SPHERICAL HOMOLOGY CLASSES IN THE BORDISM OF LIE GROUPS

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The mod torsion Hurewicz map

$$h_H : \Pi_*(G)/\mathrm{Tor} \longrightarrow H_*(G)/\mathrm{Tor}$$

for compact Lie groups provides a useful and efficient means of studying G. In effect, it measures how far G fails to be a product of spheres. For the Hopf-Samelson theorem (see Milnor-Moore [17]) tells us that $H_*(G;\mathbb{Q}) = E(x_1,\ldots,x_r)$ where $\deg x_i = 2r_i - 1$. In other words, $H_*(G;\mathbb{Q}) = H_*(\prod_{i=1}^{r} S^{2n_i-1};\mathbb{Q})$. Serre pointed out that there exists a canonical map $f : \prod_{i=1}^{r} S^{2n_i-1} \longrightarrow G$ inducing this $\mathbb{Q}$ isomorphism. Just take the generators of $\Pi_*(G)/\mathrm{Tor}$ (they lie in degrees $\{2n_i-1,\ldots,2n_r-1\}$) and multiply them together

$$f : S^{2n_i-1}\times\ldots\times S^{2n_r-1} \xrightarrow{f_1\times\ldots\times f_r} G\times\ldots\times G \longrightarrow G$$

Observe that the Hurewicz map is the study of the restrictions $H_*(S^{2n_i-1}) \longrightarrow H_*(\prod_{i=1}^{r} S^{2n_i-1}) \xrightarrow{f_*} H_*(G)$. So it provides an index of how far $f_* : H_*(\prod_{i=1}^{r} S^{2n_i-1}) \longrightarrow H_*(G)$ fails to be an isomorphism.

A great deal of information has been obtained about the map f and/or the Hurewicz map h_H. The study can, of course, be reduced to the case of p primary information through localization. The approach has been to concentrate on reasonably large primes. For such primes a complete solution has been given. The relevant concepts are regularity (see Serre [21] or Kumpel [14]) or quasi regularity (see Mimura-Toda [19], Harper [9] and Wilkerson [27]). For small primes much less is known. Among the various simple Lie groups the Hurewicz map has been calculated only for the classical groups and for G_2 and F_4. We will cite references at the appropriate places in the text.

In this paper we will study the question of a general characterization of spherical homology classes. Such a characterization would appear to be rather difficult in terms of ordinary homology. The purpose of this paper is to study whether such a characterization can be obtained using MU theory. One has a factorization

$$\begin{tikzcd} & MU_*(G)/\mathrm{Tor} \arrow[d, "T"] \\ \Pi_*(G)/\mathrm{Tor} \arrow[ur, "h_{MU}"] \arrow[r, "h_H"'] & H_*(G)/\mathrm{Tor} \end{tikzcd}$$

where the top map is the MU Hurewicz map and T is the Thom map. So the determination of h_{MU} also determines h_H. In this paper we will study whether $\mathrm{Im}\ h_{MU}$ can be characterized as the elements of $MU_*(G)/\mathrm{Tor}$ which are primitive both with respect to MU operations and with respect to the coalgebra structure of $MU_*(G)/\mathrm{Tor}$. As we have already indicated, the study of h_H and

h_{MU} can always be reduced to the p primary case through localization. Our answers, in so far as it goes (G classical or $G = G_2, F_4$), is "yes" for MU localized at an odd prime and "no" for MU theory localized at $p = 2$.

Our study of the MU Hurewicz map is related to (and, indeed, motivated by) another question about the Hurewicz map. Atiyah and Mimura asked if, in the case of Lie groups, $\operatorname{Im} h_H$ can be characterized in terms of the Chern character $ch : K_*(G) \otimes \mathbb{Q} \longrightarrow H_*(G;\mathbb{Q})$. Our answer agrees with Atiyah and Mimura's expectations. In the printed version of the conjecture (see Stasheff [23]) they expect a positive answer for all primes. However, they lates allowed the possibility of the conjecture failing for the 2 primary case. (We are grateful to J.F. Adams for this last piece of information). See §7 for a further discussion of the Atiyah-Mimura conjecture and its relation to MU theory.

This paper is divided into three parts. In Part I we study rational MU theory and define an operation P which characterizes the operation primitive elements of $MU_*(X) \otimes \mathbb{Q}$. In Part II we study how one uses the rational information to obtain information about the primitives in $MU_*(X)/\mathrm{Tor}$. One reduces to integrality problems connected with the inclusions $MU_*(X)/\mathrm{Tor} \subset MU_*(X) \otimes \mathbb{Q}$ and $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$. In Part III we study the relation between sphericals and primitives in the bordism of Lie groups.

In this paper X will denote the arbitrary space or spectrum while G will be reserved for a connected compact Lie group. Given a spectrum E we will adopt the usual convention of using $E_*(X)$ and $E^*(X)$ to denote the homology and cohomology defined by E. In particular $MU_*(X)$ and $MU^*(X)$ will be used for bordism and cobordism, respecitvely. Also $H_*(X)$ will always be homology with $\mathbb{Z}$ coefficients while $H_*(X)_{(p)}$ will denote homology localized at the prime p.

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PART I: The Operation P

§1 MU Theory

As a general reference for the material covered in Part I we refer the reader to Adams [1].

(a) $\Pi_*(MU)$

The ring $\Pi_*(MU)$ is a polynomial algebra $\mathbb{Z}[t_1, t_2, \ldots]$ (deg $t_i = 2i$). However, there is no obvious canonical choice of the generators $\{t_i\}$. When we pass to rational MU theory this problem disappears. We can write

$$\begin{aligned} \Pi_*(MU) \otimes \mathbb{Q} &= \mathbb{Q}[b_1, b_2, \ldots] \qquad (\deg b_i = 2i) \\ &= \mathbb{Q}[m_1, m_2, \ldots] \qquad (\deg m_i = 2i) \end{aligned}$$

where $\{b_i\}$ and $\{m_i\}$ are canonical. The $\{b_i\}$ are obtained as follows. There is a canonical map

$$\omega : \mathbb{C}P^\infty = MU(1) \longrightarrow MU$$

which lower degree by 2 in homology. If we write $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ and choose $\beta_i \in H_{2i}(\mathbb{C}P^\infty)$ by $\langle x^i, \beta_j \rangle = \delta_{ij}$ then $b_i = \omega_*(\beta_{i+1})$. One has $H_*(MU) = \mathbb{Z}[b_1, b_2, \ldots]$. The identity $\Pi_*(MU) \otimes \mathbb{Q} \cong H_*(MU) \otimes \mathbb{Q}$ then gives the first description of $\Pi_*(MU) \otimes \mathbb{Q}$.

The elements $\{m_i\}$ are the conjugates of the $\{b_i\}$. If we consider the power series

$$\exp(X) = \sum_{i \geq 0} b_i X^{i+1}$$

and let $\log(X)$ be the inverse power series then

$$\log(X) = \sum_{i \geq 0} m_i X^{i+1} \quad .$$

If we apply the Todd map then $\exp(X)$ and $\log(X)$ turn into the usual exp and log series. For $Td : \Pi_*(MU) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$ sends m_n to $\frac{1}{n+1}$ and b_n to $\frac{1}{n+1!}$. If we consider $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$ then we have the following integrality condition

(1.1) $$(n + 1)m_n \in \Pi_*(MU)$$

$$(n + 1!)b_n \in \Pi_*(MU) .$$

There are best possible since $Td(\Pi_*(MU) \subset \mathbb{Z}$.

(b) MU Homology and Cohomology

Both MU homology, $MU_*(X)$, and cohomology, $MU^*(X)$, are modules over $\Pi_*(MU)$. One must, however, adopt the connection that

$$MU_* = MU^{-*} = \Pi_*(MU) \quad .$$

In other words, the elements of $\Pi_*(MU)$ are considered to be negatively graded when one works in cohomology. There is a natural pairing

$$MU^*(X) \otimes MU_*(X) \longrightarrow \Pi_*(MU) \quad .$$

Provided $H_*(X)$ in torsion free then $MU^*(X)$ and $MU_*(X)$ are free $\Pi_*(MU)$ modules (the Atiyah-Hirzebruch spectral sequence collapses) and the above pairing is non-singular. In such cases we can think of $MU^*(X)$ and $MU_*(X)$ as being "dual" $\Pi_*(MU)$ modules. However, one must keep in mind the change in grading between homology and cohomology. As a result $MU_*(X)$ is always connected and of finite type whereas $MU^*(X)$ need not be either. For example let $\omega \in MU^2(\mathbb{C}p^\infty)$ be given by the map $\omega : \mathbb{C}P^\infty \longrightarrow MU$. Then

$$MU^*(\mathbb{C}P^\infty) = MU^*[[\omega]]$$

while

$$MU_*(\mathbb{C}P^\infty) = MU_*\{\beta_0, \beta_1, \beta_2, \ldots\}$$

where $\langle \omega^i, \beta_j \rangle = \delta_{ij}$. In the first case we have all formal power series in ω. In the second we have the free module generated by $\{\beta_0, \beta_1, \beta_2, \ldots\}$.

In the case of rational MU theory the situation is always simple. Both $MU^*(X) \otimes \mathbb{Q}$ and $MU_*(X) \otimes \mathbb{Q}$ are free and are "dual". The Thom map $T : MU_*(X) \otimes \mathbb{Q} \longrightarrow H_*(X;\mathbb{Q})$ is surjective with kernel = ideal $(m_1, m_2, \ldots)$.

§2 The Operation P

For each exponential sequence $E = (e_1, e_2, \ldots)$ (i.e. a sequence of non negative integers with only fintely many non zero terms) we have the Landweber-Novikov operations

$$s_E : MU^*(X) \longrightarrow MU^*(X)$$

$$s_E : MU_*(X) \longrightarrow MU_*(X)$$

which, in the cohomology case, raise degree by $|E| = 2\Sigma e_i$ and, in the homology case, lower degree by $|E|$. The action of s_E on $\Pi_*(MU)$ is difficult to describe. When we pass to $\Pi_*(MU) \otimes \mathbb{Q}$ the situation improves. We have the canonical generators $\{b_i\}$ and $\{m_i\}$ of $\Pi_*(MU) \otimes \mathbb{Q}$ given in §1. Given an exponential sequence $E = (e_1, e_2, \ldots)$ let

$$b^E = b_1^{e_1} b_2^{e_2} \ldots b_k^{e_k}$$

$$m^E = m_1^{e_1} m_2^{e_2} \ldots m_k^{e_k} .$$

Then we have

$$(2.1) \qquad s_E(b^F) = \begin{cases} 0 & \text{if } |E| \geq |F| \text{ and } E \neq F \\ 1 & \text{if } E = F \end{cases} .$$

If r_E is the conjugate of s_E defined by the recursive formula

$$(2.2) \qquad \sum_{E_1+E_2=E} r_{E_1} s_{E_2} = 0$$

then r_E acts by the rule

$$(2.3)\qquad r_E(m^F) = \begin{cases} 0 & \text{if } |E| \geq |F| \text{ and } E \neq F \\ 1 & \text{if } E = F \end{cases} .$$

We now define operations

$$\mathcal{P} : MU^*(X) \otimes \mathbb{Q} \longrightarrow MU^*(X) \otimes \mathbb{Q}$$

$$\mathcal{P} : MU_*(X) \otimes \mathbb{Q} \longrightarrow MU_*(X) \otimes \mathbb{Q}$$

by the rule

$$\mathcal{P}(X) = \sum_E m^E s_E(x)$$

where one sums over all exponential sequences. These operations have a number of useful properties. We will only state them for cohomology

(2.4) <u>Multiplicative:</u> $\mathcal{P}(xy) = \mathcal{P}(x)\mathcal{P}(y)$

(2.5) <u>Primitive Idempotent:</u> $\mathcal{P}^2 = \mathcal{P}$ where $\mathrm{Im}\ \mathcal{P}$ = the operator primitives of $MU^*(X) \otimes \mathbb{Q}$ and $\mathrm{Ker}\ \mathcal{P}$ = the ideal $(m_1, m_2, \ldots)$.

This last property has a number of consequences. We have that x is primitive if and only if $x = \mathcal{P}(y)$ for some y. Also the Thom map induces an isomorphism $\mathrm{Im}\ \mathcal{P} \cong H^*(X;\mathbb{Q})$. In other words, although an element of $H^*(X;\mathbb{Q})$ has many representatives in $MU^*(X) \otimes \mathbb{Q}$, it has an unique primitive representative. Lastly, there exists an unique factorization

$$\begin{tikzcd} MU^*(X) \otimes \mathbb{Q} \arrow[r, "P"] \arrow[d, "T"'] & MU^*(X) \otimes \mathbb{Q} \\ H^*(X) \otimes \mathbb{Q} \arrow[ur, dashed, "\bar{P}"'] & \end{tikzcd}$$

The above discussion also applies in homology. Of course the multiplicative property only holds in homology when X has a product e.g. X is a ring spectrum or a H-space.

For proofs of all the above properties consult Kane [11]. There, a BP version of the operation P was constructed and studied. Indeed, the next chapter is devoted to recalling this BP version. The arguments given in Kane [11] also apply to the present MU operator. Properties 2.4 and 2.5 are deduced from 2.1, 2.2 and 2.3.

