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by

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Infinite-dimensional p-adic groups, semigroups of double cosets, and inner functions on Bruhat–Tits buildings

Yury A. Neretin[1]

We construct p-adic analogs of operator colligations and their characteristic functions. Consider a p-adic group $\mathbf{G} = \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$, its subgroup $L = \mathrm{O}(k\infty, \mathbb{O}_p)$, and the subgroup $\mathbf{K} = \mathrm{O}(\infty, \mathbb{O}_p)$ embedded to L diagonally. We show that double cosets $\Gamma = \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ admit a structure of a semigroup, Γ acts naturally in $\mathbf{K}$-fixed vectors of unitary representations of $\mathbf{G}$. For any double coset we assign a 'characteristic function', which sends a certain Bruhat–Tits building to another building (buildings are finite-dimensional); image of the distinguished boundary is contained in the distinguished boundary. The latter building admits a structure of (Nazarov) semigroup, the product in Γ corresponds to a point-wise product of characteristic functions.

1 Introduction

1.1. Olshanski's theory. In [25], [26], [27] Olshanski proposed a formalism for representation theory of infinite-dimensional classical groups and infinite symmetric groups. This formalism incorporates earlier works of Berezin, Shale, Stinespring, Thoma, Vershik, Kerov, works of Olshanski himself and is a base for later works on infinite-dimensional harmonic analysis [29], [1], [9]. Also it is a base for some works on representation theory of the group of diffeomorphisms of the circle and its p-adic analog ([15], [14], [16], Chapter VII and Sections IX.5–IX.6).

An important element of Olshanski's technology is semigroups of double cosets. Let G be an infinite-dimensional group and $K \subset G$ be a subgroup. Quite often double cosets $K \setminus G/K$ admit a natural structure of a semigroup[2], these semigroups act in spaces of K-fixed vectors in unitary representations.

1.2. p-adic groups. As far as I know, a representation theory of infinite-dimensional classical p-adic groups in this moment does not exist, however there is one serious work of Nazarov [12] (also [13]) on this topic.

Recently it was shown that Olshanski's formalism admits an essential extension, see [18]–[21], this allows to return to the question about representations of infinite-dimensional p-adic groups. Our topic is semigroups of double cosets, we get an analog of construction of [19].

Note that in this case the analogy between Bruhat–Tits buildings and Riemannian noncompact symmetric spaces remains to be as mysterious as usual (see, e.g., [11], [23], [12], [4], [17], [8], [22]).

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[2]This phenomenon was firstly observed by Ismagilov [6], [7] for groups SL_n over *non-local* non-archimedean fields, in this case Hecke algebras degenerate to semigroups.

1.3. Inner functions. A holomorphic function $f(z)$ in a unit disk $z < 1$ is called *inner*, if $|f(z)| < 1$ for $|z| < 1$ and

$$\lim_{r \to 1^-} |f(re^{i\theta})| = 1 \qquad \text{a.s. } \theta \in [0, 2\pi], \tag{1.1}$$

where $z = re^{i\theta}$ and r, θ are real[3]. Inner functions is a classical topic of function theory, see, e.g., [2]. There are several extensions of this notion.

1) A holomorphic matrix-valued (operator-valued) function $f(z)$ in the unit disk is called *inner* if $\|f(z)\| \leqslant 1$ for $|z| < 1$ and boundary values of f on the circle are unitary.

2) More generally consider a pseudo-Euclidean space with inner product $\langle \cdot, \cdot \rangle$. We say that an operator g is an *indefinite contraction*, if $\langle gv, gv \rangle \leqslant \langle v, v \rangle$ for all v (see, e.g., [22], Section 2.7). We say that a meromorphic matrix-valued function f is *inner* if it is indefinite contractive in the disk and pseudo-unitary on the unit circle. Such functions is a classical topic of spectral theory of non-self-adjoint operators (starting works of Livshits and Potapov, end 40s–beginning of 50s, see [10], [30]).

3) Some semigroups of double cosets (as $\mathrm{O}(\infty) \setminus \mathrm{GL}(\infty + \alpha, \mathbb{R}) / \mathrm{O}(\infty)$, etc.) can be realized as semigroups of inner functions in the sense 2), see [26], [16], Section IX.4.

