma 6.8 in [KV].

6.1. Singularities of determinants. Let $A(z)$ be a holomorphic family of invertible elliptic PDOs, $z \in U \subset \mathbb{C}$. Let for a one-connected subdomain $U_1 \subset \overline{U_1} \subset U$ a family $\log \sigma(A(z))$, $z \in U_1$, be defined (e.g., using microlocal fields of spectral cuts as in Section 5.1 depending on z). We are interested in analytic behaviour of $\det(A(z))$ near ∂U_1. Using previous constructions we can find a function $f(z)$ from $U_1 \ni z$ to $\mathbb{C}^\times$, such that
1. $f(z)$ is defined by $\log \sigma(A(z))$,
2. $\det(A(z), \log \sigma(A(z)))/f(z)$ is holomorphic in a neighborhood of $\overline{U_1}$ in U.

Construction of $f(z)$**.** Fix a logarithmic symbol $x \in S_{\log}$ of order 1. It determines a splitting of the determinant Lie algebra $\tilde{\mathfrak{g}}$ via the map $\Pi_{\tilde{x}}$. (See Proof of Theorem 6.1.) So it defines a right invariant holomorphic connection on the $\mathbb{C}^\times$-bundle $\tilde{G} \xrightarrow[p]{} G$. Let $d_2(z)$ be a flat section of the pullback under the map $z \to \sigma(A(z))$ of this determinant bundle p on a neighborhood of $\overline{U_1}$. Set $f(z) := d_2(z)/d_0(\sigma(A(z)), \log \sigma(A(z)))$, $z \in U_1$.

Proposition 6.2. $\det(A(z), \log \sigma(A(z)))/f(z)$ *has a holomorphic extension to a neighborhood of* $\overline{U_1}$.

This assertion is clear because

$$\det(A(z), \log \sigma(A(z)))/f(z) = d_1(A(z))/d_2(z) \ ,$$

and the both factors on the right are holomorphic in a neighborhood of $\overline{U_1}$.

7. Odd-dimensional case

The algebra of classical elliptic PDOs contain an invariantly defined subalgebra of odd class operators.

Definition. Let $\{U_i\}$ be a cover of M by coordinate charts and let $E|_{U_i}$ be trivialized. Then $A \in CL^d$, $d \in \mathbb{Z}$, is an *odd class* PDO if its symbol $\sigma(A)$ obeys on any U_i the following condition

$$\sigma_k(A)(x,\xi) = (-1)^k \sigma_k(A)(x,-\xi) \ .$$

Here, $\sigma_k(A)$, $k \leq d$, are positive homogeneous in ξ components of $\sigma(A)$ in charts U_i. This condition is independent of a choice of local coordinates (near $x \in M$) and of a trivialization of E. It follows from the transformation formula for PDO-symbols under changing of space coordinates.

We denote by $CL^{\mathbf{Z}}_{(-1)}$ the linear space of odd class PDOs, and by $CS^{\mathbf{Z}}_{(-1)}$ the space of their symbols. By $\mathrm{Ell}^{\times}_{(-1)}$ we denote the group of odd class invertible elliptic PDOs and by $\mathrm{SEll}_{(-1)}$ the group of their symbols. $\mathrm{Ell}^{\times}_{(-1)}$ and $\mathrm{SEll}_{(-1)}$ are groups by the following Lemma.

Lemma 7.1. *1. Differential operators (DOs) are contained in $CL^{\mathbf{Z}}_{(-1)}$.*

2. Smoothing operators are contained in $CL^{\mathbf{Z}}_{(-1)}$.

3. $CL^{\mathbf{Z}}_{(-1)}$ is a subalgebra of $CL^{\mathbf{Z}}$.

4. If $A \in \mathrm{Ell}^{\mathbf{Z}}_{(-1)}$ is an invertible elliptic PDO, then $A^{-1} \in \mathrm{Ell}^{\mathbf{Z}}_{(-1)}$.

Proposition 7.1. *Let $a \in \mathrm{SEll}^{2k}_{(-1)}$, $k \in \mathbb{Z}$, be an elliptic symbol admitting a* microlocal field *θ* of spectral cuts *(Section* 5.1*), which is* projective, *i.e., $\theta(x,\xi) = \theta(x,-\xi)$ for $\xi \neq 0$. Then we have*

1. $a^{1/k}_{(\theta)} \in \mathrm{SEll}^2_{(-1)}$ for $k \neq 0$.