Remark 2.7: Although we will not need it in this paper it is probably useful to point out that P has a "dual" definition as

$$P(x) = \sum_E b^E r_E(x)$$

§3 BP Theory

As we have already mentioned the operation P has an analogue in rational Brown-Peterson theory. This operation has already been constructed in Kane [11]. Given a prime p then Brown-Peterson theory is a canonical summand of MU theory localized at p. If we rationalize then the relation between the two is easy to state. Namely $\Pi_*(BP) \otimes \mathbb{Q} \subset \Pi_*(MU) \otimes \mathbb{Q}$ via the identity

$$(3.1) \qquad \Pi_*(BP) \otimes \mathbb{Q} = \mathbb{Q}[m_{p-1}, m_{p^2-1}, \ldots]$$

Indeed Quillen defined BP so as to have precisely this property. For each exponential sequence E he also defined operations

$$r_E : BP^*(X) \longrightarrow BP^*(X)$$

$$r_E : BP_*(X) \longrightarrow BP_*(X)$$

which raise degrees and lower degrees, respectively, by $2\Sigma e_i(p^i-1)$. If we define the conjugate s_E of r_E by the recursive rule

$$(3.2) \qquad \sum_{E_1+E_2=E} s_{E_1} r_{E_2} = 0$$

then s_E covers the Steenrod operation P^E defined in Milnor [16]. In other words, we have commutative diagrams

$$(3.3)\qquad \begin{tikzcd} BP^*(X) \arrow[r, "s_E"] \arrow[d] & BP^*(X) \arrow[d] \\ H^*(X;\mathbb{F}_p) \arrow[r, "P^E"] & H^*(X;\mathbb{F}_p) \end{tikzcd} \qquad \begin{tikzcd} BP_*(X) \arrow[r, "s_E"] \arrow[d] & BP_*(X) \arrow[d] \\ H_*(X;\mathbb{F}_p) \arrow[r, "P^E"] & H_*(X;\mathbb{F}_p) \end{tikzcd} \quad .$$

The vertical maps are the Thom map followed by reduction mod p. Also $P^E : H_*(X;\mathbb{F}_p) \longrightarrow H_*(X;\mathbb{F}_p)$ is the left action defined from from the usual left action of P^E on $H^*(X;\mathbb{F}_p)$ by the rule

$$(3.4)\qquad <\chi(P^E)(x),y> = <x,P^E(y)>$$

for any $x \in H^*(X;\mathbb{F}_p)$ and $y \in H_*(X;\mathbb{F}_p)$. ($\chi(P^E)$ is the conjugate of P^E).

If we let $\hat{m}^E = m_{p-1}^{e_1} \ldots m_{p^k-1}^{e_k}$ then the operations

$$P : BP^*(X) \otimes \mathbb{Q} \longrightarrow BP^*(X) \otimes \mathbb{Q}$$

$$P : BP_*(X) \otimes \mathbb{Q} \longrightarrow BP_*(X) \otimes \mathbb{Q}$$

$$P(x) = \sum_E \hat{m}^E s_E$$

satisfies properties analogous to the previous P. Also, the factorization of P through $H_*(X;\mathbb{Q})$ implies that we have a commutative diagram

$$\begin{tikzcd} MU_*(X) \otimes \mathbb{Q} \arrow[r, "P"] & MU_*(X) \otimes \mathbb{Q} \\ BP_*(X) \otimes \mathbb{Q} \arrow[u] \arrow[r, dash, "P"] & BP_*(X) \otimes \mathbb{Q} \arrow[u] \end{tikzcd}$$

PART II: Integral Primitive Elements

§4 Integral Primitives

We have an imbedding $MU_*(X)/\mathrm{Tor} \subset MU_*(X) \otimes \mathbb{Q}$ where $\mathrm{Tor} = \{x \in MU_*(X) \mid nx = 0 \text{ for some } n \in \mathbb{Z}\}$. By the discussion in Part I, the problem of determining primitive elements in $MU_*(X)/\mathrm{Tor}$ reduces to determining $\mathrm{Im}\, P \cap MU_*(X)/\mathrm{Tor} \subset MU_*(X)/\mathrm{Tor}$. In other words, we have an integrality problem. When does $x \in \mathrm{Im}\, P \subset MU_*(X) \otimes \mathbb{Q}$ belong to $MU_*(X)/\mathrm{Tor} \subset MU_*(X) \otimes \mathbb{Q}$?

One can always reduce this integrality problem to an integrality problem concerning this inclusion $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$. Choose a $\Pi_*(MU) \otimes \mathbb{Q}$ basis $\{x_i\}$ of $MU_*(X) \otimes \mathbb{Q}$ where the $\{x_i\}$ are elements of $MU_*(X)/\mathrm{Tor}$. Expand

$$P(x) = \sum \alpha_i x_i \qquad \alpha_i \in \Pi(MU) \otimes \mathbb{Q} .$$

Then $P(x) \in MU_*(X)/\mathrm{Tor}$ if and only if $\alpha_i \in \Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$ for each α_i. In the rest of Part II we will illustrate how one can study primitive elements in $MU_*(X)/\mathrm{Tor}$ by the above method. In §5 we will give some precis integrality conditions about the inclusion $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$ which will be used in solving our problem for $P(x)$. In §6 we will give some examples where we solve the integrality question for $\mathrm{Im}\, P$ by the above method.

It should be noted that the above approach is not really practical as a general method for studying the primitive of $MU_*(X)/\mathrm{Tor}$. For it depends on being able to obtain reasonably explicit exansions of $P(x)$. Such knowledge is not always available. Even in this paper lack of knowledge of $\operatorname{Im} P$ will soon cause us to abandom the above approach. In Part III we will introduce and constantly use a cruder but more effective tool. This cruder index is the image of $\operatorname{Im} P \cap MU_*(X)/\mathrm{Tor}$ under the Thom map $T: MU(X)/\mathrm{Tor} \longrightarrow H_*(X)/\mathrm{Tor}$. This subgroup of $H_*(X)/\mathrm{Tor}$ is easier to study than $\operatorname{Im} P \cap MU_*(X)/\mathrm{Tor} \subset MU^*(X)/\mathrm{Tor}$.

§5 Integrality Conditions for $\Pi_*(MU) \otimes \mathbb{Q}$

In this section we prove that certain specific elements of $\Pi_*(MU) \otimes \mathbb{Q}$ actually belong to $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$. Our arguments are based on those used by Segal [20]. Let

$$b = 1 + b_1 + b_2 + \dots$$

$$(b)^i_j = \text{the homogeneous component of degree } 2j \text{ in } (b)^i$$

<u>Proposition 5.1</u> $\frac{r!}{2}(b)^q_{r-q} \in \Pi_*(MU)$ if $2 \leq q \leq r$

Observe that the restriction $q \geq 2$ is necessary. For $(b)^1_{r-1} = b_{r-1}$. And, as we observed in §1, one must multiply b_{r-1} by $r!$ to make it integral.

<u>Proof of Proposition</u>

We can expand

$$(b)^q = (b_0 + b_1 + \dots)^q = \Sigma(e_0,e_1,\dots,e_s)b_0^{e_0}b_1^{e_1}\dots b_s^{e_s}$$ where $(e_0,\dots,e_s)$ is the multinomial coefficients $\frac{(e_0+\dots+e_s)!}{e_0!\dots e_s!}$.

Then

$$(b)^q_{r-q} = \sum_{\substack{e_0+\dots+e_s=q \\ e_1+2e_2+\dots+se_s=r-q}} (e_0,e_1,\dots,e_s)b_0^{e_0}b_1^{e_1} \dots b_s^{e_s}$$

and

$$\frac{r!}{2}(b)^q_{r-q} = \sum_{\substack{e_0+\dots+e_s=q \\ e_1+2e_2+se_s=r-q}} \frac{r!}{2}(e_0,e_1,\dots,e_s)b_0^{e_0}b_1^{e_1}\dots b_s^{e_s}.$$

We will demonstrate that each term $\frac{r!}{2}(e_0,e_1,\dots,e_s)b_0^{e_0}b_1^{e_1} \dots b_s^{e_s} \in \Pi_*(MU)$. We consider two separate cases

(i) $e_i \geq 2$ for some i

We know that $i_1+1! \ldots i_q+1!\, b_{i_1} \ldots b_{i_q} \in \Pi_*(MU)$. Consequently, if $(i_1+1)+\ldots+(i_q+1)=r$ then $\frac{r!}{2} b_{i_1} \ldots b_{i_q}$ $= \frac{1}{2}(i_1+1! \ldots 2q+1!)(i_1+1,\ldots,i_q+1)b_{i_1} \ldots b_{i_q} \in \Pi_*(MU)$ provided $(i_1+1,\ldots,i_q+1) \equiv 0 \bmod 2$. But it follows from some simple number theory that $(i_1+1,\ldots,i_q+1) \equiv 0 \bmod 2$ if $i_a = i_b$ for any $a \neq b$. For

$$(k_1,\ldots,k_s) \equiv \prod_i (k_{1i},\ldots,k_{si}) \bmod 2$$

where $k_t = \Sigma k_{t\,i} 2^i$ is the 2-adic expansion of k_t.

(ii) $e_i \leq 1$ for all i

We have that $(e_0,e_1,\ldots,e_s) \equiv 0 \bmod 2$ since $\Sigma e_i = q \geq 2$. Thus, as in (i), $\frac{r!}{2}(e_0,\ldots,e_s)b_0^{e_0} \ldots b_s^{e_s} \in \Pi_*(MU)$.

Q.E.D.

Segal [20] made effective use of the Liulevicius version of the Hattori-Stong theorem in studying $\Pi_*(MU) \subset \Pi_*(MU) \otimes \mathbb{Q}$. Write $\Pi_*(MU) \otimes \mathbb{Q} = H_*(MU) \otimes \mathbb{Q}$. Then

(5.2) $x \in H_*(MU)$ belongs to $\operatorname{Im} \Pi_*(MU) \longrightarrow H_*(MU)$ if and only $tds_E(x) \in \mathbb{Z}$ for all E.

The following fact will be used in our study of $X = SO(2n+1)$. In harmony with BP theory let $v_1 = 2m_1$. Then

Proposition 5.3: $\frac{2k-1!}{2}(b_{2k-2} + v_1 b_{2k-3}) \in \Pi_*(MU)$

Proof: We will use criterion 5.2

(i) $E = (0,0,\ldots)$.

To show $\frac{2k-1!}{2}\, Td(b_{2k-2} + v_1 b_{2k-3}) \in \Pi_*(MU)$ it suffices to show $2k - 1!\; Td(b_{2k-2} + v_1 b_{2k-3}) \in 2\mathbb{Z}$. We have

$$2k - 1!\,Td(b_{2k-2} + v_1 b_{2k-3}) = \frac{2k-1!}{2k-1!} + \frac{2k-1!}{2k-2!}$$

$$= 1 + 2k - 1$$

$$= 2k$$

(ii) $|E| > 0$

We will consider the terms b_{2k-2} and $v_1 b_{2k-3}$ separately. First of all

$$s_E(b_{2k-2}) = \begin{cases} (b)^i_{2k-1-i} & E = \Delta_i \\ 0 & \text{otherwise.} \end{cases}$$

So, by Proposition 5.1, $\frac{2k-1!}{2}\, Tds_E(b_{2k-2}) \in \mathbb{Z}$. A slightly more complicated argument of the same type handles the case $\frac{2k-1!}{2}$ $Tds_E(v_1 b_{2k-3})$.

Q.E.D.

§6 Examples

We now demonstrate how one solves integrality problems for certain cases of both the MU and BP version of the operator $\mathcal{P}$.

(a) The Space $X = \mathbb{C}P^\infty$

For certain spaces one can obtain explicit formula for the operation $\mathcal{P} : MU_*(X) \otimes \mathbb{Q} \longrightarrow MU_*(X) \otimes \mathbb{Q}$. The space $X = \mathbb{C}P^\infty$ is the canonical example. Write $MU^*(\mathbb{C}P^\infty) = MU^*[[\omega]]$ and $MU_*(\mathbb{C}P^\infty) = MU_*\{\beta_0, \beta_1, \beta_2, \ldots\}$ as in §1. The operations $\{s_E\}$ act by the rule

$$s_E(\omega) = \begin{cases} \omega^{k+1} & E = \Delta_k \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$(6.1) \qquad \mathcal{P}(\omega) = \sum_{i \geq 0} m_i \omega^{i+1} = \log(\omega).$$

Inverting, we have

$$(6.2) \qquad \omega = \exp \mathcal{P}(\omega) = \sum_{i \geq 0} b_i \mathcal{P}(\omega)^{i+1}$$

For each $k \geq 1$ we then have

$$(6.3) \qquad \omega^k = \Sigma (b)_i^k \mathcal{P}(\omega)^{k+1}$$

where $(b)_i^k$ are the coefficients defined in §5. Since $\{\omega^i\}$ and $\{\beta_j\}$ are dual basis it follows that $\{\mathcal{P}(\omega^i)\}$ and $\{\mathcal{P}(\beta_j)\}$ are also dual basis. If we dualize 6.3 then we obtain

Proposition 6.4 $P(\beta_k) = \sum_{j \le k} (b)^j_{k-j} \beta_j$

It then follows from Proposition 5.1, plus

$k! b_{k-1} \in \Pi_*(MU)$, that

Corollary 6.5 $k! P(\beta_k) \in MU_{2k}(\mathbb{C}P^\infty)$

(b) The Space $X = Sp(2)$

We next demonstrate the usefulness of BP definition of the P operation in understanding the MU version. The result obtained is only partial. But it will play an important role in the study of the spaces $Sp(n)$ in Part III.

The problem we are dealing with at the moment is to determine the minimal integer N such that $NP(x) \in MU_*(x)/Tor \subset MU_*(X) \otimes \mathbb{Q}$. To determine the p primary factor of N it suffices to localize and work with BP theory. In other words, if p^s is the minimal power of p such that $p^s P(x) \in BP_*(X)/Tor$ then $N = p^p N'$ where $(N',p) = 1$. The advantage of BP is that even if one has no information about $P(x) \in MU_*(x) \otimes \mathbb{Q}$ one can often obtain information about $P(x) \in BP_*(X) \otimes \mathbb{Q}$. For, as explained in 3.3, the BP operations $\{s_E\}$ are related to the Steenrod operations $\{P^E\}$. So one can use knowledge of the $A^*(p)$ action on $H_*(X; \mathbb{F}_p)$ to deduce results about $P(x) = \Sigma m^E s_E(x)$ in $BP_*(X) \otimes \mathbb{Q}$. We give a simple but useful example of this process.

