CONTINUED FRACTIONS, THE GROUP $GL(2, \mathbb{Z})$, AND PISOT NUMBERS

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Abstract. We use subsemigroups of the group of invertible integer matrices to prove some properties of continued fractions, generalized golden sections, generalized Fibonacci and Lucas numbers. Some properties of special recurrent sequences are studied. Connections between continued fractions and Pisot numbers are investigated. We suggest also some unsolved problems.

Introduction

In this paper we investigate interconnections between continued fractions, some subsemigroups of the group $GL(2, \mathbb{Z})$, and Pisot numbers. The main tool of this investigation is the matrix calculus which permits from the author's point of view to explain interconnections between these notions much better. Obtained structural results on subsemigroups of the group $GL(2, \mathbb{Z})$ are used to study properties of generalized Fibonacci and Lucas numbers. Also we formulate some unsolved problems in the paper.

The paper is divided into eight sections. In the first section we give a necessary information on continued fractions and corresponding matrices, investigate some properties of these matrices. In the second section we discuss Pisot numbers, polynomials and matrices, prove some results on Pisot polynomials. In the third section we prove a series of results on recurrent sequences. On this ground we give a new proof of the classical Pisot-Vijayaraghavan theorem in the fourth section. Some properties of subsemigroups of the group $GL(2, \mathbb{Z})$, connected with continued fractions and Pisot numbers, are investigated in the fifth section. In the sixth section we study asymptotic properties of sequences of linear-fractional transformations, corresponding to infinite continued fractions. In the seventh section we find some relations between the entries of natural powers of (2×2)-matrices. The last, eighth section, is devoted to an investigation of properties of generalized golden sections, Fibonacci and Lucas numbers. Let us remark that (generalized) golden sections are Pisot numbers.

In the course of the article we use the notation $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of rational, real, and complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of integer rational numbers and nonnegative integer rational numbers respectively. The set of natural numbers is denoted by the symbol $\mathbb{N}$ (we have in mind that $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$). The symbol $\{u_k\}$ will denote a sequence, indexed by numbers from $\mathbb{Z}_+$. By A^t we denote the transposition of a matrix A, and by $\mathrm{tr}(B)$ we denote the trace of the matrix or the operator B. Another notation is explained in the corresponding parts of the text.

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1. Continued fractions and matrices generated by them

A simplest (regular) continued fraction is an expression of the form

$$q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ddots}} \tag{1}$$

corresponding to a finite or an infinite sequence $q_n, 0 \leq n < \omega + 1$, where q_0 is an integer number, while $q_1, q_2, \dots$ are natural numbers; in the case $0 < \omega < +\infty$ one suppose that $q_\omega \geq 2$. If $\omega < +\infty$, then the fraction (1) has quite definite value, denoted by the symbol $[q_0; q_1, q_2, \dots, q_\omega]$. It is known [21] that in the case of infinite sequence there is a finite limit

$$[q_0; q_1, q_2, \dots] := \lim_{n\to\infty} [q_0; q_1, \dots, q_n]. \tag{2}$$

Here the initial segment $q_0, q_1, \dots, q_n$ of a finite or an infinite sequence $q_0, q_1, q_2, \dots$ defines the *n-th convergent (of continued) fraction*

$$\frac{P_n}{Q_n} := [q_0; q_1, \dots, q_n] \tag{3}$$

of the number $[q_0; q_1, \dots]$; the numerator and the denominator of the fraction on the left are supposed to be coprime integer numbers with a positive denominator. On the ground of the Leibnitz convergent criterion for alternating series and formulas (4), (13) it follows easily the existence of the limit (2) and the fact that this limit lies between every two consequent convergent fractions. Conversely [21], any real number r is uniquely represented in the form of the fraction (1), where numbers q_n are successive non-complete quotients of the pair $(r, 1)$ by analogy with the Euclidean algorithm for a pair of natural numbers. Thus, there is one-to-one correspondence between real numbers and regular continued fractions of the form (1), which simultaneously with their value (2) are denoted by the symbol $[q_0; q_1, q_2, \dots]$.

One can read on continued fractions and their generalizations in books and papers [1], [2], [3], [4], [5], [8], [10], [12], [17], [19], [21], [22], [23], [31], [36], [38]. A simple geometric construction of convergent (and closely related with them intermediate) fractions of a given real number as vertices (respectively, integer valued points in the interior of sides) of so-called Klein polygons is given in the book [15]; there is also a picture of this construction for golden section there. Continued fractions have various applications. Connections of Pell equations with continued fractions are considered in [6]. It follows easily from the Klein construction mentioned above that convergent (sometimes intermediate) fractions are the best approximations for a given number among all rational numbers with a given upper bound for denominators. This fact is applied in music (the creation of nowadays existing the logarithmically uniform musical scale with twelve tones by German scientist and musician Andreas Werkmeister about 1700 year, which was rejected by D. Diderot and was supported by "Well tempered clavier" of J.S. Bach, see [9], [24]) and in astronomy [13]. In the book [20] the moment problem is solved with the help of functional continued fractions.

One can write well-known recurrent formulas

$$P_{n+1} = q_{n+1}P_n + P_{n-1}, \quad Q_{n+1} = q_{n+1}Q_n + Q_{n-1} \tag{4}$$

for numerators and denominators of convergent fractions in the following matrix form

$$\begin{pmatrix} P_n & Q_n \\ P_{n+1} & Q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & q_{n+1} \end{pmatrix} \begin{pmatrix} P_{n-1} & Q_{n-1} \\ P_n & Q_n \end{pmatrix}. \tag{5}$$

Continued fractions of the form (1) with $q_0 \geq 1$ or non-finishing by q_0 if $q_0 = 0$ (with the reserve mentioned above in the case of a finite fraction) represent all positive numbers (in the interval $(0, 1)$, if $q_0 = 0$). We suppose that these conditions are satisfied in what follows. Then it is convenient to suppose that

$$\begin{pmatrix} P_{-1} & Q_{-1} \\ P_0 & Q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ q_0 & 1 \end{pmatrix}. \tag{6}$$

For natural numbers $q, q_1, \dots, q_n$, we introduce the notation

$$M(q) := \begin{pmatrix} 0 & 1 \\ 1 & q \end{pmatrix}; \quad N(q) := \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, \tag{7}$$

$$M(q_1, \dots, q_n) := M(q_n) \dots M(q_1). \tag{8}$$

Evidently,

$$\begin{vmatrix} 0 & 1 \\ 1 & q \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 \\ q & 1 \end{vmatrix} = 1. \tag{9}$$

It follows from formulas (5), (6), (7), and (9) that, under condition $n \geq 0$,

$$\begin{pmatrix} P_n & Q_n \\ P_{n+1} & Q_{n+1} \end{pmatrix} = M(q_{n+1}) \dots M(q_1) \quad \text{when} \quad q_0 = 0, \tag{10}$$

$$\begin{pmatrix} P_n & Q_n \\ P_{n+1} & Q_{n+1} \end{pmatrix} = M(q_{n+1}) \dots M(q_1) N(q_0), \quad \text{when} \quad q_0 \geq 1, \tag{11}$$

$$\begin{vmatrix} P_n & Q_n \\ P_{n+1} & Q_{n+1} \end{vmatrix} = P_n Q_{n+1} - Q_n P_{n+1} = (-1)^{n+1}, \tag{12}$$

i.e. matrices (10) and (11) (with nonnegative entries) are contained in the group $GL(2,\mathbb{Z})$ of all nondegenerate integer (2×2)-matrices, and the well-known formula for the difference of successive convergent fractions is fulfilled:

$$\frac{P_n}{Q_n} - \frac{P_{n+1}}{Q_{n+1}} = \frac{(-1)^{n+1}}{Q_n Q_{n+1}}. \tag{13}$$

By transposing the matrix (10) and using relations (3) and (7), we get, as a corollary of symmetry of the matrix $M(q)$, the equality

$$\frac{Q_n}{Q_{n+1}} = [0; q_{n+1}, \dots, q_1]. \tag{14}$$

Definition 1. *We shall say that a (2×2)-matrix is non-decreasing by rows (columns), if the difference of the second and the first columns (respectively, of the second and the first rows) is a vector-column (vector-row) with nonnegative coordinates. Analogously we define a matrix which is non-increasing by rows (columns).*

The main result of this section is

Theorem 1. *An integer matrix*

$$Z := \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with nonnegative entries is a matrix of the form (10) or (11) with the uniquely determined natural numbers $q_0, q_1, ..., q_{n+1}$ if and only if the matrix Z is non-decreasing by columns and $z_{11}z_{22} - z_{12}z_{21} = \pm 1$. Moreover, the matrix Z has the form (10) if and only if it is non-decreasing by rows also.

Remark 1. *The last statement of this theorem has been proved in the paper* [34].

Remark 2. *Conditions of Theorem 1 have simple geometric sense:*

1) Every vector-column (and vector-row in the last case) of the matrix lies in the octant $0 \leq x \leq y$;

2) Vector-columns (or vector-rows) of the matrix constitute a basis of integer lattice on the plane x, y over the ring of integer numbers.

Matrices and facts indicated here are used further for simple proofs of some properties of continued fractions and generalized golden sections and also Fibonacci and Lucas numbers.

Proof of Theorem 1. Let us prove at first the necessity. The unimodularity of matrices under consideration (i.e. the property 2) of Remark 2) follows from (9). The property 1) of Remark 2 for vector-columns of a matrix follows immediately from the formula (4). Then the same property for vector-rows of a matrix of the form (10) is obtained by transposing of this matrix and with using the well-known rule for transposition of a matrix product and, also, a symmetry of the matrices $M(q)$.

Let us prove the sufficiency by induction on the nonnegative number $k = k(Z) = z_{11}z_{12}$. Let suppose that the integer (2×2)-matrix Z is unimodular and non-decreasing by columns. Let $k = 0$. Then two cases are possible.

A) $z_{11} = 0$. Then $z_{12} = z_{21} = 1$ and $Z = M(z_{22})$.

B) $z_{12} = 0$. Then $z_{11} = z_{22} = 1$ and $Z = N(z_{21})$.

Evidently, in these cases the representation of the matrix in the form (10) or (11) is unique.

Let suppose that for all matrices with $0 \leq k < l$ the statement is proved and $k(Z) = k = l$. Obviously, all entries of the matrix Z are positive and at least one of inequalities $z_{11} \leq z_{21}, z_{12} \leq z_{22}$ is strong.

Set $q = \min([z_{21}/z_{11}], [z_{22}/z_{12}])$, where by $[\alpha]$ we denote the integer part of a number α. Evidently, $q \geq 1$. Let us consider the matrix

$$Z' = M(q)^{-1}Z = \begin{pmatrix} -q & 1 \\ 1 & 0 \end{pmatrix} Z = \begin{pmatrix} z_{21} - qz_{11} & z_{22} - qz_{12} \\ z_{11} & z_{12} \end{pmatrix}.$$

It follows from the definition of the number q that all entries of the matrix Z' are nonnegative, moreover, at least one of inequalities

$$1)\, z'_{12} < z'_{22}, \quad \text{or} \quad 2)\, z'_{11} < z'_{21}.$$

is fulfilled. In the first case we have $z'_{11} \leq z'_{21}$. Otherwise

$$|detZ'| \geq (z_{21} - qz_{11})z_{12} - z_{11}(z_{22} - qz_{12}) \geq$$

$$(z_{11} + 1)(z_{22} - qz_{12} + 1) - z_{11}(z_{22} - qz_{12}) \geq z_{11} + 1 \geq 2,$$

what is impossible in view of unimodularity of the matrix Z'. In the second case we must have $z'_{12} \leq z'_{22}$. Otherwise

$$|detZ'| \geq (z_{22} - qz_{12})z_{11} - z_{12}(z_{21} - qz_{11}) \geq$$

$$(z_{12} + 1)(z_{21} - qz_{11} + 1) - z_{12}(z_{21} - qz_{11}) \geq z_{12} + 1 \geq 2,$$

what is impossible. Thus the matrix Z' is unimodular, non-decreasing by columns, and consists of nonnegative entries, furthermore $k(Z') < k(Z)$ because one of conditions 1) or 2) is satisfied. Then by the inductive assumption, Z' is uniquely represented as the product (10) or (11). Therefore, the matrix $Z = M(q)Z'$ is also represented in the form (10) or (11). The uniqueness of such representation follows from the fact that for a natural q_1 and the matrix $Z_1 = M(q_1)^{-1}Z$, at least one of its entries is negative if $q_1 > q$; if $q_1 < q$, then for the matrix Z_1 the condition of non-decreasing by columns is false.

