HYPERBOLIC SURFACES IN $\mathbb{P}^3(\mathbb{C})$

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Abstract. We show a class of pertubation X of the Fermat hypersurface such that any holomorphic curve from $\mathbb{C}$ into X is degenerate. Applying this result, we give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of arbitrary degree $d \geq 22$, and of curves of arbitrary degree $d \geq 19$ in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements.

§1. Introduction

A holomorphic curve in a projective variety X is said to be degenerate if it is contained in a proper algebraic subset of X. In 1979 ([GG]) M. Green and Ph. Griffiths conjectured that every holomorphic curve in a complex projective variety of general type is degenerate. Up to now this conjecture seems still far completly proved, but there has been some progress. M. Green ([G]) proved the degeneracy of holomorphic curves in the Fermat variety of large degree. In [N] A. M. Nadel gives a class of projective hypersurfaces for which the conjecture is valid. Using the results on degeneracy of holomorphic curves Nadel constructed some explicit examples of hyperbolic hypersurfaces in $\mathbb{P}^3$.

In this note, we first consider pertubations X of the Fermat hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ such that for some fixed $k \geq 0$ each monomial in the defining polynomial of X contains every homogeneous coordinate of power either 0, or at

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least $d-k$. We show that if d large enough with respect to n and to the number of non-zero monomials in the defining polynomial, then for such a hypersurface, any holomorphic map $\mathbb{C} \longrightarrow X$ is degenerate.

Second, we apply the above result to give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of arbitrary degree $d \geq 22$, and curves in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements of arbitrary degree $d \geq 19$. Notice that up to now all known explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ are of degree d divided by some integer > 1 (2 in Brody-Green's example, 3 in Nadel's example, 3 or 4 in Masuda-Noguchi's examples). Indeed, in [MN] it is given an algorithm to construct hyperbolic surfaces of degree $d > 54$.

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§2. DEGENERACY OF HOLOMORPHIC CURVES

Let

$$M_j = z_1^{\alpha_{j,1}} \dots z_{n+1}^{\alpha_{j,n+1}}, \qquad 1 \leq j \leq s,$$

be distinct monomials of degree d with non-negative exponents. Let X be a hypersurface of degree d of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1 + \dots c_s M_s = 0$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants. We call X *a pertubation of the Fermat hypersurface of degree* d if $s \geq n+1$ and

$$M_j = z_j^d, \quad j = 1, \dots, n+1.$$

Theorem 2.1. *Suppose that there is an integer $k \geq 0$ such that X satisfies the following conditions:*

i) For $j \geq n+2$, $m = 1, \ldots, n+1$, the exponent $\alpha_{j,m}$ is either 0, or $\alpha_{j,m} \geq d-k$.

ii) $d > k + s(s-2)$.

Then every holomorphic curve in X is degenerate.

To prove Theorem 1 let us recall Cartan's defect relation for holomorphic curves ([C], see also [MN]).

Let f be a holomorphic curve and H be a hyperplane of $\mathbb{P}^n(\mathbb{C})$ which does not contain the image of f. We denote by $\deg_z f^*H$ the degree of the pull-backed divisor f^*H at $z \in \mathbb{C}$. We say that f ramifies at least d (> 0) over H if $\deg_z f^*H \geq d$ for all $z \in f^{-1}H$. In case $f^{-1}H = 0$, we set $d = \infty$.

Lemma 2.2. (H. Cartan [C]). *Assume that f is linearly non-degenerate and ramifies at least d over H_j, $1 \leq j \leq q$, where the hyperplanes H_j are in general position. Then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) \leq n + 1.$$

Now let X be a hypersurface satisfying the hypothesis of Theorem 2.1, and let $f = (f_1, \ldots, f_{n+1}) : \mathbb{C} \longrightarrow X$ be a holomorphic curve. We are going to show that $\{f_1^d, \ldots, f_{n+1}^d, M_{n+2} \circ f, \ldots, M_s \circ f\}$ are linearly dependent. Suppose that it is not the case. Consider a holomorphic curve g in $\mathbb{P}^{s-2}(\mathbb{C})$ defined by

$$g: \; z \in \mathbb{C} \mapsto (f_1^d(z), \ldots, f_{n+1}^d(z), M_1 \circ f(z), \ldots, M_{s-1} \circ f(z)) \in \mathbb{P}^{s-2}(\mathbb{C}).$$

Take the following hyperplanes in general potion:

$$H_1 = \{z_1 = 0\}, \ldots, H_{s-1} = \{z_{s-1} = 0\}, \; H_s = \{c_1 z_1 + \cdots + c_{s-1} z_{s-1} = 0\}.$$

By the hypothesis of Theorem 2.1 we see that g ramifies at least $d-k$ over H_j for all $1 \le j \le s$. It follows from Lemma 2.2 that

$$(1) \qquad \sum_{j=1}^{s}\left(1-\frac{s-2}{d-k}\right) \le s-1.$$

Hence $d \le k + s(s-2)$, a contradiction. Then the image of f is contained in the proper algebraic subset of X defined by the following equation

$$a_1 z_1^d + \cdots + a_{n+1} z_{n+1}^d + a_{n+2} M_{n+2} + \cdots + a_{s-1} M_{s-1} = 0.$$

where not all a_j are zeros. Theorem 1 is proved.