2. $a^{s}_{(\theta)}$ belongs to $\mathrm{SEll}^0_{(-1)}$ for $k = 0$.

3. $\log_{(\theta)} a$ belongs to $CS^0_{(-1)}$ for $k = 0$.

Lemma 7.2. *We have* $\exp a \in \mathrm{SEll}^0_{(-1)}$ *for $a \in CS^0_{(-1)}$.*

Proposition 7.2. *Let elliptic symbols $a_1 \in \mathrm{SEll}^{2k_1}_{(-1)}$ and $a_2 \in \mathrm{SEll}^{2k_2}_{(-1)}$, $k_j \in \mathbb{Z} \setminus 0$, admit projective fields of spectral cuts θ_1 and θ_2. Then*

$$\frac{\log_{(\theta_1)} a_1}{k_1} - \frac{\log_{(\theta_2)} a_2}{k_2} \in CS^0_{(-1)} \ . \tag{7.1}$$

For a and θ as in Proposition 7.1 we denote by $a^s_{(\theta),2ks-j}(x,\xi)$, $j \in \mathbb{Z}_+ \cup 0$, the homogeneous components of $a^s_{(\theta)}$. Then assertions of Propositions 7.1 and 7.2 follow from the equalities

$$a^s_{(\theta),2ks-j}(x,\xi) = (-1)^j a^s_{(\theta),2ks-j}(x,-\xi) \ , \qquad (7.2)$$
$$a^0_{(\theta)} = \mathrm{Id} \ .$$

The equality (7.2) is a direct consequence of the integral representation (5.3) for $a^s_{(\theta)}$ and of explicit formulas for the symbol $(a-\lambda)^{-1}$ (with $\deg\lambda = \operatorname{ord} a$).

From now on we suppose that M is odd-dimensional.

Lemma 7.3. *For $A \in CL^{\mathbf{Z}}_{(-1)}$ we have*

$$\operatorname{res}\sigma(A) = 0 \ . \qquad (7.3)$$

This formula follows immediately from the definition of res because $\sigma_{-n}(A)(x,\xi)$ is odd in ξ, $n := \dim M$.

Now we have tools for investigation of the multiplicative anomaly in the odd class. Let A and B be odd class invertible elliptic PDOs of nonzero even orders, $\operatorname{ord} A + \operatorname{ord} B \neq 0$, such that the symbols of A, B, and of AB admit projective fields of spectral cuts θ_1, θ_2, θ_3. The multiplicative anomaly is defined (in this case) as

$$F(A,B) := \frac{\det\left(AB, \log_{(\theta_3)}\sigma(AB)\right)}{\det\left(A, \log_{(\theta_1)}\sigma(A)\right)\det\left(B, \log_{(\theta_2)}\sigma(B)\right)} \ .$$

Theorem 7.1. *1. $F(A,B)$ is locally constant in A, B (for given admissible θ_j).*

2. For the principal symbols of A and B sufficiently close to positive definite self-adjoint ones and for fields θ_j close to π we have

$$F(A,B) = 1 \ . \qquad (7.4)$$

The multiplicative property (7.4) for zeta-regularized determinants of positive self-adjoint deifferential operators on closed odd-dimensional manifolds is a new one.

Our proof of Theorem 7.1 is based on a general variation formula analogous to (1.3) (valid without assumptions that A, B are of odd class and that $\dim M$ is odd)

$$\frac{\partial}{\partial t}\log F(A_t, B) = -\left(\sigma\left(\frac{\partial}{\partial t}A_t \cdot A_t^{-1}\right), \frac{\log_{(\theta_3)}\sigma(AB)}{\operatorname{ord} A + \operatorname{ord} B} - \frac{\log_{(\theta_1)}\sigma(A)}{\operatorname{ord} A}\right)_{\mathrm{res}} .$$

According to Proposition 7.2 and to Lemmas 7.1, 7.3 the right hand side is equal to zero.

