Generating functions for modular graphs and Burgers equation.

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1 Introduction.

A Deligne-Mumford stable pointed curve is an algebraic curve having at most nodal singularities and $n \geq 0$ ordered nonsingular points. Stability means that each rational irreducible component has at least three singular or marked points, and each elliptic component has at least one such a point. The dual modular graph of a pointed nodal curve X is a graph whose set of vertices is the set of irreducible components of the curve X, the set of edges is the set of its nodal points and the set of half-edges is the set its of marked points. For each modular graph Γ consider the moduli space M_Γ of curves whose dual modular graph is Γ. Then the Deligne-Mumford compactification $\overline{M}_{g,n}$ of the moduli space $M_{g,n}$ of n-pointed genus g curves has the stratification

$$\overline{M}_{g,n} = \bigcup_{\substack{\text{all genus } g \\ \text{modular graphs } \Gamma \\ \text{with } n \text{ half-edges}}} M_\Gamma. \tag{1.1}$$

Formally speaking a modular graph Γ may be defined by the following data $(V, \overrightarrow{E}, i, \overrightarrow{E}_-, s, g, l)$, where:

(1) V is the finite set of *vertices* of Γ;

(2) $\overrightarrow{E}$ is the finite set of *oriented edges* of the modular graph Γ;

(3) $i\colon \overrightarrow{E} \to \overrightarrow{E}$ is an *orientation-changing involution*, (that is fixed point free).

(4) $\overrightarrow{E}_-$ is the set of *outgoing oriented edges* of the modular graph Γ; it is claimed that $\overrightarrow{E} = \overrightarrow{E}_- \cup i(\overrightarrow{E}_-)$, so that each edge is either incoming or outgoing;

(5) $s\colon \overrightarrow{E}_- \to V$ is a surjective *source map*, assigning to every outgoing edge from $\overrightarrow{E}_-$ its source vertex.

(6) $g\colon V \to \{0,1,2,3,\ldots\}$ — the genus function;

(7) the set $\overrightarrow{H}_- = \overrightarrow{E}_- \setminus i(\overrightarrow{E}_-)$ is the set of outgoing half-edges (incident to only one vertex), for a nonempty $\overrightarrow{H}_-$ the bijection $l\colon \overrightarrow{H}_- \to \{1,2,3,\ldots,n\}$ defines the ordering of the set of half-edges of the modular graph Γ.

The *set of non-oriented edges* of a modular graph Γ is the quotient set $E = (\overrightarrow{E} \cap i(\overrightarrow{E}))/i$. An isomorphism of two modular graphs is given by a pair of bijections between the corresponding sets of vertices and corresponding sets of oriented edges, preserving all the described data. Note that a non-trivial automorphism of a modular graph may be identic both on the set of vertices and on the set of non-oriented edges. The number $\nu(v) = |s^{-1}(v)|$ of outgoing oriented edges incident to a given vertex v is called the *valence* of the vertex v. A modular graph is called *stable* if $2(g(v)-1)+\nu(v) > 0$ for any vertex $v \in V(\Gamma)$. The number $g(\Gamma) = \sum_{v\in V(\Gamma)} g(v) + \dim H_1(\Gamma)$ is the *genus* of the connected modular graph Γ; $n(\Gamma)$ is the number of half-edges of the modular graph Γ. Suppose that $g(v) = 0$ for all vertices of the modular graph Γ, such graphs will be called *combinatorial graphs* or simply *graphs*; stability in this case means that the valency of every vertex is at least three. In this article all the graphs are supposed to be connected.

Let $\{\mu_{g,n}\ ,\ 2(g-1)+n>0\}$ be a set of (commutative) variables, Γ — a modular graph. Consider the monomial:

$$\mu(\Gamma) = \frac{1}{|\operatorname{Aut}\Gamma|} \prod_{v\in V(\Gamma)} \mu_{g(v),\nu(v)}, \tag{1.2}$$

where $\operatorname{Aut}\Gamma$ is the automorphism group of the modular graph Γ.

Denote by $\mathcal{G}^k_{g,n}$ the set of all genus g modular graphs with k edges and n half-edges, consider the polynomials

$$\mu^k_{g,n} = \sum_{\Gamma\in\mathcal{G}^k_{g,n}} \mu(\Gamma) \tag{1.3}$$

and the generating functions

$$\Psi(s,t,\hbar)=\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{3g-3+n}\mu_{g,n}^{k}\frac{t^n}{n!}s^k\hbar^{g-1} \tag{1.4}$$

and

$$\Phi(s,t,\hbar)=\frac{\partial\Psi(s,t,\hbar)}{\partial t}=\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{3g-3+n}\mu_{g,n}^{k}\frac{t^{n-1}}{(n-1)!}s^k\hbar^{g-1}. \tag{1.5}$$

Note that

$$\Psi(1,t,\hbar)=\sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\Gamma\in\mathcal{G}_{g,n}}\mu(\Gamma)\frac{t^n}{n!}\hbar^{g-1} \tag{1.6}$$

is the partition function, usually considered in the quantum field theory (here $\mathcal{G}_{g,n}=\bigcup_k\mathcal{G}_{g,n}^k$ is the set of all genus g modular graphs with n half-edges).

We prove that the functions Φ and Ψ satisfy the Burgers equation:

Theorem 1.1 *The function $\Psi(s,t,\hbar)$ satisfies the potential form of the Burgers equation:*

$$\frac{\partial\Psi}{\partial s}=\frac{\hbar}{2}\left[\frac{\partial^2\Psi}{\partial t^2}+\left(\frac{\partial\Psi}{\partial t}\right)^2\right]; \tag{1.7}$$

and the function $\Phi(s,t,\hbar)$ satisfies the Burgers equation:

$$\frac{\partial\Phi}{\partial s}=\frac{\hbar}{2}\left[\frac{\partial^2\Phi}{\partial t^2}+2\Phi\frac{\partial\Phi}{\partial t}\right]. \tag{1.8}$$

Note that the initial condition $\Psi(0,t,\hbar)$ for the Burgers equation is the sum over the set of all edgeless graphs $\mathcal{G}_{g,n}^0$. For each pair (g,n) such that $2(g-1)+n>0$ the set $\mathcal{G}_{g,n}^0$ has only one element — the modular tree $S_{g,n}$ that haves one genus g vertex and n half-edges. This tree corresponds to the moduli space of all nonsingular n-pointed curves: $M_{S_{g,n}}=M_{g,n}$. Therefore

$$\Psi(0,t,\hbar)=\sum_{\substack{g\geq 0\\ n\geq 0\\ 2(g-1)+n>0}}\mu_{g,n}\frac{t^n}{n!}\hbar^{g-1}. \tag{1.9}$$

There are many ways of specializing the variables $\{\mu_{g,n}\}$ that provide interesting generating functions Ψ (or Φ).

(1) **Counting functions for combinatorial graphs of definite type.**

(a) For an integer $d \geq 3$ put

$$\mu_{g,n} = \begin{cases} 1 & \text{if } g = 0 \quad n = d \\ 0 & \text{either} \end{cases} \tag{1.10}$$

Then Ψ is the counting functions for all d-valent (combinatorial) graphs:

$$\Psi(s,t,\hbar) = \sum_{g,n,k} \left[\sum_{\substack{\text{All genus } g \ d\text{-valent} \\ \text{graphs } \Gamma \text{ with } k \text{ edges} \\ \text{and } n \text{ half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|} \right] \frac{t^n}{n!} s^k \hbar^{g-1} \tag{1.11}$$

(Note that the sum in brackets is nonzero only for $(d-2)k = n + d(g-1)$.)

In this case the initial condition is:

$$\Psi(0,t,\hbar) = \frac{t^d}{d!\hbar} \quad \text{or} \quad \Phi(0,t,\hbar) = \frac{t^{d-1}}{(d-1)!\hbar}. \tag{1.12}$$

Below we present explicit formulas for the most interesting case of trivalent graphs ($d = 3$).

(b) Put

$$\mu_{g,n} = \begin{cases} 1 & \text{if } g = 0 \\ 0 & \text{either} \end{cases} \tag{1.13}$$

We obtain the counting series for all stable (combinatorial) graphs:

$$\Psi(s,t,\hbar) = \sum_{g,n,k} \left[\sum_{\substack{\text{All genus } g \\ \text{stable graphs } \Gamma \\ \text{with } k \text{ edges and} \\ n \text{ half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|} \right] \frac{t^n}{n!} s^k \hbar^{g-1} \tag{1.14}$$

Initial condition for this case is:

$$\Psi(0,t,\hbar) = (e^t - \frac{t^2}{2} - t - 1)\frac{1}{\hbar}. \tag{1.15}$$

(c) Putting $\mu_{g,n} = 1$ for all g, n provides the counting function for all modular graphs:

$$\Psi(s,t,\hbar) = \sum_{g,n,k} \left[\sum_{\substack{\text{all modular graphs } \Gamma \\ \text{genus } g \quad \text{with } k \text{ edges} \\ \text{and } n \text{ half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|} \right] \frac{t^n}{n!} s^k \hbar^{g-1} \tag{1.16}$$

Initial condition for this case is:

$$\Psi(0,t,\hbar) = (e^t - \frac{t^2}{2} - t - 1)\frac{1}{\hbar} - 1 + \frac{e^t}{1-\hbar}. \tag{1.17}$$

(2) **Virtual motivic measure of** $M_{g,n}$

Choose some motivic measure v, attaching to every nonsingular algebraic variety X an element $v(X)$ of a certain commutative $\mathbb{Q}$-algebra, satisfying the following conditions:

(a) $v(X \setminus Y) + v(Y) = v(X)$ for any closed nonsingular subvariety $Y \subset X$;

(m) $v(X \times Z) = v(X)v(Z)$.