Recall that

$$MU_*(Sp(2)) = E(x_3, x_7) ,$$

where $\Pi_*(MU)$ is the coefficient ring. We have $x_3 = P(x_3)$ is primitive. On the other hand, it is not clear for what coefficient N we have $NP(x_7) \in MU_*(Sp(2))$ we now obtain an upper board.

<u>Proposition 6.6</u> $3!P(x_7) \in MU_*(Sp(2))$.

In §8 we will demonstrate that this is a best possible result. We will prove the proposition by using the BP version of the P operation. For each prime p we have $BP_*(Sp(2)) = E(x_3, x_7)$ where $\Pi_*(BP)$ is the coefficient ring. It suffices to show

(i) for $p = 2$ $2P(x_7)$ $BP_*(Sp(2))$

(ii) for $p = 3$ $3P(x_7)$ $BP_*(Sp(2))$

(iii) for $p \geqq 5$ $P(x_7)$ $BP_*(Sp(2))$

<u>Proof of (i)</u>

For $p = 2$ we have

$$P(x_7) = x_7 + m_1 s_1(m_7) + m_1^2 s_2(x_7)$$

Since $2m_1 \in \Pi_*(BP)$ we have

$$2m_1 s_1(x_7) \in BP_*(Sp(2))$$

Since $Sq^4: H_7(Sp(2); \mathbb{F}_2) \longrightarrow H_3(Sp(2); \mathbb{F}_2)$ is trivial ($A^*(2)$ acts unstably !) we must have $s_2(x_7) = 2\alpha x_3$ for some $\alpha \in \mathbb{Z}_{(2)}$. Thus

$$2m_1^2 s_2(x_7) = (2m_1)(2m_1)\alpha x_3 \in MU_*(Sp(2))$$

<u>Proof of (ii) and (iii)</u>

For $p = 3$ we can write $P(x_7) = x_7 + m_2 s_1(x_7)$ and $3m_2 \in \Pi_*(BP)$. For $p \geq 5$ we have $P(x_7) = x_7$.

<u>Remark 6.7:</u> Observe how, in the case $p = 2$, we used the relation between BP operations and Steenrod operations to deduce a fact about $P(x_7)$ from our knowledge of the $A^*(2)$ action on $H_*(Sp(2);\mathbb{F}_2)$.

PART III: Primitive versus Spherical Classes

§7 Primitive and Spherical Classes

So far we have only discussed homology classes which are primitive with respect to cohomology operations. However a homology theory also has a coalgebra structure induced by the diagonal map $\Delta : X \longrightarrow X \times X$. And there is also the concept of a homology class being primitive with respect to this coalgebra structure. Given a coalgebra C with coproduct $\Delta : C \longrightarrow C \otimes C$, an element of C is said to be (coalgebra) primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$. We will use the symbol $P(X)$ to denote such elements (The word "primitive" will be reserved for operation primitives so far as that is possible.)

In the remainder of this paper we will study spherical homology classes in the bordism of Lie groups. It is well known that spherical homology classes are always primitive in both senses of the word. The question is to what extent being biprimitive characterizes spherical homology classes mod torsion. More exactly, let

$$\mathcal{S}_{MU} = \operatorname{Im} h_{MU} : \Pi_*(G)/\mathrm{Tor} \longrightarrow MU_*(G)/\mathrm{Tor}$$
$$\mathcal{P}_{MU} = PMU_*(G)/\mathrm{Tor} \cap \operatorname{Im} P \subset MU_*(G)/\mathrm{Tor}$$

Then $\mathcal{S}_{MU} \subset \mathcal{P}_{MU}$ and our question is, to what extent, $\mathcal{S}_{MU} = \mathcal{P}_{MU}$.

The conjecture that $\mathcal{S}_{MU} = \mathcal{P}_{MU}$ is related to another conjecture about spherical homology classes in Lie groups called the Atiyah-Mimura conjecture. Let $ch : K_4(X) \otimes \mathbb{Q} \longrightarrow H_*(X;\mathbb{Q})$ be the Chern character isomorphism

<u>Atiyah-Mimura Conjecture:</u> $x \in PH_*(G)/\mathrm{Tor}$ is spherical if and only if $ch^{-1}(x) \in K_*(G) \subset K_*(G) \subset K_*(G) \otimes \mathbb{Q}$.

This conjecture implies that $\mathcal{S}_{MU} = \mathcal{P}_{MU}$. The main point is that we have a commutative diagram

$$\begin{tikzcd}
MU_*(X) \otimes \mathbb{Q} \arrow[r, "\mathcal{P}"] \arrow[d, "T"] & MU_*(X) \otimes \mathbb{Q} \arrow[d, "CF"] \\
H_*(X) \otimes \mathbb{Q} \arrow[r, "ch^{-1}"] & K_*(X) \otimes \mathbb{Q}
\end{tikzcd}$$

where CF is the Conner-Floyd map (see Kane [11]). Consider $x \in \mathcal{P}_{MU}$. We want to show $x \in \mathcal{S}_{MU}$. Since $\mathcal{P}^2 = \mathcal{P}$ we have $\mathcal{P}(x) = x$. Let $\overline{x} = T(x)$. Then $ch^{-1}(\overline{x}): CF\ \mathcal{P}(x) = CF(x)$. Since $x \in MU_*G/\mathrm{Tor}$ we have $ch^{-1}(\overline{x}) \in K_*(G)$. So, by the Atiyah-Mimura conjecture, $\overline{x}$ is spherical. By the commutativity of the diagram

$$\begin{tikzcd}
 & PMU_*(G)/\mathrm{Tor} \arrow[d, "T"] \\
\Pi_*(G)/\mathrm{Tor} \arrow[ur, "h_{MU}"] \arrow[r, "h_H"'] & PH_*(G)/\mathrm{Tor}
\end{tikzcd}$$

$\overline{x}$ has a spherical representative y in $MU_*(G)/\mathrm{Tor}$. But $x,y \in \mathcal{P}_{MU}$. Since we have an isomorphism $T : \mathrm{Im}\ \mathcal{P} \cong H_*(X;\mathbb{Q})$ the relation $T(x) = \overline{x} = T(y)$ forces $x = y$.

We should also note that, although we have not been able to prove the reverse implication, in practical terms, the two conjectures are equivalent. All our arguments and results in MU theory have appropriate analogues.

As we indicated in §4 it can be quite difficult to determine P_{MU} is an explicit manner. Fortunately, one can simplify the study of the inclusion $S_{MU} \subset P_{MU}$ by passing to ordinary homology. Let

$$P_H = \text{the image of } P_{MU} \text{ under } T : PMU_*(G)/Tor \longrightarrow PH_*(G)/Tor$$

$$S_H = \text{Im } h_H : \Pi_*(G)/Tor \longrightarrow PH_*(G)/Tor$$

We have an inclusion $S_H \subset P_H$. Moreover the study of $S_H \subset P_H$ is equivalent to the study of $S_{MU} \subset P_{MU}$. For, as we observed after 2.5, T is injective when restricted to P_{MU}. So we have a commutative diagram

$$\begin{array}{ccc} S_{MU} & \subset & P_{MU} \\ \| \wr & & \| \wr \\ S_H & \subset & P_H \end{array} .$$

We will usually study $S_H \subset P_H$. For it is much easier to determine P_H rather than P_{MU}. Consequently, it is easier to prove that $S_H = P_H$ or $S_H \neq P_H$ rather than $S_{MU} = P_{MU}$ or $S_{MU} \neq P_{MU}$. In this manner we will often be able to settle the question $S_{MU} = P_{MU}$ without any explicit knowledge of P_{MU}.

We will study the question $S_H \overset{?}{=} P_H$ for the classical groups plus the exceptional Lie groups G_2 and F_4. First, we do the infinite Lie groups SU, Sp and SO. These results follow in a fairly pleasant fashion. From these results the answer for SU(n) and Sp(n) are automatic. However, the case SO(n) demands a great deal more work. The result for SO does not simply desuspend.

Similarly, G_2 and F_4 involve a great deal of effort.

Most of our energy will be expended on $\mathcal{P}_H$ rather than $\mathcal{S}_H$. For $\mathcal{S}_H$ we will basically rely on the calculations of $\Pi_*(G)/\mathrm{Tor} \longrightarrow PH_*(G)/\mathrm{Tor}$ as obtained from various sources. We will concentrate on calculating $\mathcal{P}_H$. We can isolate two basic techniques which will be utilized in this study. We might describe the techniques as giving upper bound and lower bound results. For example, let us suppose that we want to prove $\mathcal{P}_H \subset PH_*(G)/\mathrm{Tor}$ is given in degree k by $N\mathbb{Z} \subset \mathbb{Z}$. The inclusion $\mathcal{P}_H \subset N\mathbb{Z}$ is the upper bound result while the inclusion $N\mathbb{Z} \subset \mathcal{P}_H$ is the lower bound result.

(a) <u>Representations</u>

Once we have the answer for SU we can use representations to deduce upper bound results for other groups. As we will see, $\mathcal{P}_H \subset PH_*(SU)$ is given in degree $2m+1$ by $n!\mathbb{Z} \subset \mathbb{Z}$. If we have a representation $\rho : G \longrightarrow SU$ such that $\rho_* : P_{2n+1}H_*(G)/\mathrm{Tor} \rightarrow P_{2n+1}H_*(SU)$ is of the form $\mathbb{Z} \xrightarrow{xk} \mathbb{Z}$ then $\mathcal{P}_H$ for the case G must satisfy $\mathcal{P}_H \subset \frac{n!}{2}\mathbb{Z}$. For the commutative diagram

$$\begin{tikzcd} \mathcal{P}_H \arrow[d, "\rho_*"'] & \subset & P_{2n+1}H_*(G)/\mathrm{Tor} \arrow[d, "\rho_*"] \\ \mathcal{P}_H & \subset & P_{2n+1}H_*(SU) \end{tikzcd}$$

is of the form

$$\begin{tikzcd} \mathcal{P}_H \arrow[d] & \subset & \mathbb{Z} \arrow[d, "\times k"] \\ n!\mathbb{Z} & \subset & \mathbb{Z} \end{tikzcd}$$

(b) Generating Varieties

This technique is useful for the groups $G = SU(n)$, $SO(n)$, G_2 and F_4 in obtaining lower bounds. Bott [3] demonstrated that, for each compact Lie group G, there exists a (non unique) finite complex V and a map $f : V \longrightarrow \Omega_0 G$ so that $H_*(\Omega_0 G)$ is generated, as an algebra, by $\operatorname{Im} f_*$. In other words, $f_* : H_*(V) \longrightarrow QH_*(\Omega_0 G)$ is surjective. Both $H_*(V)$ and $H_*(\Omega_0 G)$ are torsion free. Consequently, the Atiyah-Hirzebruch spectral sequence collapses in both cases and $MU_*(V) \longrightarrow QMU_*(\Omega_0 G)$ is surjective. The map $\Sigma\Omega_0 G \longrightarrow G$ induces the "loop" maps.

$$\Omega_* : QH_*(\Omega_0 G) \longrightarrow PH_*(G)/\mathrm{Tor}$$

$$\Omega_* : QMU_*(\Omega_0 G) \to PMU_*(G)/\mathrm{Tor}$$

By using the composite

$$MU_*(V) \longrightarrow QMU_*(\Omega_0 G) \longrightarrow PMU_*(G)/\mathrm{Tor}$$

One can reduce the study of $P_{MU} \subset PMU_*(G)/\mathrm{Tor}$ to the study of primitive elements in $MU_*(V)$. In the cases $G = SU(n)$, $SO(n)$, G_2 and F_4 the complex V is simple enough to enable one to obtain detailed information about the primitives of $MU_*(V)$. On the other hand, we have found no generating variety for $Sp(n)$ whose bordism $MU_*(V)$ is effectively computable (in terms of the action of the operations). So it is fortunate that we can simply deduce our answer for $Sp(n)$ from the stable case Sp.

We might also remark that the generating variety only appears explicitly in the cases $G = SO(n)$ and $G = G_2$. In the SU case we use the "infinite" generating variety $\mathbb{C}P^\infty \subset \Omega SU$. In the F_n case the generating variety appears implicitly in our appeal to the calculations of Watanabe [26].

§8 The Groups $G = SU$, Sp and SO

We begin our study with the infinite Lie groups $SU = \lim_{n\to\infty} SU(n)$, $Sp = \lim_{n\to\infty} Sp(n)$ and $SO = \lim_{n\to\infty} SO(n)$. As we will observe at the end of this chapter, our results for these groups automatically extend to certain other groups, namely $SU(n)$, $Sp(n)$ and $Spin = \lim_{n\to\infty} Spin\ (n)$.

(a) The Group $G = SU$

Recall that $H_*(SU) = E(x_3,x_5,x_7,\ldots)$ and $PH_*(SU)$ has a $\mathbb{Z}$ basis $\{x_3,x_5,x_7,\ldots\}$. So we must study the inclusion $\mathcal{S}_H \subset \mathcal{P}_H$ in degrees $3,5,7,\ldots$ Our result is

$\mathcal{S}_H$ $\subset$	$\mathcal{P}_H$ $\subset$	$P_{2n+1}H_*$
$n!\mathbb{Z}$	$n!\mathbb{Z}$	$\mathbb{Z}$

for each $n \geq 1$. So $\mathcal{S}_H = \mathcal{P}_H$ in this case.