If the matrix $Z \in GL(2,\mathbb{Z})$ is non-decreasing by columns and rows simultaneously, then by the proved above, it is uniquely represented in the form of the product (10) or (11). Moreover, the last variant is impossible. Otherwise the transposed matrix Z^t is represented in the form of the product

$$Z^t = N(q_0)^t M(q_{n+1}) \dots M(q_1), q_0 \geq 1.$$

Hence the matrix Z^t is not non-decreasing by columns, which is impossible. The theorem is proved. ■

2. Pisot numbers, polynomials, and matrices

In this section we consider some facts, connected with one special class of real numbers, namely, Pisot numbers. Besides an account of a part of the classical theory, we prove also some new results, which have an independent interest and are necessary for further presentations.

Definition 2. *An integer algebraic number $\lambda > 1$ is called a Pisot number if all its conjugates, different from λ, lie in the open disc $|z| < 1$ on the complex plane.*

We note that Pisot numbers are also called Pisot-Vijayaraghavan or P-V numbers. The foundations of the theory of these numbers have been established in papers of Ch. Pisot (1910-1984), T. Vijayaraghavan (1902-1955), and R. Salem (1898-1963). A classical presentation of this theory one can find in books [14] and [26].

Later on we shall need also the following two definitions.

Definition 3. *A polynomial $f(z) = z^n + a_{n-1}z^n + \cdots + a_1 z + a_0$ with integer rational coefficients is called a Pisot polynomial, if it is irreducible and one of its roots η has the property that η or $-\eta$ is a Pisot number.*

It is clear that any Pisot number η is a root of the unique Pisot polynomial $f(z)$, because every common root of two polynomials with rational coefficients is also a root of their greatest common divisor. Moreover, the number $-\eta$ is a root of another Pisot polynomial $\tilde{f}(z) = (-1)^{\deg f(z)} f(z)$. Thus there is an evident one-to-one correspondence between Pisot numbers and pairs of Pisot polynomials.

Definition 4. *A square matrix A with integer rational entries is called a Pisot matrix, if its characteristic polynomial is a Pisot polynomial.*

Every Pisot polynomial is characteristic polynomial of some (non-unique) Pisot matrix. This follows, for example, from Lemma 3 below.

The significance of Pisot numbers in the theory of diophantine approximations one can see from results below. For a real number y, we shall denote by $\|y\|$ the distance from y to a nearest integer number, i.e.

$$\|y\| = \min\{|y-k| \mid k \in \mathbb{Z}\}.$$

Some information on the arithmetic nature of a real number $\lambda > 1$ one can obtain by the investigation of asymptotic properties of the sequences $\{a\lambda^n\}$, $n \in \mathbb{Z}_+$, for various real numbers a.

Definition 5. *For a real number $\lambda > 1$, by a Pisot set of this number we mean the set*

$$\mathcal{P}_\lambda = \{a \in \mathbb{R} \mid \|a\lambda^n\| \to 0,\ n \to \infty\}.$$

For a given algebraic number λ, we consider the finite extension $\mathbb{Q}(\lambda)$ of the field $\mathbb{Q}$ and denote by $\mathrm{Tr}(y)$ *algebraic trace* of a number $y \in \mathbb{Q}(\lambda)$, i.e. the trace of the linear map $x \to x \cdot y$ in the linear space $\mathbb{Q}(\lambda)$ over the field $\mathbb{Q}$ ([7], Chapter VI, §47). An effective way for calculating of this trace is indicated in Lemma 7.

Theorem (Pisot-Vijayaraghavan). *An algebraic number $\lambda > 1$ is a Pisot number if and only if $\mathcal{P}_\lambda \neq \{0\}$. In this case*

$$\mathcal{P}_\lambda = \{a \in \mathbb{Q}(\lambda) \mid \mathrm{Tr}(a\lambda^n) \in \mathbb{Z} \textit{ for sufficiently large } n\}.$$

Thus, this theorem gives a complete description of algebraic numbers $\lambda > 1$ with the set $\mathcal{P}_\lambda \neq \{0\}$.

In 1935 J.F. Koksma [32] had shown that for almost all numbers $\lambda > 1$, the sequence $\{\lambda^n\}$ is uniformly distributed by $(mod\,1)$ (about uniformly distributed sequences see, for example, [10]). Pisot numbers $\lambda > 1$ in view of the theorem above (which have been proved in the pioneer work [37]) are exceptional in this sense, because one can easily prove that $1 \in \mathcal{P}_\lambda$ (see the proof of the Pisot-Vijayaraghavan theorem in the section 4).

It is interesting the following

Question 1 (Ch. Pisot). *Are there transcendental numbers $\lambda > 1$ with the set $\mathcal{P}_\lambda \neq \{0\}$?*

Some results in this direction have been established by Ch. Pisot himself and others [14, 26]. Nevertheless, there is no final answer to this question up to now.

It is appropriate to give here three astonishing characterizations of Pisot numbers, which are not connected with any algebraic condition.

The first one is connected with the harmonic analysis. A subset $E \subset (0,1)$ is called a *set of uniqueness* (for a trigonometric series), if condition

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n x + b_n \sin 2\pi n x) = 0, \quad x \in (0,1), \quad x \notin E,$$

for an ordinary trigonometric series with real coefficients implies that $a_n = b_n = 0$ for all n. For a real number $\lambda > 2$, consider (generalized) Cantor set (of measure zero)

$$E_\lambda := \left\{ x = (\lambda - 1) \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k} \,\middle|\, \varepsilon_k = 0, 1 \right\}.$$

In the case $\lambda = 3$ one get the famous Cantor set. The Salem-Zigmund theorem in [26] states that the set E_λ is a set of uniqueness if and only if λ is a Pisot number.

The second characterization is connected with numbers represented by polynomials of the number λ with bounded integer nonnegative coefficients. For a natural number k denote by $0 =: y_0^{(k)} < y_1^{(k)} < \dots$ an increasing sequence of all real numbers y, which admit a representation of the form

$$y = \varepsilon_1 \lambda + \dots + \varepsilon_i \lambda^i; \quad i \geq 1; \quad \varepsilon_1, \dots, \varepsilon_i \in \{0, 1, \dots, k\}.$$

Next we define the difference sequence of these numbers $u_n^{(k)} := y_{n+1}^{(k)} - y_n^{(k)}, n \geq 0$. Theorem 1 in [29] states that a real number $\lambda \in (1,2)$ is a Pisot number if and only if $\varliminf_{n \to \infty} u_n^{(k)} > 0$ for all natural numbers k.

The third characterization is connected with the finite automaton theory. Let us given a real number λ, a natural number c, and set of integer numbers $C = \{-c, \dots, 0, \dots, c\}$. Consider set $Z(\lambda, c)$ of infinite words in $C^{\mathbb{Z}_+}$, corresponding to 0, i.e.

$$Z(\lambda, c) = \left\{ s = \{s_n\}_{n \geq 0} \in C^{\mathbb{Z}_+} \,\middle|\, \sum_{n=0}^{\infty} s_n \lambda^{-n} = 0 \right\}.$$

It is proved in paper [25] that the number λ is a Pisot number if and only if the set $Z(\lambda, c)$ is recognizable by a finite automaton for every natural number c.

The structure of the set S of all Pisot numbers at this time is thoroughly investigated [26]. By the R. Salem theorem, the set S is closed [14]. It is proved in paper [27] that the order type of the set S is the ordered sum $\sum_{n=1}^{\infty} a_n$, where $a_1 = \omega + 1 + \omega^*$, and by induction, $a_{n+1} = a_n \omega + 1 + (a_n \omega)^*$. The smallest Pisot number θ is a positive root of the equation $z^3 - z - 1 = 0$. Nevertheless, there are many unsolved problems, connected with Pisot numbers. For example, there is no convenient criterion, which permits to define by coefficients of some polynomial, whether it is a Pisot polynomial or not. We prove some results in this direction.

We associate with arbitrary Pisot polynomial $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ the vector $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n$. Thus, the set of Pisot polynomials of degree n corresponds to some subset of vectors in $\mathbb{Z}^n$, which we denote by $\mathcal{P}(n)$. Further we shall describe sets $\mathcal{P}(2)$ and $\mathcal{P}(3)$.

By symbols $\Re(z)$ and $\Im(z)$ we shall denote respectively the real and the imaginary parts of a complex number $z \in \mathbb{C}$, the symbol $\operatorname{sign}(x)$ denotes the sign of a number $x \in \mathbb{R}$.

Lemma 1. *A polynomial $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ with integer rational coefficients is a Pisot polynomial if and only if $a_0 \neq 0$ and $f(z)$ has one root in the set $|z| > 1$ and $n-1$ roots in the disc $|z| < 1$.*

Proof. If $f(z)$ is a Pisot polynomial, then it is irreducible over the field of rational numbers, in particular, $a_0 \neq 0$. Moreover, one of its roots λ ($|\lambda|$ is Pisot number) is greater than 1 by its absolute value. By definition of Pisot number, the roots of $f(z)$, which are not equal to λ, lie in the disc $|z| < 1$.

Suppose now that the polynomial $f(z)$ has one root in the set $|z|>1$ and $n-1$ roots in the disc $|z|<1$, and besides this, $a_0\neq 0$. We shall show that $f(z)$ is a Pisot polynomial. For this it's enough to show that $f(z)$ is irreducible over the field $\mathbb{Q}$. Really, if $f(z)$ is irreducible, then the absolute value of its unique root lying in the set $|z|>1$, is a Pisot number. By the condition of the lemma $f(0)=a_0\neq 0$, hence all roots of the polynomial $f(z)$ are not equal to 0. Recall that every polynomial with integer rational coefficients is irreducible over rational numbers if and only if it is irreducible over integer numbers ([7], Chapter V, §30). Let $f(z)=f_1(z)\cdot f_2(z)$, where $f_i(z)$ have integer coefficients. Evidently, absolute values of older coefficients of these polynomials are equal to 1. Since one of polynomials $f_i(z)$ has only roots z_k such that $0<|z_k|<1$, then its constant term is not an integer number. ■

Theorem 2. *A polynomial $f(z)=z^n+a_{n-1}z^n+\cdots+a_1z+a_0$ with integer rational coefficients is a Pisot polynomial if and only if $a_0\neq 0$ and polynomial*

$$g(t)=b_nt^n+b_{n-1}t^{n-1}+\cdots+b_1t+b_0=(t-1)^nf\left(\frac{t+1}{t-1}\right)$$

with integer rational coefficients has one root in the half-plane $\Re(t)>0$ and $n-1$ roots in the half-plane $\Re(t)<0$.

Proof. The linear-fractional map $z=F(t)=\frac{t+1}{t-1}$ is an involution on the extended complex plane and translates sets $\Re(t)<0$ and $\Re(t)>0$ to sets $|z|<1$ and $|z|>1$ respectively.

Now let $f(z)=z^n+a_{n-1}z^n+\cdots+a_1z+a_0$ be a Pisot polynomial. According to Lemma 1, the condition $a_0\neq 0$ is fulfilled. Furthermore, $f(z)$ has $n-1$ roots in the disc $|z|<1$, and one root (whose absolute value is a Pisot number) in the set $|z|>1$. Taking into account properties of the map F, we obtain that the polynomial

$$g(t)=(t-1)^nf\left(\frac{t+1}{t-1}\right)$$

has one root in the half-plane $\Re(t)>0$ and $n-1$ roots in the half-plane $\Re(t)<0$.