The multiplicative property (7.4) provides us with a possibility to define $\det(A)$ for any invertible elliptic A from $\mathrm{Ell}^0_{(-1)}$ close to positive definite self-adjoint ones. Namely, define $\det A$ as

$$\det(A) := \det_\zeta(AB)/\det_\zeta(B) \tag{7.5}$$

for an arbitrary positive self-adjoint invertible $B \in \mathrm{Ell}^{2k}_{(-1)}$, $k > 0$. Here, spectral cuts are close to $\mathbb{R}_-$. Independence of the expression on the right in (7.5) of B follows from the equality (7.4).

Lemma 7.4. *For an elliptic DO A of zero order (i.e., for $A \in \mathrm{Aut}\, E$) sufficiently close to positive definite ones, its determinant (7.5) is equal to* 1.

This statement can be proved by the remark that the map

$$q\colon\ \mathrm{End}\, E \ni f \to \partial_s \log\det(\exp(sf))|_{s=0} \in \mathbb{C}$$

is a homomorphism of Lie algebras. Here, det is defined by (7.5). The map q is invariant under the adjoint action of the group $\mathrm{Diff}(M,E)$ of diffeomorphisms of the total space of E which are linear maps between fibers. It is clear that the only $\mathrm{Diff}(M,E)$-invariant continuous linear functional on $\mathrm{End}(E)$ is $q \equiv 0$.

7.1. Determinant Lie group for odd class operators. We define the Lie groups

$$G_{(-1)} := \mathrm{Ell}^\times_{(-1)}/H = \mathrm{SEll}_{(-1)}, \quad \tilde{G}_{(-1)} := \mathrm{Ell}^\times_{(-1)}/H^{(1)} ,$$

analogous to (2.4), (2.3). We call $\tilde{G}_{(-1)}$ the determinant Lie group for the odd class. (To remind, M is a closed odd-dimensional manifold.)

By Proposition 7.1 the Lie algebra $\mathfrak{g}_{(-1)}$ of $G_{(-1)}$ is $CS^0_{(-1)}$. The group $\tilde{G}_{(-1)}$ is a central $\mathbb{C}^\times$-extension of $G_{(-1)}$ by analogy with (2.5). Here again, the identification $\mathbb{C}^\times = H/H^{(1)}$ is defined by $\det_{Fr}$.

Proposition 7.3. *The Lie algebra $\tilde{\mathfrak{g}}_{(-1)}$ of $\tilde{G}_{(-1)}$ is canonically splitted into the direct sum of the Lie algebras*

$$\tilde{\mathfrak{g}}_{(-1)} = \mathbb{C} \oplus \mathfrak{g}_{(-1)} \ . \tag{7.6}$$

This splitting is invariant with respect to the adjoint action of $\tilde{G}_{(-1)}$ on $\tilde{\mathfrak{g}}_{(-1)}$.

For the splitting (7.6) we can use any elliptic symbol $a \in \mathrm{SEll}^{2k}_{(-1)}$, $k \in \mathbb{Z} \setminus 0$, admitting a projective microlocal field of spectral cuts θ. Set $x := (1/2k)\log_{(\theta)} a \in \mathfrak{g}$. We embed $\mathfrak{g}_{(-1)}$ into $\tilde{\mathfrak{g}}_{(-1)}$ by the map $\Pi_{\tilde{x}}$. (See Proof of Theorem 6.1.) We claim:

1. $\Pi_{\tilde{x}}$ is a Lie algebra homomorphism.
2. $\Pi_{\tilde{x}}$ is independent of a, θ.

The first assertion follows from Lemma 7.1, *3*, Lemma 7.3, and from formula (6.4). The second assertion is a consequence of (6.6) and (7.1).

The splitting (7.6) defines a bi-invariant flat connection ∇ on the $\mathbb{C}^\times$-bundle $\tilde{G}_{(-1)} \underset{p}{\rightarrow} G_{(-1)}$.

Proposition 7.4. *The image of the monodromy map for ∇,*

$$\mathrm{Mon}_\nabla \colon \pi_1\left(G_{(-1)}, \mathrm{Id}\right) \rightarrow \mathbb{C}^\times \ ,$$

is a finite cyclic group consisting of roots of unity of order 2^m, where $0 \leq m \leq [n/2]^2$, $n := \dim M$.