We begin with the space $X = \Sigma\mathbb{C}P^\infty$. Our study of $MU_*(\mathbb{C}P^\infty)$ in §6 also applies to $MU_*(\Sigma\mathbb{C}P^\infty)$ with the obvious change of degree. We will use the same symbol to denote corresponding elements in $MU_*(\Sigma\mathbb{C}P^\infty)$. So $PMU_*(\Sigma\mathbb{C}P^\infty) = MU_*(\Sigma\mathbb{C}P^\infty)$ is a free $\Pi_*(MU)$ module with basis $\{\beta_k\}$ and $\mathcal{P}_{MU} \subset PMU_*(\Sigma\mathbb{C}P^\infty)$ is a free $\mathbb{Z}$ module with basis $\{n!\mathcal{P}(\beta_n)\}$. So, in degree $2n+1$ $\mathcal{P}_H \subset PH_*(\Sigma\mathbb{C}P^\infty)$ is the inclusion $n!\mathbb{Z} \subset \mathbb{Z}$. But we claim that $\mathcal{S}_H \subset PH_*(\Sigma\mathbb{C}P^\infty)$ is also given in degree $2n+1$ by $n!\mathbb{Z} \subset \mathbb{Z}$. Consider $f : S^3 \longrightarrow K(\mathbb{Z},3)$ representing the generator of $\Pi_3(K(\mathbb{Z},3) = \mathbb{Z}$. The map $\Omega f : \Omega S^3 \longrightarrow \mathbb{C}P^\infty$ is multiplication by $n!$ in degree $2n$. $(H_*(\Omega S^3$ is a divided polynomial algebra while $H^*(\mathbb{C}P^\infty)$ is a polynomial algebra). Consequently, the map $S^{2n+1} \subset \bigvee_{n\geq 2} S^{2n+1} = \Sigma\Omega S^3 \longrightarrow \Sigma\mathbb{C}P^{\infty}$ is multiplication by $n!$ in degree $2n+1$.

We have a canonical map $\Sigma\mathbb{C}P^{\infty} \longrightarrow SU$ which induces an isomorphism $PMU_*(\Sigma\mathbb{C}P^{\infty}) \cong PMU_*(SU)$. So our treatment of $X = \Sigma\mathbb{C}P^{\infty}$ extends to $X = SU$ as well.

(b) <u>The Groups $G = Sp$ and $G = SO$</u>

We now study the relation between $\mathcal{S}_{MU}$ and $\mathcal{P}_{MU}$ for the spaces $G = Sp$ and $G = SO$. Because of the Bott periodicity between Sp and SO ($\Omega_0^4 Sp = SO$, $\Omega_0^4 SO = Sp$) it is advantageous to treat these cases simultaneously. Recall that

$$H_*(Sp) = H_*(SO)/\mathrm{Tor} = E(x_3, x_7, x_{11}, \ldots)$$

and $\quad PH_*(Sp) = PH_*(SO)/\mathrm{Tor}$ has a $\mathbb{Z}$ basis $\{x_3, x_7, x_{11}, \ldots\}$

We will demonstrate that the inclusions $\mathcal{S}_H \subset \mathcal{P}_H \subset PH_*$ are given, in degree $4k-1$, by the following charts.

			$\mathcal{S}_H$ $\subset$	$\mathcal{P}_H$ $\subset$	$P_{4k-1}H_*$
k	even	Sp	$2(2k-1!)\mathbb{Z}$	$2k-1!\,\mathbb{Z}$	$\mathbb{Z}$
		SO	$\frac{2k-1!}{2}\mathbb{Z}$	$\frac{2k-1!}{2}\mathbb{Z}$	$\mathbb{Z}$

			$\mathcal{S}_H$ $\subset$	$\mathcal{P}_H$ $\subset$	$P_{4k-1}H_*$
k	odd	Sp	$2k-1!\,\mathbb{Z}$	$2k-1!\,\mathbb{Z}$	$\mathbb{Z}$
		SO	$2k-1!\,\mathbb{Z}$	$\frac{2k-1!}{2}\mathbb{Z}$	$\mathbb{Z}$

So $\mathcal{S}_H$ depends on $k \pmod 2$. And the equality $\mathcal{S}_H = \mathcal{P}_H$ has a similar dependence. We now verify the charts

(i) <u>Spherical Classes</u>

Consider the commutative diagrams

$$\begin{tikzcd}
\Pi_{4k-1}(Sp) \arrow[r] \arrow[d] & \Pi_{4k-1}(SU) \arrow[d] \\
P_{4k-1}H_*(SP) \arrow[r] & P_{4k-1}H_*(SU)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{4k-1}(SO)/\mathrm{Tor} \arrow[r] \arrow[d] & \Pi_{4k-1}(SU) \arrow[d] \\
P_{4k-1}H_*(SO)/\mathrm{Tor} \arrow[r] & \Pi_{4k-1}H_*(SU)
\end{tikzcd}$$

The horizontal maps are induced by the standard inclusions $Sp \subset SU \subset SO$. Let

$$a_k = \begin{cases} 1 & k \text{ odd} \\ 2 & k \text{ even} \end{cases} \qquad b_k = \begin{cases} 2 & k \text{ odd} \\ 1 & k \text{ even} \end{cases}$$

Then the above diagrams are of the form

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times a_k"] \arrow[d] & \mathbb{Z} \arrow[d, "\times 2k-1!"] \\
\mathbb{Z} \arrow[r, "\times 1"] & \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[r, "\times b_k"] \arrow[d] & \mathbb{Z} \arrow[d, "\times 2k-1!"] \\
\mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z}
\end{tikzcd}$$

For the horizontal maps see Kervaire [13] and Cartan [5].

(ii) <u>Primitive Classes</u>

We will consider $\mathcal{P}_H \subset PH_*$ and write $P_{4k-1}H_* = \mathbb{Z}$. First of all, we have

$$(*) \begin{cases} \mathcal{P}_H \subset 2k-1!\,\mathbb{Z} & \text{in the case } G = Sp \\ \mathcal{P}_H \subset \dfrac{2k-1!}{2}\,\mathbb{Z} & \text{in the case } G = SO \end{cases}$$

For the canonical maps $SO \longrightarrow SU$ and $Sp \longrightarrow SU$ induce commutative diagrams

$$\begin{array}{ccc} P_H(Sp) & \longrightarrow & P_H(SU) \\ \downarrow & & \downarrow \\ P_{4k-1}H_*(Sp) & \longrightarrow & P_{4k-1}H_*(SU) \end{array} \qquad \begin{array}{ccc} P_H(SO) & \longrightarrow & P_H(SU) \\ \downarrow & & \downarrow \\ P_{4k-1}H_*(SO) & \longrightarrow & P_{4k-1}H_*(SU) \end{array}$$

which are known to be of the form

$$\begin{array}{ccc} P_H(Sp) & \longrightarrow & 2k-1!\mathbb{Z} \\ \cap\downarrow & & \cap\downarrow \\ \mathbb{Z} & \xrightarrow{\times 1} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} P_H(SO) & \longrightarrow & 2k-1!\mathbb{Z} \\ \cap\downarrow & & \cap\downarrow \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \end{array}$$

For the bottom map see Cartan [5].

Secondly, in degree $4k-1$

$$(**) \qquad \begin{cases} 2k-1!\mathbb{Z} \subset P_H & \text{in the case} \quad G = Sp \\ \frac{2k-1!}{2}\mathbb{Z} \subset P_H & \text{in the case} \quad G = SO \end{cases}$$

For the Bott periodicity equivalences $\Omega_0^4 Sp = SO$, $\Omega_0^4 SO = Sp$, $\Omega_0^4 U = U$ induce a commutative diagram

$$\begin{array}{ccc} P_{4k-1}H_*(Sp) & \longrightarrow & P_{4k-3}H_*(SO)/Tor \\ \downarrow & & \downarrow \\ P_{4k-1}H_*(SU) & \longrightarrow & P_{4k-3}H_*(SU) \\ \uparrow & & \uparrow \\ P_{4k-1}H_*(SO)/Tor & \longrightarrow & P_{4k-3}H_*(Sp) \end{array}$$

which is of the form

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "\times 1"'] & \mathbb{Z} \arrow[d, "\times 2"] \\
\mathbb{Z} \arrow[r, "\times (2k)\cdot(2k+1)"] & \mathbb{Z} \\
\mathbb{Z} \arrow[r] \arrow[u, "\times 2"] & \mathbb{Z} \arrow[u, "\times 1"']
\end{tikzcd}$$

For the middle horizontal map see Corollary 16.23 of Switzer [24]. We can deduce from the above that

$$\Omega^4_* : P_{4k-1}H_*(Sp) \longrightarrow P_{4k+3}H_*(SO)/Tor \quad \text{is multiplication by } \frac{(2k)(2k+1)}{2}$$

$$\Omega^4_* : P_{4k-1}H_*(SO)/Tor \longrightarrow P_{4k+3}H_*(Sp) \quad \text{is multiplication by } 2(2k)(2k+1).$$

We now prove (**) by induction on degree. Obviously (**) holds in degree 3. By the example $G = Sp(2)$ treated in §6 we can assume (**) holds for $G = Sp$ in degree 7. We now proceed by induction. A flow chart for the argument is as follows

$$\begin{tikzcd}[column sep=small]
 & 3 & 7 & 11 & 15 & 19 & \cdots\cdots \\
SO & & \frac{3!}{2} \arrow[drr] & \frac{5!}{2} \arrow[drr] & \frac{7!}{2} \arrow[drr] & \frac{9!}{2} & \cdots\cdots \\
Sp & 1 \arrow[ur] & 3! \arrow[ur] & 5! \arrow[ur] & 7! \arrow[ur] & 9! & \cdots\cdots
\end{tikzcd}$$

(c) The Groups $G =$ SU(n), Sp(n) and S pin

We close §8 by observing that the preceeding results for SU, Sp and SO pass to SU(n), Sp(n) and Spin respectively. In the first two cases the inclusions $SU(n) \subset SU$ and $Sp(n) \subset Sp$ induce homotopy equivalences in the range of degrees in which the algebra generators of $H_*(SU(n))$ and $H_*(Sp(n))$ lie. In the last case one can simply replace SO by Spin in all the preceeding argument.

On the other hand, the results for SO(n) and Spin (n) cannot be easily deduced from those for SO and Spin . For the inclusions $SO(n) \subset SO$ and Spin $(n) \subset$ Spin are not homotopy equivalences in a sufficient range of dimensions.

§9 The Groups $G = SO(n)$ and $Spin(n)$

The study of these groups constitutes the major calculation of this paper. We will study these groups via the generating variety approach described in §7. For the presence of 2 torsion in $H_*(SO(n))$ and $H_*(Spin(n))$ means that the structure of $MU_*(SO(n))$ and $MU_*(Spin(n))$ is complicated. So the indirect approach of studying $MU_*(\Omega_0 SO(n))$ and $MU_*(\Omega\ Spin(n))$ is quite useful in this case.

We will concentrate on $G = SO(n)$. The arguments and results for $G = Spin(n)$ are similar and will be indicated at the end of the chapter.

Before studying $S_H \subset P_H \subset PH_*(SO(n))/Tor$ we first study the relation of $SO(n)$ to the generating variety $V_n \subset \Omega_0 SO(n)$.

(a) Generating Variety V_n

It was shown in Bott [3] that we can define the generating variety $V_n \subset \Omega_0 SO(n)$ to be

$$V_n = SO(n)/SO(2) \times SO(n-2)$$

The structure of $H^*(V_n)$ is slightly different for n odd and n even. $H^*(V_{2n+1})$ has a basis

$$\{1, A, \ldots, A^{n-1}, A^n/2, \ldots A^{2n-1}/2\}$$

while $H^*(V_{2n+2})$ has a basis

$$\{1, A, \ldots, A^n, A^{n+1}/2, \ldots, A^{2n}/2, B\}$$

where $\deg A = 2$ and $\deg B = 2n$. B is uniquely determined by the requirement that

$$AB = A^{n+1}/2$$

If we dualize then $H_*(V_{2n+1})$ has a basis

$$\{\delta_0, \delta_1, \ldots, \delta_{n-1}, 2\delta_n, \ldots, 2\delta_{2n-1}\}$$

while $H_*(V_{2n+2})$ has a basis

$$\{\delta_0, \delta_1, \ldots, \delta_n, 2\delta_{n+1}, \ldots, 2\delta_{2n}, \lambda\}$$

The inclusion $SO(n) \subset SO(n+1)$ induces a map $V_n \to V_{n+1}$. Our notation is chosen so that elements with the same name correspond under the induced maps in homology and cohomology. Also, λ has the property of generating $\operatorname{Ker}\{H_{2n}(V_{2n+2}) \longrightarrow H_{2n}(V_{2n+3})\}$ while $A^n - 2B$ has the property of generating $\operatorname{Ker}\{H^{2n}(V_{2n+2}) \longrightarrow H^{2n}(V_{2n+1})\}$.

We next study the relation between $H^*(V_n)$ and $H^*(SO(n))/\mathrm{Tor}$ and then between $H_*(V_n)$ and $H_*(SO(n))/\mathrm{Tor}$.