The corresponding virtual motivic measure $\tilde{v}$ of an orbifold X is defined by $\tilde{v}(X) = v(\tilde{X})/N$, where $\tilde{X} \to X$ is an unramified covering of orbifolds and X is nonsingular. (It is sufficient to have such a covering for each strata of some stratification of X). Denote $\mu_{g,n} = \tilde{v}(M_{g,n})$. Then it is not hard to deduce that

$$\mu(\Gamma) = \tilde{v}(M_\Gamma) \tag{1.18}$$

and

$$\mu^k_{g,n} = \tilde{v}(M^k_{g,n}), \tag{1.19}$$

where $M^k_{g,n}$ is the moduli space of Deligne-Mumford stable n-pointed curves, having exactly k nodal points. Note that for fixed g and n the spaces $M^k_{g,n}$ form a stratification of $\overline{M}_{g,n}$ and $\operatorname{codim}_{\overline{M}_{g,n}} M^k_{g,n} = k$. So the generating functions (1.4) for this case is

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{3g-3+n} \tilde{v}(M^k_{g,n})\frac{t^n}{n!}s^k\hbar^{g-1}. \tag{1.20}$$

Thus the partition function (1.6) for this case is the generating function for the values of the virtual motivic measure $\tilde{v}(\overline{M}_{g,n})$ of the compactified moduli space $\overline{M}_{g,n}$:

$$\Psi(1,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\tilde{v}(\overline{M}_{g,n})\frac{t^n}{n!}\hbar^{g-1}, \tag{1.21}$$

and the initial condition (1.9) is the generating function for the values of the virtual motivic measure $\tilde{v}(M_{g,n})$ of the moduli space of nonsingular curves $M_{g,n}$:

$$\Psi(0,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\tilde{v}(M_{g,n})\frac{t^n}{n!}\hbar^{g-1}. \tag{1.22}$$

For such virtual motivic measure $\tilde{v}$ we may take the virtual Poincare polynomial of X (see [8] or [7]), or the virtual Euler characteristic of X, or the number of points of $X(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. But an explicit formula for the initial condition is known only for the case of virtual Euler characteristic. It is given by the well-known result by Harer-Zagier [9]: for $g > 0$

$$\tilde{\chi}(M_{g,n}) = (-1)^n\frac{(2g-3+n)!(2g-1)}{(2g)!}B_{2g} \tag{1.23}$$

for $g \geq 2$, $n \geq 0$ or $g = 1$, $n \geq 1$. Adding the $g = 0$ case (see [7] or section 6), we obtain the generating functions

$$\Psi(0,t,\hbar) = \frac{2(1+t)^2\ln(1+t) - 2t - 3t^2}{4\hbar} - \frac{B_2}{2}\ln(1+t) + \sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\frac{\hbar^{g-1}}{(1+t)^{2g-2}} \tag{1.24}$$

and

$$\Phi(0,t,\hbar) = \frac{(1+t)\ln(1+t)-t}{\hbar} - \sum_{g=1}^{\infty} \frac{B_{2g}}{2g} \frac{\hbar^{g-1}}{(1+t)^{2g-1}}. \tag{1.25}$$

In all the described cases we need to solve the Cauchy problem for the Burgers equation with the initial condition given by one of the formulas (1.12), (1.15), (1.17) or (1.25).

The equation (1.7) may be linearized by the Cole-Hopf transform (see [5], [6]):

$$\Psi = \ln F. \tag{1.26}$$

Substituting in (1.7) we obtain the heat equation

$$\frac{\partial F}{\partial s} = \frac{\hbar}{2} \frac{\partial^2 F}{\partial t^2} \tag{1.27}$$

with the initial condition

$$F(0,t,\hbar) = e^{\Psi(0,t,\hbar)} = e^{\sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \mu_{g,n} \frac{t^n}{n!} \hbar^{g-1}}. \tag{1.28}$$

The solution of the Cauchy problem for (1.27) with the initial condition (1.28), formally expressed by the Poisson integral is known for $s = 1$ as Wick's theorem (see [1]):

$$\Psi(s,t,\hbar) = \ln \left(\frac{1}{\sqrt{2\pi\hbar s}} \int_{-\infty}^{\infty} e^{\Psi(0,\xi,\hbar) - \frac{(t-\xi)^2}{2s\hbar}} d\xi \right). \tag{1.29}$$

Of course (1.29) should be considered as an equality of formal Laurent series in $\hbar$, but unfortunately the usage of the Poisson integral can not be justified because the initial conditions (1.12), (1.15), (1.17) and (1.25) are unbounded, so that (1.29) diverges. Moreover, A.N.Tykhonov in 1935 (see [11]) has proved that the solution of the Cauchy problem for the heath equation with the initial condition growing faster than e^{t^2} is no longer unique. That is just the case for all our examples. For instance for the virtual number of trivalent graphs we have the following integral:

$$\Psi(s,t,\hbar) = \ln \left(\frac{1}{\sqrt{2\pi\hbar s}} \int_{-\infty}^{\infty} e^{\frac{1}{\hbar}\left[\frac{\xi^3}{6} - \frac{(t-\xi)^2}{2s}\right]} d\xi \right). \tag{1.30}$$

In (1.57) we present an explicit formula for a one-parametric family of solutions of (1.27) with the initial condition $\frac{t^3}{6\hbar}$.

Of course (1.29) may be considered as a distribution but this can hardly help us to get the coefficients of the generating function. Instead of that we may try to expand the solution by the powers of $\hbar$:

$$\Phi(s,t,\hbar) = \sum_{g=0}^{\infty} \Phi_g(s,t)\hbar^{g-1}, \tag{1.31}$$

and

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty} \Psi_g(s,t)\hbar^{g-1}, \tag{1.32}$$

where $\Phi_g = \frac{\partial \Psi_g}{\partial t}$, and then try to find a recursive formula for the functions Φ_g or Ψ_g.

In this way we get quasi-linear equation for Φ_0:

$$\frac{\partial \Phi_0}{\partial s} = \Phi_0 \frac{\partial \Phi_0}{\partial t}. \tag{1.33}$$

and recursive quasi-linear equation for Φ_g and Ψ_g for $g > 0$:

$$\frac{\partial \Phi_g}{\partial s} = \frac{1}{2}\frac{\partial^2 \Phi_{g-1}}{\partial t^2} + \Phi_0 \frac{\partial \Phi_g}{\partial t} + \Phi_g \frac{\partial \Phi_0}{\partial t} + \sum_{i=1}^{g-1} \Phi_i \frac{\partial \Phi_{g-i}}{\partial t}. \tag{1.34}$$

$$\frac{\partial \Psi_g}{\partial s} = \frac{1}{2}\frac{\partial^2 \Psi_{g-1}}{\partial t^2} + \Phi_0 \frac{\partial \Psi_g}{\partial t} + \frac{1}{2}\sum_{i=1}^{g-1} \frac{\partial \Psi_i}{\partial t}\frac{\partial \Psi_{g-i}}{\partial t}. \tag{1.35}$$

Solving (1.33) with the initial condition $\Phi_0(0,t) = \Phi(0,t,0)$ we obtain the following description of generating functions for modular trees ($g = 0$). Note that the moduli space $\overline{M}_{0,n}$ is smooth and modular trees have no automorphisms, so for $g = 0$ we obtain the decent Poincare polynomials or Euler characteristics or the number of trees.

Theorem 1.2 *The formal series*

$$\alpha_s(t) = t - s\Phi_0(0,t) = t - s\sum_{n=3}^{\infty} v(M_{0,n})\frac{t^{n-1}}{(n-1)!} \tag{1.36}$$

and

$$\beta_s(t) = t + s\Phi_0(s,t) = t + s\sum_{n=3}^{\infty}\left(\sum_{k=0}^{n-3} v(M_{0,n}^k)s^k\right)\frac{t^{n-1}}{(n-1)!} \tag{1.37}$$

are inverse to each other with respect to the composition of functions; the function $\Phi_0(s,t)$ *satisfies the functional equations*

$$\Phi_0(s,t) = \Phi_0(0, t + s\Phi_0(s,t)) \tag{1.38}$$

and

$$\Phi_0(0,t) = \Phi_0(s, t - s\Phi_0(0,t)). \tag{1.39}$$

Corollary 1.1 *(1) The counting function for the number of trivalent trees is*

$$\Phi_0(s,t) = \frac{1 - st - \sqrt{1-2st}}{s^2} \quad \textit{and} \quad \Phi_0(1,t) = 1 - t - \sqrt{1-2t} \tag{1.40}$$

(2) The counting function for the number of stable trees $\Phi_0(s,t)$ *satisfies the functional equation*

$$e^{t+s\Phi_0(s,t)} = 1 + t + (1+s)\Phi_0(s,t) \tag{1.41}$$

and the differential equation

$$(\Phi_0(s,t))'_t = \frac{t + (s+1)\Phi_0(s,t)}{1 - s[t + (s+1)\Phi_0(s,t)]}. \tag{1.42}$$

(3) The generating function $\Phi_0(s,t)$ *for the Poincare polynomial of* $\overline{M}_{0,n}$ *satisfies the functional equation*

$$(1 + t + s\Phi_0(s,t))^q = q(q+s-1)\Phi_0(s,t) + qt + 1, \tag{1.43}$$

and the differential equation

$$(\Phi_0(s,t))' = \frac{t + (q+s)\Phi_0(s,t)}{1 + t - st - s(q+s-1)\Phi_0(s,t)}. \tag{1.44}$$

(4) The generating function $\Phi_0(s,t)$ *for the Euler characteristic of* $\overline{M}_{0,n}$ *satisfies the functional equation*

$$\ln(1+t+s\Phi_0(s,t)) = \frac{t+(s+1)\Phi_0(s,t)}{1+t+s\Phi_0(s,t)} \tag{1.45}$$

and the differential equation

$$(\Phi_0(s,t))'_t = \frac{t+(s+1)\Phi_0(s,t)}{1+t-st-s^2\Phi_0(s,t)}. \tag{1.46}$$

The equations for Poincare polynomial and Euler characteristic are well-known for $s=1$ (see [7] or [2].

The function $\Phi_0(s,t)$ is essentially used in the recursive formulas for the solutions of the equations (1.35) for $g>0$ based on the following integral representation.

Theorem 1.3 *The solution*

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty} \Psi_g(s,t)\hbar^{g-1}. \tag{1.47}$$

of the potential Burgers equation (1.7) with the initial condition $\Psi(0,t,\hbar)$ *is given by*

$$\Psi(s,t,\hbar) = \Psi(0,t+s\Phi_0(s,t)),\hbar)+ \\ +\frac{\hbar}{2}\int_0^s \left[\frac{\partial^2\Psi}{\partial t^2} + \left(\frac{\partial(\Psi-\Psi_0)}{\partial t}\right)^2\right](\sigma, t+(s-\sigma)\Phi_0(s,t),\hbar)\,d\sigma \tag{1.48}$$

As the result we obtain explicit recursive formulas.

Corollary 1.2 *For* $g>0$

$$\Psi_g(s,t) = \Psi_g(0,t+s\Phi_0(s,t)))+ \\ +\frac{1}{2}\int_0^s \left[\frac{\partial^2\Psi_{g-1}}{\partial t^2} + \sum_{i=1}^{g-1}\frac{\partial\Psi_i}{\partial t}\frac{\partial\Psi_{g-i}}{\partial t}\right](\sigma, t+(s-\sigma)\Phi_0(s,t))\,d\sigma \tag{1.49}$$

Now we are in position to apply (1.49) to any case, for which we are able to find $\Phi_0(s,t)$ and the initial condition $\Psi(0,t,\hbar)$.

1.1 Trivalent graphs.

We have seen that for stable trivalent graphs $\Phi_0(s,t) = \frac{1-st-\sqrt{1-2st}}{s^2}$ and $\Psi(0,t,\hbar) = t^3/6\hbar$.

In this case we present a one-parametric family of explicit solutions of the equations (1.34) in terms of modified Bessel functions $I_\nu(z)$ or Airy functions $Ai(z)$ and $Bi(z)$ (see the definitions in [4]). All these solutions are analytic outside the line $s=0$, and have infinitely many derivatives on this line, and each solution provides the same (divergent) expansion in s and t.