The origin of Proposition 7.4 lies in the K-theory. Namely, we can take the direct sum of E with another vector bundle E_1 such that $E \oplus E_1$ is a trivial vector bundle on M.

The group $G^0_{(-1)}$ (corresponding to zero order operators) for the trivial N-dimensional bundle $\mathbf{1}_N$ on M is homotopy equivalent to the space of continuous maps

$$P^*M \rightarrow GL(N, \mathbb{C}) \sim U(N) \ ,$$

where $P^*M := S^*M/\{\pm 1\}$. By the Bott periodicity the fundamental group of the space $\mathrm{Map}(P^*M, U(N))$ stabilizes in the limit $N \rightarrow +\infty$ to $K^0(P^*M)$.

Lemma 7.5. *The monodromy* Mon_∇ *is trivial for loops* $\exp(2\pi i t\sigma(p))$, $0 \leq t \leq 1$, *in* $G^0_{(-1)}$, *where p is a projector in* $\mathrm{End}\, E$, $p^2 = p$ *(considered as a zero order differential operator).*

The proof of Lemma 7.5 is based on analytic facts proved below in this Section. (See Remark 7.3.)

Hence Mon_∇ is defined by a homomorphism

$$\mathrm{mon}_\nabla \colon K^0(P^*M)/\pi^* K^0(M) \rightarrow \mathbb{C}^\times \ , \tag{7.7}$$

$\pi \colon P^*M \rightarrow M$ is the natural projection map.

The orders of elements of the group $K^0(P^*M)/\pi^*K^0(M)$ are divisors of $2^{([n/2]^2)}$ by the Atiyah-Hirzebruch spectral sequence and by the fact that $\widetilde{K^0}(\mathbb{R}P^{n-1}) := K^0(\mathbb{R}P^{n-1})/\pi^*K^0(pt)$ is the cyclic group $\mathbb{Z}/2^{[n/2]}\mathbb{Z}$ (see [KV], Lemma 4.5).

7.2. Absolute value and holomorphic determinants. For any invertible elliptic operator $A \in \mathrm{Ell}^k_{(-1)}$, $k \in \mathbb{Z} \setminus 0$, its absolute value determinant is defined by

$$|\det|A = (\det_\zeta(A^*A))^{1/2} \ . \tag{7.8}$$

Here, A^* is the adjoint to A operator with respect to a Hermitian structure on E and to a positive smooth density on M. The square root on the right in (7.8) is positive. (The determinant $\det_\zeta(A^*A)$ of a positive self-adjoint operator A^*A is the usual one, i.e., it is taken with respect the spectral cut $\mathbb{R}_-$.)

Proposition 7.5. *1.* $|\det|A$, *(7.8), is independent of a Hermitian structure and of a positive density.*

2. $|\det|(AB) = |\det|A \cdot |\det|B$ *for* $A, B \in \mathrm{Ell}_{(-1)}^{\mathbb{Z}+,\times}$. *Hence by multiplicativity we can extend* $|\det|$ *to a homomorphism from* $\mathrm{Ell}_{(-1)}^{\times}$ *to* $\mathbb{R}_{+}^{\times}$.

3. The function $(|\det|A)^2$ *on* $\mathrm{Ell}_{(-1)}^{\times}$ *can be locally presented as* $|f(A)|^2$, *where* f *is holomorphic in* A.

The assertions 1. and 2. are consequences of Theorem 7.1 and of Lemma 7.4. The assertion 3. is obtained by consideration of a holomorphic function $(A, B) \to \det_\zeta(AB)$ for pairs (A, B) close to (A, A^*). By the multiplicativity property, Theorem 7.1, (7.4), this function possesses the property

$$\det_\zeta(A_1B_1)\det_\zeta(A_2B_2) = \det_\zeta(A_1B_2)\det_\zeta(A_2B_1) \ ,$$

i.e., the matrix $(f_{A,B})$, $f_{A,B} := \det_\zeta(AB)$, has rank one. Hence locally $f_{A,B} = f_1(A)f_2(B)$, where f_1 and f_2 are holomorphic. The restriction of $f_{A,B}$ to the diagonal $\{(A, A^*)\}$ is real-valued. Multiplying f_1, f_2 by appropriate positive constants we obtain the assertion 3..