(b) Cohomology

First of all the mod 2 cohomology of $SO(n)$ can be described in terms of a simple system of generators as

$$H^*(SO(n);\mathbb{F}_2) = \Delta(x_1,x_2, \dots ,x_{n-1})$$

$$S_q^i(x_j) = \binom{j}{i} x_{i+j} \quad .$$

(To obtain the complete algebra structure of $H^*(SO(n);\mathbb{F}_2)$ one must replace each x_{2k} by x_k^2) . Let $\{B_r\}$ be the Bockstein spectral sequence for 2 torsion in $H^*(SO(n))$. Then

$$B_1 = H^*(SO(n);\ \mathbb{F}_2)$$

$$B_2 = H^*(SO(n))/\mathrm{Tor} \otimes \mathbb{F}_2$$

Since $d_1 = S_q^1$ we can calculate

$x = SO(2n+1) \qquad B_2 = E(y_3,y_7, \dots ,y_{4n-1})$

$x = SO(2n+2) \qquad B_2 = E(y_3,y_7, \dots ,y_{4n-1}) \otimes E(z)$

where

$$y_{4k-1} = \{x_{4k-1} + x_{2k-1}x_{2k}\}$$

$$z = \begin{cases} \{x_n x_{n+1}\} & n \text{ odd} \\ \\ \{x_{2n+1}\} & n \text{ even} \end{cases}$$

(In the description of y_{4k-1} we are assuming that $x_{4k-1} = 0$ when $4k-1 > 2n+1$). So we can write

$$H^*(SO(2n+1))/\mathrm{Tor} = E(Y_3, Y_7, \ldots, Y_{4n-1})$$

$$H^*(SO(2n+2))/\mathrm{Tor} = E(Y_3, Y_7, \ldots, Y_{4n-1}) \otimes E(Z)$$

where $\{Y_i\}$ and Z reduce mod 2 to $\{y_i\}$ and z . Our notation is consistent with the maps $SO(n) \longrightarrow SO(n+1)$ in that symbols with the same name map to each other. Observe, also, that Z maps to Y_{2n+1} under the map $H^*(SO(2n+2))/\mathrm{Tor} \longrightarrow H^*(SO(2n+1))/\mathrm{Tor}$ when n is odd.

Now consider the loop map

$$\Omega^* : QH^*(SO(n))/\mathrm{Tor} \longrightarrow PH^*(\Omega_0 SO(n))$$

The assertion that $H_*(V_n) \longrightarrow QH_*(\Omega_0 SO(n))$ is surjective dualizes to give $PH^*(\Omega_0 SO(n)) \subset H^*(V_n)$ is a direct summand. We will describe $\mathrm{Im}\,\Omega^*$ in terms of $H^*(V_n)$. We have

Proposition 9.1:

(i) $\Omega^*(Y_{2i+1}) = A^i$ for $i = 1,3,5, \dots ,2n-1$

(ii) $$\Omega^*(Z) = \begin{cases} 2B & n \text{ odd} \\ 2B - A^n & n \text{ even} \end{cases}$$

PROOF: For (i) we need only consider $SO(2n+1)$. We have a commutative diagram

$$\begin{tikzcd}
Q^{4i-1}H^*(SO(2n+2k+1))/\mathrm{Tor} \arrow[r, "\cong"] \arrow[d, "\Omega^*"', "\cong"] & Q^{4i-1}H^*(SO(2n+1))/\mathrm{Tor} \arrow[d, "\Omega^*"] \\
P^{4i-2}H^*(\Omega_0 SO(2n+2k+1)) \arrow[r] \arrow[d, "\cong"] & P^{4i-2}H^*(\Omega_0 SO(2n+1)) \arrow[d] \\
H^{4i-2}(V_{2n+2k+1}) \arrow[r, "\cong"] & H^{4i-2}(V_{2n+1})
\end{tikzcd}$$

(Assume $k >> 0$ and $1 \leq i \leq n$).

We have already justified all the isomorphisms except for the left vertical isomorphism involving Ω^*. It follows from the fact that $\Omega^*: Q^{odd}H^*(SO(2n+2k+1)_j \mathbb{F}_2) \to P^{even}H^*(\Omega_0 SO(2n+2k+1)_j \mathbb{F}_2)$ is injective (see, for example CLARK [6]). Since $k >> 0$, Y_{4i+1} is represented mod 2 by $x_{4i-1} + x_{2i-1}x_{2i}$ where $x_{4i-1} = 0$. So $\Omega^*(x_{4i-1}) \neq 0$ forces $\Omega^*(Y_{4i-1}) \neq 0 \bmod 2$.

The fact that the right hand composition in the diagram must also be an isomorphism now gives us property (i).

Regarding (ii) we must treat n odd and n even separately. When n is odd we can choose Y_{2n+1} and Z so that $Y_{2n+1} - Z \in \mathrm{Ker}\{H^*(SO(2n+2))/\mathrm{Tor} \to H^*(SO(2n+1))/\mathrm{Tor}\}$. But then $\Omega^*(Y_{2n+1} - Z) \in \mathrm{Ker}\{H^*(V_{2n+2}) \to H^*(V_{2n+1})\}$. So $\Omega^*(Y_{2n+1} - Z) = A^n - 2B$. (In particular we already know that it is non zero mod 2). Since $\Omega^*(Y_{2n+1}) = A^n$ we have $\Omega^*(Z) = 2B$. When n is even we can choose Z from $\mathrm{Ker}\{H^*(SO(2n+2))/\mathrm{Tor} \to H^*(SO(2n+1))/\mathrm{Tor}\}$. We then obtain $\Omega^*(Z) = A^n - 2B$.

Q.E.D.

(c) <u>Homology</u>

If we dualize the above description then we obtain

$$H_*(SO(2n+1))/\mathrm{Tor} = E(\alpha_3, \alpha_7, \ldots, \alpha_{4n-1})$$

$$H_*(SO(2n+2))/\mathrm{Tor} = E(\alpha_3, \alpha_7, \ldots, \alpha_{4n-1}) \otimes E(\beta)$$

where $\{\alpha_i\} \cup \{\beta\}$ is a basis of $PH_*(SO(n))/\mathrm{Tor}$ and elements with the same symbol correspond under the maps $SO(n) \to SO(n+1)$.

The map $\Omega_* : QH_*(SO(n))/Tor \rightarrow PH_*(SO(n))/Tor$ is described by

$$\Omega_*(\delta_i) = \alpha_{2i+1} \qquad i = 1,3,5, \ldots ,2n-1$$

$$\Omega_*(\lambda) = 2\beta$$

Remember, of course, that, for $n \leq i \leq 2n-1$, $Q_{2i}H_*(\Omega_0 SO(n))$ is not generated by δ_i but by $2\delta_i$. So, in those degrees, Ω_* is multiplication by 2.

We will also need to know a little about the relation between $H_*(SO(n))/Tor$ and $H_*(SO(n);\mathbb{F}_2)$. If we dualize our description of $H_*(SO(n);\mathbb{F}_2)$ then we can write

$$H_*(SO(n);\mathbb{F}_2) = E(\gamma_1,\gamma_2, \ldots ,\gamma_{n-1})$$

This time the identity is as algebras, not just with respect to a simple a simple system of generators. Let D^q = the q fold decomposables of $H_*(SO(n);\mathbb{F}_2)$ and let
$\rho : H_*(SO(n)) \longrightarrow H_*(SO(n);\mathbb{F}_2)$ be the mod 2 reduction map.
Our main result is

<u>Proposition 9.2</u>: Let $\beta \in H_*(SO(2n+2))$ be any representative for $\beta \in H_*(SO(2n+2))/Tor$

Then

$$\rho(\beta) = \begin{cases} \gamma_{2n+1} \bmod D^2 & \text{for } n \text{ even} \\ \gamma_n\gamma_{n+1} + \sum\limits_{i<n} \varepsilon_{ij}\gamma_i\gamma_{2n+1-i} \bmod D^3 & \text{for } n \text{ odd} \end{cases}$$

Let $\{B^r\}$ be the homology Bockstein spectral sequence with respect to 2 torsion in $H_*(SO(n))$. It is dual to the spectral sequence $\{B_r\}$ considered in part (b). So it follows from the calculations in part (b) that $B^2 = H^*(SO(n))/\mathrm{Tor} \otimes \mathbb{F}_2$ is an exterior algebra on odd degree generators. However, it is difficult to explicitly calculate B^2. For, although $d^1\gamma_{2k-1} = 0$, the $\{\gamma_1, \ldots, \gamma_{n-1}\}$ are not invariant under the action of d^1. However we can use the results from part (b) to deduce that, in the case $X = SO(2n+2)$,

(*) one can choose $\{\gamma_{2n+1}\}$ (n even) and $\{\gamma_n\gamma_{n+1} + ?\}$ where $? \in D^3$ (n odd) among exterior algebra generators of B^2.

We need to show that the classes γ_{2n+1} and $\gamma_n\gamma_{n+1} + ? \in \mathrm{Ker}\, d^1$ and that they pair off non trivially with the cohomology elements x_{2n+1} and x_nx_{n+1} respectively. The only fact which needs comment is that we can choose a class of the form $\gamma_n\gamma_{n+1} + ?$ in $\mathrm{Ker}\, d_1$. If we filter $B^1 = H_*(SO(2n+2); \mathbb{F}_2)$ by $\{D^q\}$ then, as in Browder [4] we obtain a spectral sequence converging to B^2. The action of $d^1 = Sq^1$ on $E_1 = E_0B^1 = E(\gamma_1,\gamma_2, \ldots ,\gamma_{2n-1})$ is $Sq^1(\gamma_{2i}) = \gamma_{2i-1}$. So $E_2 = E(\{\gamma_1\gamma_2\},\{\gamma_3\gamma_4\}, \ldots ,\{\gamma_{2n-1}\gamma_{2n}\}) \otimes E(\gamma_{2n+1})$. Therefore $E_2 = E_\infty = E_0B_2$. In particular $\{\gamma_n\gamma_{n+1}\}$ (n odd) survives the spectral sequence.

We can rephrase (*) as stating that the canonical map $\hat{\rho}\colon H_*(SO(2n+2)) \longrightarrow H_*(SO(2n+2))/\mathrm{Tor} \otimes \mathbb{F}_2$ satisfies $\hat{\rho}(\beta) = \{\gamma_{2n+1}\}$ or $\{\gamma_n\gamma_{n+1} + ?\}$. This determines $\rho(\beta)$ for $\rho : H_*(SO(2n+2)) \longrightarrow H_*(SO(2n+2)_i\, \mathbb{F}_2)$ modulo the indeterminancy $\mathrm{Im}\, d^1$. However, $\mathrm{Im}\, d^1$ is spanned by the monomials of D^2 distinct from $\gamma_n\gamma_{n+1}$. So Proposition 9.2 follows.

(d) Spherical Classes

As we will see the Hurewicz map for $SO(n)$ is roughly the same as for SO. Some added complications arise, however.

(i) The Case $X = SO(2n+1)$

The Hurewicz map has been determined by Barratt-Mahowald [2], Kervaire [13] and Lundell [15]. If we ignore $k = 1,2,4$ then, for $k \leq 2n$, we have a commutative diagram

$$\begin{array}{ccc}
\Pi_{4k-1}(SO(2n+1))/Tor & \xrightarrow{\cong} & \Pi_{4k-1}(SO)/Tor = \mathbb{Z} \\
\downarrow & & \downarrow \qquad \downarrow 2k-1! \\
P_{4k-1}H_*(SO(2n+1))/Tor & \xrightarrow{\cong} & P_{4k-1}H_*(SO)/Tor = \mathbb{Z}
\end{array}$$

So, in those cases, $S_H \subset P_{4k-1}H_*(SO(2n+1))$ is given by $2k-1!\ \mathbb{Z} \subset \mathbb{Z}$. When $k = 1$ then, in certain cases, the map $\Pi_{4k-1}(SO(2n+1))/Tor \longrightarrow P_{4k-1}H_*(SO(2n+1))/Tor$ is not an isomorphism. The following diagrams describes these cases.

$$\begin{array}{ccc}
\Pi_3(SO(3))/Tor & \xrightarrow{x2} & \Pi_3(SO)/Tor \\
\downarrow x2 & & \downarrow \cong \\
P_3H(SO(3))/Tor & \xrightarrow[\cong]{} & P_3H(SO)/Tor
\end{array}$$

$$\begin{array}{ccccccc}
\Pi_7(SO(5))/Tor & \xrightarrow{x2} & \Pi_7(SO(6))/Tor & \xrightarrow{x2} & \Pi_7(SO(7))/Tor & \xrightarrow{x2} & \Pi_7(SO)/Tor \\
\downarrow 24 & & \downarrow 12 & & \downarrow 6 & & \downarrow 3 \\
P_7H(SO(5))/Tor & \xrightarrow{\cong} & P_7H(SO(6))/Tor & \xrightarrow{\cong} & P_7H(SO(7))/Tor & \xrightarrow{\cong} & P_7H_*(SO)/Tor
\end{array}$$

$$\begin{array}{ccccccccc}
\Pi_{15}(SO(9))/Tor & \xrightarrow{\cong} & \Pi_{15}(SO(10))/Tor & \xrightarrow{\cong} & \Pi_{15}(SO(11))/Tor & \xrightarrow{\cong} & \Pi_{15}(SO(12))/Tor & \xrightarrow{x2} & \Pi_{15}(SO)/Tor \\
\downarrow 7! & & \downarrow 7! & & \downarrow 7! & & \downarrow 7! & & \downarrow 7!/2 \\
P_{15}H(SO(9))/Tor & \xrightarrow{\cong} & P_{15}H(SO(10))/Tor & \xrightarrow{\cong} & P_{15}H(SO(11))/Tor & \xrightarrow{\cong} & P_{15}H(SO(12))/Tor & \xrightarrow{\cong} & P_{15}H(SO)/Tor
\end{array}$$

(ii) <u>The Case $X = SO(2n+2)$</u>

Again, the Hurewicz map for $X = SO(2n+2)$ is similar to that for $X = SO$. We have the deviation between the two already noted above in degrees 7 and 15. We also have the added complication that, in degree $2n+1$, $\Pi_{2n+1}(SO(2n+2))/Tor \longrightarrow \Pi_{2n+1}(SO)/Tor$ and $P_{2n+1}H_*(SO(2n+2))/Tor \longrightarrow P_{2n+1}H_*(SO)/Tor$ have non trivial kernels. We know that the homology kernel is $\mathbb{Z}$ generated by β. At the moment we show

<u>Proposition 9.3</u>: $2\beta \in S_H$

Of course, it is possible that $\beta \in S_H$. We will later show that $\beta \notin P_H$. So $S_H \subset P_{2n+1}H_*(SO(2n+2)) = \mathbb{Z} \oplus \mathbb{Z}$ is givenly $\frac{n!}{2}\mathbb{Z} \oplus 2\mathbb{Z}$ with the exceptions noted in degrees 7 and 15.