4) In the recent work [19], [20] it was shown that quite general semigroups of double cosets can be realized as semigroups of multivariate inner functions. In fact we get holomorphic maps sending Hermitian symmetric spaces to Hermitian symmetric spaces such that Shilov boundaries fall to Shilov boundaries.

In the present paper we extend the last construction to p-adic case. For a double coset we assign a map from a Bruhat–Tits building Ω to a Bruhat–Tits building Ξ such that image of the distiguished boundary is contained in the distinguished boundary. We also have a structure of a semigroup on the set of vertices of building Ξ (the Nazarov semigroup) and the product of double cosets corresponds to pointwise product of maps $\Omega \to \Xi$.

1.4. Notation. Let
— A^t be the transposed matrix;
— 1_α, 1_V be the unit matrix of order α, the unit operator in a space V;
— $\mathbb{Q}_p$ be the p-adic field;
— $\mathbb{O}_p$ be the ring of p-adic integers;
— $\mathbb{Q}_p^\times$, $\mathbb{C}^\times$ be multiplicative groups of $\mathbb{Q}_p$, $\mathbb{C}$.
We denote the standard character $\mathbb{Q}_p \to \mathbb{C}^\times$ by $\exp\{2\pi i a\}$,

$$\exp\{2\pi i a\} = \exp\Bigl\{2\pi i \sum_{j \geqslant -N} a_j p^j\Bigr\} := \exp\Bigl\{2\pi i \sum_{j: -1 \geqslant j \geqslant -N} a_j p^j\Bigr\}$$

Below we define:

[3]We can not write $z \to e^{i\theta}$, an inner function can be discontinuous in all points of the circle.

— $\mathrm{GL}(n,\mathbb{Q}_p)$, $\mathrm{Sp}(2n,\mathbb{Q}_p)$, $\mathrm{Sp}(2n,\mathbb{Q}_p)$, $\mathrm{O}(n,\mathbb{O}_p)$, $\mathrm{GL}(\infty,\mathbb{O}_p)$, $\mathrm{Sp}(2\infty,\mathbb{Q}_p)$, etc., Subsection 2.1;

— $\mathbf{G}$, $\mathbf{K}$, Subsection 2.2;

— $\mathfrak{g}\star\mathfrak{h}$, Subsection 2.2;

— $\mathfrak{g}^*$, Subsection 2.5;

— $R_\downarrow$, $R^\uparrow$, , Subsection 3.1;

— $R_j \nearrow R$, 3.2;

— $\mathrm{LMod}(V)$, $\mathrm{LLat}(V)$, $\mathrm{LGr}(V)$, Subsection 3.3;

— $\Delta(V)$, $\mathrm{Bd}(V)$, Subsections 3.4, 3.5;

— $P : V \rightrightarrows W$, $\ker P$, $\mathrm{indef}\, P$, $\mathrm{dom}\, P$, $\mathrm{im}\, P$, $P^\square$, Subsection 3.6;

— Naz, $\overline{\mathrm{Naz}}$, **Naz**, Subsections 3.9; 3.10;

— We, Subsection 3.12;

— $\chi_\mathfrak{g}(Q,T)$, Subsection 4.1.

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2 Multiplication of double cosets

2.1. Groups. By $V=\mathbb{Q}_p^n$ we denote linear spaces over $\mathbb{Q}_p$. Denote by $\mathrm{GL}(n,\mathbb{Q}_p)=\mathrm{GL}(V)$ the group of invertible linear operators in $\mathbb{Q}_p^n$; by $\mathrm{GL}(n,\mathbb{O}_p)$ the group of all matrices g with integer elements, such that g^{-1} have integer elements.

Consider a space $V=\mathbb{Q}_p^{2n}$ equipped with a non-degenerate skew-symmetric bilinear form B_V, say $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. The symplectic group $\mathrm{Sp}(2n,\mathbb{Q}_p)$ is the group of matrices preserving this form, $\mathrm{Sp}(2n,\mathbb{O}_p)$ is the group of symplectic matrices with integer elements.