Remark 7.1. The assertion of Proposition 7.5, *1.*, in the case of elliptic differential operators (of positive orders) was obtained in [Sch]. This result was one of the origins of the present subsection.

Lemma 7.6. *1. For* $A \in H$ *we have*

$$|\det|A = |\det_{Fr}(A)| \ .$$

2. For $A \in \mathrm{Aut}\, E$ *we have* $|\det|A = 1$.

We use here $|\det|$ defined by multiplicativity (Proposition 7.5, *2.*) for operators from $\mathrm{Ell}_{(-1)}^{0}$.

Remark 7.2. By this lemma and by Proposition 7.5 there is a Hermitian metric $\|\cdot\|_{\det}$ with *zero curvature* on the line bundle L over $G_{(-1)}$ associated with the $\mathbb{C}^\times$-bundle $\widetilde{G}_{(-1)}$ over $G_{(-1)}$ such that

$$|\det|A = \|d_1(A)\|_{\det} \ .$$

Proposition 7.6. *The holomorphic flat connection* $\nabla_{|\det|}$ *on* L *associated with the metric* $\|\cdot\|_{\det}$ *coincides with the connection* ∇ *defined before Proposition* 7.4.

One of possible proofs of this assertion follows from the invariance of the connections $\nabla_{|\det|}$ and ∇ with respect to the natural action on L of the group $\mathrm{Diff}(M, E)$.[2] This group is introduced in the proof of Lemma 7.4. Namely, any bi-invariant flat connection ∇_1 on the $\mathbb{C}^\times$-bundle $\widetilde{G}_{(-1)} \xrightarrow[p]{} G_{(-1)}$ defines a homomorphism of Lie algebras $a(\nabla_1) : \tilde{\mathfrak{g}}_{(-1)} \to \mathbb{C}$ such that $a(\nabla_1)|_{\mathbb{C}} = \mathrm{Id}$ (where $\mathbb{C} \xhookrightarrow[j]{} \tilde{\mathfrak{g}}_{(-1)}$, (6.2), is the central Lie subalgebra), and vice versa.

Thus $a\left(\nabla_{|\det|}\right) - a(\nabla)$ defines a homomorphism

$$b\colon \mathfrak{g}_{(-1)} \approx CS^0_{(-1)} \to \mathbb{C}\ .$$

This homomorphism vanishes on $CS^{-(n+1)}_{(-1)}$, $n := \dim M$. Let us proof by induction (in k) that there are no nonzero continuous linear functionals on $CS^{-k}_{(-1)}/CS^{-(k+1)}_{(-1)}$ invariant with respect to the natural action of $\mathrm{Diff}(M, E)$. The step of this induction is the assertion that the space of invariant continuous linear functionals

$$\left(\left(CS^{-k}_{(-1)}/CS^{-(k+1)}_{(-1)}\right)^*_{cont}\right)^{\mathrm{Diff}(M,E)}$$

is zero.

The quotient space $CS^{-k}_{(-1)}/CS^{-(k+1)}_{(-1)}$ is naturally isomorphic to the space of $\mathbb{R}^\times$-homogeneous functions (with the values in $\mathrm{End}\, E$) on $T^*M\backslash M$ of order $-k$. The space of continuous linear functionals on it is isomorphic via pairing by the noncommutative residue $(,)_{\mathrm{res}}$ to the space of $\mathbb{R}^\times_+$ homogeneous generalized functions f of order $k-n$ such that $f(x, -\xi) = (-1)^{k-n+1} f(x, \xi)$. (Here, $(-1)^{k-n+1} = (-1)^k$ as n is odd.) But there is only one (up to a constant factor) $\mathrm{Diff}(M, E)$-invariant generalized function with the values in $\mathrm{End}\, E$ on $T^*M\backslash M$. It is the identity operator Id. However, it corresponds to $k = n$ and has no appropriate central reflection symmetry. Hence there are no nonzero $\mathrm{Diff}(M, E)$-invariant continuous linear functionals on $CS^0_{(-1)}/CS^{-(n+1)}_{(-1)}$. Thus $b = 0$ and $\nabla = \nabla_{|\det|}$.