Let us prove the converse statement. If the polynomial $g(t)$ has one root in the half-plane $\Re(t)>0$ and $n-1$ roots in the half-plane $\Re(t)<0$, then taking one more into account properties of the map F, we get that the polynomial $f(z)$ has one root in the set $|z|>1$ and $n-1$ roots in the disc $|z|<1$. Besides this, $a_0\neq 0$, therefore, by Lemma 1, $f(z)$ is a Pisot polynomial. ■

The theorem mentioned above gives an effective tool for searching Pisot polynomials, because the problem of finding the quantity of complex roots of the polynomial $g(t)$, satisfying the condition $\Re(t)>0$ (and condition $\Re(t)<0$, if we apply involutive substitution $s=-t$), is well-known, and is solved by classical Routh-Hurwitz theorem ([11], Chapter XVI). Let us apply Theorem 2 for a description of Pisot polynomials of degrees 2 and 3.

Theorem 3. *Let us consider the set*

$$\mathcal{P}(2)=\{(p,q)\in\mathbb{Z}^2\,|\,f(z)=z^2+pz+q \text{ is a Pisot polynomial}\}.$$

Then

$$\mathcal{P}(2)=\{(p,q)\in\mathbb{Z}^2\,|\,|p|>|q+1|,\ q\neq 0\}.$$

Proof. According to Theorem 2, a polynomial $f(z)=z^2+pz+q$ is a Pisot polynomial if and only if $q\neq 0$, and the polynomial

$$g(t)=(t-1)^2f\left(\frac{t+1}{t-1}\right)=(1+p+q)t^2+(2-2q)t+(1-p+q)$$

has one root in both half-planes $\Re(t)>0$ and $\Re(t)<0$. But the last condition is equivalent to the inequality $(1+p+q)(1-p+q)<0$, whence we get the statement of the theorem. ■

Theorem 4. *Let us consider the set*

$$\mathcal{P}(3) = \{(p,q,r) \in \mathbb{Z}^3 \,|\, f(z) = z^3 + pz^2 + qz + r \text{ is a Pisot polynomial}\}.$$

Then $\mathcal{P}(3) = M_1 \cup M_2 \cup M_3$, *where*

$$M_1 = \{(p,q,r) \in \mathbb{Z}^3 \,|\, p = 3r,\, q = 3,\, |r| > 1\},$$

$$M_2 = \{(p,q,r) \in \mathbb{Z}^3 \,|\, (1+p+q+r)(3-p-q+3r) < 0,\, |p+r| > |q+1|,\, r \neq 0\},$$

$$M_3 = \{(p,q,r) \in \mathbb{Z}^3 \,|\, (1+p+q+r)(3-p-q+3r) > 0,\, |p+r| > |q+1|,\, 1-q+pr-r^2 > 0,\, r \neq 0\}.$$

One could prove the stated theorem with the help of the Routh-Hurwitz theorem, but we shall prove it by using the Cauchy index for rational real-valued functions ([11], Chapter XVI).

For every real-valued rational function $R(x)$, there is the expansion ([7], Chapter V, §36):

$$R(x) = \sum_{k=1}^{p} \sum_{l=1}^{n_k} \frac{A_l^k}{(x-\alpha_k)^l} + R_1(x), \tag{15}$$

where α_k are real poles of the function $R(x)$, n_k is the degree of the pole α_k $(1 \leq k \leq p)$, A_l^k are some real numbers, and $R_1(x)$ is some rational function without real poles.

The Cauchy index of a real-valued rational function between limits $-\infty$ and $+\infty$ is defined by the formula

$$I_{-\infty}^{+\infty} R(x) := \sum_{k=1}^{p} \sum_{l=1}^{n_k} \frac{1-(-1)^l}{2} \operatorname{sign}(A_l^k). \tag{16}$$

Let suppose that the polynomial

$$g(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0, \quad b_l \in \mathbb{R},\, 0 \leq l \leq n,$$

has no roots in the imaginary axis $\Re(t) = 0$ and has k roots in the half-plane $\Re(t) > 0$. Then the following formula is fulfilled ([11], Chapter XVI, §3):

$$I_{-\infty}^{+\infty} R(\omega) = n - 2k, \text{ where } R(\omega) = \frac{g_2(\omega)}{g_1(\omega)}, \tag{17}$$

while $g_1(\omega)$ and $g_2(\omega)$ are polynomials with real coefficients, defined by the equality

$$\frac{1}{i^n} g(i\omega) = g_1(\omega) - i g_2(\omega), \quad i = \sqrt{-1}.$$

With usage of the formula (17), we shall prove the following auxiliary statement.

Lemma 2. *A polynomial* $g(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0$, *where* $b = (b_0, b_1, b_2, b_3) \in \mathbb{R}^4$, *has* 1 *root in the half-plane* $\Re(t) > 0$ *and* 2 *roots in the half-plane* $\Re(t) < 0$ *if and only if* $b \in \widetilde{M_1} \cup \widetilde{M_2} \cup \widetilde{M_3}$, *where*

$$\widetilde{M_1} = \{b \in \mathbb{R}^4 \,|\, b_1 = 0,\, b_2 = 0,\, b_0 b_3 < 0\},$$

$$\widetilde{M_2} = \{b \in \mathbb{R}^4 \,|\, b_1 b_3 < 0,\, b_0 b_3 < 0\},$$

$$\widetilde{M_3} = \{b \in \mathbb{R}^4 \,|\, b_1 b_3 > 0,\, b_0 b_3 < 0,\, b_1 b_2 - b_0 b_3 > 0\}.$$

Proof. If the polynomial $g(t) = b_3 t^3 + b_2 t^2 + b_1 t + b_0$ has 1 root in the half-plane $\Re(t) > 0$ and 2 roots in the half-plane $\Re(t) < 0$, then $b_3 \neq 0$ and $g(t)$ has no roots in the imaginary axis $\Re(t) = 0$. Let

$$g_1(\omega) = b_3 \omega^3 - b_1 \omega, \quad g_2(\omega) = b_2 \omega^2 - b_0.$$

It is clear that

$$i \cdot g(i\omega) = g_1(\omega) - i g_2(\omega), \quad \text{where } i = \sqrt{-1}.$$

The polynomial $g(t)$ has a root in the imaginary axis $\Re(t) = 0$ if and only if polynomials $g_1(\omega)$ and $g_2(\omega)$ have a common real root. It is easily to see that considered polynomials have a common root $\omega = 0$ if and only if $b_0 = 0$. Let us assume that they have a common real root $\omega \neq 0$. Hence the polynomial $g_3(\omega) = b_3 \omega^2 - b_1$ has the same root. Since $b_3 \neq 0$, then polynomials $g_2(\omega)$ and $g_3(\omega)$ have a common (maybe, complex) root if and only if their resultant $\mathrm{Res}(g_2, g_3)$ is equal to 0 ([7], Chapter V, §34). Direct calculations show that $\mathrm{Res}(g_2, g_3) = (b_1b_2 - b_0b_3)^2$. In order that the common root $\omega \neq 0$ of the polynomials $g_1(\omega)$ and $g_2(\omega)$ will be real in addition it is necessary and sufficient that the condition $b_1b_3 > 0$ is satisfied. Thus the polynomial $g(t)$ has no purely imaginary root if and only if the following two conditions are satisfied:

$$1)\, b_0 \neq 0 \quad \text{and} \quad 2)\, b_1b_2 - b_0b_3 \neq 0 \text{ when } b_1b_3 > 0.$$

According to the formula (17), the polynomial $g(t)$ (when it has no roots on imaginary axis) has exactly one root on the half-plane $\Re(t) > 0$ if and only if the equality

$$I_{-\infty}^{+\infty} \frac{g_2(\omega)}{g_1(\omega)} = 1\,,$$

is fulfilled, which, in turn, is equivalent to the equality

$$I_{-\infty}^{+\infty} R(\omega) = \mathrm{sign}(b_3)\,, \text{ where } R(\omega) = \frac{b_2\omega^2 - b_0}{\omega(\omega^2 - L)},\ L = b_1/b_3. \tag{18}$$

Next we consider successively next three cases: $L = 0$, $L < 0$ and $L > 0$.

The equality $L = 0$ is equivalent to $b_1 = 0$. The condition that the polynomial $g(t)$ has no purely imaginary roots is equivalent in this case to the inequality $b_0 \neq 0$. It is clear that

$$R(\omega) = \frac{b_2\omega^2 - b_0}{\omega(\omega^2 - L)} = \frac{b_2\omega^2 - b_0}{\omega^3} = \frac{b_2}{\omega} - \frac{b_0}{\omega^3}\,,$$

Hence the equality (18) is equivalent in this case to the equation

$$\mathrm{sign}(b_2) - \mathrm{sign}(b_0) = \mathrm{sign}(b_3)\,.$$

The last equation is fulfilled if and only if $b_2 = 0$ and $b_0b_3 < 0$. Thus in the case $L = 0$ the polynomial $g(t)$ has 1 root in the half-plane $\Re(t) > 0$ and 2 roots in the half-plane $\Re(t) < 0$ if and only if

$$b \in \{b \in \mathbb{R}^4 \mid b_1 = 0,\ b_2 = 0,\ b_0b_3 < 0\}.$$

The inequality $L < 0$ is equivalent to $b_1b_3 < 0$. The condition that the polynomial $g(t)$ has no purely imaginary roots, as well as in the previous case, is equivalent to the inequality $b_0 \neq 0$. It is not difficult to get the expansion

$$R(\omega) = \frac{b_2\omega^2 - b_0}{\omega(\omega^2 - L)} = \frac{b_0/L}{\omega} + R_1(\omega)\,,$$

where $R_1(\omega)$ is a rational function without real poles. Therefore, the equality (18) is equivalent in this case to the equation

$$\mathrm{sign}\left(\frac{b_0}{L}\right) = -\,\mathrm{sign}(b_0) = \mathrm{sign}(b_3)\,.$$

The last equality is satisfied if and only if $b_0b_3 < 0$. Consequently, in the case $L < 0$ the polynomial $g(t)$ has 1 root in the half-plane $\Re(t) > 0$ and 2 roots in the half-plane $\Re(t) < 0$ if and only if

$$b \in \{b \in \mathbb{R}^4 \mid b_0b_3 < 0\}.$$

Now consider the case $L > 0$, which is equivalent to $b_1b_3 > 0$. The condition that the polynomial $g(t)$ has no purely imaginary root is equivalent to the fulfillment of the next two inequalities: $b_0 \neq 0$ and $b_1b_2 - b_0b_3 \neq 0$. Let $L = s^2$, where $s > 0$. Direct calculations show that

$$R(\omega) = \frac{b_2\omega^2 - b_0}{\omega(\omega^2 - L)} = \frac{A_1}{\omega} + \frac{A_2}{\omega - s} + \frac{A_3}{\omega + s},\ \text{where } A_1 = b_0/L,\ A_2 = A_3 = \frac{b_2L - b_0}{2L}\,.$$

Therefore, the equality (18) in the case $L > 0$ is equivalent to the equation

$$\mathrm{sign}(b_0) + 2\,\mathrm{sign}(b_2L - b_0) = \mathrm{sign}(b_3)\,.$$

This equation is satisfied if and only if we have two inequalities:

$$b_0b_3 < 0 \text{ and } (b_2L - b_0)b_3 = b_2b_1 - b_0b_3 > 0.$$

Consequently, in the case $L > 0$ the polynomial $g(t)$ has 1 root in the half-plane $\Re(t) > 0$ and 2 roots in half-plane $\Re(t) < 0$ if and only if

$$b \in \{b \in \mathbb{R}^4 \mid b_0b_3 < 0,\, b_1b_2 - b_0b_3 > 0\}.$$

Combining three cases, considered above, we obtain the statement of the lemma. ∎

Proof of Theorem 4. According to Theorem 2, the polynomial $f(z) = z^3+pz^2+qz+r$ with integer rational coefficients is a Pisot polynomial if and only if $r \neq 0$ and the polynomial

$$g(t) = (t-1)^3 f\left(\frac{t+1}{t-1}\right) = b_3t^3 + b_2t^2 + b_1t + b_0 =$$

$$= (1+p+q+r)t^3 + (3+p-q-3r)t^2 + (3-p-q+3r)t + (1-p+q-r)$$

has one root in the half-plane $\Re(t) > 0$ and 2 roots in the half-plane $\Re(t) < 0$. Now it is sufficient to apply Lemma 2 to get the required result. Note only that under assumption of the theorem equalities $b_1b_3 = (3-p-q+3r)(1+p+q+r)$ and $b_1b_2 - b_0b_3 = 8(1-q+pr-r^2)$ are fulfilled, while the inequality $b_0b_3 = (1+q)^2 - (p+r)^2 < 0$ is equivalent to the inequality $|p+r| > |q+1|$. Moreover, conditions $b_1 = b_2 = 0$ and $b_0b_3 < 0$ are satisfied if and only if $q = 3$, $p = 3r$ and $|r| > 1$. ∎

The following theorem gives an easily verified sufficient criterion for Pisot polynomials of higher degrees.