Theorem 1.4 *The counting function Φ for trivalent graphs*

$$\Phi(s,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{3g-3+n}\left[\sum_{\substack{\text{genus } g \text{ trivalent} \\ \text{graphs } \Gamma \text{ with } k \text{ edges} \\ \text{and } n \text{ labelled half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|}\right]\frac{t^{n-1}}{(n-1)!}s^k\hbar^{g-1}$$

is the asymptotic expansion of

$$\begin{aligned}
&\frac{1-st}{s^2\hbar} - \frac{\sqrt{1-2st}}{s^2\hbar}\left[\frac{C_1 I_{-2/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right) + C_2 I_{2/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)}{C_1 I_{1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right) + C_2 I_{-1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)}\right] = \\
&\quad = \frac{1-st}{s^2\hbar} - \frac{2^{1/3}}{s\hbar^{2/3}}\left[\frac{C_1' Ai'\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right) + C_2' Bi'\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)}{C_1' Ai\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right) + C_2' Bi\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)}\right] = \\
&\quad = \frac{1-st-\sqrt{1-2st}}{s^2\hbar} + \frac{s}{1-2st}W\left(\frac{s^3\hbar}{(\sqrt{1-2st})^3}\right), \qquad (1.50)
\end{aligned}$$

where C_1, C_2, C_1', C_2' are arbitrary constants ($C_1' = \sqrt{3}(C_2 - C_1)$ and $C_2' = C_2 + C_1$), and

$$W(u) = \frac{1}{u}\left[1 - \frac{C_1 I_{-2/3}\left(\frac{1}{3u}\right) + C_2 I_{2/3}\left(\frac{1}{3u}\right)}{C_1 I_{1/3}\left(\frac{1}{3u}\right) + C_2 I_{-1/3}\left(\frac{1}{3u}\right)}\right].$$

The counting function for trivalent graphs with one half-edge (see also table 1) *is the asymptotic expansion of $W(u)$:*

$$W(u) \sim \sum_{g=1}^{\infty} \left[\sum_{\substack{\text{genus } g \text{ trivalent} \\ \text{graphs } \Gamma \text{ with} \\ \text{one half-edge}}} \frac{1}{|\operatorname{Aut}\Gamma|} \right] u^{g-1} =$$

$$= \frac{1}{u}\left[1 - \frac{1 - \frac{(16/9-1)}{1!}\frac{3}{8}u + \frac{(16/9-1)(16/9-9)}{2!}(\frac{3}{8}u)^2 - \frac{(16/9-1)(16/9-9)(16/9-25)}{3!}(\frac{3}{8}u)^3 + \dots}{1 - \frac{(4/9-1)}{1!}\frac{3}{8}u + \frac{(4/9-1)(4/9-9)}{2!}(\frac{3}{8}u)^2 - \frac{(4/9-1)(4/9-9)(4/9-25)}{3!}(\frac{3}{8}u)^3 + \dots}\right] =$$

$$= \frac{\sum_0^\infty \frac{(6n+1)!}{2(2n)!(3n)!}\frac{u^n}{288^n}}{\sum_0^\infty \frac{(6n)!}{(2n)!(3n)!}\frac{u^n}{288^n}} = \frac{1}{2} + \frac{5}{8}u + \frac{15}{8}u^2 + \frac{1105}{128}u^3 + \frac{1695}{32}u^4 + \frac{414125}{1024}u^5 + \dots. \tag{1.51}$$

Denote

$$W(u) = \sum_{g=1}^{\infty} \tau_g u^{g-1}, \tag{1.52}$$

then[1]

$$\sum_{\substack{\text{genus } g \text{ trivalent} \\ \text{graphs } \Gamma \text{ with } k \text{ edges} \\ \text{and } n \text{ labelled half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|} = \begin{cases} (2n-5)!! & \text{for } g=0,\ n \geq 3 \\ \frac{1}{2}(2n-2)!! & \text{for } g=1,\ n \geq 1 \\ \tau_g \frac{(3g+2n-5)!!}{(3g-3)!!} & \text{for } g \geq 2, \end{cases} \tag{1.53}$$

For $g > 1$ the numbers $\tau_g \quad g > 1$ satisfy the following recurrence:

$$\tau_g = \frac{1}{2}\left((3g-4)\tau_{g-1} + \sum_{i=1}^{g-1} \tau_i \tau_{g-i}\right). \tag{1.54}$$

[1]Don Zagier has noticed that the right side of (1.53) may be uniformly written in the form $\tau_g \frac{(3g+2n-5)!!}{(3g-3)!!}$ for all such g and n such that $n + 2g > 2$, putting $\tau_1 = \frac{1}{2} \quad \tau_0 = -1$, since it is natural to extend $0!! = 1$ and $(-3)!! = -1$.

Corollary 1.3 *The counting function Ψ for trivalent graphs*

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{3g-3+n}\left[\sum_{\substack{\text{genus } g \text{ trivalent} \\ \text{graphs } \Gamma \text{ with } k \text{ edges} \\ \text{and } n \text{ labelled half-edges}}} \frac{1}{|\operatorname{Aut}\Gamma|}\right]\frac{t^n}{n!}s^k\hbar^{g-1}$$

is the asymptotic expansion of

$$\frac{1}{6s^3\hbar}\left[-2+6st-3s^2t^2\right]-\frac{1}{6}\ln(s^3\hbar)+\ln\left[C_1'Ai\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)+C_2'Bi\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)\right]=$$
$$=\frac{1}{6s^3\hbar}\left[-2+6st-3s^2t^2\right]-\frac{1}{2}\ln(s^3\hbar)+\frac{1}{2}\ln(1-2st)-\ln 3-\frac{1}{3}\ln 2+$$
$$+\ln\left[C_1I_{1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)+C_2I_{-1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)\right]=$$
$$=\frac{1}{6s^3\hbar}\left[2\left(\sqrt{1-2st}\right)^3-2+6st-3s^2t^2\right]-\frac{1}{4}\ln(1-2st)+V\left(\frac{s^3\hbar}{(\sqrt{1-2st})^3}\right), \tag{1.55}$$

where

$$V(u)=\ln\left[\frac{C_1I_{1/3}\left(\frac{1}{3u}\right)+C_2I_{-1/3}\left(\frac{1}{3u}\right)}{3\sqrt[3]{2}\sqrt{u}}\;e^{-\frac{1}{3u}}\right].$$

The counting function for trivalent graphs without half-edges (see also table 2)

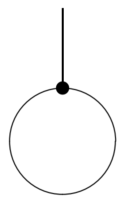
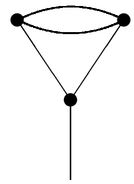
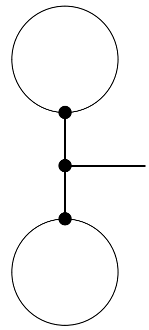
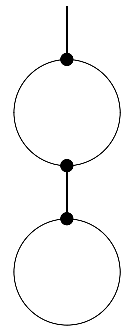
	$g=1$	$g=2$		
Graph Γ				
$\lvert Aut\Gamma \rvert$	2	4	8	4
τ_g	$\frac{1}{2}$	$\frac{1}{4}$ +	$\frac{1}{8}$ +	$\frac{1}{4} = \frac{5}{8}$

Table 1: First two terms of the expansion $W(u) = \frac{1}{2} + \frac{5}{8}u + \cdots$

is the asymptotic expansion of $V(u)$*:*

$$V(u) \sim \sum_{g=2}^{\infty} \left[\sum_{\substack{\textit{genus } g \textit{ trivalent} \\ \textit{graphs } \Gamma}} \frac{1}{|\operatorname{Aut}\Gamma|} \right] u^{g-1} =$$

$$= \ln \left[1 - \frac{(\frac{4}{9}-1)}{1!} \frac{3}{8} u + \frac{(\frac{4}{9}-1)(\frac{4}{9}-9)}{2!} \left(\frac{3}{8}u\right)^2 - \frac{(\frac{4}{9}-1)(\frac{4}{9}-9)(\frac{4}{9}-25)}{3!} \left(\frac{3}{8}u\right)^3 + \ldots \right] =$$

$$= \ln \left[\sum_0^{\infty} \frac{(6n)!}{(2n)!(3n)!} \frac{u^n}{288^n} \right] = \sum_{g=2}^{\infty} \frac{\tau_g}{3g-3} u^{g-1} = \frac{5}{24}u + \frac{5}{16}u^2 + \frac{1105}{1152}u^3 + \frac{565}{128}u^4 + \frac{82825}{3072}u^5 + \ldots, \tag{1.56}$$

where τ_g*,* C_i *and* C_i' *are the same as in the theorem 1.4.*[2]

Causally we have constructed a one-parametric family of solutions of the Cauchy problem for the heath equation (1.27) with the initial condition $F(0,t) = e^{t^3/6\hbar}$. Non-uniqueness of the solutions of the Cauchy problem

[2] Don Zagier communicated to me a nice direct combinatorial proof of the formula $V(u) = \ln \left[\sum_0^{\infty} \frac{(6n)!}{(2n)!(3n)!} \frac{u^n}{288^n} \right]$

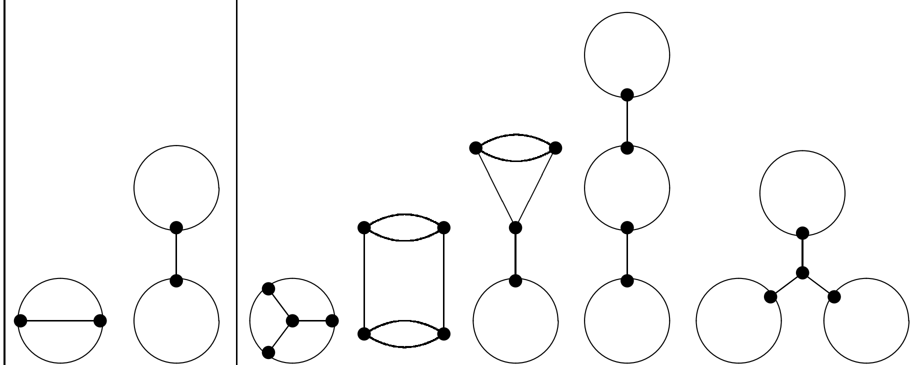
	$g=2$		$g=3$				
Graph Γ							
$\lvert Aut\Gamma \rvert$	12	8	24	16	8	16	48
$\tau_g/(3g-3)$	$\frac{1}{12}$ +	$\frac{1}{8} = \frac{5}{24}$	$\frac{1}{24}$ +	$\frac{1}{16}$ +	$\frac{1}{8}$ +	$\frac{1}{16}$ +	$\frac{1}{48} = \frac{5}{16}$

Table 2: First two terms of the expansion $V(u) = \frac{5}{24}u + \frac{5}{16}u^2 + \cdots$

for the heath equation with rapidly increasing initial conditions (greater than e^{t^2}) was observed by A.N.Tikhonov in 1935 [11].

$$F(s,t) = \sqrt{\frac{1-2st}{s^3\hbar}}e^{\frac{-2+6st-3s^2t^2}{6s^3\hbar}}\left[C_1 I_{1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right) + C_2 I_{-1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)\right] =$$

$$= \frac{e^{\frac{1}{6s^3\hbar}\left[2\left(\sqrt{1-2st}\right)^3 - 2+6st-3s^2t^2\right]}}{\sqrt[4]{1-2st}} \times \frac{C_1 I_{1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right) + C_2 I_{-1/3}\left(\frac{(\sqrt{1-2st})^3}{3s^3\hbar}\right)}{\sqrt{\frac{s^3\hbar}{(\sqrt{1-2st})^3}}} e^{-\frac{(\sqrt{1-2st})^3}{3s^3\hbar}}. \tag{1.57}$$

The second representation in (1.57) is given to show that F is defined and infinitely differentiable on the real axe $s=0$: this is evident about the first product, and the second is $V(\frac{s^3\hbar}{(\sqrt{1-2st})^3})$ (see (1.56)).