Remark 7.3. Let L_1 be the pullback of L to the complex subgroup $\mathrm{Aut}\, E$ of elliptic operators of zero order. Then L_1 is trivialized by the section d_1 and the norm of this section with respect to the pullback of $\|\cdot\|_{\det}$ is identically equal to 1 by Lemma 7.6. Hence the monodromy of L_1 with respect to ∇ is trivial by Proposition 7.6. This is equivalent to the assertion of Lemma 7.5.

7.3. Trace type functional for odd class operators. For $A \in CL^{\mathbb{Z}}_{(-1)}$ define $\mathrm{TR}_{(-1)}(A)$ as the value at zero of the function $\zeta_{A,C}(s) = \mathrm{TR}(AC^{-s})$, where $C \in \mathrm{Ell}^{2k}_{(-1)}$, $k \in \mathbb{Z}_+$, is sufficiently close to a positive definite self-adjoint one. The correctness of the definition above is the content of the following lemma.

[2]Another proof of Proposition 7.6 is contained in [KV], Proposition 6.12 (p. 108 of the preprint).

Lemma 7.7. *1. $\zeta_{A,C}(s)$ is holomorphic at $s=0$.*
2. $\zeta_{A,C}(0)$ is independent of C.

The assertion 1. follows from the equalities

$$\mathrm{Res}_{s=0}\left(\zeta_{A,C}(s)\right)=\frac{1}{2k}\,\mathrm{res}(A)=0\ .$$

(These equalities hold by Theorem 3.1 and by Lemma 7.3.) The assertion 2. is a consequence of Theorem 3.1, (3.2), applied to a holomorphic at $s=0$ family $\left(AC^{s/\operatorname{ord}C}-AB^{s/\operatorname{ord}B}\right)/s$. Namely

$$\left(\zeta_{A,C}(s)-\zeta_{A,B}(s)\right)|_{s=0}=\left(\sigma(A),\frac{\log_{(\theta)}\sigma(C)}{\operatorname{ord}C}-\frac{\log_{(\theta)}\sigma(B)}{\operatorname{ord}B}\right)_{\mathrm{res}}\ . \tag{7.9}$$

Here, $L_{(\theta)}$ are admissible spectral cuts close to $\mathbb{R}_-$. The difference of the logarithms in (7.9) belongs to $CS^0_{(-1)}$ by Proposition 7.2, (7.1). Thus $\zeta_{A,C}(0)=\zeta_{A,B}(0)$.

Lemma 7.8. *1. The linear functional $\mathrm{TR}_{(-1)}$ coincides with the usual trace Tr on $CL^{-(n+1)}_{(-1)}$. Hence it is equal to zero on $\mathfrak{h}^{(1)}\subset CL^{-(n+1)}_{(-1)}$.*
2. The induced by $\mathrm{TR}_{(-1)}$ linear functional on $\tilde{\mathfrak{g}}_{(-1)}$ is equal to the splitting homomorphism $a(\nabla)\colon\tilde{\mathfrak{g}}_{(-1)}\to\mathbb{C}$ defined in the proof of Proposition 7.6, see also (7.6).
3. For $A,B\in CL^{\mathbf{Z}}_{(-1)}$ we have

$$\mathrm{TR}_{(-1)}([A,B])=0\ , \tag{7.10}$$

i.e., $\mathrm{TR}_{(-1)}$ is a trace type functional on $CL^{\mathbf{Z}}_{(-1)}$.

The first assertion follows from Theorem 3.1, *1.* (i.e., from the equality $\mathrm{TR}=\mathrm{Tr}$ on trace class operators). The proof of the second assertion is the same as the proof of Proposition 7.6. (In that proof we use only the invariance under the natural action of $\mathrm{Diff}(M,E)$.)

The trace property (7.10) in the case $A,B\in CL^0_{(-1)}$ follows from the assertion 2. as $a(\nabla)$ is a Lie algebra homomorphism. (We don't give here a proof for A or B of positive orders and don't use in this text the property (7.10) for general A, B.)