Theorem 5. *Let suppose that for the polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ with integer rational coefficients the inequalities*

$$|a_{n-1}| > 1 + \sum_{k=0}^{n-2} |a_k|, \quad a_0 \neq 0$$

are fulfilled. Then $f(z)$ is a Pisot polynomial.

Proof. We shall prove the theorem with the help of the complex analysis. Let us denote by Γ the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ in the complex plane $\mathbb{C}$. Let us consider the function $g(z) = a_{n-1}z^{n-1}$. It is clear that $f(z) - g(z) = z^n + \sum\limits_{k=0}^{n-2} a_kz^k$, and for every number $z \in \Gamma$ we have the inequality

$$|f(z) - g(z)| = \left| z^n + \sum_{k=0}^{n-2} a_kz^k \right| \leq 1 + \sum_{k=0}^{n-2} |a_k| < |a_{n-1}| = |g(z)|.$$

In particular, it follows from the last formula that $f(z)$ has no roots on the circle Γ. According to the Rouche theorem, functions $f(z)$ and $g(z)$ have one and the same number of roots in the disc $|z| < 1$. Therefore, $f(z)$ has $n-1$ roots inside the disc $|z| < 1$ (they are nonzero in view of the condition $a_0 \neq 0$) and one root in the exterior of this disc. According to Lemma 1, $f(z)$ is a Pisot polynomial. ∎

Remark 3. *Obviously, Theorem 5 implies the well-known statement on the existence of Pisot polynomials of any given degree $n \in \mathbb{N}$. Note that one can say more about the set of all Pisot polynomials. Let us consider for a real number $d > 0$ the cube*

$$K(d) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid |x_i| \leq d\,,\, i = 1 \dots n\}\,,$$

and denote by $\mathcal{Z}_n(d)$ the quantity of points in $\mathbb{Z}^n \cap K(d)$, and let $\mathcal{P}_n(d)$ be the quantity of the points $(a_0, a_1, \dots, a_{n-1}) \in \mathbb{Z}^n \cap K(d)$ such that the polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ is a Pisot polynomial. It is not difficult to deduce from Theorem 5 the inequality

$$\varliminf_{d \to \infty} \frac{\mathcal{P}_n(d)}{\mathcal{Z}_n(d)} > 0$$

for every $n \in \mathbb{N}$. Let us note also that Theorem 3 implies the equality $\lim\limits_{d\to\infty} \frac{\mathcal{P}_2(d)}{\mathcal{Z}_2(d)} = \frac{1}{2}$.

At the end of this section we formulate two unsolved questions. For a Pisot polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, let us denote by $M(f)$ the set of $(n \times n)$-matrices with integer entries, which has $f(z)$ as the characteristic polynomial. Thus, $M(f)$ is the set of Pisot matrices with the common characteristic polynomial $f(z)$.

Question 2. *Let $f(z)$ be a Pisot polynomial such that all its roots are real. Is there a symmetric matrix in the set $M(f)$?*

Let us define the equivalence relation $\sim_{GL(n,\mathbb{Z})}$ on the set $M(f)$ (a definition of the group $GL(n,\mathbb{Z})$ is given in the fifth section):

$$M_1 \sim_{GL(n,\mathbb{Z})} M_2 \Longleftrightarrow \exists A \in GL(n,\mathbb{Z}),\ M_1 = AM_2A^{-1}$$

for $M_1, M_2 \in M(f)$. Thus $M_1 \sim_{GL(n,\mathbb{Z})} M_2$ if and only if matrices M_1 and M_2 are $GL(n,\mathbb{Z})$-similar.

Question 3. *How many equivalence classes contains the quotient set $M(f)/_{\sim_{GL(n,\mathbb{Z})}}$ for a given Pisot polynomial?*

3. Some properties of recurrent sequences

We shall obtain some results on recurrent sequences in this section. Besides the fact that these results have independent interest, we shall use them for a proof of the Pisot-Vijayaraghavan theorem in the next section.

Later on we shall need the simple

Lemma 3. *For any polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with coefficients in $\mathbb{K}$ (where $\mathbb{K}$ is one of the fields $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$ or the ring of integer numbers $\mathbb{Z}$), there is a $(n\times n)$-matrix A with elements in $\mathbb{K}$ such that its characteristic polynomial is equal to $f(z)$.*

Proof. One can take as a matrix A the matrix of the system of linear homogeneous differential equations of the first order, to which the linear differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$ is reduced by changing $y_1 = y$, $y_2 = y^{(1)}$, $y_3 = y^{(2)}, \ldots, y_n = y^{(n-1)}$. The same matrix will be used in the proof of Lemma 7. ■

Theorem 6. *Let A be a square matrix with complex entries, $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is its characteristic polynomial, $\{u_k\}$ is a sequence of complex numbers, satisfying the recurrent equation*

$$u_{k+n} + a_{n-1}u_{k+n-1} + \cdots + a_1 u_{k+1} + a_0 u_k = 0, \quad k \in \mathbb{Z}_+.$$

Let us suppose that $f(z)$ has no multiple roots. Then the following statements are fulfilled.

1) There is the unique polynomial

$$g(z) = \sum_{l=0}^{n-1} b_l z^l, \quad b_l \in \mathbb{C}\,,$$

such that for all $k \in \mathbb{Z}_+$ the equation

$$u_k = \operatorname{tr}(A^k \cdot g(A)) = \sum_{m=1}^{n} g(z_m)(z_m)^k,$$

is satisfied, where z_m are roots of the polynomial $f(z)$, $m = 1, \ldots, n$.

2) If all entries of the matrix A and all members of the sequence $\{u_n\}$ belong to the field $\mathbb{R}$ or to the field $\mathbb{Q}$, then also all coefficients of the polynomial $g(z)$ belong to $\mathbb{R}$ or $\mathbb{Q}$ respectively.

3) If all entries of the matrix A belong to the field $\mathbb{Q}$, then there are integer rational numbers (depending only on entries of the matrix A) $F \neq 0$ and F_l^s $(0 \leq l \leq n-1$, $1 \leq s \leq n)$, such that coefficients of the polynomial $g(z)$ are expressed by formulas

$$b_l = F^{-1} \cdot P_l(u_0, u_1, \dots, u_{n-1}), \text{ where } P_l(y_1, y_2, \dots, y_n) = \sum_{s=1}^{n} F_l^s y_s.$$

Proof. Let us prove the first statement of the theorem. According to the Hamilton-Cayley theorem, the matrix A satisfies its characteristic equation. Thus, the equality

$$A^{n+k} + \sum_{m=0}^{n-1} a_m A^{m+k} = 0.$$

is fulfilled. By multiplying both sides of the last equation on the right by arbitrary matrix C and calculating the trace of both sides, we obtain that the sequence $\{v_k\}$, where $v_k = \operatorname{tr}(A^k \cdot C)$, satisfies recurrent equation

$$v_{k+n} + a_{n-1}v_{k+n-1} + \dots + a_1 v_{k+1} + a_0 v_k = 0, \quad k \in \mathbb{Z}_+.$$

Now we shall indicate a matrix C such that $\{v_k\} = \{u_k\}$. We shall seek it in the form $C = \sum\limits_{l=0}^{n-1} b_l A^l$, where b_l are some complex numbers. To prove the coincidence of the sequences $\{v_k\}$ and $\{u_k\}$, it is enough to establish the coincidence of their first n members. I.e., the following equalities

$$u_m = \sum_{l=0}^{n-1} b_l \cdot \operatorname{tr}(A^{m+l}), \quad m \in \{0, \dots, n-1\}$$

must be satisfied. One can write these equalities in the form of a system of linear equations

$$\begin{pmatrix} \operatorname{tr} A^0 & \operatorname{tr} A^1 & \cdots & \operatorname{tr} A^{n-1} \\ \operatorname{tr} A^1 & \operatorname{tr} A^2 & \cdots & \operatorname{tr} A^n \\ \cdots & \cdots & \cdots & \cdots \\ \operatorname{tr} A^{n-1} & \operatorname{tr} A^n & \cdots & \operatorname{tr} A^{2n-2} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \cdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \\ \cdots \\ u_{n-1} \end{pmatrix}. \tag{19}$$

Remark now that the matrix of this system of equations has the form $V \cdot V^t$, where

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{pmatrix},$$

while z_i are roots of the polynomial $f(z)$. Thus, V is a Vandermonde matrix with the determinant $\prod\limits_{i<j}(z_j - z_i) \neq 0$. It means that the determinant of the matrix equation (19) is equal to $D = \prod\limits_{i<j}(z_j - z_i)^2 \neq 0$, therefore, the system (19) is solvable relatively to the variables b_i. Let us note also that the number D is the discriminant of the polynomial $f(z)$. Since for roots of the polynomial $f(z)$ $(z_m,\ m = 1, \dots, n)$ the equality $\operatorname{tr} A^k = \sum\limits_{m=1}^{n} (z_m)^k$ $(k \in \mathbb{Z}_+)$ is fulfilled, we get that

$$u_k = \operatorname{tr}\left(A^k \cdot \sum_{l=0}^{n-1} b_l A^l\right) = \sum_{l=0}^{n-1} b_l \operatorname{tr} A^{k+l} = \sum_{l=0}^{n-1} b_l \sum_{m=1}^{n} (z_m)^{k+l} =$$

$$= \sum_{m=1}^{n} \left(\sum_{l=0}^{n-1} b_l (z_m)^l\right) \cdot (z_m)^k = \sum_{m=1}^{n} g(z_m)(z_m)^k.$$

Consequently, we proved the existence of the required b_i; the uniqueness of the polynomial $g(z)$ follows from the fact that the system (19) has the unique solution.

The second statement of the theorem is practically evident. One needs only to remark that in the case when all entries of the matrix A and all u_k are real or rational, the obtained coefficients b_i of the polynomial $g(z)$ will be also real or rational.

The third statement of the theorem follows immediately from Cramer's rule for a solution of systems of linear equations. ■

Lemma 4. *Let $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial with rational coefficients, which is irreducible over $\mathbb{Q}$, while $\{u_k\}$ is a nonzero sequence of rational numbers, satisfying the recurrent equation*

$$u_{k+n} + a_{n-1}u_{k+n-1} + \cdots + a_1 u_{k+1} + a_0 u_k = 0, \quad k \in \mathbb{Z}_+.$$

Then there are complex numbers $C_m \neq 0$ $(1 \le m \le n)$ such that for all $k \in \mathbb{Z}_+$ the equality

$$u_k = \sum_{m=1}^{n} C_m (z_m)^k$$

is satisfied, where z_i are roots of the polynomial $f(z)$. Moreover, $C_m \in \mathbb{Q}(z_m)$.

Proof. According to Lemma 3, there is a square matrix A with rational entries, which has the characteristic polynomial $f(z)$. By Theorem 6, there is a polynomial

$$g(z) = \sum_{l=0}^{n-1} b_l z^l, \quad b_l \in \mathbb{Q}\,,$$

such that for all $k \in \mathbb{Z}_+$ the equality

$$u_k = \operatorname{tr}(A^k \cdot g(A)) = \sum_{m=1}^{n} g(z_m)(z_m)^k,$$

is fulfilled, where z_m, $m = 1, \ldots, n$ are roots of the polynomial $f(z)$. Thus, one can take $g(z_m)$ as C_m. It remains to show that for every $m \in \{1, \ldots, n\}$, the inequality $C_m \neq 0$ is fulfilled. But if $g(z_m) = 0$ for some number, then this contradicts to irreducibility of the polynomial $f(z)$. Really, $g(z)$ is the polynomial with rational coefficients and its degree is less than the degree of the polynomial $f(z)$. Therefore, the existence of a common root of these polynomials implies the reducibility of $f(z)$ over the field $\mathbb{Q}$. Finally, since $C_m = g(z_m)$, then $C_m \in \mathbb{Q}(z_m)$. ■

Lemma 5. *Let suppose that a sequence of complex numbers $\{v_k\}$ satisfies the equation*

$$v_k = \sum_{l=1}^{m} C_l (z_l)^k,$$

where $C_l \in \mathbb{C}$, z_l are mutually distinct complex numbers and $|z_l| \ge 1$ for all $l = 1, \ldots, m$. Hence, if $v_k \to 0$ when $k \to \infty$, then $C_l = 0$ for all l.