1.2 Counting series for all combinatorial graphs.

In this case the function Φ_0 satisfies the functional equation (1.41) and the differential equation (1.42) The initial condition for the Burgers equation (1.7) is given by (1.15).

Theorem 1.5 *Terms Ψ_g of the expansion of the counting function*

$$\Psi(s,t,\hbar)=\sum_{g=0}^{\infty}\Psi_g(s,t)\hbar^{g-1}. \tag{1.58}$$

for all (combinatorial) graphs are expressed as follows:

- *for $g=1$*

$$\Psi_1(s,t)=-\frac{1}{2}\ln(1-s(t+(s+1)\Phi_0(s,t))); \tag{1.59}$$

- *for $g>1$*

$$\Psi_g(s,t)=\frac{s^g(1+t+(s+1)\Phi_0(s,t))}{(1-s(t+(s+1)\Phi_0(s,t)))^g} \times P_g\left(\frac{s(1+t+(s+1)\Phi_0(s,t))}{(1-s(t+(s+1)\Phi_0(s,t)))}\right), \tag{1.60}$$

  where P_g is a polynomial of degree $2g-2$, satisfying the following recurrence[3]*:*

$$\begin{aligned} gP_g(u)+uP_g'(u)= \\ =\frac{1}{2}\Big[u^2(u+1)^2P_{g-1}''(u)+u(u+1)\left[(2g+1)u+3\right]P_{g-1}'(u)+ \\ +\left[(g^2-1)u^2-(3g-2)u+1\right]P_{g-1}(u)+ \\ +u\sum_{i=1}^{g-1}\left[u(u+1)P_i'(u)+(iu+1)P_i(u)\right]\left[u(u+1)P_{g-i}'(u)+((g-i)u+1)P_{g-i}(u)\right]\Big] \end{aligned} \tag{1.61}$$

Here we present first three functions Ψ_g, calculated using the package MAPLE.

$$\begin{aligned} \Psi_1(s,t)=\frac{1}{2}st+\left(\frac{1}{4}s+\frac{1}{2}s^2\right)t^2+ \\ +\left(\frac{7}{12}s^2+\frac{2}{3}s^3+\frac{1}{12}s\right)t^3+\left(\frac{59}{48}s^3+\frac{3}{8}s^2+s^4+\frac{1}{48}s\right)t^4+ \\ +\left(\frac{8}{5}s^5+\frac{121}{48}s^4+\frac{19}{16}s^3+\frac{41}{240}s^2+\frac{1}{240}s\right)t^5+O\left(t^6\right)) \end{aligned} \tag{1.62}$$

[3]Here we formally put $P_1(u)=\frac{\ln(u+1)}{2u}$.

$$\Psi_2(s,t) = \left(\frac{1}{8}s^2 + \frac{5}{24}s^3\right) + \left(\frac{5}{8}s^4 + \frac{2}{3}s^3 + \frac{1}{8}s^2\right)t+$$
$$+\left(\frac{41}{48}s^3 + \frac{1}{16}s^2 + \frac{25}{16}s^5 + \frac{109}{48}s^4\right)t^2+\left(\frac{175}{48}s^6 + \frac{53}{8}s^5 + \frac{133}{36}s^4 + \frac{47}{72}s^3 + \frac{1}{48}s^2\right)t^3+$$
$$+\left(\frac{3419}{192}s^6 + \frac{1885}{144}s^5 + \frac{15}{4}s^4 + \frac{525}{64}s^7 + \frac{203}{576}s^3 + \frac{1}{192}s^2\right)t^4+$$
$$+\left(\frac{7943}{192}s^6 + \frac{1}{960}s^2 + \frac{53}{360}s^3 + \frac{7901}{2880}s^4 + \frac{593}{36}s^5 + \frac{1091}{24}s^7 + \frac{1155}{64}s^8\right)t^5+O\left(t^6\right) \tag{1.63}$$

$$\Psi_3(s,t) = \left(\frac{11}{48}s^4 + \frac{1}{48}s^3 + \frac{25}{48}s^5 + \frac{5}{16}s^6\right)+\left(\frac{25}{48}s^4 + \frac{1}{48}s^3 + \frac{15}{8}s^7 + \frac{185}{48}s^6 + \frac{119}{48}s^5\right)t+$$
$$+\left(\frac{241}{48}s^5 + \frac{9}{16}s^4 + \frac{1}{96}s^3 + \frac{15}{2}s^8 + \frac{1745}{96}s^7 + \frac{727}{48}s^6\right)t^2+$$
$$+\left(\frac{4595}{144}s^6 + \frac{295}{48}s^5 + \frac{113}{288}s^4 + \frac{1}{288}s^3 + \frac{20357}{288}s^7 + 25\,s^9 + \frac{2225}{32}s^8\right)t^3+$$
$$+\left(\frac{40465}{144}s^8 + 75\,s^{10} + \frac{30075}{128}s^9 + \frac{6365}{144}s^6 + \frac{184495}{1152}s^7 + \frac{1}{1152}s^3 + \frac{29}{144}s^4 + \frac{6101}{1152}s^5\right)t^4+$$
$$+\left(\frac{794353}{1152}s^8 + \frac{93555}{128}s^{10} + 210\,s^{11} + \frac{385291}{384}s^9 + \frac{258589}{5760}s^6 + \frac{31815}{128}s^7 +\right.$$
$$\left. + \frac{1}{5760}s^3 + \frac{157}{1920}s^4 + \frac{20159}{5760}s^5\right)t^5 + O\left(t^6\right) \tag{1.64}$$

1.3 Virtual Euler characteristic $\overline{M}_{g,n}$.

In this case the function Φ_0 satisfies the functional equation (1.45) and the differential equation (1.46) The initial conditions for the Burgers equation (1.7) or (1.8) are given by (1.24) (1.25).

Theorem 1.6 *The terms Ψ_g of the expansion of the generating function*

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty} \Psi_g(s,t)\hbar^{g-1}. \tag{1.65}$$

for the virtual euler characteristic of may be expressed as follows:

- *for $g=1$*

$$\Psi_1(s,t) = -\frac{1}{2}\ln(1+t-s(t+s\Phi_0(s,t)))+\frac{5}{12}\ln(1+t+s\Phi_0(s,t)); \quad (1.66)$$

- *for $g>1$*

$$\Psi_g(s,t) = \frac{1}{(1+t+s\Phi_0(s,t))^{2g-2}}\Psi_g\left(\frac{s(1+t+s\Phi_0(s,t))}{1+t-s(t+s\Phi_0(s,t))}, 0\right), \quad (1.67)$$

and $\Psi_g(s,0)$ are polynomials in s of degree $3g-3$, satisfying the following recurrence:

$$\Psi_{g+1}(s,0) = \\ = \frac{B_{2g}}{2g(2g+2)} + \frac{1}{2}\int_0^s \left[\sigma^4 \frac{\partial^2\Psi_g}{\partial s^2}(\sigma,0) + \sigma^2(3\sigma+3-4g)\frac{\partial\Psi_g}{\partial s}(\sigma,0) - \right. \\ -2(g-1)(\sigma-2g+1)\Psi_g(\sigma,0)+\sum_{i=1}^{g}\left(\sigma^2\frac{\partial\Psi_i}{\partial s}(\sigma,0)-2(i-1)\Psi_i(\sigma,0)\right)\times \\ \left.\times\left(\sigma^2\frac{\partial\Psi_{g-i+1}}{\partial s}(\sigma,0)-2(g-i)\Psi_{g-i+1}(\sigma,0)\right)\right]d\sigma \quad (1.68)$$

The coefficient of the leading term of $\Psi_g(s,0)$ equals $\frac{\tau_g}{3g-3}$ (see the definition of τ_g in the theorem 1.4).

In section 6 we present the results of calculations based on these formulas, performed with the package MAPLE.

2 Cutting and clutching modular graphs.

Consider the set $\mathcal{G}^k_{g,n}$ of genus g stable modular graphs with k edges and n half-edges. For $k>0$, $n>0$ and $g\geq 0$ there is the uniquely defined clutching map

$$\& : \mathcal{G}^{k-1}_{g-1,n+2} \to \mathcal{G}^k_{g,n}, \quad (2.1)$$

gluing together the first and the last half-edges of the modular graph $\Gamma \in \mathcal{G}^{k-1}_{g-1,n+2}$ into one edge $e^{\&}$ of a new modular graph $\&(\Gamma) \in \mathcal{G}^{k-1}_{g-1,n+2}$ (the ordering of the remaining $n-2$ half-edges is inherited from Γ). Note that the edge $e^{\&}$ possess a uniquely defined orientation (directed from the first half-edge to the last one), so we have defined the clutching map

$$\tilde{\&}:\mathcal{G}^{k-1}_{g-1,n+2}\to\tilde{\mathcal{G}}^{k}_{g,n}, \tag{2.2}$$

where $\tilde{\mathcal{G}}^{k}_{g,n}$ is the set of genus g stable modular graphs having k edges, n half-edges and one marked oriented edge. The mapping $\tilde{\&}$ is injective: we may reconstruct Γ by cutting the marked edge of $\tilde{\&}(\Gamma)$.