<u>PROOF</u>: The map $H_*(V_{2n+2}) \longrightarrow H_*(V_{2n+3})$ has kernel $\mathbb{Z}$ generated by λ. It follows that $\lambda \in H_{2n+1}(\Sigma V_{2n+2})$ is spherical (look at the cofibre sequence $V_{2n+2} \to V_{2n+3} \to K \to \Sigma V_{2n+1} \to \Sigma V_{2n+3}$). Since $\Omega_*(\lambda) = 2\beta$ we have 2β is spherical.

(e) <u>Primitive Classes for $X = SO(2n+1)$</u>

We have to study the submodule $P_H \subset P_{4k-1}H_*(SO(2n+1))/Tor$ for $1 \leq k \leq n$. We will obtain the same answer as for the stable case $X = SO$. Because of the homotopy equivalence between the $2n-1$ skeletons of $SO(2n+1)$ and SO this is automatic when $2k \leq n$. But, for $n+1 \leq 2k \leq 2n$, we must produce an entirely new argument. The case $n = 1$ is easy. For $MU_*(SO(3))/Tor = E(x_3)$.

So $\mathcal{P}_H = P_3 H_*(SO(3))/\mathrm{Tor}$. So we can assume that n (and hence k) ≥ 2 . Our goal is to prove.

<u>Proposition 9.4</u>: Let $n \geq 2$ and $n+1 \leq 2k \leq 2n$. Then

$$\mathcal{P}_H \subset P_{4k-1} H_*(SO(2n+1))/\mathrm{Tor} \quad \text{is given by}$$

$$\frac{2k-1!}{2}\,\mathbb{Z} \subset \mathbb{Z} \ .$$

Write $P_{4k-1} H_*(SO(2n+1))/\mathrm{Tor} = \mathbb{Z}$. Then we want to prove $\mathcal{P}_H = \frac{2k-1!}{2}\,\mathbb{Z}$. The inclusion $\mathcal{P}_H \subset \frac{2k-1!}{2}\,\mathbb{Z}$ is easy. For the diagram

$$\begin{tikzcd}
\mathcal{P}_H \arrow[d] & \subset & P_{4k-1} H_*(SO(2n+1))/\mathrm{Tor} \arrow[d] \\
\mathcal{P}_H & \subset & P_{4k-1} H_*(SO)/\mathrm{Tor}
\end{tikzcd}$$

is of the form

$$\begin{tikzcd}
\mathcal{P}_H \arrow[d, hook] & \subset & \mathbb{Z} \arrow[d, equal, "\wr"] \\
\frac{2k-1!}{2}\,\mathbb{Z} & \subset & \mathbb{Z}
\end{tikzcd}$$

The reverse inclusion $\frac{2k-1!}{2}\,\mathbb{Z} \subset \mathcal{P}_H$ demands all the work. We will use the generating variety $V(= V_{2n+1})$ described in part (a).

Because of the isomorphisms

$$
\begin{aligned}
P_{4k-1}H_*(SO(2k+1))/\mathrm{Tor} &\cong P_{4k-1}H_*(SO(2k+3))/\mathrm{Tor} \\
&\;\vdots \\
&\cong P_{4k-1}H_*(SO(2k+5))/\mathrm{Tor}
\end{aligned}
$$

we can reduce to the case

$$k = n$$

We first remark that we will defer our treatment of the case $k = n = 3$ until §10. The argument we are about to give fails in this case. (At the end of §10 we will indicate the nature of the failure). However, our treatment of the exceptional group G_2 in §10 will handle the case $k = n = 3$. We want to show that $P_H \subset P_{11}H_*(SO(7))/\mathrm{Tor}$ satisfies $\frac{5!}{2}\mathbb{Z} \subset P_H$. Now, the canonical maps $G_2 \to \mathrm{Spin}(7) \to SO(7)$ induce isomorphisms $P_{11}H_*(G_2)/\mathrm{Tor} \cong P_{11}H_*(\mathrm{Spin}(7))/\mathrm{Tor} \cong P_{11}H_*(SO(7))/\mathrm{Tor}$. So it suffices to show that $P_H \subset P_{11}H_*(G_2)/\mathrm{Tor}$ is given by $\frac{5!}{2}\mathbb{Z} \subset \mathbb{Z}$. This will be done in §10.

We now set about treating the cases $n = 2$ and $n \geq 4$.

Let $\{\Sigma_0, \Sigma_1, \Sigma_2, \ldots, \Sigma_{2n-1}\}$ be a $\Pi_*(MU)$ basis of $MU_*(V)$. The map $MU_*(V) \longrightarrow QMU_*(\Omega_0 SO(2n+1))$ is surjective. We will also use Σ_i to denote the image of Σ_i in $QMU_*(\Omega_0 SO(2n+1))$. In $QMU_*(\Omega_0 SO(2n+1))$ we have the relation

$$2\Sigma_2 = v_1\Sigma_1 \quad \text{where} \quad v_1 = 2m_1 .$$

(The arguments in Kane [10] establish that relations of this sort exist). In order to prove $\frac{2n-1!}{2}\mathbb{Z} \subset P_H$ it suffices to prove

<u>Proposition 9.5</u>: $3P(\Sigma_3) \in QMU_*(\Omega_0 SO(5))$

$$\frac{2n-1!}{4} P(\Sigma_{2n-1}) \in QMU_*(\Omega_0 SO(2n+1)) \quad \text{for} \quad n \geq 4 .$$

To see the sufficiency of this proposition consider the commutative diagram

$$\begin{tikzcd}
QMU_*(\Omega_0 SO(2n+1)) \arrow[r, "\Omega_*"] \arrow[d, "T"] & PMU_*(SO(2n+1)) \arrow[d, "T"] \\
QH_*(\Omega_0 SO(2n+1)) \arrow[r, "\Omega_*"] & PH_*(SO(2n+1))
\end{tikzcd}$$

Since $T(\Sigma_{2n-1}) = 2\delta_{2n-1}$ and $\Omega_*(\delta_{2n-1}) = \alpha_{4n-1}$ we have $T\Omega_*(\Sigma_{2n-1}) = \Omega_* T(\Sigma_{2n-1}) = 2\alpha_{4n-1}$. On the other hand, the proposition implies that $\frac{2n-1!}{4} P\Omega_*(\Sigma_{2n-1}) \in PMU_*(SO(2n+1))$. Consequently, $\frac{2n-1!}{2}\alpha_{4n-1} \in P_H$. In other words, $\frac{2n-1!}{2}\mathbb{Z} \subset P_H$ as required.

To prove the proposition expand

$$P(\Sigma_{2n-1}) = \Sigma_{2n-1} + \sum_{i\leq 2n-2} c_i \Sigma_i .$$

For the moment assume that we are dealing with the case $n \geq 4$. So we want to show that $\frac{2n-1!}{4} c_i \in \Pi_*(MU) \subset \Pi_*(MU) \otimes Q$ for each $1 \leq i \leq 2n-2$. We will divide our argument into two cases

(i) $i \geq n+1$

(ii) $i \leq n$

(i) <u>The Case $i \geq n+1$</u>

Given $k = \Sigma k_s 2^s$ (2-adic expansion) let

$$\alpha(k) = \Sigma k_s$$

$$\gamma_2(k) = \text{maximal power of } 2 \text{ dividing } k$$

It is easy to prove

<u>LEMMA 9.6</u>: $\gamma_2(k!) = k - \alpha(k)$

<u>LEMMA 9.7</u>: $\gamma_2(k_1+1!k_2+1! \ldots k_r+1!) \leq \Sigma k_s$

Since $n \geq 4$ we have $n \geq \alpha(2n-1)+1$. Thus $i \geq n+1 \geq \alpha(2n-1)+2$. It follows that

$$\frac{\deg c_i}{2} \leq 2n-1-(\alpha(2n-1)+2) = \gamma_2\left(\frac{2n-1!}{4}\right)$$

Thus c_i can be expanded in terms of the monomials $b_{k_1} \ldots b_{k_r}$ where $\Sigma k_s \leq \gamma_2\left(\frac{2n-1!}{4}\right)$. Now by 1.1,

$$k_1+1! \ldots k_r+1! b_{k_1} \ldots b_{k_r} \in \Pi_*(MU)_{(2)} .$$

So, by lemma 6.7, $\frac{2k-1!}{4} b_{k_1} \ldots b_{k_1} \in \Pi_*(MU)_{(2)}$.

(ii) <u>The Case $i \leq n$</u>

Before handling these cases we put some restrictions on the coefficients c_i. As before write $MU_*(\mathbb{C}P^{2n-1}) = \Pi_*(MU)\{\beta_0,\beta_1, \ldots ,\beta_{2n-1}\}$. There exists a map $f:V \to \mathbb{C}P^{2n-1}$ such that $f_*:MU_*(V) \to MU_*(\mathbb{C}P^{2n-1})$ satisfies

$$f_*(\Sigma_i) = \beta_i \qquad i \leq n-1$$

$$(*) \qquad f_*(\Sigma_n) = 2\beta_n \qquad i = n$$

$$f_*(\Sigma_i) = 2\beta_i + ? \qquad i \geq n+1$$

We use this map to prove

<u>LEMMA 9.8</u>: For $i \leq n-1$ one can assume $c_i = 2(b)^i_{2n-i-1}$

For $i = n$ one can assume $c_n = (b)^n_{n-1}$

PROOF: Since $f_*(\Sigma_{2n-1}) = \beta_{2n-1} + ?$ and since $\mathcal{P}$ annihilates $(m_1, m_2, \ldots)$ we have

$$f_* \mathcal{P}(\Sigma_{2n-1}) = 2\mathcal{P}(\beta_{2n-1})$$

Expanding both sides we obtain

$$\sum_{i \leq 2n-1} c_i f_*(\Sigma_i) = \sum_{i \leq 2n-1} (b)^i_{2n-1-i} \beta_i$$

If we replace each $f_*(\Sigma_i)$ by its expression in the $\{\beta_i\}$ and collect the coefficients of $\{\beta_1, \ldots, \beta_n\}$ then we have

$$i = n \qquad 2c_n + ? = 2(b)^{2n-1}_n$$

$$i \leq n-1 \qquad c_i + ? = 2(b)^i_{2n-1-i}$$

It follows from our discussion of the case $i \geq n+1$ that $\frac{2n-1!}{4}(?) \in \Pi_*(MU)_{(2)}$. Consequently, to prove $\frac{2n-1!}{4} c_i \in \Pi_*(MU)_{(2)}$ for $i \leq n$ it suffices to reduce to the cases given in the lemma. Q.E.D.

In the case $i = 1$ we actually want to make a further modification in c_1.

LEMMA 9.9: We can assume $c_1 = 2b_{2n-2} + 2v_1 b_{2n-3}$.

<u>PROOF</u>: By lemma 9.8 we have already reduced our expansion of $\mathcal{P}(\Sigma_{2n-1})$ to the form

$$\mathcal{P}(\Sigma_{2n-1}) = \Sigma_{2n-1} + \dots + 2(b)^{2}_{2n-3}\ \Sigma_2 + 2(b)^{1}_{2n-2}\ \Sigma_1$$

Now

$$(b)^{2}_{2n-3} = 2b_{2n-3} + \dots$$

$$(b)^{1}_{2n-2} = b_{2n-2}$$

By the relation $2\Sigma_2 = v_1\Sigma_1$ in $QMU_*(\Omega_0 SO(2n+1))$ we can replace $4b_{2n-3}\beta_2$ (in the expansion of $\mathcal{P}(\Sigma_{2n-1})$) by $2v_1 b_{2n-3}\beta_1$. Q.E.D.

We can now set about showing that $\frac{2n-1!}{4}\ c_i \in \Pi_*(MU)_{(2)}$. For $i = 1$ and $2 \leq i \leq n-1$ we appeal to Propositions 5.3 and 5.1 respectively. Regarding $i = n$ the argument given in part (i) for the case $i \geq n+1$ also covers the case $i = n \geq 5$. For the argument given there actually applies to the cases $i \geq \alpha(2n-1)+2$. Regarding $i = n = 4$ it follows from Lemma 9.8 that

$$c_4 = (b)^{4}_{3} = 4b_1^3 + 4b_3 + 2b_1 b_2 \ .$$

Since $(k+1)!b_k \in \Pi_*(MU)$ it follows that $\frac{7!}{4}\ (b)^{4}_{3} \in \Pi_*(MU)$.

We have now finished our proof of Proposition 9.5 for the $n \geq 4$ case. For $n = 2$ we have, by Lemma 9.8,

$$P(\Sigma_3) = \Sigma_3 + (b)_1^2 \Sigma_2 + 2(b)_2^1 \Sigma_1$$

$$= \Sigma_3 + 2b_1 \Sigma_2 + 2b_2 \Sigma_1$$

Also $2b_1 \in \Pi_*(MU)$ while $3!b_2 \in \Pi_*(MU)$. Thus $3P(\Sigma_3) \in \Pi_*(MU)$.