Proof. At first we shall prove the lemma under additional condition that absolute values of all numbers z_l are equal to 1. Let us consider for this goal the discrete dynamical system on the torus $\mathbb{T} = (\mathbb{S}^1)^m$ $(\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z}),)$ supplied with invariant probability Lebesgue measure. This system is defined by the map $A : \mathbb{T} \to \mathbb{T}$,

$$A(\varphi_1, ..., \varphi_m) = (\varphi_1 + \alpha_1, ..., \varphi_m + \alpha_m),$$

where $\alpha_l = \operatorname{Arg}(z_l)$. The map A preserves volumes. Let us consider function $F : \mathbb{T} \to \mathbb{C}$, defined by formula

$$F(\varphi) = \sum_{i=1}^{m} C_l e^{\varphi_l \sqrt{-1}}.$$

Suppose, that $\mathbb{O} = (0, \ldots, 0)$. By the condition of the lemma, $F(A^k(\mathbb{O})) \to 0$ when $k \to \infty$. If for some m the inequality $|F(A^m(\mathbb{O}))| > \epsilon > 0$ is satisfied, then the same inequality is satisfied for all points x in some neighborhood U of the point $A^m(\mathbb{O})$ on the torus $\mathbb{T}$. By the

Poincare recurrence theorem, members of the sequence $A^k(\mathbb{O})$ get into the neighborhood U infinitely many times. Thus, the condition $F(A^k(\mathbb{O})) \to 0$ is not satisfied when $k \to \infty$. It follows from the obtained contradiction that for every natural number m, the equality $F(A^m(\mathbb{O})) = 0$ is fulfilled. Further, it is enough to note that the matrix of the homogeneous system of linear equations $\sum\limits_{l=1}^{m} C_l(z_l)^k = 0$, $k = 0, \dots, n-1$, is a nondegenerate Vandermonde matrix, and therefore, all the numbers C_l must be equal to zero.

Let us prove now the lemma in general case. Suppose that the lemma is not true. Then for some sequence of complex numbers $\{v_k\}$ such that $v_k \to 0$ when $k \to \infty$, for some mutually distinct z_l ($|z_l| \geq 1$) and for some complex numbers $C_l \neq 0$, $l = 1, \dots, m$, we have the equality $v_k = \sum\limits_{l=1}^{m} C_l(z_l)^k$. Let us order z_l by decreasing of their absolute values $|z_l|$ and let us use the notation $t = |z_1|$. Let s ($1 \leq s \leq m$) be such that $|z_l| = t > 1$ when $l = 1, \dots, s$, while $|z_l| < t$ when $l > s$. Consider the sequence $\{\tilde{v}_k\}$, defined by the equation

$$\tilde{v}_k = \frac{v_k}{t^k} - \sum_{l=s+1}^{m} C_l \left(\frac{z_l}{t}\right)^k.$$

Obviously, $\tilde{v}_k \to 0$ when $k \to \infty$. Furthermore,

$$\tilde{v}_k = \sum_{l=1}^{s} C_l(\tilde{z}_l)^l,$$

where $\tilde{z}_l = z_l/t$, $|\tilde{z}_l| = 1$ if $l = 0, \dots, s$. But according to the first part of the proof, we get that $C_l = 0$ when $1 \leq l \leq s$, which contradicts our hypothesis. Thus the lemma is completely proved. ∎

Lemma 6 ([14], Chapter VIII, §2, Lemma 3). *Let us suppose that there is a number $\gamma \neq 0$ in the field $\mathbb{Q}(\lambda)$ of an algebraic number λ (of degree r) such that all the numbers γ, $\gamma\lambda$, $\gamma\lambda^2, \dots$ are equal to polynomials of degree $r-1$ with integer rational coefficients relatively to λ. Then λ is an integer algebraic number.*

Theorem 7. *Let $\{u_k\}$ be an integer nonzero sequence, which satisfies the recurrent equation*

$$u_{k+n} + a_{n-1}u_{k+n-1} + \dots + a_1 u_{k+1} + a_0 u_k = 0, \quad k \in \mathbb{Z}_+,$$

where $a_l \in \mathbb{Q}$, $l = 0, \dots, n-1$, and the polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$$

is irreducible over $\mathbb{Q}$. Then all the numbers a_l, $0 \leq l \leq n-1$, are integer rational numbers.

Proof. Since any irreducible polynomial has no multiple roots, then according to Lemma 6, we have the equality

$$u_k = \sum_{m=1}^{n} g(z_m)(z_m)^k, \quad k \in \mathbb{Z}_+ \tag{20}$$

for some polynomial $g(z)$ with rational coefficients. Let us consider a natural number p and the sequence $\{\tilde{u}_k\}$ such that $\tilde{u}_k = u_{k+p}$ for $k \in \mathbb{Z}_+$. It is clear that such sequence also satisfies the recurrent equation

$$\tilde{u}_{k+n} + a_{n-1}\tilde{u}_{k+n-1} + \dots + a_1\tilde{u}_{k+1} + a_0\tilde{u}_k = 0, \quad k \in \mathbb{Z}_+.$$

By the statement 3) of Theorem 6,

$$\tilde{u}_k = \sum_{m=1}^{n} \tilde{g}(z_m)(z_m)^k, \quad k \in \mathbb{Z}_+,$$

for some polynomial

$$\tilde{g}(z) = \sum_{l=0}^{n-1} b_l z^l\,,$$

and there are integer rational numbers $F \neq 0$ and F_l^s $(0 \leq l \leq n-1,\ 1 \leq s \leq n)$ such that coefficients of the polynomial $\tilde{g}(z)$ are expressed by formulas

$$b_l = F^{-1} \cdot P_l(u_p, u_{p+1}, \dots, u_{n+p-1}), \text{ where } P_l(y_1, y_2, \dots, y_n) = \sum_{s=1}^{n} F_l^s y_s.$$

Let us examine numbers $A_l = P_l(u_p, u_{p+1}, \dots, u_{n+p-1})$. Evidently, they are integer rational numbers. Therefore, we have the equality

$$F \cdot \tilde{u}_k = \sum_{m=1}^{n} \left(\sum_{l=0}^{n-1} A_l (z_m)^l \right) (z_m)^k, \quad k \in \mathbb{Z}_+ \tag{21}$$

Since $\tilde{u}_k = u_{k+p}$, then we obtain from the equalities (20) and (21) that

$$F \cdot \tilde{u}_k - F \cdot u_{k+p} = \sum_{m=1}^{n} \widetilde{C}_m (z_m)^k = 0, \quad k \in \mathbb{Z}_+, \tag{22}$$

where $\widetilde{C}_m = \sum\limits_{l=0}^{n-1} A_l (z_m)^l - F \cdot g(z_m)(z_m)^p$. It is not difficult to understand that $\widetilde{C}_m = 0$ for all $1 \leq m \leq n$. Indeed, if we consider the system of equations (22) relatively to variables $\widetilde{C}_m$ where $0 \leq k \leq n-1$, then the matrix of this system is a nondegenerate Vandermonde matrix. If λ is any root z_m of the polynomial $f(z)$, we get the equality

$$F \cdot g(\lambda) \cdot \lambda^p = \sum_{l=0}^{n-1} A_l \lambda^l.$$

It follows from this equality and Lemma 6 that λ is an integer algebraic number. In fact, the number $p \in \mathbb{N}$ has been taken arbitrarily, $\gamma = F \cdot g(\lambda) \in \mathbb{Q}(\lambda)$ and $g(\lambda) \neq 0$, as it follows from the proof of Lemma 4. Further, since λ is a root of the polynomial $f(z)$, which is irreducible over the field of rational numbers, it means that all coefficients of the last polynomial must be integer rational numbers. ■

Remark 4. *The irreducibility condition for the polynomial $f(z)$ in the statement of Theorem 7 is essential. In order to illustrate this, let us consider the sequence $\{u_n = 2^n\}$. For every rational number $p \neq 0$ this sequence satisfies the recurrent equation*

$$u_{k+2} - \frac{2p+1}{p} u_{k+1} + \frac{2}{p} u_k = 0, \quad k \in \mathbb{Z}_+,$$

We have such effect, because the polynomial

$$z^2 - \frac{2p+1}{p} z + \frac{2}{p} = \left(z - \frac{1}{p} \right)(z-2)$$

is reducible.

4. Proof of the Pisot-Vijayaraghavan theorem

A classical proof of the Pisot-Vijayaraghavan theorem is given, for example, in the book [14]. We consider as possible to give another proof in this paper, which in our view describes better interconnections of Pisot numbers theory with other branches of Mathematics such as the theory of recurrent sequences and the theory of dynamical systems. We shall get the proof on the ground of properties of recurrent sequences from the previous section.

Let us consider a number λ, being a root of a polynomial

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

with rational coefficients, irreducible over $\mathbb{Q}$. For $\theta \in \mathbb{Q}(\lambda)$, let us indicate an effective way to calculate the algebraic trace $\mathrm{Tr}(\theta \lambda^p)$. We have the following

Lemma 7. *Let us consider the equation* $\theta = h(\lambda) = \sum\limits_{l=0}^{n-1} c_l \lambda^l$ *with rational coefficients* c_l. *Then the following statements are fulfilled.*

1) For any $p \in \mathbb{Z}_+$, *the equality*

$$\mathrm{Tr}(\theta\lambda^p) = \sum_{m=1}^{n} h(z_m)(z_m)^p,$$

is satisfied, where z_m *are roots of the polynomial* $f(z)$, $m = 1, \dots, n$.

2) Moreover, if λ *is an integer algebraic number, then all the numbers* $\mathrm{Tr}(\theta\lambda^p)$ *are rational with denominators not greater than the least common multiple of the denominators of the coefficients* c_l, $l = 0, \dots, n-1$.

Proof. Let us prove the first statement of the lemma. Let us choose a basis $\{\xi_l\}_{l=1}^n$ in $\mathbb{Q}(\lambda)$ such that $\xi_l = \lambda^{l-1}$. Consider the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix},$$

of the linear map

$$L : \mathbb{Q}(\lambda) \to \mathbb{Q}(\lambda), \quad L(y) = y \cdot \lambda$$

in the chosen basis $\{\xi_l\}_{l=1}^n$ ([7], Chapter VI, §47). It is clear that $\mathrm{Tr}(\lambda) = \mathrm{tr}(M)$. One can easily see that $\mathrm{Tr}(\lambda^q) = \mathrm{tr}(M^q) = \sum\limits_{m=1}^{n} (z_m)^q$ for $q \in \mathbb{Z}_+$. Therefore,

$$\mathrm{Tr}(\theta\lambda^p) = \mathrm{Tr}\left(\lambda^p \cdot \sum_{l=0}^{n-1} c_l \lambda^l\right) = \sum_{l=0}^{n-1} c_l \,\mathrm{Tr}\, \lambda^{p+l} = \sum_{l=0}^{n-1} c_l \,\mathrm{tr}\, M^{p+l} =$$

$$= \sum_{l=0}^{n-1} c_l \sum_{m=1}^{n} (z_m)^{p+l} = \sum_{m=1}^{n} \left(\sum_{l=0}^{n-1} c_l (z_m)^l\right) \cdot (z_m)^p = \sum_{m=1}^{n} h(z_m)(z_m)^p.$$

Let us prove now the second statement of the lemma. Since λ is an integer algebraic number, then all coefficients of the polynomial $f(z)$ and all entries of the matrix M are integer rational numbers. Now it is enough to refer to the equality

$$\mathrm{Tr}(\theta\lambda^p) = \sum_{l=0}^{n-1} c_l \,\mathrm{tr}\, M^{p+l}.$$

■

Proof of the Pisot-Vijayaraghavan theorem. At first we shall show that for any Pisot number $\lambda > 1$ we have $\mathcal{P}_\lambda \neq \{0\}$. Let us verify, for example, that $1 \in \mathcal{P}_\lambda$. According to the statement 2) of Lemma 7, we get that $\mathrm{Tr}(\lambda^p) \in \mathbb{Z}$ for all $p \in \mathbb{Z}_+$. On the other hand, according to the statement 1) of lemma 7, we have the equality

$$\mathrm{Tr}(\lambda^p) = \sum_{m=1}^{n} (z_m)^p$$

for all $p \in \mathbb{Z}_+$, where $\lambda = z_1, z_2, \dots, z_m$ are roots of the polynomial $f(z)$. According to the definition of Pisot number, $|z_m| < 1$ for $m \geq 2$. Therefore, $\lambda^p - \mathrm{Tr}(\lambda^p) = o(1)$ when $p \to \infty$. Since $\mathrm{Tr}(\lambda^p) \in \mathbb{Z}$, we get that $1 \in \mathcal{P}_\lambda$.