Now for $k>0$, $n\geq 0$, $g\geq 0$ fix some nonnegative integers n_1,n_2,k_1,k_2,g_1,g_2, such that $n_1+n_2=n$, $k_1+k_2=k-1$, $g_1+g_2=g$ and some partition (I_1,I_2) of the set $\{1,2,\ldots,n\}=I_1\sqcup I_2$ such that $|I_1|=n_1$, $|I_2|=n_2$. Put $I_1=\{i_1,i_2,\ldots,i_{n_1}\}$ and $I_2=\{j_1,j_2,\ldots,j_{n_2}\}$, where $i_1<i_2<\ldots<i_{n_1}$ $j_1<j_2<\ldots<j_{n_2}$. Choose two modular graphs $\Gamma_1\in\mathcal{G}^{k_1}_{g_1,n_1+1}$ and $\Gamma_2\in\mathcal{G}^{k_2}_{g_2,n_2+1}$ and glue together the first half-edge of the modular graph Γ_1 and the last half-edge of the modular graph Γ_2. Define the labelling of the half-edges of the joint graph $\Gamma_1\&\Gamma_2$ as follows: m-th half-edge of the modular graph Γ_1 becomes i_{m-1}-th half-edge of the modular graph $\Gamma_1\&\Gamma_2$ for $2\leq m\leq n_1+1$ and m-th half-edge of the modular graph Γ_2 becomes j_m-th half-edge of the modular graph $\Gamma_1\&\Gamma_2$ for $1\leq m\leq n_2$. Thus we have defined the clutching maps:

$$\&:\mathcal{G}^{k_1}_{g_1,n_1+1}\times\mathcal{G}^{k_2}_{g_2,n_2+1}\times\mathcal{P}_{n_1,n_2}\to\mathcal{G}^{k}_{g,n}, \tag{2.3}$$

and

$$\tilde{\&}:\mathcal{G}^{k_1}_{g_1,n_1+1}\times\mathcal{G}^{k_2}_{g_2,n_2+1}\times\mathcal{P}_{n_1,n_2}\to\tilde{\mathcal{G}}^{k}_{g,n}, \tag{2.4}$$

where $\mathcal{P}_{n_1,n_2}$ is the set of all partitions. Repeating the above arguments it is easy to see that the map (2.4) is injective. For fixed n and g we may arrange all the clutching maps (2.1), (2.3) (and, respectively (2.2), (2.4)) into one map

$$\&_{g,n}:\mathcal{G}^{k-1}_{g-1,n+2}\cup\bigcup_{\substack{k_1+k_2=k-1\\ g_1+g_2=g\\ n_1+n_2=n}}\left(\mathcal{G}^{k_1}_{g_1,n_1+1}\times\mathcal{G}^{k_2}_{g_2,n_2+1}\times\mathcal{P}_{n_1,n_2}\right)\to\mathcal{G}^{k}_{g,n}, \tag{2.5}$$

and, respectively

$$\tilde{\&}_{g,n}:\mathcal{G}^{k-1}_{g-1,n+2}\cup\bigcup_{\substack{k_1+k_2=k-1\\ g_1+g_2=g\\ n_1+n_2=n}}\left(\mathcal{G}^{k_1}_{g_1,n_1+1}\times\mathcal{G}^{k_2}_{g_2,n_2+1}\times\mathcal{P}_{n_1,n_2}\right)\to\tilde{\mathcal{G}}^{k}_{g,n}. \tag{2.6}$$

The last mapping $\tilde{\&}_{g,n}$, is obviously bijective: for $(\vec{e},\tilde{\Gamma}) \in \tilde{\mathcal{G}}^k_{g,n}$ the inverse mapping is $\tilde{\&}^{-1}_{g,n}$ given by cutting the marked oriented edge $\vec{e}$ of the graph $\tilde{\Gamma} \in \mathcal{G}^k_{g,n}$ into two half-edges. The ordering of the half-edges is inherited from the graph $\tilde{\Gamma}$, and the two new half-edges get the first and the last number according to the orientation of the marked edge.

Consider the projection

$$\pi_{g,n}\colon \tilde{\mathcal{G}}^k_{g,n} \to \mathcal{G}^k_{g,n}. \tag{2.7}$$

Choose a modular graph $\tilde{\Gamma} \in \mathcal{G}^k_{g,n}$; the group $\operatorname{Aut}(\tilde{\Gamma})$ acts on the set of its oriented edges $\vec{E}(\tilde{\Gamma})$. There is one to one correspondence between the set of orbits of this action and the set of the pairs $(\vec{e},\tilde{\Gamma}) \in \pi^{-1}_{g,n}(\tilde{\Gamma})$. Choose one representative $\{\vec{e}_\alpha\}$ from each orbit, then

$$2k = |\vec{E}(\tilde{\Gamma})| = \sum_{\vec{e}_\alpha} |\operatorname{Aut}(\tilde{\Gamma})\cdot\vec{e}_\alpha| = \sum_{\vec{e}_\alpha}\Big(\operatorname{Aut}(\tilde{\Gamma}) : \operatorname{Aut}(\tilde{\Gamma})_{\vec{e}_\alpha}\Big) = \sum_{\vec{e}_\alpha}\frac{\operatorname{Aut}(\tilde{\Gamma})}{\operatorname{Aut}(\tilde{\Gamma})_{\vec{e}_\alpha}}, \tag{2.8}$$

where $\operatorname{Aut}(\tilde{\Gamma})_{\vec{e}_\alpha}$ is the stabilizer of the oriented edge $\vec{e}_\alpha$. Each pair $(e_\alpha,\tilde{\Gamma})$ belongs to a uniquely defined image of one of the mappings $\tilde{\&}$: (2.4) if the edge e_α disconnects the modular graph $\tilde{\Gamma}$, or (2.2) if it does not. In the first case

$$\operatorname{Aut}(\tilde{\Gamma})_{\vec{e}_\alpha} \cong \operatorname{Aut}(\Gamma_1)\times\operatorname{Aut}(\Gamma_2), \tag{2.9}$$

and in the second

$$\operatorname{Aut}(\tilde{\Gamma})_{\vec{e}_\alpha} \cong \operatorname{Aut}(\Gamma). \tag{2.10}$$

($\tilde{\Gamma} = \Gamma_1\&\Gamma_2$ for (2.9) and $\tilde{\Gamma} = \&(\Gamma)$ for (2.10).) Combining (2.10), (2.9) and (2.8) we obtain

$$\frac{2k}{|\operatorname{Aut}(\tilde{\Gamma})|} = \sum_{\Gamma\in\&^{-1}_{g,n}(\tilde{\Gamma})}\frac{1}{|\operatorname{Aut}(\Gamma)|} + \sum_{(\Gamma_1,\Gamma_2,(I_1,I_2))\in\&^{-1}_{g,n}(\tilde{\Gamma})}\frac{1}{|\operatorname{Aut}(\Gamma_1)|}\frac{1}{|\operatorname{Aut}(\Gamma_2)|} \tag{2.11}$$

Let $\{\mu_{g,n}\ ,\ 2(g-1)+n>0\}$ be a set of (commutative) variables. In (1.2) and (1.3) we have defined the monomials

$$\mu(\Gamma) = \frac{1}{|\operatorname{Aut}\Gamma|}\prod_{v\in V(\Gamma)}\mu_{g(v),\nu(v)}. \tag{2.12}$$

and the polynomials

$$\mu^k_{g,n} = \sum_{\Gamma \in \mathcal{G}^k_{g,n}} \mu(\Gamma). \tag{2.13}$$

Each of the modular graphs (or each of the pairs of the modular graphs) in $\tilde{\&}^{-1}_{g,n}(\tilde{\Gamma})$ has the same collection of vertices with the same valences, therefore multiplying (2.11) by $\prod_{v \in V(\tilde{\Gamma})} \mu_{g(v),\nu(v)}$we obtain:

$$2k\mu(\tilde{\Gamma}) = \sum_{\Gamma \in \tilde{\&}^{-1}_{g,n}(\tilde{\Gamma})} \mu(\Gamma) + \sum_{(\Gamma_1,\Gamma_2,(I_1,I_2)) \in \tilde{\&}^{-1}_{g,n}(\tilde{\Gamma})} \mu(\Gamma_1)\mu(\Gamma_2). \tag{2.14}$$

Taking the sum (2.14) over all $\tilde{\Gamma} \in \mathcal{G}^k_{g,n}$we obtain:

$$2k\mu^k_{g,n} = \mu^{k-1}_{g-1,n+2} + \sum_{\substack{k_1+k_2=k-1 \\ g_1+g_2=g \\ n_1+n_2=n}} \binom{n}{n_1} \mu^{k_1}_{g_1,n_1}\mu^{k_2}_{g_2,n_2}. \tag{2.15}$$

Using the definition of the generating function (1.4)

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{3g-3+n} \mu^k_{g,n}\frac{t^n}{n!}s^k\hbar^{g-1}, \tag{2.16}$$

and multiplying(2.15) by $\frac{1}{2}\frac{t^n}{n!}s^{k-1}\hbar^{g-1}$, we obtain the potential Burgers equation:

$$\frac{\partial \Psi}{\partial s} = \frac{\hbar}{2}\left[\frac{\partial^2 \Psi}{\partial t^2} + \left(\frac{\partial \Psi}{\partial t}\right)^2\right]. \tag{2.17}$$

Theorem 1.1 is proved.

Similar arguments may be used to prove the formula (1.18): for any virtual motivic measure $\tilde{v}$

$$\tilde{v}(M_\Gamma) = \frac{1}{|\operatorname{Aut}\Gamma|}\prod_{v \in V(\Gamma)} \tilde{v}(M_{g(v),\nu(v)}). \tag{2.18}$$

This is evident for the case when Γ is a tree; any modular graph may be constructed from a tree by a sequence of clutching maps (2.1). So to complete the proof it is sufficient to compare $\tilde{v}(M_\Gamma)$ and $\tilde{v}(M_{\&(\Gamma)})$ in (2.1). Put

$(\vec{e}, \tilde{\Gamma}) = \tilde{\&}(\Gamma)$ (this simply means that $\tilde{\Gamma} = \&(\Gamma)$ and $\vec{e}$ is the marked oriented edge, obtained by gluing two half-edges together). Consider the moduli space $\tilde{M}_{\tilde{\Gamma}}$ parameterizing Deligne-Mumford stable nodal pointed curves with one marked branch of one of its nodal points, whose dual graph is $\tilde{\Gamma}$. The projection $\pi : \tilde{M}_{\tilde{\Gamma}} \to M_{\tilde{\Gamma}}$ is a $2k$-fold unramified covering of orbifolds. The space $\tilde{M}_{\tilde{\Gamma}}$ splits into disjoint union of components, corresponding to the orbits of the action of the group $\mathrm{Aut}(\tilde{\Gamma})$ on the set of oriented edges $\vec{E}(\tilde{\Gamma})$ of the modular graph $\tilde{\Gamma}$. The component corresponding to the orbit of $\vec{e}$ will be denoted by $\tilde{M}_{\tilde{\Gamma},\vec{e}}$, then $\pi : \tilde{M}_{\tilde{\Gamma},\vec{e}} \to M_{\tilde{\Gamma}}$ is an unramified covering of orbifolds of degree $(\mathrm{Aut}(\tilde{\Gamma}) : \mathrm{Aut}(\tilde{\Gamma})_{\vec{e}}) = \frac{|\,\mathrm{Aut}(\tilde{\Gamma})|}{|\,\mathrm{Aut}(\tilde{\Gamma})_{\vec{e}}|}$. Therefore

$$\tilde{v}(\tilde{M}_{\tilde{\Gamma},\vec{e}}) = \frac{|\,\mathrm{Aut}(\tilde{\Gamma})|}{|\,\mathrm{Aut}(\tilde{\Gamma})_{\vec{e}}|}\tilde{v}(\tilde{M}_{\tilde{\Gamma}}). \tag{2.19}$$

The clutching maps (2.1) and (2.2) define the clutching maps

$$\&_\Gamma \colon M_\Gamma \longrightarrow M_{\tilde{\Gamma}} \tag{2.20}$$

and

$$\tilde{\&}_\Gamma \colon M_\Gamma \to \tilde{M}_{\tilde{\Gamma}}, \tag{2.21}$$

$\tilde{M}_{\tilde{\Gamma},\vec{e}}$ is the image of $\tilde{\&}_\Gamma$, and

$$\tilde{\&}_\Gamma \colon M_\Gamma \cong \tilde{M}_{\tilde{\Gamma},\vec{e}}, \tag{2.22}$$

is an isomorphism, hence $\tilde{v}(M_\Gamma) = \tilde{v}(\tilde{M}_{\tilde{\Gamma},\vec{e}})$. This completes the proof of (1.18).