Corollary 2.3. (M. Green [G]) *Let X be the Fermat hypersurface*

$$X: \; z_1^d + \cdots + z_{n+1}^d = 0,$$

and let $f = (f_1, \dots, f_{n+1})$ be a holomorphic curve in X. If $d > n^2 - 1$, there is a decomposition of indices $\{1, \dots, n+1\} = \cup I_\xi$ such that:

i) If $i, j \in I_\xi$, $f_i/f_j = const.$

ii) $\sum_{i \in I_\xi} f_i^d = 0$ for any ξ.

Proof. It suffices to take $k = 0$, $s = n + 1$ in Theorem 2.1, and apply Theorem 2.1 repeatedly. Corollary then is proved by induction. Notice that the hypothesis of Theorem 2.1 is fulfielled after every step of induction.

The following more precise form of Theorem 2.1 is very useful in applications to surfaces in $\mathbb{P}^3(\mathbb{C})$.

Theorem 2.4. *Let X be a hypersurface satisfying the hypothesis of Theorem 2.1, where the inequality ii) is replaced by an weaker one:*

$$\frac{(n+1)(s-2)}{d} + \frac{(s-2)(s-n-1)}{d-k} < 1.$$

Then any holomorphic curve in X is degenerate.

Proof. We can repeat the proof of Theorem 2.1, but instead of (1) we use the following inequality

$$(2) \qquad \sum_{j=1}^{n+1}(1-\frac{s-2}{d}) + \sum_{j=n+2}^{s}(1-\frac{s-2}{d-k}) \leq s-1.$$

§3. Hyperbolic surfaces

In this section we use Theorems 2.1 and 2.4to give explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ and of curves in $\mathbb{P}^2(\mathbb{C})$ with hyperbolic complements.

Theorem 3.1. *Let X be a surface in $\mathbb{P}^3(\mathbb{C})$ of degree d defined by the equation*

$$(3) \qquad X: \ z_1^d + z_2^d + z_3^d + z_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $c \neq 0$, $\sum_{i=1}^{3} \alpha_i = d, \alpha_i \geq 7$. Then X is hyperbolic if $d \geq 22$,

Proof. Take $k = d-7$. Then X satisfies the hypothesis of Theorem 2.4, and every holomorphic curve in X is degenerate.

Now let $f = (f_1, f_2, f_3, f_4): \ \mathbb{C}_p \longrightarrow X$ be a holomorphic curve in X. Consider the following possible cases:

1) For some $i = 1, 2, 3$, $f_i \equiv 0$. then f is a constant map by Corollary 2.3.

2) $f_4 \equiv 0$. Then the image of (f_1, f_2, f_3) is contained in the curve defined by the following equation

$$Y: \ z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

From the proof of Theorem 2.1 it follows that $\{f_1^d, f_2^d, f_3^d\}$ are linearly dependent. Then at least two of $\{f_1, f_2, f_3\}$, say, f_1 and f_2, have a constant ratio. Substitute this relation into (3) we can show that f is a constant map (note that $\alpha_i \neq 0$ for all $i = 1, 2, 3$).

3) Assume that any $f_i \not\equiv 0$. From the proof of Theorem 2.1 it follows that $\{f_1^d, \ldots, f_4^d\}$ are linearly dependent:

$$a_1 f_1^d + \cdots + a_4 f_4^d \equiv 0,$$

where not all a_i are zeros. Consider the following possible cases:

i) $a_i \neq 0,\ i = 1, \ldots, 4$. By Corollary 2.3, f is a constant map, or we can assume that $f_1 = c_1 f_2,\ f_3 = c_2 f_4$. Then we can substitute this relation to (3) and show that f is a constant map.

ii) Only one of $a_i = 0$, say $a_4 = 0$. Then (f_1, f_2, f_3) is a constant map by Corollary 2.3, and it is easy to show that f is a constant map.

iii) If $a_4 \neq 0$, and two coefficients, say, $a_1 = a_2 = 0$. Then we have $f_3 = c_3 f_4$. Substitute this relation into (3) we obtain

$$f_1^d + f_2^d + \varepsilon_1 f_3^d + \varepsilon_2 f_1^{\alpha_1} f_2^{\alpha_2} f_3^{\alpha_3} \equiv 0, \tag{4}$$

where $\varepsilon_2 \neq 0$. We return to the case 2).

iv) $a_4 = 0$, and one of a_1, a_2, a_3, say, $a_1 = 0$. Then f_2/f_3 is a constant, and we obtain:

$$f_1^d + A f_3^d + f_4^d + B f_1^{\alpha_1} f_3^{\alpha_2 + \alpha_3} = 0,$$

where $B \neq 0$. If $A \neq 0$, then $\{f_1^d, f_3^d, f_4^d\}$ are linearly dependent, again by the proof of Theorem 2.1 and we return to the case similar to 2).