Now let $\lambda > 1$ be an algebraic number of degree n, i.e. there is a polynomial

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,\ a_n > 0,$$

with integer coefficients, which is irreducible over rational numbers and has λ as one of its roots; and $\mathcal{P}_\lambda \neq \{0\}$. Consequently, there is $a \in \mathbb{R}$ such that $a \neq 0$ and

$$\|a\lambda^n\| \to 0 \quad \text{when} \quad n \to \infty. \tag{23}$$

We shall show that λ is a Pisot number and $a \in \mathbb{Q}(\lambda)$. We obtain from the equality (23) that

$$a\lambda^k = u_k + o(1) \quad \text{when} \quad k \to \infty$$

for some integer sequence $\{u_k\}$. Evidently, this sequence is nonzero and for $0 \le i \le n$ the equalities

$$a \cdot a_i \lambda^{k+i} = a_i u_{k+i} + o(1) \quad \text{when} \quad k \to \infty,$$

are fulfilled, by summing of which we get

$$0 = a\lambda^k \left(\sum_{i=0}^{n} a_i \lambda^i \right) = \sum_{i=0}^{n} a_i u_{k+i} + o(1) \quad \text{when} \quad k \to \infty.$$

Consequently, it is necessary that for sufficiently large indices k the equality

$$\sum_{i=0}^{n} a_i u_{k+i} = 0$$

is satisfied. According to Theorem 7, it must be $a_n = 1$. Since the polynomial $f(z)$ is irreducible, then according to Lemma 4, for sufficiently large k, we have the expansion

$$u_k = \sum_{i=1}^{n} C_i (z_i)^k$$

for appropriate (maybe, complex) numbers $C_i \neq 0$, where $\lambda = z_1, z_2, \ldots, z_n$ are roots of the polynomial $f(z)$. Since

$$a\lambda^k = \sum_{i=1}^{n} C_i (z_i)^k + o(1) \quad \text{when} \quad k \to \infty,$$

then, if for some $i \ge 2$ inequality $|z_i| \ge 1$ is satisfied, then, according to Lemma 5, we have $C_i = 0$, which contradicts to Lemma 4. Therefore, the inequality $|z_i| < 1$ is satisfied for all $i \ge 2$. Thus, the polynomial $f(z)$ is a Pisot polynomial, while λ is a Pisot number. Moreover, it is clear that $a = C_1$. Applying Lemma 4 again, we get that $a \in \mathbb{Q}(\lambda)$.

Now let us give the complete characterization of the set $\mathcal{P}_\lambda$ for the Pisot number λ. As we have showed above, $\mathcal{P}_\lambda \subset \mathbb{Q}(\lambda)$. Let us consider now an arbitrary $a \in \mathcal{P}_\lambda \subset \mathbb{Q}(\lambda)$. Then there is a polynomial $h(z) = \sum\limits_{l=0}^{n-1} c_l z^l$ with coefficients in $\mathbb{Q}$ such that $a = h(\lambda)$. According to Lemma 7, for any $p \in \mathbb{Z}_+$ we have the equality

$$\mathrm{Tr}(a\lambda^p) = \sum_{m=1}^{n} h(z_m)(z_m)^p,$$

where z_m are roots of the polynomial $f(z)$, $m = 1, \ldots, n$, $z_1 = \lambda$. Since $|z_m| < 1$ for $m \ge 2$ and $h(\lambda) = a$, then

$$\mathrm{Tr}(a\lambda^p) = a\lambda^p + o(1) \quad \text{when} \quad p \to \infty.$$

Since $\|a\lambda^p\| \to 0$ when $p \to \infty$, then, according to the statement 2) of Lemma 7, we get that $\mathrm{Tr}(a\lambda^p) \in \mathbb{Z}$ for sufficiently large p.

Conversely, if $\mathrm{Tr}(a\lambda^p) \in \mathbb{Z}$ for sufficiently large p, then $\|a\lambda^p\| \to 0$ when $p \to \infty$, because

$$a\lambda^p = \mathrm{Tr}(a\lambda^p) + o(1) \text{ when } p \to \infty.$$

Thus, for the Pisot number λ we get

$$\mathcal{P}_\lambda = \{a \in \mathbb{Q}(\lambda) \mid \mathrm{Tr}(a\lambda^n) \in \mathbb{Z} \text{ for sufficiently large } n\}.$$

∎

Remark 5. *It follows immediately from the statement 2) of Lemma 7 and the Pisot-Vijayaraghavan theorem that for any Pisot number λ, a number θ, satisfying the equality $\theta = \sum\limits_{l=0}^{n-1} c_l\lambda^l$ for some integer rational numbers c_l $(l = 0, \dots, n-1)$, belongs to the set $\mathcal{P}_\lambda$.*

5. General information on the group $GL(2,\mathbb{Z})$ and some of its subsemigroups

The group $GL(n,\mathbb{Z})$ of all invertible integer $(n \times n)$-matrices is naturally identified with the group of all invertible linear maps of n-dimensional (real) vector space, preserving a fixed lattice of this space. It consists of integer $(n \times n)$-matrices of determinant ± 1. Therefore, some authors (see, for example, [16]) call it the unimodular group. Equally often by the unimodular group one means its subgroup $SL(n,\mathbb{Z})$ of index 2, consisting of matrices of determinant 1. The group $GL(n,\mathbb{Z})$ (and its subgroup $SL(n,\mathbb{Z})$ in the case of even n) has the center C of order 2, generated by the *central inversion*, inverting the sign of each coordinate. Its quotient group $PGL(n,\mathbb{Z}) := GL(n,\mathbb{Z})/C$ is the *projective (unimodular) group.* If n is odd, then the pairs of groups $PGL(n,\mathbb{Z})$ and $SL(n,\mathbb{Z})$, $GL(n,\mathbb{Z})$ and $SL(n,\mathbb{Z}) \times C$ are naturally isomorphic; for even n, the quotient group $PGL(n,\mathbb{Z})$ contains the subgroup $PSL(n,\mathbb{Z}) := SL(n,\mathbb{Z})/C$ of index 2. The last group in the case $n = 2$ is isomorphic to the (discrete) (so-called *modular*) group Γ of linear-fractional transformations

$$z' = \frac{az+b}{cz+d} \tag{24}$$

of the upper complex half-plane (serving as the Poincare model of the Lobachevskii plane) with integer coefficients, satisfying the condition $ac - bd = 1$.

We shall denote by $f(A)$ the linear-fractional transformation (24) *of the whole complex plane* $\mathbb{C}$, corresponding to the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $GL(2,\mathbb{Z})$. For $n \geq 2$, the group $GL(n,\mathbb{Z})$ is generated by three matrices for odd n and by two matrices for even n, [16]. Let us write generators and relations for $n = 2$. The group $GL(2,\mathbb{Z})$ has genetic code (see [16])

$$\Big\{ R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, R_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$R_1^2 = R_2^2 = R_3^2 = E, (R_1R_2)^2 = (R_1R_3)^2 = Z, Z^2 = E \Big\}.$$

The group $SL(2,\mathbb{Z})$ has genetic code

$$\Big\{ S := R_1R_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, T := R_1R_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

$$S^3 = T^2 = Z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, Z^2 = E \Big\}.$$

The modular group Γ has genetic code

$$\{\sigma, \tau; \sigma^3 = \tau^2 = 1\}$$

and it is a free product of two cyclic groups of orders 3 and 2.

As generators of the group $GL(2,\mathbb{Z})$ one can take matrices

$$u := R_1, \quad v := R_1R_3R_2R_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$R_2 = v^{-1}uv, \quad R_3 = uvuv^{-1}uv.$$

Theorem 8. *The subsemigroup $GL_+(2,\mathbb{Z})$ of the group $GL(2,\mathbb{Z})$, consisting of matrices with nonnegative entries, has genetic code*

$$\{u, v; u^2 = E.\}$$

Proof. At first let us prove that the semigroup, generated by elements u, v, coincides with $GL_+(2,\mathbb{Z})$. Evidently, $u, v \in GL_+(2,\mathbb{Z})$. Rearranging rows and (or) columns of any matrix A in $GL_+(2,\mathbb{Z})$, i.e. multiplying, if it's necessary, the matrix A by the matrix u on the left and (or) on the right, one can achieve that the maximal entry of the obtained matrix Z will be $z_{22} > 0$. Moreover, if $z_{22} = z_{12}$ (or $z_{22} = z_{21}$) then, rearranging, if it's necessary, rows and columns, one can assume that $z_{11} \le z_{21}$ (respectively $z_{11} \le z_{12}$), i.e. Z is non-decreasing by columns and rows. Otherwise, if the condition of non-decreasing for Z by rows and columns be violated, one can easily verify that $Z = E$. If $Z = E$, then $A = E$ or $A = u$. If Z is non-decreasing by columns and rows, then by Theorem 1 the matrix Z is uniquely represented in the form of the product (10). In turn, every factor of this product, corresponding to the natural number $q_i, i = 1, ..., n+1$, is the matrix uv^{q_i}.

The uniqueness of a representation of the matrix A in the form of the product of powers of elements u, v with nonnegative exponents, where exponents of powers for element u do not exceed 1, follows from the above arguments and Theorem 1. ■

It follows from Theorems 8 and 1 the following

Corollary 1. *The set of integer (2×2)-matrices with nonnegative entries, non-decreasing by rows and columns, or, which is the same, of matrices of the form (10), constitutes the free (non-commutative) subsemigroup $GL_{+,\le}(2,\mathbb{Z})$ of the semigroup $GL_+(2,\mathbb{Z})$ with countable set of generators of the form $uv^n, n \in \mathbb{N}$, which does not admit a finite generating set.*

Analogously, the set of all matrices in $GL_+(2,\mathbb{Z})$, non-decreasing by rows and columns, constitutes the free (non-commutative) semigroup $GL_{+,\ge}(2,\mathbb{Z})$ with a countable set of generators of the form $v^n u$, $n \in \mathbb{N}$, which does not admit a finite generating set.

The subsemigroup $SGL_+(2,\mathbb{Z}) - E$ of the semigroup $GL_+(2,\mathbb{Z})$, consisting of matrices of determinant 1 and different from the unit matrix, is a free semigroup with the generators v and uvu.

On the ground of Theorem 8 and Corollary 1 we get successively all the statements of Corollary 2.

Corollary 2. *For any element A of semigroup $GL_+(2,\mathbb{Z})$, it is satisfied exactly one alternative:*

1) $A = E$;
2) $A = u$;
3) $A = v^n$, $n \in \mathbb{N}$;
4) $A \in GL_{+,\le}(2,\mathbb{Z})$;
5) $A \in GL_{+,\ge}(2,\mathbb{Z})$;
6) $A \in GL_{+,\le}(2,\mathbb{Z})u$;
7) $A \in GL_{+,\ge}(2,\mathbb{Z})v^n$, $n \in \mathbb{Z}$.

Furthermore, every set G_i, consisting of matrices A with the property i), besides G_2, is a subsemigroup in $GL_+(2,\mathbb{Z})$, while the set $G_{1,2} := G_1 \cup G_2$ is a two-element group. Thus, the semigroup $GL_+(2,\mathbb{Z})$ is an union of six mutually disjoint subsemigroups $G_{1,2}$ and G_k, where $k = 3, \dots, 7$; moreover, only G_3, G_4, G_5 are free semigroups.