3 Solving the Burgers equation.

In this section we solve the Burgers equations (1.7) or (1.8) using the expansions (1.32) and (1.31):

$$\Psi(s,t,\hbar) = \sum_{g=0}^{\infty} \Psi_g(s,t)\hbar^{g-1}, \tag{3.1}$$

$$\Phi(s,t,\hbar) = \sum_{g=0}^{\infty} \Phi_g(s,t)\hbar^{g-1}. \tag{3.2}$$

Substituting (1.7) into (1.8), we get a quasi-linear equation for Φ_0:

$$\frac{\partial \Phi_0}{\partial s} = \Phi_0 \frac{\partial \Phi_0}{\partial t}. \tag{3.3}$$

and recursive quasi-linear equation for Φ_g and Ψ_g for $g > 0$:

$$\frac{\partial \Phi_g}{\partial s} = \frac{1}{2}\frac{\partial^2 \Phi_{g-1}}{\partial t^2} + \Phi_0 \frac{\partial \Phi_g}{\partial t} + \Phi_g \frac{\partial \Phi_0}{\partial t} + \sum_{i=1}^{g-1} \Phi_i \frac{\partial \Phi_{g-i}}{\partial t}. \tag{3.4}$$

$$\frac{\partial \Psi_g}{\partial s} = \frac{1}{2}\frac{\partial^2 \Psi_{g-1}}{\partial t^2} + \Phi_0 \frac{\partial \Psi_g}{\partial t} + \frac{1}{2}\sum_{i=1}^{g-1} \frac{\partial \Psi_i}{\partial t}\frac{\partial \Psi_{g-i}}{\partial t}. \tag{3.5}$$

For $g = 0$ we have only the quasi-linear equation (3.3). The equations for the characteristics are

$$\frac{ds}{1} = -\frac{dt}{\Phi_0} = \frac{d\Phi_0}{0}. \tag{3.6}$$

The two first integrals for (3.6) are

$$\Phi_0 = C_1 \qquad t + s\Phi_0 = C_2. \tag{3.7}$$

Then the general solution of (3.3) is

$$f_0(\Phi_0, t + s\Phi_0) = 0, \tag{3.8}$$

for some function f_0. Using the initial conditions $\Phi_0(0,t)$, we see that for $s = 0$ $f_0(\Phi_0(0,t), t) = 0$; this means that $f_0(a,b) = 0$ is equivalent to $a = \Phi_0(0,b)$. Thus the equation (3.8) provides the functional equation for $\Phi_0(s,t)$:

$$\Phi_0(s,t) = \Phi_0(0, t + s\Phi_0(s,t)). \tag{3.9}$$

Put

$$\alpha_s(t) = t - s\Phi_0(0,t) \quad \text{and} \quad \beta_s(t) = t + s\Phi_0(s,t). \tag{3.10}$$

From (3.9) we obtain $t + s\Phi_0(s,t) - s\Phi_0(0, t + s\Phi_0(s,t)) = t$, so the function α_s is inverse to the β_s with respect to the composition of functions:

$$\alpha_s(\beta_s(t)) = t \quad \text{and} \quad \beta_s(\alpha_s(t)) = t. \tag{3.11}$$

Thus the theorem 1.2 is proved.

Now let us study the quasi-linear equation (3.5). The equations for the characteristics for (3.5) are

$$\frac{ds}{1} = -\frac{dt}{\Phi_0} = \frac{d\Psi_g}{\frac{1}{2}\frac{\partial^2 \Psi_{g-1}}{\partial t^2} + \frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \Psi_i}{\partial t}\frac{\partial \Psi_{g-i}}{\partial t}}. \tag{3.12}$$

The equations for the characteristics for (3.4) are

$$\frac{ds}{1} = -\frac{dt}{\Phi_0} = \frac{d\Phi_g}{\Phi_g \frac{\partial \Phi_0}{\partial t} + \frac{1}{2}\frac{\partial^2 \Phi_{g-1}}{\partial t^2} + \sum_{i=1}^{g-1}\Phi_i \frac{\partial \Phi_{g-i}}{\partial t}}. \tag{3.13}$$

Let us denote the denominator in (3.12) by

$$H_g(s,t) = \frac{1}{2}\frac{\partial^2 \Psi_{g-1}}{\partial t^2} + \frac{1}{2}\sum_{i=1}^{g-1}\frac{\partial \Psi_i}{\partial t}\frac{\partial \Psi_{g-i}}{\partial t}, \tag{3.14}$$

note that H_g depends only on Ψ_i for $i < g$. We have already found one first integral for (3.13) and (3.12) for all g:

$$\Phi_0(s,t) = C_1. \tag{3.15}$$

Substituting into (3.9) we see that

$$\Phi_0(0, t + sC_1) = C_1. \tag{3.16}$$

Let us denote one of the branches of the function inverse to $\Phi_0(0,t)$ by φ, then $t = \varphi(C_1) - sC_1$. Now the second first integral for (3.5) may be (recursively) found by simple integration:

$$\Psi_g - \int H_g(s, \varphi(C_1) - sC_1)ds = C_2. \tag{3.17}$$

Eliminating C_1 we obtain:

$$\Psi_g - \Xi_g(s,t) = C_2 \tag{3.18}$$

where

$$\Xi_g(s,t) = \int_0^s H_g(\sigma, t + (s-\sigma)\Phi_0(s,t))d\sigma. \tag{3.19}$$

Note that we choose the integration constant so that $\Xi_g(0,t) = 0$. Thus the general solution of (3.5) may be written as:

$$\Psi_g(s,t) = \Xi_g(s,t) + U_g(\Phi_0(s,t)) \tag{3.20}$$

for an arbitrary function U_g. The function U_g then may be determined from the initial condition:

$$U_g(\Phi_0(0,t)) = \Psi_g(0,t). \tag{3.21}$$

Substituting $t + s\Phi_0(s,t)$ instead of t in $\Phi_0(0,t)$ and using (3.9) we obtain the following recurrence formula for the solution of the Burgers equation.

$$\Psi_g(s,t) = \Psi_g(0, t + s\Phi_0(s,t))) + \\ + \frac{1}{2}\int_0^s \left[\frac{\partial^2 \Psi_{g-1}}{\partial t^2} + \sum_{i=1}^{g-1} \frac{\partial \Psi_i}{\partial t}\frac{\partial \Psi_{g-i}}{\partial t}\right](\sigma, t + (s-\sigma)\Phi_0(s,t))\, d\sigma \tag{3.22}$$

The theorem 1.3 is proved.

4 $g = 0$.

In this section we use Theorem 1.2 to obtain in a uniform way functional equations for $\Phi_0(s,t)$ for all the cases we have discussed.

1) Counting functions for trivalent trees.

In this case $\Phi_0(0,t) = \frac{t^2}{2}$ (see (1.12) for $d = 3$). The inverse function for $\alpha_s(t) = t - s\frac{t^2}{2}$ is the solution of the quadratic equation $\beta_s(t) - s\frac{\beta_s(t)^2}{2} = t$. The solution is $\beta_s(t) = \alpha_1^{-1}(t) = \frac{1-\sqrt{1-2st}}{s}$ and therefore

$$\Phi_0(s,t) = \frac{1 - st - \sqrt{1-2st}}{s^2} \quad \text{and} \quad \Phi_0(1,t) = 1 - t - \sqrt{1-2t}. \tag{4.1}$$

This is a well-known generating function for the number of trivalent trees with labelled half-edges.

2) Counting functions for all stable trees.

In this case (see (1.15))

$$\Phi_0(0,t) = e^t - t - 1. \tag{4.2}$$

Substituting into (1.38), we obtain the functional equation (1.41). The differential equation (1.42) is deduced from it in a standard way.

3) The Poincare polynomial and the Euler characteristic for $\overline{M}_{0,n}$.

The Poincare polynomial for $M_{0,n}$ in variable y coincides with the number of points in $M_{0,n}(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$ (after the substitution $q = y^2$). There are $q+1$ point on the projective line; the first three of them we may send to 0, 1 and ∞ by some projective automorphism; the remaining $q-3$ points may be chosen in

$$(q-2)(q-3)\dots(q-n+2) \tag{4.3}$$

ways. Hence the generating function is

$$\Phi_0(0,t) = \sum_{n=3}^{\infty} \frac{(q-2)!}{(n-1)!(q-n+1)!} t^{n-1} = \frac{(1+t)^q - qt - 1}{q(q-1)}. \tag{4.4}$$

Substituting into (1.38) we obtain the functional equation (1.43). The differential equation (1.44) is deduced from it in a standard way.

For the Euler characteristic we may simply put $q = 1$ in (4.4).

5 Counting function for trivalent graphs.

In this case it is better to begin with the equation (1.8) on the function $\Phi(s,t,\hbar)$. For any genus g trivalent graph with k edges and n half-edges $k = 3g-3+n$, therefore Φ contains only monomials

$$s^{3g-3+n} t^{n-1} \hbar^{g-1} = s(s^3\hbar)^g (st)^{n-1} \tag{5.1}$$

and hence

$$\Phi(s,t,\hbar) = sZ(s^3\hbar, st) \tag{5.2}$$

for some function $Z(x,y)$. Substituting (5.2) into (1.8) we obtain the equation

$$Z + 3x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = \frac{x}{2}\frac{\partial^2 Z}{\partial y^2} + xZ\frac{\partial Z}{\partial y}. \tag{5.3}$$

Similarly to (1.31) consider the expansion

$$Z(x,y) = \sum_{g=0}^{\infty} Z_g(y) x^{g-1}. \tag{5.4}$$

Then for $Z_0(y)$ we have the homogeneous equation

$$-2Z_0 + yZ_0' = Z_0 Z_0'. \tag{5.5}$$

The solution we need (with the initial condition $Z_0(0) = 0$) is

$$Z_0(y) = 1 - y - \sqrt{1-2y}, \tag{5.6}$$

which of course coincides with (1.40). For $g > 0$ (5.3) provides the following recursive linear equation for $Z_g(y)$, which is equivalent to (1.34):

$$-2Z_g + 3gZ_g + yZ_g' = \frac{1}{2}Z_{g-1}'' + Z_0Z_g' + Z_gZ_0' + \sum_{i=1}^{g-1} Z_i'Z_{g-i}. \tag{5.7}$$