Also, consider a space $\mathbb{Q}_p^n$ equipped with the standard symmetric bilinear form $(v,w)=\sum v_j w_j$. We denote by $\mathrm{O}(n,\mathbb{Q}_p)$ the group of all matrices preserving this form[4].

By $\mathrm{GL}(\infty,\mathbb{Q}_p)$ we denote the group of all infinite invertible matrices over $\mathbb{Q}_p$ such that $g-1$ has only finite number of non-zero elements. We call such matrices *finite*. We define $\mathrm{GL}(\infty,\mathbb{O}_p)$, $\mathrm{Sp}(2\infty,\mathbb{Q}_p)$, $\mathrm{Sp}(2\infty,\mathbb{O}_p)$, $\mathrm{O}(\infty,\mathbb{O}_p)$ in the same way.

2.2. Multiplication of double cosets. Let

$$\mathbf{G}:=\mathrm{GL}(\infty,\mathbb{Q}_p):=\mathrm{GL}(\alpha+k\infty,\mathbb{Q}_p)$$

be the group of finite block $(\alpha+\infty+\dots+\infty)\times(\alpha+\infty+\dots+\infty)$- matrices (there are k copies of ∞). By $\mathbf{K}$ we denote the group

$$\mathbf{K}=\mathrm{O}(\infty,\mathbb{O}_p)$$

[4]There are several non-equivalent non-degenerate quadratic forms and several different orthogonal groups over $\mathbb{Q}_p$, however we consider only this group.

embedded to $\mathbf{G}$ by the rule

$$\mathfrak{I} : u \mapsto \begin{pmatrix} 1_\alpha & 0 & \dots & o \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix}, \qquad (2.1)$$

where 1_α denotes the unit matrix of order α.

We wish to define a structure of a semigroup on double cosets $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$.

Set

$$\Theta_N := \begin{pmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \qquad (2.2)$$

Let $\mathfrak{g}$, $\mathfrak{h} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$. Choose their representatives g, $h \in \mathbf{G}$. Consider the sequence

$$f_N := g\mathfrak{I}(\Theta_N)h$$

and double coset $\mathfrak{f}_N$ containing f_N.

Theorem 2.1 a) *The sequence $\mathfrak{f}_N$ is eventually constant.*

b *The limit $\mathfrak{f} := \lim_{N\to\infty} \mathfrak{f}_N$ does not depend on a choice of representatives g, h.*

c) *The product $\mathfrak{g} \star \mathfrak{h}$ in $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ obtained in this way is associative.*

These statements are simple, see proofs of parallel real statements in [20]. Also, it is easy to write an explicit formula for the product. For definiteness, set $k = 2$. Then

$$\begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} \star \begin{pmatrix} a' & b'_1 & b'_2 \\ c'_1 & d'_{11} & d'_{12} \\ c'_2 & d'_{21} & d'_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_\infty & 0 & 0 \\ 0 & 1_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_\infty \\ 0 & 0 & 0 & 1_\infty & 0 \end{pmatrix} \begin{pmatrix} a' & b'_1 & 0 & b'_2 & 0 \\ c'_1 & d'_{11} & 0 & d'_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c'_2 & d'_{21} & 0 & d'_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since a result is double coset, we can write the final matrix in different forms, say

$$\mathfrak{f} = \begin{pmatrix} aa' & | & b_1 & ab'_1 & & b_1 & ab'_1 \\ - & + & - & - & - & - & - \\ c_1a' & | & d_{11} & c_1b'_1 & & d_{12} & c_1b'_2 \\ c'_1 & | & 0 & d'_{11} & & 0 & d'_{12} \\ & | & & & & & \\ c_2a' & | & d_{21} & c_2b'_1 & & d_{22} & c_2b'_2 \\ c'_2 & | & 0 & d'_{21} & & 0 & d'_{22} \end{pmatrix} \qquad (2.3)$$

or

$$\mathfrak{f}=\begin{pmatrix} aa' & | & ab_1' & b_1 & & ab_2' & b_2\\ - & + & - & - & - & - & \\ c_1a' & | & c_1b_1' & d_{11} & & c_1b_2' & d_{12}\\ c_1' & | & d_{11}' & 0 & & d_{12}' & 0\\ & | & & & & & \\ c_2a' & | & c_2b_1' & d_{21} & & c_2b_2' & d_{22}\\ c_2' & | & d_{21}' & 0 & & d_{22}' & 0 \end{pmatrix}.$$