The conjugation of the group $GL(2,\mathbb{Z})$ by the element u is an involutive automorphism φ. The automorphism φ translates semigroups $GL_+(2,\mathbb{Z}), G_{1,2}$ to themselves, interchanges semigroups G_4,G_5, and maps semigroups G_3 and G_7 onto some semigroups $\varphi(G_3)$ and $\varphi(G_7)$, composing a partition of the semigroup G_6; φ maps subsemigroups $\varphi(G_3)$ and $\varphi(G_7)$ of the semigroup G_6 onto semigroups G_3 and G_7.

Thus, the semigroup $GL_+(2,\mathbb{Z})$ splits into seven mutually disjoint semigroups $G_{1,2}$; G_3, $\varphi(G_3)$; G_5, G_4; G_7, $\varphi(G_7)$, where pairs of semigroups, separated by the comma, are pairwise isomorphic by means of the automorphism φ and the union of all first (respectively, second) *semigroups of these pairs also constitute some semigroup G_{odd} ($G_{even} = G_4 \cup G_6$). The semigroup $GL_+(2,\mathbb{Z}) - G_{1,2}$ splits into subsemigroups G_{odd} and G_{even}, pairwise isomorphic by means of φ. Sets $G_3 \cup G_7$, $G_5 \cup G_7$, $G_4 \cup G_7$, $G_5 \cup G_6$ and their images under the automorphism φ are semigroups.*

The antiautomorphism φ_- of the group $GL(2,\mathbb{Z})$, changing the order of letters in the alphabet $\{u, v\}$ to the opposite, commutes with φ, acts identically on $G_{1,2}$, G_3 and $\varphi(G_3)$, maps semigroups $\varphi(G_7)$ and G_7 onto themselves, and also interchanges semigroups G_5 and G_4.

Let $GL^+(2,\mathbb{Z})$ be a subsemigroup of the group $GL(2,\mathbb{Z})$, consisting of matrices with positive entries. On the ground of Corollary 2 we get

Corollary 3. *The semigroups $GL^+(2,\mathbb{Z})$ is a two-sided ideal of the semigroup $GL_+(2,\mathbb{Z})$, containing in the union of (disjoint) semigroups G_4, G_5, G_6, G_7; moreover,*

$$\begin{array}{lclcl} G_4^+ & := & GL^+(2,\mathbb{Z}) \cap G_4 & = & G_4G_4, \\ G_5^+ & := & GL^+(2,\mathbb{Z}) \cap G_5 & = & G_5G_5, \\ G_6^+ & := & GL^+(2,\mathbb{Z}) \cap G_6 & = & \varphi(G_7), \\ G_7^+ & := & GL^+(2,\mathbb{Z}) \cap G_7 & = & G_7. \end{array}$$

All these intersections, as well as $GL^+(2,\mathbb{Z})$, are non-free. The automorphism φ from Corollary 2 interchanges pairs of (isomorphic) semigroups G_4^+,G_5^+ and G_6^+,G_7^+. The subsemigroup G_4^+ of the free semigroup G_4 has the unique smallest countable generating set, consisting of matrices, which are indecomposable into a product in G_4^+, and admits a countable (but does not admit a finite) set of defining relations with the genetic code (see (8)*)*

$$\{M(p,q), M(r,s,t),\ p,\ q,\ r,\ s,\ t \in \mathbb{N}\,; M(p,q)M(r,s,t) = M(t,p,q)M(r,s),$$

$$M(r,s,t)M(r_1,s_1,t_1) = M(s,t)M(t_1,r)M(r_1,s_1)\}.$$

The subsemigroup $SGL_{+,\leq}(2,\mathbb{Z})$ of the semigroup $GL_{+,\leq}(2,\mathbb{Z})$, consisting of matrices of determinant 1*, is contained in G_4^+ and has a system of the free generators $\{M(p,q);\, p,q \in \mathbb{N}\}$.*

At the end of the section, we give the statement on a connection between semigroups of the group $GL(2,\mathbb{Z})$, investigated above, and Pisot numbers.

Proposition 1. *Each element of one of the semigroup $GL_{+,\leq}(2,\mathbb{Z})$, $GL_{+,\geq}(2,\mathbb{Z})$ or $GL^+(2,\mathbb{Z})$ is a Pisot matrix.*

Proof. Let $Z = (z_{ij})$ $(i,j = 1,2)$ be any matrix of one of the above semigroup, and let

$$\lambda^2 + p\lambda + q = 0 \tag{25}$$

be its characteristic equation. Then $q = \pm 1$. According to Theorem 3, it is enough to prove the inequality $|p| > |1+q|$. If $q = -1$, then the condition $|p|(= z_{11} + z_{22}) > |1+q| = 0$ is evidently satisfied. Let suppose that $q = 1$. Then $|1+q| = 2$. In the case $Z \in GL^+(2,\mathbb{Z})$, equalities $z_{11} = z_{22} = 1$ are impossible; therefore $|p| = z_{11}+z_{22} > 2$. When $Z \in GL_{+,\leq}(2,\mathbb{Z})$, it is possible only the case $z_{11} \geq 1$; then $z_{22} \geq 2$ and $|p| > 2$. The statement is also fulfilled in the case $Z \in GL_{+,\geq}(2,\mathbb{Z})$, because the symmetry relatively to the secondary diagonal translates a matrix in $GL_{+,\geq}(2,\mathbb{Z})$ to a matrix in $GL_{+,\leq}(2,\mathbb{Z})$, and does not change p, q. ■

6. Asymptotic properties of linear-fractional transformations

Let us consider the question: when an infinite sequence of linear-fractional transformations $z_n := f(M(q_1,\dots,q_n))(z)$ (see (8) and (24)) of a point $z \in \mathbb{C}$, corresponding to a selected sequence of natural numbers $\{q_n\}$, converges to a point in $\mathbb{C}$ (in usual topology) and what are possible limits of the sequences z_n? The main role in the further reasoning play numbers

$$\mu(q) = [q; q, q, \dots] = q + \cfrac{1}{q + \cfrac{1}{q + \ddots}}\,. \tag{26}$$

The number $\mu := \mu(1)$ is known as *the golden section.* Therefore it is appropriate to call the number $\mu(q)$ by a *generalized golden section.* The answer to the above question gives the following

Theorem 9. *For any point $z \in \mathbb{C}$, $z \neq -(1/[0; q_1, q_2, \dots])$, the sequence of points $\{z_n = f(M(q_1, \dots, q_n))(z)\}$ is convergent if and only if the sequence $\{q_n\}$ converges to infinity or stabilizes. In both two cases $\lim\limits_{n\to\infty} z_n$ does not depend on a choice of the point z, is equal to zero in the first case and to infinite (periodic) continued fraction*

$$\frac{1}{\mu(q)} = [0; q, q, \dots]$$

in the second case, if the sequence $\{q_n\}$ stabilizes on the number $q \in \mathbb{N}$.

Proof. On the ground of formulas (10) and (24), we have

$$z_n = f(M(q_1, \dots, q_{n+1}))(z) = \frac{P_n z + Q_n}{P_{n+1} z + Q_{n+1}} = \frac{Q_n}{Q_{n+1}} \frac{\frac{P_n}{Q_n} z + 1}{\frac{P_{n+1}}{Q_{n+1}} z + 1}.$$

As a consequence of (2),(3), this sequence has a limit when $n \to \infty$ for any $z \neq -(1/[0; q_1, q_2, \dots])$ if and only if there is a limit of the sequence (Q_n/Q_{n+1}); moreover, these limits coincide.

The formula (14) gives the value of the fraction $a_{n+1} := (Q_n/Q_{n+1})$. It is known (see [21]), that for $n \geq 3$, inequalities

$$[0; q_{n+1}, q_n] < a_{n+1} < [0; q_{n+1}] \tag{27}$$

are satisfied. It follows from here that there are zero limits of sequences a_n, z_n, if the sequence q_n tends to $+\infty$.

If the sequence q_n is bounded from above by a number q and does not stabilize when $n \to \infty$, then for every natural number k there are natural numbers $m = n + 1 > n > k$, such that $q_m < q_n \leq q$. Then, on the ground of (27), we get inequalities

$$\begin{aligned} a_n < [0; q_n] \leq [0; q_{n+1} + 1] &< [0; q_{n+1}, q_{n+1} + 1] \\ &\leq [0; q_{n+1}, q_n] < a_{n+1} = a_m. \end{aligned}$$

Consequently,

$$|a_m - a_n| > [0; q_m, q_m + 1] - [0; q_m + 1] =$$
$$\frac{q_m}{(q_m + 1)(q_m(q_m + 1) + 1)} \geq \frac{1}{(q + 1)(q(q + 1) + 1)}$$

and the sequence a_n can not converge.

If q_n is not bounded from above and does not tend to $+\infty$, then there is a number $q > 0$ with the property that for every natural k there are natural numbers $n, m > k$, such that $q_n \leq q < q + 1 < q_m$. Then in view of (27),

$$a_m < [0; q_m] < [0; q_n + 1] \leq [0; q_n, q_{n-1}] < a_n,$$
$$|a_n - a_m| > [0; q_n + 1] - [0; q_m] > \frac{1}{q + 1} - \frac{1}{q + 2} = \frac{1}{(q + 1)(q + 2)}.$$

Therefore, the sequence a_n can not converge.

It remains the case, when the sequence q_n stabilizes, say, on q, beginning with a index k. Then for every number $n > k$ we get

$$|a_n - [0; q, q, \dots]| \leq |a_n - [0; q_n, \dots, q_k]| + |[0; q_n, \dots, q_k] - [0; q, q, \dots]|.$$

Under this $[0; q_n, \dots, q_k]$ is simultaneously the $(n-k)$-th convergent fraction P_{n-k}/Q_{n-k} both for a_n and for $[0; q, q, \dots]$. It follows from formulas (13) and (4) that every positive summand in the right part of the last inequality is less than the number $1/(Q_{n-k})^2$. Consequently, there is

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n = [0; q, q, \dots],$$

which completes the proof. ∎

Remark 6. *In addition to Theorem 9, let us remark that in the case, when $q_i \equiv q$, we have $z_n \equiv z$, if $z = -\frac{1}{[0;q_1,q_2,\dots]} = -[q; q, q, \dots] = -\mu(q)$ (or $z = \frac{1}{\mu(q)}$), because $f(M(q))(z) = z$ for these z. Restrictions of transformations $f(M(q)^{2n})$ to the Lobachevskii plane (that is the upper complex half-plane) constitute the semigroup of hyperbolic motions of the plane with fixed points $-\mu(q)$ and $\frac{1}{\mu(q)}$ on the ideal boundary of the plane.*

7. Relations between elements of natural powers of (2×2)-matrices

Now we turn to the description problem for interrelation between coefficients of matrices of the form Z^n.

Theorem 10. *Let us suppose that a (2×2)-matrix $Z = (z_{ij})$ has distinct (maybe, complex) nonzero eigenvalues, and $z_{12} \neq 0$. Then entries of the matrix*

$$Z^{n+1} = \begin{pmatrix} P_n & Q_n \\ R_n & T_n \end{pmatrix}$$

satisfy relations

$$Q_n = \frac{z_{12}(\lambda^{n+1} - \eta^{n+1})}{\lambda - \eta}, \quad R_n = \frac{z_{21}}{z_{12}} Q_n, \tag{28}$$

$$T_n - P_n = \frac{z_{22} - z_{11}}{z_{12}} Q_n, \quad T_n + P_n = \lambda^{n+1} + \eta^{n+1}, \tag{29}$$

where λ and η are distinct roots of the characteristic equation of the matrix Z.