For an operator $C\in CL^0_{(-1)}$ we can define an entire function

$$\zeta_C^{\mathrm{TR}_{(-1)}}(s):=\mathrm{TR}_{(-1)}(\exp(-sC))\ . \tag{7.11}$$

(Here we use Lemma 7.2.) The determinant of $\exp C\in\mathrm{Ell}^0_{(-1)}$ corresponding to this zeta-function is equal to

$$\det{}_{(-1)}(\exp C,C):=\exp\left(-\partial_s\zeta_C^{\mathrm{TR}_{(-1)}}(s)|_{s=0}\right)=\exp\left(\mathrm{TR}_{(-1)}C\right)\ . \tag{7.12}$$

The last equality in (7.12) follows from Theorem 3.1, *3.*, applied to a holomorphic in two variables family $\exp(-zC)\cdot B^{-s}$ where $B\in\mathrm{Ell}^2_{(-1)}$ is positive definite. Namely, by this theorem $\operatorname{TR}(\exp(-zC)B^{-s})$ is holomorphic in (s,z) for $0<|s|<1/2$. Also by this thorem and by Lemma 7.3, (7.3), there are no singularities at $s=0$. So

$$\partial_z \operatorname{TR}_{(-1)}(\exp(-zC))|_{z=0} = \left(\partial_z \operatorname{TR}\left(\exp(-zC)B^{-s}\right)|_{z=0}\right)\Big|_{s=0} = -\operatorname{TR}_{(-1)} C \ .$$

Remark 7.4. The following statement holds . Let for $A \in \mathrm{Ell}^{2k}_{(-1)}$, $k \neq 0$, its principal symbol $\sigma_{2k}(A)$ possess a spectral cut θ (i.e., let the Agmon-Nirenberg condition hold for A). Then for $m \in \mathbb{Z}$

$$\zeta_{A,\theta}(-m) = \operatorname{TR}_{(-1)}(A^m) \ .$$

Let $A \in \mathrm{Ell}^0_{(-1)}$ and let $\log \sigma(A) =: x \in CS^0_{(-1)}$ exist. Then a canonical determinant of A is defined by

$$\det{}_{(-1)}(A,x) = d_1(A) d_{0,(-1)}(\sigma(A),x)^{-1} \ , \tag{7.13}$$

where

$$d_{0,(-1)}(\sigma(A),x) := d_1(\exp X)\exp\left(-\operatorname{TR}_{(-1)} X\right)$$

for any $X \in CL^0_{(-1)}$ with $\sigma(X)=x$. The proof of independence of $d_{0,(-1)}(\sigma(A),x)$ of a choice of X (with given x) uses the equality

$$d_1(\exp X \exp(-X_1)) = \det{}_{Fr}(\exp X \exp(-X_1))$$

for $\sigma(X) = \sigma(X_1) = x$, and the assertion for $\det_{(-1)}$ analogous to (2.1), Lemma 2.1. (See [KV], Lemma 6.12.)

Remark 7.5. 1. The projective microlocal Agmon-Nirenberg condition (Section 5.1) is sufficient for the existence of $\log\sigma(A)$ in $CS^0_{(-1)}$.

2. An element $d_{0,(-1)}(\sigma(A),x)$ (in (7.13) can be defined by any homotopy class of paths in $G_{(-1)}$ from Id to $\sigma(A)$, namely, using the flat connection ∇ on the $\mathbb{C}^\times$-bundle $\tilde{G}_{(-1)} \underset{p}{\rightarrow} G_{(-1)}$.

Lemma 7.9. *The determinant* $\det_{(-1)}(A,x)$ *for an operator* $A \in \mathrm{Ell}^0_{(-1)}$ *close to positive definite ones and for* x *defined by a spectral cut* $\mathbb{R}_-$ *is equal to the determinant* $\det(A)$ *introduced in (7.5) with the help of the multiplicative property (7.4).*

This assertion follows from Lemma 7.8.

8. Open problems

1. To construct the determinant group $\tilde{G}$ globally in terms of symbols (i.e., in particular, without using analytic continuations and the Fredholm determinants).