2.3. Multiplicativity theorem. Let ρ be a unitary representation of $\mathbf{G}$, denote by $H^{\mathbf{K}}$ the subspace of all $\mathbf{K}$-fixed vectors. Denote by $P^{\mathbf{K}}$ the operator of projection to $H^{\mathbf{K}}$. For $g\in\mathbf{G}$ consider the operator $\overline{\rho}(g):H^{\mathbf{K}}\to H^{\mathbf{K}}$ given by

$$\overline{\rho}(g):=P^{\mathbf{K}}\rho(g)$$

Obviously, $\overline{\rho}(g)$ is a function on double cosets $\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$, therefore we can write $\overline{\rho}(\mathfrak{g})$.

Theorem 2.2 *For any unitary representation ρ, for all $\mathfrak{g}$, $\mathfrak{h}\in\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$ the following equality* (*the "multiplicativity theorem"*) *holds,*

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(\mathfrak{h})=\overline{\rho}(\mathfrak{g}\star\mathfrak{h}).$$

We give a proof in Section 6.

2.4. Sphericity. Set $\alpha=0$.

Proposition 2.3 *The pair $(\mathbf{G},\mathbf{K})$ is spherical, i.e., for any irreducible unitary representation of $\mathbf{G}$ the dimension of the space of $\mathbf{K}$-fixed vectors is $\leqslant 1$.*

We omit a proof, it is the same as for infinite-dimensional real classical groups, see [20].

2.5. Involution. The map $g\mapsto g^{-1}$ induces an involution $\mathfrak{g}\mapsto\mathfrak{g}^*$ on $\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$. Evidently,

$$(\mathfrak{g}\star\mathfrak{h})^*=\mathfrak{h}^*\star\mathfrak{g}^*.$$

Also, for any unitary representation ρ of $\mathbf{G}$ we have

$$\overline{\rho}(\mathfrak{g}^*)=\overline{\rho}(\mathfrak{g})^*.$$

2.6. Purpose of the work. Our aim is to describe this multiplication in more usual terms. More precisely, we wish to get p-adic analogs of multivariate characteristic functions constructed in [20], [19].

2.7. Structure of the paper. In Section 2 we define multiplication of double cosets. Section 3 contains preliminaries (lattices, Bruhat–Tits buildings, relations, Weil representation of Nazarov category). A main construction (characteristic functions of double cosets and their properties) is contained in Section 4. Proofs are given in Section 5. In Section 6 we prove the multiplicativity theorem, Section 7 contains some simple results on representations.

3 Preliminaries. Modules, relations, buildings, Nazarov category, and Weil representation

3.1. Modules. Below the term *submodule* means an $\mathbb{O}_p$-submodule in a linear space $V = \mathbb{Q}_p^n$. For each submodule $R \subset \mathbb{Q}_p^n$ there is a (non-canonical) basis $e_i \in \mathbb{Q}_p^n$ such that

$$R = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_j \oplus \mathbb{O}_p e_{j+1} \oplus \cdots \oplus \mathbb{O}_p e_l.$$

If $j = n$ we have a linear subspace. If $j = 0$, $l = n$, then we get a *lattice*. A formal definition is: a lattice R is a compact $\mathbb{O}_p$-submodule such that $\mathbb{Q}_p R = \mathbb{Q}_p^n$. For details, see, e.g., [32].

Denote by $\mathrm{Mod}(V)$ the set of all submodules in V, by $\mathrm{Lat}(V)$ the space of all lattices. It is easy to see that

$$\mathrm{Lat}(V) \simeq \mathrm{GL}(V, \mathbb{Q}_p)/\mathrm{GL}(V, \mathbb{O}_p).$$

For any submodule R denote by $R_\downarrow$ the maximal linear subspace in R. By $R^\uparrow$ we denote the minimal linear subspace containing R. The image of R in $R^\uparrow/R_\downarrow$ is a lattice.