Proof. Let us consider matrices

$$C := \begin{pmatrix} \frac{1}{\lambda-\eta} & z_{12} \\ \frac{\eta - z_{11}}{z_{12}(\lambda-\eta)} & \lambda - z_{11} \end{pmatrix}, \quad B := \begin{pmatrix} \lambda - z_{11} & -z_{12} \\ -\frac{\eta - z_{11}}{z_{12}(\lambda-\eta)} & \frac{1}{\lambda-\eta} \end{pmatrix}.$$

It is not difficult to verify that vector-columns of the matrix C are eigenvectors of the matrix Z, corresponding to eigenvalues η and λ, while $B = C^{-1}$. Consequently,

$$\begin{pmatrix} P_n & Q_n \\ R_n & T_n \end{pmatrix} = Z^{n+1}CC^{-1} = Z^{n+1}CB =$$

$$\begin{pmatrix} \eta^{n+1}\frac{1}{\lambda-\eta} & \lambda^{n+1} z_{12} \\ \eta^{n+1}\frac{\eta - z_{11}}{z_{12}(\lambda-\eta)} & \lambda^{n+1}(\lambda - z_{11}) \end{pmatrix} \begin{pmatrix} \lambda - z_{11} & -z_{12} \\ -\frac{\eta - z_{11}}{z_{12}(\lambda-\eta)} & \frac{1}{\lambda-\eta} \end{pmatrix}.$$

We get from here by direct calculations the required relations for entries of the matrix Z^{n+1}. ∎

Let us remark being easily proved claim.

Lemma 8. *Every quadratic equation of the form $\lambda^2 + p\lambda + q = 0$ with integer rational coefficients p and q is the characteristic equation of the unique integer (2×2)-matrix $Z = (z_{ij})$ with $z_{11} = 0$ and $z_{12} = 1$.*

Proof. Evidently, the characteristic equation of every (2×2)-matrix $Z = (z_{ij}); i, j = 1, 2$, has the form $\lambda^2 + p\lambda + q = 0$, where

$$p = -(z_{11} + z_{22}), \quad q = z_{11}z_{22} - z_{12}z_{21},$$

and for given integer numbers p and q, integer numbers z_{ij} are defined by equalities $z_{11} = 0$, $z_{22} = -p$, $z_{12} = 1$, $z_{21} = -q$. ∎

Further we apply the statement of Theorem 10 to matrices $Z = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$, where $p, q \in \mathbb{Z}$, $p^2 - 4q \neq 0$. One can easily deduce from Theorem 10 the following

Corollary 4. *Let us consider an integer (2×2)-matrix $Z = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$ with the condition $p^2 - 4q \neq 0$. Then the entries of the matrice*

$$Z^{n+1} = \begin{pmatrix} P_n & Q_n \\ R_n & T_n \end{pmatrix}$$

satisfy relations

$$R_n = P_{n+1}, \quad T_n = Q_{n+1}, \quad Q_n = \frac{\lambda^{n+1} - \eta^{n+1}}{\lambda - \eta}, \quad P_{n+1} = -qQ_n, \tag{30}$$

$$Q_{n+1} - P_n = -pQ_n, \quad Q_{n+1} + P_n = \lambda^{n+1} + \eta^{n+1}, \tag{31}$$

where λ and η are distinct roots of the characteristic equation $\lambda^2 + p\lambda + q = 0$ of the matrix Z.

8. Generalized Fibonacci and Lucas numbers

Definition 6. *The number $F_{q,n}$, which is equal to the number P_n in formulas (4) and (6), where $q_0 = 0$, while all the numbers $q_i, i \geq 1$, are equal to q, is called a generalized Fibonacci number (see* [28]*). The number $L_{q,n} = F_{q,n-1} + F_{q,n+1}$ will be called a generalized Lucas number.*

Remark 7. *The numbers $F_n = F_{1,n}$ ($L_n = L_{1,n}$) are classical Fibonacci (Lucas) numbers. The numbers $F_{2,n}$ ($L_{2,n}$) are said to be Pell (Pell-Lucas) numbers, see* [35], [33]*. Evidently,*

$$\lim_{n\to\infty} \frac{F_{q,n+1}}{F_{q,n}} = \lim_{n\to\infty} \frac{L_{q,n+1}}{L_{q,n}} = \mu(q) = [q; q, q, \dots].$$

Besides Theorem 9 from the section 5, generalized golden sections and Fibonacci and Lucas numbers have some other properties, which are common for all $q \in \mathbb{N}$. We discuss in this section some of these properties.

Obviously,

$$\mu(q) - \frac{1}{\mu(q)} = [q; q, q, \dots] - [0; q, q, \dots] = q,$$

that is $\lambda := \mu(q)$ is a root of the quadratic equation

$$\lambda - \frac{1}{\lambda} = q \iff \lambda^2 - q\lambda - 1 = 0. \tag{32}$$

It is clear that this equation is an equation of the form (25) (where one needs to set $p := -q, q := -1$), satisfying conditions of Theorem 3. It is easily to see that $\lambda^2 - q\lambda - 1$ is the characteristic polynomial of the matrix $Z := M(q)$ (see (7)) with the roots

$$\lambda_{1,2} = \frac{q \pm \sqrt{q^2+4}}{2}, \lambda_1 = \lambda, \lambda_2 = -\frac{1}{\lambda}. \tag{33}$$

Formulas (30) and (31) in the case under consideration imply (taking into account that $P_n = Q_{n-1}$) the equalities

$$F_{q,n} = \frac{\lambda^n - \left(-\frac{1}{\lambda}\right)^n}{\lambda + \frac{1}{\lambda}} = \frac{1}{\sqrt{q^2+4}} \left(\mu(q)^n - \left(-\frac{1}{\mu(q)} \right)^n \right), \tag{34}$$

$$L_{q,n} = \lambda^n + \left(-\frac{1}{\lambda}\right)^n = \mu(q)^n + \left(-\frac{1}{\mu(q)}\right)^n. \tag{35}$$

The formula (34) is well-known in the particular case $q=1$ (see the book [30]). It implies that

$$\left|F_{q,n}-\frac{1}{\sqrt{q^2+4}}\mu(q)^n\right|=\frac{1}{\sqrt{q^2+4}}\,\frac{1}{\mu(q)^n} \tag{36}$$

very quickly tends to zero and always less than $1/2$. Analogously,

$$|L_{q,n}-\mu(q)^n|=\frac{1}{\mu(q)^n} \tag{37}$$

As a corollary, we get the rule:

Proposition 2. *The generalized Fibonacci number $F_{q,n}$ (respectively, the generalized Lucas number $L_{q,n}$ where $n\geq 2$ or $q\geq 2$) is equal to the nearest integer number to the number $\frac{1}{\sqrt{q^2+4}}\mu(q)^n$ (respectively, $\mu(q)^n$).*

From the formula (34) we get its another version:

$$F_{q,n}=\frac{\mu(q)^n-\left(-\frac{1}{\mu(q)}\right)^n}{\mu(q)+\frac{1}{\mu(q)}}=\sum_{k=0}^{n-1}(\mu(q))^{n-1-k}\left(-\frac{1}{\mu(q)}\right)^k. \tag{38}$$

In particular, it follows from here

Proposition 3. *The generalized Fibonacci number $F_{q,n}$ is the sum of a geometric progression of the length n with the initial member $\mu(q)^{n-1}$ and with the denominator $-\left(\frac{1}{\mu(q)}\right)^2$.*

Formulas (34) and (35) show that all the numbers $f_{\lambda,n}:=F_{q,n}$ (respectively, $l_{\lambda,n}:=L_{q,n}$) and $\lambda:=\mu(q), q\in\mathbb{N}$, satisfy common relations

$$f_{\lambda,n}=\frac{\lambda^n-\left(-\frac{1}{\lambda}\right)^n}{\lambda+\frac{1}{\lambda}}\in\mathbb{N}\,,\quad \lambda>1, \tag{39}$$

$$l_{\lambda,n}=\lambda^n+\left(-\frac{1}{\lambda}\right)^n\in\mathbb{N}\,,\quad \lambda>1. \tag{40}$$

Only generalized golden sections and generalized Fibonacci and Lucas numbers, corresponding to them by formulas (39) and (40), satisfy all these relations, because these formulas for $n=2$ and $n=1$ give respectively

$$f_{\lambda,2}=\frac{\lambda^2-\frac{1}{\lambda^2}}{\lambda+\frac{1}{\lambda}}=\lambda-\frac{1}{\lambda}\in\mathbb{N};\quad l_{\lambda,1}=\lambda-\frac{1}{\lambda}\in\mathbb{N}\,,$$

i.e. the equation (32) for $q=f_{\lambda,2}=l_{\lambda,1}$, and the equality $\lambda=\mu(q)$.

Formulas (7) and (10) for $q_i=q$, $i\geq 1$ imply the matrix equality

$$\begin{pmatrix}P_{n+m+1} & Q_{n+m+1}\\ P_{n+m+2} & Q_{n+m+2}\end{pmatrix}=\begin{pmatrix}P_n & Q_n\\ P_{n+1} & Q_{n+1}\end{pmatrix}\begin{pmatrix}P_m & Q_m\\ P_{m+1} & Q_{m+1}\end{pmatrix}.$$

Therefore,

$$P_{n+m+1}=P_nP_m+Q_nP_{m+1}=P_nP_m+P_{n+1}P_{m+1}$$

or

$$F_{q,n+m+1}=F_{q,n}F_{q,m}+F_{q,n+1}F_{q,m+1}. \tag{41}$$

This formula for the case $q=1$ is given in the book [30].

It's naturally to regard that for a natural number q,

$$\begin{pmatrix}F_{q,-2} & F_{q,-1}\\ F_{q,-1} & F_{q,0}\end{pmatrix}=\begin{pmatrix}P_{-2} & Q_{-2}\\ P_{-1} & Q_{-1}\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & q\end{pmatrix}^{-1}=\begin{pmatrix}-q & 1\\ 1 & 0\end{pmatrix}.$$

Then

$$\begin{pmatrix}L_{q,-1} & L_{q,0}\\ L_{q,0} & L_{q,1}\end{pmatrix}=\begin{pmatrix}-q & 2\\ 2 & q\end{pmatrix},$$

$$\begin{pmatrix} L_{q,n} & L_{q,n+1} \\ L_{q,n+1} & L_{q,n+2} \end{pmatrix} = M(q)^{n+1} \begin{pmatrix} -q & 2 \\ 2 & q \end{pmatrix}, \tag{42}$$

$$\begin{pmatrix} L_{q,n+m+1} & L_{q,n+m+2} \\ L_{q,n+m+2} & L_{q,n+m+3} \end{pmatrix} = M(q)^{n+1} \begin{pmatrix} L_{q,m} & L_{q,m+1} \\ L_{q,m+1} & L_{q,m+2} \end{pmatrix},$$

$$M(q)^{n+1} = \begin{pmatrix} F_{q,n} & F_{q,n+1} \\ F_{q,n+1} & F_{q,n+2} \end{pmatrix}.$$

We get from here the formula

$$L_{q,n+m+1} = F_{q,n}L_{q,m} + F_{q,n+1}L_{q,m+1}. \tag{43}$$

Further, we get from formulas (12) and (42) respectively

$$F_{q,n}^2 = F_{q,n-1}F_{q,n+1} + (-1)^{n+1}, \tag{44}$$

$$L_{q,n}^2 = L_{q,n-1}L_{q,n+1} + (-1)^n(q^2+4). \tag{45}$$

Remark 8. *Formulas* (30), (31), (34), *and* (35) *show that one can consider the numbers* λ, Q_n, *and* $P_n + Q_{n+1}$ *in formulas* (30) *and* (31), *corresponding to a Pisot matrix* Z, *as further generalizations of golden sections and Fibonacci and Lucas numbers.*

For $q = 1, 2$, it is known (see [18]) a connection of the number $\mu(q)$ with the regular star-liked (q^2+4)-gon. One can describe this connection as follows. Continuations of sides of the regular convex 5-gon together with its sides bounds 5 equal isosceles triangles with the proportion of (non-equal) sides $\mu(1) = \frac{1+\sqrt{5}}{2}$. Continuations of sides of the regular convex 8-gon bounds 8 equal symmetric (relative their diagonal) 4-gons with the proportion of (non-equal) sides $\mu(2) = \frac{2+\sqrt{8}}{2} = 1 + \sqrt{2}$.

Question 4. *Is it true that for every natural number* $q \geq 3$, *one can find symmetric quadrangles or triangles, bounded by continuations of sides of a regular convex* (q^2+4)*-gon and having sides with the proportion* $\mu(q)$*?*

Remark 9. $F_5 = 5 = 1^2 + 4, \quad F_6 = 8 = 2^2 + 4, \quad F_7 = 13 = 3^2 + 4.$

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