It is not hard to find Z_1 and the general form of Z_g (as well we could use (1.49), which would lead to a bit cumbersome transformations):

$$Z_1 = \frac{1}{2(1-2y)} \quad \text{and} \quad Z_g = \frac{\tau_g}{(\sqrt{1-2y})^{3g-1}} \tag{5.8}$$

for some constants τ_g, where $\tau_1 = \frac{1}{2}$. The equation (5.7) provides the recursive formula for τ_g, $g > 1$:

$$\tau_g = \frac{1}{3g-2}\left(\frac{1}{2}(3g-2)(3g-4)\tau_{g-1} + \sum_{i=1}^{g-1}(3i-1)\tau_i\tau_{g-i}\right), \tag{5.9}$$

It is not hard to transform (5.9) to a better form:

$$\tau_g = \frac{1}{2}\left((3g-4)\tau_{g-1} + \sum_{i=1}^{g-1}\tau_i\tau_{g-i}\right). \tag{5.10}$$

Here are the four first values of τ_g:

$$\tau_1 = \frac{1}{2}; \quad \tau_2 = \frac{5}{8}; \quad \tau_3 = \frac{15}{8}; \quad \tau_4 = \frac{1105}{128}. \tag{5.11}$$

Substituting (5.8) into (5.4) we express of the solution Z in the following form :

$$\begin{aligned} Z = Z_0 + \sum_{g=1}^{\infty}\tau_g\frac{x^g}{(\sqrt{1-2y})^{3g-1}} &= Z_0 + \frac{x}{1-2y}\sum_{g=1}^{\infty}\tau_g\left(\frac{x}{(\sqrt{1-2y})^3}\right)^{3g-1} = \\ &= Z_0 + \frac{x}{1-2y}W(\left(\frac{x}{(\sqrt{1-2y})^3}\right), \end{aligned} \tag{5.12}$$

where W is some function in one variable. Substituting into (5.3), we get an ordinary differential the equation for $W(u)$:

$$1 + (8u-2)W + (27u^2 - 6x)W' + 9u^3W'' + 4uW^2 + 6u^2W'W = 0. \tag{5.13}$$

Multiplying by $u^{-2/3}$ and integrating we get:

$$-2u^{1/3}W(u)+2u^{4/3}W(u)+u^{4/3}W(u)^2+3u^{7/3}W'(u)+u^{1/3}+C=0 \quad (5.14)$$

Since W is regular at $u=0$ then $C=0$. Dividing by $u^{1/3}$ we get the Riccati equation

$$2(u-1)W(u)+uW(u)^2+1+3u^2W'(u)=0. \quad (5.15)$$

For the equation (5.15) $u=0$ is a singular point and it has the unique formal series solution

$$W\left(u\right)=\frac{1}{2}+\frac{5}{8}u+\frac{15}{8}u^2+\frac{1105}{128}u^3+\frac{1695}{32}u^4+\frac{414125}{1024}u^5+O\left(u^6\right). \quad (5.16)$$

The general solution of (5.15) may be expressed analytically via modified Bessel functions:

$$W(u)=\frac{1}{u}\left(1-\frac{C_1I_{-2/3}(\frac{1}{3u})+C_2I_{2/3}(\frac{1}{3u})}{C_1I_{1/3}(\frac{1}{3u})+C_2I_{-1/3}(\frac{1}{3u})}\right) \quad (5.17)$$

for any C_1, C_2. Using asymptotic expansion of Bessel functions (see [4]) the solution (for any C_1 and C_2) may be represented as a quotient of two power series:

$$W(u)=$$
$$=\frac{1}{u}\left(1-\frac{1-\frac{(16/9-1)}{1!}\frac{3}{8}u+\frac{(16/9-1)(16/9-4)}{2!}(\frac{3}{8}u)^2-\frac{(16/9-1)(16/9-4)(16/9-25)}{3!}(\frac{3}{8}u)^3+\dots}{1-\frac{(4/9-1)}{1!}\frac{3}{8}u+\frac{(4/9-1)(4/9-4)}{2!}(\frac{3}{8}u)^2-\frac{(4/9-1)(4/9-4)(4/9-25)}{3!}(\frac{3}{8}u)^3+\dots}\right). \quad (5.18)$$

Now we can present the answer:

$$\Phi(s,t,\hbar)=\frac{1}{s^2\hbar}\left[1-st-\sqrt{1-2st}\right]+\frac{1}{s^2}\sum_{g=1}^{\infty}\tau_g\frac{s^{3g}\hbar^{g-1}}{(\sqrt{1-2st})^{3g-1}}=$$
$$=\frac{1-st-\sqrt{1-2st}}{s^2\hbar}+\frac{s}{1-2st}\sum_{g=1}^{\infty}\tau_g\left(\frac{s^3\hbar}{(\sqrt{1-2st})^3}\right)^{g-1}=$$
$$=\frac{1-st-\sqrt{1-2st}}{s^2\hbar}+\frac{s}{1-2st}W\left(\frac{s^3\hbar}{(\sqrt{1-2st})^3}\right). \quad (5.19)$$

It is also useful to express the solution $\Phi(s,t,\hbar)$ by the Airy functions (see [4]):

$$\Phi(s,t,\hbar)=\frac{1-st}{s^2\hbar}-\frac{2^{1/3}}{s\hbar^{2/3}}\left[\frac{C_1'Ai'\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)+C_2'Bi'\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)}{C_1'Ai\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)+C_2'Bi\left(\frac{1-2st}{2^{2/3}s^2\hbar^{2/3}}\right)}\right], \quad (5.20)$$

where $C_1'=\sqrt{3}(C_2-C_1)$ and $C_2'=C_2+C_1$. Now it is easy to find the analytical expression (1.55) for $\Psi(s,t,\hbar)$ by integration; the integration constant (depending on s and $\hbar$) can be found from the Burgers equation (1.7).

The proof of Theorem 1.4 and Corollary 1.3 is completed.

6 Virtual Euler characteristic of $\overline{M}_{g,n}$.

The main step in the proof of theorem 1.6 is to notice that the solutions $\Psi_g(s,t)$ may be represented in the following form:

$$\Psi_g(s,t)=\frac{1}{(1+t+s\Phi_0(s,t))^{2g-2}}P_g\left(\frac{s(1+t+s\Phi_0(s,t))}{1+t-s(t+s\Phi_0(s,t))}\right), \quad (6.1)$$

where P_g is some polynomial. It is sufficient for that to find $\Psi_g(s,t)$ using formulas (1.49) for several first values of g, and then prove the statement by induction, using (1.33). (Use (1.46) to find the derivatives of Φ_0.) After that we only need to notice that for $t=0$ the equation (6.1) provides $\Psi_g(s,0)=P_g(s)$.

Note that for the leading coefficients of $P_g(s)$ the recurrence (1.68) gives exactly the recurrence (1.54), defining the numbers τ_g. This has a clear geometric explanation: the leading coefficients of $P_g(s)$ are exactly the coefficients of $s^{3g-3}\hbar^{g-1}$ in the expansion of $\Psi_g(s,t)$. But these coefficients represent the contribution to the Euler characteristic of the 0-dimensional strata $M_{g,n}^{3g-3}$ corresponding to the discrete set of maximally degenerated curves. These are exactly the curves, whose dual graph is trivalent and all the irreducible components are rational (ll-curves in terms of A.N.Tyurin's book [3]).

Here we present the results of calculations based on formulas (1.68), performed with the package MAPLE.

Polynomials $\Psi_g(s,0)\quad g=2,3,4,5,6$**.**

$$\Psi_2(s,0)=\frac{5}{24}s^3-\frac{1}{6}s^2+\frac{13}{288}s-\frac{1}{240} \quad (6.2)$$

$$\Psi_3(s,0) = \frac{5}{16}\,s^6 - \frac{55}{96}\,s^5 + \frac{35}{72}\,s^4 - \frac{2539}{10368}\,s^3 + \frac{1307}{17280}\,s^2 - \frac{19}{1440}\,s + \frac{1}{1008} \quad (6.3)$$

$$\Psi_4(s,0) = \frac{1105}{1152}\,s^9 - \frac{1045}{384}\,s^8 + \frac{8549}{2304}\,s^7 - \frac{66773}{20736}\,s^6 + \frac{182341}{92160}\,s^5 - \frac{2235257}{2488320}\,s^4 + \\ + \frac{187051}{622080}\,s^3 - \frac{17063}{241920}\,s^2 + \frac{6221}{604800}\,s - \frac{1}{1440} \quad (6.4)$$

$$\Psi_5(s,0) = \frac{565}{128}\,s^{12} - \frac{26015}{1536}\,s^{11} + \frac{145883}{4608}\,s^{10} - \frac{3182161}{82944}\,s^9 + \\ + \frac{2805265}{82944}\,s^8 - \frac{229328099}{9953280}\,s^7 + \frac{374564131}{29859840}\,s^6 - \frac{578872613}{104509440}\,s^5 + \\ + \frac{114641981}{58060800}\,s^4 - \frac{667199}{1209600}\,s^3 + \frac{32821}{290304}\,s^2 - \frac{181}{12096}\,s + \frac{1}{1056} \quad (6.5)$$

$$\Psi_6(s,0) = \frac{82825}{3072}\,s^{15} - \frac{400565}{3072}\,s^{14} + \frac{1266935}{4096}\,s^{13} - \\ - \frac{159107029}{331776}\,s^{12} + \frac{241682111}{442368}\,s^{11} - \frac{9702562787}{19906560}\,s^{10} + \frac{253843871663}{716636160}\,s^9 - \\ - \frac{1079372228279}{5016453120}\,s^8 + \frac{835339878797}{7524679680}\,s^7 - \frac{614429790997}{12541132800}\,s^6 + \frac{6419764103}{348364800}\,s^5 - \\ - \frac{3031168109}{522547200}\,s^4 + \frac{106613887}{72576000}\,s^3 - \frac{24719227}{88704000}\,s^2 + \frac{441541}{12700800}\,s - \frac{691}{327600} \quad (6.6)$$

First terms of expansion of the functions $\Psi_g(s,t)$ for $g = 1, 2, 3$.