(f) <u>Primitive Classes for $X = SO(2n+2)$</u>

First of all, in degrees $\neq 2n+1$, our description of P_H for $X = SO(2n+1)$ applies for $X = SO(2n+2)$ as well.

<u>Proposition 9.10</u>: Given $1 \leq k \leq n$ where $4k-1 \neq 2n+1$ then $P_H \subset P_{4k-1}H_*(SO(2n+1))/Tor$ is given by $\frac{2k-1!}{2}\mathbb{Z} \subset \mathbb{Z}$.

<u>PROOF</u>: Consider the maps

$$P_iH_*(SO(2n+1))/Tor \xrightarrow{f} P_iH_*(SO(2n+2))/Tor \xrightarrow{g} P_iH_*(SO(2n+3))/Tor$$

It follows from our description of homology in part (c) that f is surjective in degrees $\neq 2n+1$ while g in injective in degrees $\neq 2n+1$. Because of Proposition 9.4 we can use f to force $\frac{2k-1!}{2}\mathbb{Z} \subset P_H$ in degree $4k-1$ and g to force $P_H \subset \frac{2k-1!}{2}\mathbb{Z}$ in degree $4k-1$. Q.E.D.

On the other hand, in degree $2n+1$, the difference between $X = SO(2n+1)$ and $X = SO(2n+2)$ appears. For $H^*(SO(2n+2))/\mathrm{Tor} \cong H^*(SO(2n+1))/\mathrm{Tor} \otimes E(\beta)$ where β generates $\ker\{P_{2n+1}H_*(SO(2n+2))/\mathrm{Tor} \longrightarrow P_{2n+1}H_*(SO)/\mathrm{Tor}\}$. We have already shown in Proposition 9.3 that $2\beta \in S_H$. The other key result about β is

<u>Proposition 9.11</u>: $\beta \notin P_H$

<u>PROOF</u>: We begin with n odd. If $\beta \in P_H$ then it follows from Proposition 9.2 that $\mathrm{Im}\,\{\rho T: MU_{2n+1}(SO(2n+2)) \to H_{2n+1}(SO(2n+2))\}$ contains an element of the form $\sum_{i+j=2n+1} \varepsilon_{ij}\gamma_i\gamma_j + ?$ where $? \in D^3$. We claim that this is not possible. For $\mathrm{Im}\,\rho T \subset \bigcap_{k\geq 1} Sq^{\Delta_k}$ while such an element does not belong to $\mathrm{Ker}\, Sq^{\Delta_1} \cap \mathrm{Ker}\, Sq^{\Delta_2}$. Filter $H_*(SO(2n+2);\mathbb{F}_2)$ by the decomposables $\{D^q\}$ and pass to the associated graded object $E_0H_*(SO(2n+2);\mathbb{F}_2)$. So we can ignore $?$. Now Sq^1 and Sq^{01} act by the rule

$$Sq^1(\gamma_{2i}) = \gamma_{2i-1} \qquad (i \geq 1)$$

$$Sq^{01}(\gamma_{2i}) = \gamma_{2i-3} \qquad (i \geq 2)$$

Consequently, the elements of the form $\sum \varepsilon_{ij}\gamma_i\gamma_j$ belonging to $\mathrm{Ker}\, Sq^1$ are spanned by

$$\gamma_{2i}\gamma_{2j-1} + \gamma_{2i-1}\gamma_{2j}$$

while such elements belonging to Ker Sq^{01} are spanned by

$$\gamma_{2i}\gamma_{2j-3} + \gamma_{2i-3}\gamma_{2j}$$

Consequently, an element $x = \Sigma\, \varepsilon_{ij}\gamma_i\gamma_j$ can belong to Ker $Sq^1 \cap$ Ker Sq^{01} only if n is even and $x = \gamma_1\gamma_{2n} + \gamma_2\gamma_{2n-1} + \cdots + \gamma_n\gamma_{n+1}$.

Now consider n even. Suppose $T(\omega) = \beta$. (If $B \notin \operatorname{Im} T$ then, as above, we are done). We will show that we must have $s_1(\omega) \neq 0$ in $MU_*(SO(2n+2))/\mathrm{Tor}$. In particular, ω is not primitive. So $\beta \notin P_H$.

It suffices to show $Ts_1(\omega) \neq 0$ in $H_*(SO(2n+2))/\mathrm{Tor} \otimes \mathbb{F}_2$. Let $\{B^r\}$ be the homology Bockstein spectral sequence studied in part (c). Consider $\rho Ts_1(\omega) \in B^1 = H_*(SO(2n+2)_{;}\, \mathbb{F}_2)$. Since $\operatorname{Im} \rho T \subset \operatorname{Ker} S_q^1$ we have $Sq^1\rho Ts_1(\omega) = 0$. So $\{\rho Ts_1(\omega)\} \in \beta^2 = H_*(SO(2n+2))/\mathrm{Tor} \otimes \mathbb{F}_2$ is defined. To see $\{\rho Ts_1(\omega)\} \neq 0$ we use the equations

$$\rho Ts_1(\omega) = Sq^2\rho T(\omega) \qquad \text{(by 3.3)}$$

$$= Sq^2\rho(\beta)$$

$$= Sq^2(\gamma_{2n+1}) \bmod D^2 \qquad \text{(by 9.2)}$$

$$= \gamma_{2n-1} \bmod D^2$$

The last equality is based on the fact that, by Thomas [25], $S_q^2(\gamma_{2n+1}) = \gamma_{2n-1}$ for n even. Lastly, since $\gamma_{2n-1} + ?$ pairs off non trivialy with the cohomology class x_{2n-1} and $\{x_{\alpha n-1}\} \neq 0$ in $B_2 = H^*(SO(2n+2))/Tor \otimes \mathbb{F}_2$ it follows that $\{\gamma_{2n-1} + ?\} \neq 0$ in B^2. Q.E.D.

We can now determine P_H (as well as S_H) in degree $2n+1$.

<u>n even</u>

We have $P_{2n+1}H_*(SO(2n+2))/Tor = \mathbb{Z}$ generated by β. We have $S_H \subset P_H \subset P_{2n+1}H_*(SO(2n+2))/Tor$ is given by $2\mathbb{Z} = 2\mathbb{Z} \subset \mathbb{Z}$. This follows from the already demonstrated relations $2\mathbb{Z} \subset S_H$ and $P_H \subset 2\mathbb{Z}$.

<u>n odd</u>

In this case we have $P_{2n+1}H_*(SO(2n+2))/Tor = \mathbb{Z} \oplus \mathbb{Z}$ generated by α_{2n+1} and β. We claim that $P_H \subset P_{2n+1}H_*(SO(2n+2))/Tor$ is givenly $\frac{n!}{2}\mathbb{Z} \oplus 2\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}$. The $\frac{n!}{2}\mathbb{Z}$ factor arises as in the case of $\deg \neq 2n+1$. The $2\mathbb{Z}$ factor arises in a similar fashion to the case of $\deg = 2n+1$ and n even. This time we do not have $S_H = P_H$. For the spherical classes contained in the $\frac{n!}{2}\mathbb{Z}$ factor have the variation described in part (d).

(g) The Case $X = \mathrm{Spin}\ (n)$

We finish §9 by describing $S_H \subset P_H \subset PH_*(\mathrm{Spin}\ (n))/\mathrm{Tor}$. Pick s where $2^s < n \leq 2^{s+1}$. Then our answer for $X = \mathrm{Spin}\ (n)$ is the same as for $X = SO(n)$ except in degree $2^{s+1} - 1$. In that degree we must divide our answer by a factor of 2. This result is based on the commutative diagram

$$\begin{tikzcd}
\Pi_i(\mathrm{Spin}(n))/\mathrm{Tor} \arrow[rr, "\cong"] \arrow[d, "h"'] & & \Pi_i(SO(n))/\mathrm{Tor} \arrow[d, "h"] \\
P_iH_*(\mathrm{Spin}(n))/\mathrm{Tor} \arrow[rr, "g"] & & P_iH_*(SO(n))/\mathrm{Tor} \\
& H_*(V) \arrow[ul, "\Omega_*"] \arrow[ur, "\Omega_*"'] &
\end{tikzcd}$$

plus the fact that

Proposition 9.12: The map g is an isomorphism except in degree $2^{s+1} - 1$. In that degree g is injective but has cokernel $= \mathbb{Z}/2$. Equivalently, $\Omega_* : H_{2^{s+1}-1}(V) \longrightarrow P_{2^{s+1}-1}H_*(\mathrm{Spin}(n))$ is an isomorphism.

PROOF: g is a $\mathbb{Q}$ isomorphism. By using a Bockstein spectral sequence argument we can show that f is a mod 2 isomorphism in degrees $\neq 2^{s+1}-1$ while, in degree $2^{s+1}-1$, $f \otimes \mathbb{F}_2$ has kernel = cokernel = $\mathbb{F}_2$. (We have already written down $H^*(SO(n);\mathbb{F}_2)$. On the other hand, $H^*(Spin(n);\mathbb{F}_2)$ $= \Delta(x_i(3 \leq i \leq n-1,\ 1 \neq 2^j) \otimes \Delta(x_{2^{\Delta+1}-1}))$ So the proposition is proved for f except that, in degree $2^{s+1}-1$, f is only known to be of the form $\mathbb{Z}/_{2}k$ for some $k \geq 1$. We now use the bottom triangle of the above diagram plus our knowledge of $\Omega_* : H_*(V) \longrightarrow PH_*(SO(n))/Tor$ from part (c) to deduce that f can be multiplication by at most 2 and that $\Omega_* : H_*(V) \longrightarrow PH_*(Spin(n))/Tor$ is an isomorphism in degree $2^{s+1}-1$. Q.E.D.

The only remarks we might add is that, in the case when $n = 2^{s+1}$ and, so, $P_{2^{s+1}-1}H_*(SO(n))/Tor = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\alpha_{\alpha^{s+1}-1}$ and β, it is the factor corresponding to β which is altered by 2. In other words, $\beta \in S_H$ instead of $2\beta \in S_H$ as before.

§10 The Group $G = G_2$

Now $H_*(G_2)/\mathrm{Tor} = E(X_3, X_{11})$. Of course, in $P_3H_*(G_2)/\mathrm{Tor} = \mathbb{Z}$ we have $S_H = P_H = \mathbb{Z}$. We now show

Proposition 10.1: Write $P_{11}H_*(G_2)/\mathrm{Tor} = \mathbb{Z}$. Then

$$S_H \subset P_H \subset P_{11}H_*(G_2)/\mathrm{Tor} \quad \text{is given by}$$

$$5!\,\mathbb{Z} \subset \frac{5!}{2}\,\mathbb{Z} \ .$$

All of §10 will be devoted to the proof of this proposition.

(i) Spherical Elements

We will reduce to the case $G = SO$. We have a diagram

$$\begin{array}{ccccccc}
\Pi_{11}(G_2)/\mathrm{Tor} & \rightarrow & \Pi_{11}(\mathrm{Spin}(7))/\mathrm{Tor} & \rightarrow & \Pi_{11}(\mathrm{Spin})/\mathrm{Tor} & \rightarrow & \Pi_{11}(SO)/\mathrm{Tor} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_{11}H_*(G_2)/\mathrm{Tor} & \rightarrow & P_{11}H_*(\mathrm{Spin}\ (7))/\mathrm{Tor} & \rightarrow & P_{11}H_*(\mathrm{Spin})/\mathrm{Tor} & \rightarrow & P_{11}H_*(SO)/\mathrm{Tor}
\end{array}$$

where the vertical maps are the Hurewicz maps. The horizontal maps are all incuced from standard maps. In particular, the fibration $G_2 \rightarrow \mathrm{Spin}\ (7) \rightarrow S^7$ gives rise to the first square. The bottom horizontal maps are all isomorphisms. (Use

Bockstein spectral sequence arguments). The top maps imbed $\Pi_*(G_2)/\mathrm{Tor}$ as a direct summand of $\Pi_*(SO)/\mathrm{Tor}$. For the first map we use the fact that, for $p = 2$, $\mathrm{Spin}(7)_{(\infty)} \underset{(2)}{\simeq} G_2 \times S^7$ while, for p odd, $\Pi_{11}(S^7)_{(p)} = 0$. The fact that the second map is an isomorphism was established in part (d) of §9.

So, since $S_H = 5!\,\mathbb{Z}$ for $G = SO$, the same result holds for $G = G_2$.

(ii) Primitive Elements

First of all, the map $P_{11}H_*(G_2)/\mathrm{Tor} \cong P_{11}H_*(SO)/\mathrm{Tor}$ tells us that $P_H \subset \frac{5!}{2}\mathbb{Z}$. To prove that $\frac{5!}{2}\mathbb{Z} \subset P_H$ we use a generating variety. Let $V \subset \Omega G_2$ be the generating variety of G_2 given in Bott [3]. Then $H^*(V)$ has an additive basis

$$\{1\,,\,x,\,\frac{x^2}{3}\,,\,\frac{x^3}{2.3}\,,\,\frac{x^4}{2.3^2}\,,\,\frac{x^5}{2.3^2}\}\ \deg x = 2$$

(This basis is due to Clarke [7] and corrects the one given by Bott). Let $\{\delta_0,\delta_1,\delta_2,\delta_3,\delta_4,\delta_5\}$ be the dual basis of $H_*(V)$.