This problem is not solved even for the connected component $\tilde{G}_c$ of Id $\in \tilde{G}$. By Theorem 6.1 we know a description of the Lie algebra of $\tilde{G}_c$ in terms of symbols. Thus we can reconstruct the universal cover $\mathbf{G}$ of $\tilde{G}_c$,

$$\tilde{G}_c \simeq \mathbf{G}/\Gamma \ ,$$

where Γ is a discrete subgroup of the center $Z(\mathbf{G})$. We have some information about Γ because $\tilde{G}_c/\mathbb{C}^\times = \mathrm{SEll}_0$ is the connected component of Id of the group of elliptic symbols. It is enough to have a description of the restrictions of the $\mathbb{C}^\times$-central extension $\tilde{G}_c \underset{p}{\to} G_c$ to one-parameter compact subgroups $T_q := \exp(2\pi i t q)$, where $q \in CS^0 = \mathfrak{g}_0 \subset \mathfrak{g}$ is a projector in symbols, $q^2 = q$, and $0 \le t \le 1$.

These subgroups are generators of the fundamental group $\pi_1(G_c, \mathrm{Id})$. The preimage $\tilde{T}_q := p^{-1}(T_q)$ of T_q in $\tilde{G}_c$ is a 3-dimensional abelian Lie group,

$$1 \to \mathbb{C}^\times \to \tilde{T}_q \underset{p}{\to} S^1 (= T_q) \to 1 \ .$$

To describe this central extension, we choose an element q_1 of the Lie algebra Lie $\left(\tilde{T}_q\right) \subset \tilde{\mathfrak{g}}$ of $\tilde{T}_q$, $q_1 = q \mod \mathbb{C}$. Then the nonzero complex number

$$\exp(2\pi i q_1) =: c = c(q_1) \in \mathbb{C}^\times \subset \tilde{T}_q$$

defines $\tilde{T}_q$ because of the identification

$$\tilde{T}_q \simeq \mathbb{C}^\times \times \mathbb{R}/(c, 1) \cdot \mathbb{Z} \ .$$

To define such an element q_1, it is enough to choose $x \in \mathfrak{g} = S_{\log}$ with $\operatorname{ord} x = 1$ and set $q_1 := \Pi_{\tilde{x}} q$. (See Proposition 6.1, *1.*, and the proof of Theorem 6.1.)

The number $c(\Pi_{\tilde{x}} q)$ can be expressed in terms of a spectral invariant of a pair (P, X) of a PDO-projector P with $\sigma(P) = q$ (such a projector always exists) and of an element $X \in \mathfrak{ell}$ with $\sigma(X) = x$. Namely

$$c(\Pi_{\tilde{x}} q) = \exp(-2\pi i f(P, X)) \ , \tag{8.1}$$

where $f(P, X) = \mathrm{TR}(P \exp(sX))|_{s=0} (\mod \mathbb{Z})$. (Note that by the equality (8.1) the element $f(P, X) \in \mathbb{C}/\mathbb{Z}$ is independent of choices of P and X.) The equality (8.1) is proved in [KV], Proposition 7.1. Hence $f(P, X)$ is a function of $q = \sigma(P)$, $x = \sigma(X)$. We call it the *generalized spectral asymmetry.* If $\exp X$ is self-adjoint and P is the orthogonal projector to the linear subspace spanned by the eigenvectors of $\exp X$ with positive eigenvalues, this invariant is simply expressed via the spectral asymmetry of $\exp X$, [APS]. However $f(P, X)$ cannot be obtained as a value (taken modulo $\mathbb{Z}$) of an integral of a local in symbols $\sigma(P)$, $\sigma(X)$ expression. Thus the description of $\tilde{G}_c$

in terms of symbols reduces to computation of the generalized spectral asymmetry in terms of symbols (and without using of liftings to PDOs).

2. To generalize constructions and results of this paper to the case of elliptic complexes.

3. To compute the homomorphism (7.7) for odd-dimensional manifolds. (This is the monodromy of the holomorphic determinant for odd class operators.)

Estimates of Proposition 7.4 for the torsion of $\mathrm{Image}(\mathrm{mon}_\nabla)$ are probably not the best possible. They were produced by bounding the orders for the elements in $K^0(P^*M)/\pi^*K^0(M)$.

4. To investigate analytic properties of entire functions $\zeta_C^{\mathrm{TR}_{(-1)}}(s)$ for $C \in CL^0_{(-1)}$ defined in (7.11).

Do these functions have representations in terms of Dirichlet series?

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