$$\Psi_1(s,t) = \left(1/2\,s - \frac{1}{12}\right)t + \left(1/2\,s^2 - \frac{7}{24}\,s + \frac{1}{24}\right)t^2 + \\ + \left(2/3\,s^3 - 5/8\,s^2 + 2/9\,s - \frac{1}{36}\right)t^3 + \left(s^4 - \frac{41}{32}\,s^3 + \frac{199}{288}\,s^2 - 3/16\,s + \frac{1}{48}\right)t^4 + \\ + \left(8/5\,s^5 - \frac{83}{32}\,s^4 + \frac{89}{48}\,s^3 - \frac{533}{720}\,s^2 + 1/6\,s - \frac{1}{60}\right)t^5 + O\left(t^6\right) \quad (6.7)$$

$$\Psi_2(s,t) = \left(\frac{5}{24}\,s^3 - 1/6\,s^2 + \frac{13}{288}\,s - \frac{1}{240}\right) +$$
$$+ \left(5/8\,s^4 - 3/4\,s^3 + \frac{109}{288}\,s^2 - \frac{13}{144}\,s + \frac{1}{120}\right)t +$$
$$+ \left(\frac{25}{16}\,s^5 - \frac{39}{16}\,s^4 + \frac{325}{192}\,s^3 - \frac{379}{576}\,s^2 + \frac{67}{480}\,s - \frac{1}{80}\right)t^2 +$$
$$+ \left(\frac{175}{48}\,s^6 - \frac{167}{24}\,s^5 + \frac{3497}{576}\,s^4 - \frac{677}{216}\,s^3 + \frac{4393}{4320}\,s^2 - \frac{7}{36}\,s + \frac{1}{60}\right)t^3 +$$
$$+ \left(\frac{525}{64}\,s^7 - \frac{3547}{192}\,s^6 + \frac{44519}{2304}\,s^5 - \frac{9439}{768}\,s^4 + \frac{17933}{3456}\,s^3 - \frac{5065}{3456}\,s^2 + \frac{23}{90}\,s - \frac{1}{48}\right)t^4 +$$
$$+ \left(\frac{1155}{64}\,s^8 - \frac{1123}{24}\,s^7 + \frac{14579}{256}\,s^6 - \frac{5485}{128}\,s^5 + \right.$$
$$\left. + \frac{11887}{540}\,s^4 - \frac{3833}{480}\,s^3 + \frac{5801}{2880}\,s^2 - \frac{97}{300}\,s + \frac{1}{40}\right)t^5 + O\left(t^6\right) \quad (6.8)$$

$$\Psi_3(s,t) = \left(\frac{5}{16}\,s^6 - \frac{55}{96}\,s^5 + \frac{35}{72}\,s^4 - \frac{2539}{10368}\,s^3 + \frac{1307}{17280}\,s^2 - \frac{19}{1440}\,s + \frac{1}{1008}\right) +$$
$$+\left(\frac{15}{8}\,s^7 - \frac{395}{96}\,s^6 + \frac{305}{72}\,s^5 - \frac{9259}{3456}\,s^4 + \frac{29311}{25920}\,s^3 - \frac{341}{1080}\,s^2 + \frac{19}{360}\,s - \frac{1}{252}\right) t +$$
$$+\left(15/2\,s^8 - \frac{3665}{192}\,s^7 + \frac{4415}{192}\,s^6 - \frac{119495}{6912}\,s^5 + \right.$$
$$\left. + \frac{928913}{103680}\,s^4 - \frac{171107}{51840}\,s^3 + \frac{1819}{2160}\,s^2 - \frac{15}{112}\,s + \frac{5}{504}\right) t^2 +$$
$$+\left(25\,s^9 - \frac{4625}{64}\,s^8 + \frac{28655}{288}\,s^7 - \frac{1790105}{20736}\,s^6 + \frac{1360669}{25920}\,s^5 - \frac{1821137}{77760}\,s^4 + \right.$$
$$\left. + \frac{66737}{8640}\,s^3 - \frac{27491}{15120}\,s^2 + \frac{415}{1512}\,s - \frac{5}{252}\right) t^3 +$$
$$+\left(75\,s^{10} - \frac{62075}{256}\,s^9 + \frac{864475}{2304}\,s^8 - \frac{1131595}{3072}\,s^7 + \frac{35398361}{138240}\,s^6 - \frac{5165251}{38880}\,s^5 + \right.$$
$$\left. + \frac{16351757}{311040}\,s^4 - \frac{141479}{8960}\,s^3 + \frac{69679}{20160}\,s^2 - \frac{2995}{6048}\,s + \frac{5}{144}\right) t^4 +$$
$$+\left(210\,s^{11} - \frac{192115}{256}\,s^{10} + \frac{247385}{192}\,s^9 - \frac{13001167}{9216}\,s^8 + \frac{76135781}{69120}\,s^7 - \frac{3737291}{5760}\,s^6 + \right.$$
$$\left. + \frac{11534753}{38880}\,s^5 - \frac{165193453}{1555200}\,s^4 + \frac{17647}{600}\,s^3 - \frac{911023}{151200}\,s^2 + \frac{155}{189}\,s - \frac{1}{18}\right) t^5 + O\left(t^6\right)) \tag{6.9}$$

Values $\tilde{\chi}(\overline{M}_{g,0})$ and $\tilde{\chi}(\overline{M}_{g,1})$, $g \le 20$. Note that for $g \le 20$ the Euler characteristic grows approximately as $C\frac{(g-1)!}{2^{g-1}}$, and the quotient

$$\frac{\tilde{\chi}(\overline{M}_{g,1})}{\tilde{\chi}(\overline{M}_{g,0})(2g-2)}$$

grows from 1.025 for $g = 3$ to 1.038 for $g = 20$.

g	$\tilde{\chi}(\overline{M}_{g,0})$	
2	$\frac{119}{1440}$	≈ 0.0826
3	$\frac{8027}{181440}$	≈ 0.0442
4	$\frac{2097827}{43545600}$	≈ 0.0482
5	$\frac{150427667}{1916006400}$	≈ 0.0785
6	$\frac{31966432414753}{188305108992000}$	≈ 0.170
7	$\frac{21067150021261}{46115536896000}$	≈ 0.457
8	$\frac{27108194937436478387}{18438836272496640000}$	≈ 1.470168428
9	$\frac{12253091020103495716943}{2225676001833123840000}$	≈ 5.505334564
10	$\frac{41107639746528672580958364833}{1748045931839735463936000000}$	≈ 23.51633844
11	$\frac{18149470500315527186930400759373}{160820225729255662682112000000}$	≈ 112.8556462
12	$\frac{19004221040884074685037446900552041691}{31610823569342493056795934720000000}$	≈ 601.1934804
13	$\frac{1335395944593790109991624206528868880873}{3793298828321099166815512166400000000}$	≈ 3520.407975
14	$\frac{2697359250099761465877837488047416054790459}{1200061811141584100047089303552000000000}$	≈ 22476.83599
15	$\frac{17628737527982037548325073368636345668379043678957}{113436082710520465373771125836113510400000000}$	≈ 155406.7904
16	$\frac{61187507009333322043736181893289455692441208195878609}{52893624852448399854284136389867785420800000000}$	≈ 1156803.059
17	$\frac{71372306743070002809491037076029984614872395664643491}{7737834356680016697596425386187554816000000000}$	≈ 9223809.073
18	$\frac{17198235432952170987858390769814893434655150721674671445771265141}{219267898302032160155302114911031967019769528320000000000}$	≈ 78434807.68
19	$\frac{13050435425469643163551878925079739017685769865160451968198706727723}{184185034573707014530453776525266852296606403788800000000000}$	≈ 708550260.6
20	$\frac{137014760506364785741048203429669320537974177259444567259217133497233731}{2021925775076881060119301903762158630079581860659200000000000}$	≈ 6776448582.0

g	$\tilde{\chi}(\overline{M}_{g,1})$	
1	$\frac{5}{12}$	≈ 0.4166666667
2	$\frac{247}{1440}$	≈ 0.1715277778
3	$\frac{13159}{72576}$	≈ 0.1813133818
4	$\frac{5160601}{17418240}$	≈ 0.2962756857
5	$\frac{1060344499}{1642291200}$	≈ 0.6456495042
6	$\frac{43927799939987}{25107347865600}$	≈ 1.749599367
7	$\frac{25578458051299001}{4519322615808000}$	≈ 5.659799095
8	$\frac{71323310082487963309}{3352515685908480000}$	≈ 21.27456417
9	$\frac{482708908140083875850272 69}{5297108884362834739200 00}$	≈ 91.12686159
10	$\frac{15320139468462439557133157769 17}{34960918636794709278720000 00}$	≈ 438.2075777
11	$\frac{225588984176891190148454846952 7387}{96492135437553397609267200000 00}$	≈ 2337.900215
12	$\frac{28883289261481518538841759906455 1131741}{21073882379561662037863956480000 000}$	≈ 13705.72766
13	$\frac{66447212654413192038655941663348291926069}{758659765664219833363102433280000000}$	≈ 87584.99615
14	$\frac{12307009699630853132382998154930866963085 9857}{20308738342396038616181511290880000000 00}$	≈ 605995.7784
15	$\frac{1462811819677743837384975294499934432804425416 90511}{3241030934586299010679175023888957440000000 00}$	≈ 4513415.173
16	$\frac{133309147159236466453784033068792506720345957334028501807}{370255373967138798979988954729074497945600000000 0}$	≈ 36004648.83
17	$\frac{2721690926359201802650400830738540838572166621421649160557}{8886128975211331175519734913497787950694400000000 0}$	≈ 306285327.8
18	$\frac{123136066030368677688394485156439501180080883329398977909415435779}{444891387859195687271627479529630078011126579200000000000}$	≈ 2767778145.0
19	$\frac{975371306046856089312443646349848042163131041618448148820157981265999}{368370069147414029060907553050533704593212807577600000000000}$	≈ 26478028150.0
20	$\frac{18378243829728231044942829473669253595348748451258655601647380429955511412 7}{6874547635261395604405626472791339342270578326241280000000000000}$	≈ 267337500700.0

Values $\tilde{\chi}(M_{g,n})$, $2 \le g \le 7$, $2 \le n \le 6$.

g	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
2	$\frac{413}{720}$ ≈ 0.5736111111	$\frac{89}{32}$ ≈ 2.781250000	$\frac{12431}{720}$ ≈ 17.26527778	$\frac{189443}{1440}$ ≈ 131.5576389	$\frac{853541}{720}$ ≈ 1185.473611
3	$\frac{179651}{181440}$ ≈ 0.9901399912	$\frac{495611}{72576}$ ≈ 6.828855269	$\frac{684641}{12096}$ ≈ 56.60061177	$\frac{199014019}{362880}$ ≈ 548.4292852	$\frac{1103123803}{181440}$ ≈ 6079.826957
4	$\frac{97471547}{43545600}$ ≈ 2.238378780	$\frac{1747463783}{87091200}$ ≈ 20.06475721	$\frac{9056350741}{43545600}$ ≈ 207.9739570	$\frac{71024755987}{29030400}$ ≈ 2446.564842	$\frac{1402182822991}{43545600}$ ≈ 32200.33305
5	$\frac{357631300 21}{5748019200}$ ≈ 6.221818121	$\frac{1579280415 17}{2299207680}$ ≈ 68.68802801	$\frac{701735503159}{821145600}$ ≈ 854.5810915	$\frac{135972856739213}{11496038400}$ ≈ 11827.80120	$\frac{115110462356893}{638668800}$ ≈ 180234.9862
6	$\frac{350875518979697}{17118646272000}$ ≈ 20.49668609	$\frac{14466239894532961}{53801459712000}$ ≈ 268.8819220	$\frac{105018494553645499}{26900729856000}$ ≈ 3903.927333	$\frac{4680800827073885069}{75322043596800}$ ≈ 62143.83736	$\frac{15587244161672916947}{14485008384000}$ ≈ 1076094.935
7	$\frac{5346168720992921}{68474585088000}$ ≈ 78.07522622	$\frac{766050649843508339}{645617516544000}$ ≈ 1186.539445	$\frac{44501877704266668461}{2259661307904000}$ ≈ 19694.04775	$\frac{1601797289485334976137}{4519322615808000}$ ≈ 354433.0480	$\frac{3106681102072897118941}{451932261580800}$ ≈ 6874218.475

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