Now suppose that $A = 0$. Then the image of the map (f_1, f_3, f_4) is contained in the following curve in $\mathbb{P}^2(\mathbb{C})$ (with homogeneous coordinates (z_1, z_3, z_4)):

$$Y:\ z_1^d + z_4^d + B z_1^{\alpha_1} z_3^{\alpha_2 + \alpha_3} = 0.$$

We are going to show that under the hypothesis of Theorem 3.1, the genus of Y is at least 2.

The genus of Y is equal to the number of integer points in the triangle with the vertices $(d, 0), (0, d)$ and $(\alpha_1, 0)$ (see, for example, [Ho]). It is easy to see that this triangle contains at least two integer points, if $\alpha_1 < d-2$. Here, by the hypothesis we have $\alpha_1 = d - (\alpha_2 + \alpha_3) \leq d - 14$. The proof is completed.

Remark 1. In [MN] K. Masuda and J. Noguchi proved that for every n there is a number $d(n)$ such that for every $d \geq d(n)$ there are hyperbolic hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C})$. They pointed that $d(3) \leq 54$. In [N] M. Nadel gives explicit examples of hyperbolic surfaces in $\mathbb{P}^3(\mathbb{C})$ of degree $d = 3e, e \geq 7$. From Theorem 3.1 it follows that $d(3) \leq 22$. Combining Theorem 3.1 with Nadel's results ([N]) we have $d(3) \leq 21$.

Remark 2. It is clear that we can take in the equation (3) $cz_i^{\alpha_i} z_j^{\alpha_j} z_l^{\alpha_l}$ instead of $cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}$, for any triple (z_i, z_j, z_l) from (z_1, z_2, z_3, z_4).

Remark 3. From the proof of Theorem 3.1 it follows that the following surfaces are hyperbolic:

$$X : \; z_1^d + z_2^d + z_3^d + z_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0, \tag{5}$$

where $c \neq 0, \sum_{i=1}^{4} \alpha_i = d, \alpha_i \geq 6, d > 24$. In fact, it suffices to take $k = d - 6$ and repeat the proof of Theorem 3.1

Theorem 3.2. *Let X be a curve in $\mathbb{P}^2(\mathbb{C})$ defined by the following equation*

$$X : \; z_1^d + z_2^d + z_3^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $c \neq 0$, $\alpha_i \geq 6$. If $d \geq 19$, then $\mathbb{P}^2(\mathbb{C}) \setminus X$ is complete hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$.

Proof. Due to R. Brody and M. Green ([BG]), X is hyperbolic and the complement $\mathbb{P}^2(\mathbb{C}) \setminus X$ is complete hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$ if and only if neither X nor $\mathbb{P}^2(\mathbb{C})$ does not admit a non-constant holomorphic curve from $\mathbb{C}$. By the proof of Theorem 3.1 it suffices to prove that any holomorphic curve $f: \mathbb{C} \longrightarrow \mathbb{P}^2 \setminus X$ is constant.

Let $f = (f_1, f_2, f_3)$ be such a curve. Consider the surface Y defined by the following equation:

$$Y: \; z_1^d + z_2^d + z_3^d + z_4^d + cz_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0.$$

Let

$$\varphi : Y \setminus \{z_4 = 0\} \to \mathbb{P}^2(\mathbb{C}) \setminus X$$

be the projection of the first three homogeneous coordinates. Then φ is an unramified covering, and f may be lifted to $\tilde{f} = (f_1, f_2, f_3, f_4) : \mathbb{C} \to Y \setminus \{z_4 = 0\}$.

Now we will show that under the hypothesis of Theorem 3.2, $\tilde{f}$ is degenerate in Y. In fact, if it is not the case, then we take $k = d - 6$ and repeat the proof of Theorem 2.4. Note that $f_4 \neq 0$, and making use of Lemma 2.2, we take $d_4 = \infty$. Therefore, instead of the inequality (2) we obtain

$$3(1 - \frac{5-2}{d}) + 1 + (1 - \frac{5-2}{6}) \leq 5 - 1.$$

It is impossible when $d \geq 19$.

Hence, by the proof of Theorem 3.1 Y is hyperbolic, then $\tilde{f}$ is constant, so is f. Theorem is proved.

Remark 4. M. G. Zaidenberg proved that for $d \geq 5$ there are hyperbolic curves of degree d such that their complements are complete hyperbolic and hyperbolically imbedded into $\mathbb{P}^2(\mathbb{C})$. In [MN] K. Masuda and J. Noguchi give the construction of such curves with $d \geq 48$. Here we have examples with $d \geq 19$.

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