Let $\{\Sigma_0,\Sigma_1,\ \ldots\ ,\Sigma_5\}$ be a $\Pi_*(MU)$ basis of $MU_*(V)$ we will use the same symbols to denote the image of these elements in $QMU_*(\Omega G_2)$. We have

LEMMA 10.2: $\Sigma_3 = 0$ in $QMU_*(\Omega G_2)_{(2)}$

PROOF: By the argument given at the end of part (e) of §9 we have $P(\Sigma_3) \in MU_*(V)_{(2)}$. Since Σ_3 is any representative for δ_3 we might as well assume that $\Sigma_3 = P(\Sigma_3)$. Now

$$MU_*(\Omega G_2) = \Pi_*(MU)[\Sigma_1,\Sigma_2,\Sigma_5]/(2\Sigma_2 - v_1\Sigma_1)$$

Consequently

$$QMU_*(\Omega G_2) = \Pi_*(MU)\ \{\Sigma_1,\Sigma_2,\Sigma_5\}/(2\Sigma_2 - v_1\Sigma_1) \ .$$

So $\Sigma_3 = x\Sigma_1 + y\Sigma_2$ where $x,y \in \Pi_*(MU)$. But, by 2.5, $P(x) = P(y) = 0$. So $\Sigma_3 = P(\Sigma_3) = P(x)P(\Sigma_1) + P(y)P(\Sigma_2) = 0$.

Q.E.D.

We want to show that

$$\frac{5!}{4} P(\Sigma_5) \in QMU_*(\Omega G_2)$$

Since $\Omega_* : Q_{10}H_*(\Omega G_2) \longrightarrow P_{11}H_*(G_2)/\mathrm{Tor}$ is of the form $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ (see the argument in parts (b) and (c) of §9) this result will suffice to show that $\frac{5!}{2}\,\mathbb{Z} \subset P_H$.

We will localize and work at each prime separately. This is more out of convenience than necessity. In particular we will be able to make use of lemma 10.2. But we are not localizing, as in some of our previous arguments, to make use of BP theory.

<u>$p \geq 7$</u> : If we write $P(\Sigma_5) = \sum_i c_i \Sigma_i$ then the $c_i \in \Pi_*(MU) \otimes \mathbb{Q} = \mathbb{Q}[b_1, b_2, \ldots]$ are polynomials in $\{b_1, b_2, b_3, b_4\}$. By 1.1 $b_1, b_2, b_3, b_4 \in \Pi_*(MU)_{(p)}$. So $P(\Sigma_5) \in MU_*(v)_{(p)}$.

<u>$p \geq 5$</u> : This time we have $5b_1, b_2, b_3, b_4 \in \Pi_*(MU)_{(5)}$. So $5P(\Sigma_5) \in MU_*(V)_{(5)}$.

<u>$p = 3$</u> : Since c_2, c_3, c_4 are polynomials in b_1, b_2, b_3 and since $b_1, 3b_2, 3b_3 \in \Pi_*(MU)_{(3)}$ we have $3c_2, 3c_3, c_4 \in \Pi_*(MU)_{(3)}$. Regarding c_1 we can reduce, as in lemma 9.8, to the case $c_1 = 2b_4$. And $3b_4 \in \Pi_*(MU)_{(3)}$. So $3P(\Sigma_5) \in MU_*(V)_{(3)}$.

<u>$p = 2$</u> : Since c_4 only involves b_1 we obviously have $2c_4 \in \Pi_*(MU)_{(2)}$. Regarding c_1, c_2 and c_3 we can reduce to the cases

$$c_1 = 2b_4 + 2v_1 b_3 \qquad \text{(by 9.9)}$$

$$c_2 = 2(b)_3^2 \qquad \text{(by 9.8)}$$

$$c_3 = 0 \qquad \text{(by 10.2)}$$

By Propositions 5.1 and 5.3 we have $2c_1, 2c_2 \in \Pi_*(MU)_{(2)}$.

So $2P(\Sigma_5) \in QMU_*(\Omega G_2)_{(2)}$.

REMARK : As in 9.8 we could have reduced c_3 to $c_3 = (b)_2^3$. However $2(b)_2^3 \notin \Pi_*(MU)_{(2)}$. Rather $4(b)_2^3 \in \Pi_*(MU)_{(2)}$. Our way out of this obstruction was to appeal to 10.2.

Now this same obstruction arises if we attempt to determine $P_H \subset P_{11}H_*(SO(7))/Tor$ by the argument in part (e) of §9. Moreover, we do not know how to prove 10.2 for $SO(7)$. It was for these reasons that we reduced our study of $P_H \subset P_{11}H_*(SO(7))/Tor$ in part (e) of §9 to the study of $P_H \subset P_{11}H_*(G_2)/Tor$.

§11 The Group $G = F_4$

We will localize and work one prime at a time. Localizing will enable us to often decompose the space F_4 into simpler factors. In particular, the space $B_n(p)$ will often appear as a factor. By $B_n(p)$ we mean the total space of the bundle with base $S^{2n+2p-1}$ and fibre S^{2n+1} such that

$$H^*(B_n(p), \mathbb{F}_p) = E(x_{2n+1}, P^1(x_{2n+1}))$$

Then $H_*(B_n(p)) = E(y_{2n+1}, y_{2n+2p-1})$ and it is easy to show that in degree $2n+2p-1$ the inclusion $S_H \subset P_H \subset P_{2n+2p-1}H_*(B_n(p)) = \mathbb{Z}$ is given by $S_H = P_H = p\,\mathbb{Z}$. (Consult the study of $G = S_p(2)$ in §6).

We have

$$H_*(F_4)/\mathrm{Tor} = E(x_3, x_{11}, x_{15}, x_{23})$$

So we must study $S_H \subset P_H \subset PH_*(F_K)/\mathrm{Tor}$ in degrees 3, 11, 15 and 23. The relations are summarized in the following chart

$$\deg \quad S_H \subset P_H \subset PH_*(F_4)/\mathrm{Tor}$$

(11.1)

3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
11	$2^3.5\ \mathbb{Z}$	$2^2.5\ \mathbb{Z}$	$\mathbb{Z}$
15	$2^3.3.7\ \mathbb{Z}$	$2^3.3.7\ \mathbb{Z}$	$\mathbb{Z}$
23	$2^7.3^2.5.7.11\ \mathbb{Z}$	$2^7.3^2.5.7.11\ \mathbb{Z}$	$\mathbb{Z}$

<u>$p \geq 5$</u> : For such primes F_4 is quasi-regular. Here we are using the results of Mimura-Toda [19]. They show

$$F_{4\,(\tilde{5})}\ B_1(5) \times B_7(5)$$

$$F_{4\,(\tilde{7})}\ B_1(7) \times B_5(7)$$

$$F_{4\,(\widetilde{11})}\ B_1(11) \times S^{11} \times S^{15}$$

$$F_{4\,(\tilde{p})}\ S^3 \times S^{11} \times S^{15} \times S^{23} \qquad (p \geq 13)$$

<u>$p = 3$</u> Harper [8] has shown that

$$F_{4\,(\tilde{3})}\ K \times B_5(3)$$

where

$$H^*(K;\mathbb{F}_3) = E(x_3,x_7) \otimes \mathbb{F}_3[x_8]/(x_8^3) \qquad \mathcal{P}^1(x_3) = x_7$$
$$\delta(x_7) = x_8$$

So $H_*(K)/\mathrm{Tor} = E(y_3,y_{23})$. This time we claim that $S_H \subset P_H \subset P_{23}H_*(K)/\mathrm{Tor} = \mathbb{Z}$ is given by $3^2\,\mathbb{Z} = 3^2\,\mathbb{Z} \subset \mathbb{Z}$. It suffices to show $3^2\,\mathbb{Z} \subset S_H$ and $P_H \subset 3^2\,\mathbb{Z}$.

Sphericals

We use connective coverings of K . Consider the fibration

$$F \xrightarrow{f} K \xrightarrow{g} K(\mathbb{Z}_{(p)},3)$$

where g represents a generator of $H^3(K)_{(3)} = \mathbb{Z}_{(3)}$. It is easy to calculate that, in degree ≤ 24

$$H^*(F;\mathbb{F}_3) = E(u_{19},u_{23}) \otimes \mathbb{F}_3[u_{18}] \qquad \delta(u_{18}) = u_{19}$$
$$\mathcal{P}^1(u_{19}) = u_{23}$$

$$H_*(F)/\mathrm{Tor} = E(W_{23})$$

The relation $3^2\,\mathbb{Z} \subset S_H$ follows from the commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} = \Pi_{23}(F)/\mathrm{Tor} & \xrightarrow{x3} & P_{23}H_*(F)/\mathrm{Tor} = \mathbb{Z} \\
\wr\| S \downarrow & & \downarrow x3 \\
\mathbb{Z} = \Pi_{23}(K)/\mathrm{Tor} & \longrightarrow & P_{23}H_*(K)/\mathrm{Tor} = \mathbb{Z}
\end{array}$$

It follows from Smith [22] that the right map is multiplication by 3 while it is easy to calculate that the top map is multiplication by 3.

<u>Primitives</u>

Consider the representation $\lambda : F_4 \longrightarrow SU(26)$ studied by Watanabe [26]. We have a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} = P_{23}H_*(F_4)/\mathrm{Tor} & \xrightarrow{\lambda_*} & P_{23}H_*(SU(26)) = \mathbb{Z} \\
\Omega_* \uparrow & & \uparrow \Omega_* \\
\mathbb{Z} = Q_{22}H_*(\Omega F_4)/\mathrm{Tor} & \xrightarrow{(\Omega\lambda)_*} & Q_{22}H_*(\Omega SU(26)) = \mathbb{Z}
\end{array}$$

Watanabe proved that $(\Omega\lambda)_*$ is multiplication by 3^3. Also Ω_4 is an isomophism for $SU(26)$ while Ω_* is multiplication by 3^k where $k \geq 1$ for F_4. It now follows from the diagram that λ_* is multiplication by 3^ℓ where $\ell \leq 2$.

Since $11! = 3^4 N$ where $(N,3) = 1$ it follows from §8 that $P_H \subset P_{23}H_*(SU(26))$ is given by $3^4\,\mathbb{Z} \subset \mathbb{Z}$. The commutative diagram

$$\begin{array}{ccccc} P_H & \subset & P_{23}H_*(F_4)/\mathrm{Tor} & = & \mathbb{Z} \\ \downarrow & & \downarrow {\scriptstyle \times 3^{\ell}} & & \\ 3^4\,\mathbb{Z} = P_H & \subset & P_{23}H_*(SU(26)) & = & \mathbb{Z} \end{array}$$

now faces $P_H \subset P_{23}H_*(F_4)/\mathrm{Tor}$ to satisfy $P_H \subset 3^2\,\mathbb{Z}$.

<u>p = 2</u> : In degrees 3 and 11 the relation $S_H \subset P_H \subset PH_*(F_4)/\mathrm{Tor}$ is the same as in the G_2 case. For $G_2 \subset F_4$ is a mod 2 homotopy equivalence in degree ≤ 14.
$(H^*(F_4; \mathbb{F}_2) = E(x_5, x_{15}, x_{23}) \otimes \mathbb{F}_2[x_3]/(x_3^4)$. In degree 15 and 23 it suffices to prove

$$2^3\,\mathbb{Z} \subset S_H \subset P_H \subset 2^3\,\mathbb{Z} \qquad \text{for deg } 15$$

$$2^7\,\mathbb{Z} \subset S_H \subset P_H \subset 2^7\,\mathbb{Z} \qquad \text{for deg } 23$$

<u>Primitives</u>

The relations $P_H \subset 2^3\,\mathbb{Z}$ in degree 15 and $P_H \subset 2^7\,\mathbb{Z}$ in degree 23 follow from an argument similar to that used above in the $p = 3$ case. It is based on two facts. Watanabe has calculated that $\Omega\lambda$ gives maps

$$(\Omega\lambda)_* : Q_{14}H_*(\Omega F_4) \xrightarrow{x2} Q_{14}H_*(\Omega SU(26))$$

$$(\Omega\lambda)_* : Q_{22}H_*(\Omega F_4) \xrightarrow{x2} Q_{22}H_*(\Omega SU(26))$$

Also $P_H \subset PH_*(SU(26))$ is given by $7!\ \mathbb{Z} \subset \mathbb{Z}$ in degree 15 and $11!\ \mathbb{Z} \subset \mathbb{Z}$ in degree 23. (In terms of 2 primary information these become $2^4\ \mathbb{Z} \subset \mathbb{Z}$ and $2^8\ \mathbb{Z} \subset \mathbb{Z}$) .

Sphericals:

The relations $2^3\ \mathbb{Z} \subset S_H$ in degree 15 and $2^7\ \mathbb{Z} \subset S_H$ in degree 23 follows from the information obtained by Mimura [18] regarding the space F_4/G_2 . One has $H_*(F_4/G_8) = E(y_{15},y_{23})$. For both $k = 15$ and $k = 23$ we have a commutative diagram

$$\begin{array}{ccccc}
\Pi_k(F_4) & \longrightarrow & \Pi_k(F_4/G_2) & \longrightarrow & \Pi_{k-1}(G_2) = \mathbb{Z}/8 \otimes \mathbb{Z}/2 \\
\downarrow h & & \downarrow h & & \\
\mathbb{Z} = P_kH_*(F_4)/\mathrm{Tor} & \xrightarrow{\cong} & P_kH_*(F_4/G_2) = \mathbb{Z} & &
\end{array}$$

Since $h : \Pi_k(F_4/G_2)/\mathrm{Tor} \longrightarrow P_kH_*(F_4/G_2)$ is of the form

$$\mathbb{Z} \cong \mathbb{Z} \qquad k = 15$$

$$\mathbb{Z} \xrightarrow{x16} \mathbb{Z} \qquad k = 23$$

we conclude that $P_H \subset P_kH_*(F_4)/\mathrm{Tor}$ satisfies $P_H \subset 2^3\ \mathbb{Z}$ and $P_H \subset 2^7\ \mathbb{Z}$ in degrees 15 and 23 respectively.

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