Conversely, let $L \subset M$ be a pair of subspaces and $P \subset M/L$ be a lattice. Then $P + L$ is a submodule and all submodules have such form.

3.2. Convergence on Mod**.** Let $V = \mathbb{Q}_p^n$. We define a norm on V as

$$\|x\| = \max_j |x_j|.$$

Denote by $B(p^l)$ the ball with center at 0 of radius p^l.

The space $\mathrm{Mod}(V)$ admits a natural topology of a compact space, we say that R_j converges to R if for each m we have a convergence $B(p^l) \cap R_j \to B(p^l) \cap R$ in the sense of Hausdorff metric. In this topology the space $\mathrm{Lat}(V)$ is a discrete dense subset in $\mathrm{Mod}(V)$.

We need an analog of the radial limit (1.1) and prefer another convergence.

We say that a sequence R_j *converges* to R (notation $R_j \nearrow R$) if

— for any compact subset $S \subset R$ we have $S \subset R_j$ starting some place.

— for each $\varepsilon > 0$, for sufficiently large j the set R_j is contained in the ε-neighborhood of R.

Lemma 3.1 *If $R_j \nearrow R$, then $(R_j)_\downarrow \subset R_\downarrow$ and $(R_j)^\uparrow \supset R^\uparrow$ starting some j.*

In particular, a $\nearrow$-convergent sequence of linear subspaces is eventually constant.

Lemma 3.2 a) *Let $L \subset V$ be a linear subspace. If $R_j \nearrow R$, then $(L \cap R_j) \nearrow (L \cap R)$.*

b) *Let $M \subset V$ be a linear subspace, denote by π the natural map $V \to V/M$. If $R_j \nearrow R$ then $\pi(R_j) \nearrow \pi(R)$.*

This is obvious.

3.3. Self-dual modules. For details, see [22], Sections 10.6–10.7. Consider a p-adic linear space $V \simeq \mathbb{Q}_p^{2n}$ equipped with a nondegenerate skew-symmetric bilinear form $B_V(\cdot,\cdot)$ (as above). We say that a subspace L is isotropic if $B_V(v,w) = 0$ for all v, $w \in V$. By $\mathrm{LGr}(V)$ we denote the set of all maximal isotropic (Lagrangian) subspaces in V (their dimensions $= n$). By $L^\perp$ we denote the orthocomplement of a subspace L.

If P is a submodule, denote by $P^{\perp\!\!\!\perp}$ the *dual submodule*, i.e., the set of vectors w such that $B(v,w) \in \mathbb{O}_p$ for all $v \in P$. If P is a subspace, then $P^{\perp\!\!\!\perp} = P^\perp$.

We say that a submodule $R \subset V$ is *isotropic* if $B_V(v,w) \in \mathbb{O}_p$ for all v, $w \in R$.

EXAMPLE. If R is a linear subspace, then R is isotropic in the usual sense. On the other hand the lattice $\mathbb{O}^{2n}$ is isotropic (and self-dual, see below). $\square$

We say that a submodule R is *self-dual* if it is a maximal isotropic submodule in V. Equivalently, $P^{\perp\!\!\!\perp} = P$. Denote by $\mathrm{LMod}(V)$ the set of self-dual submodules, by $\mathrm{LLat}(V)$ the set of all self-dual lattices. It is easy to show that

$$\mathrm{LLat}(V) = \mathrm{Sp}(2n, \mathbb{Q}_p)/\mathrm{Sp}(2n, \mathbb{O}_p).$$

For any self-dual submodule R the subspace $R_\downarrow$ is isotropic, and $R^\uparrow$ is orthocomplement of $R_\downarrow$ with respect to the bilinear form. The submodule R has the form $R_\downarrow + S$, where S is a self-dual lattice in $R^\uparrow/R_\downarrow$.

It is convenient to reformulate a definition. Define a bicharacter β on $V \times V$ by

$$\beta(x,y) = \exp\{2\pi i B(x,y)\}$$

We say that a module P is *isotropic* if $\beta(x,y) = 1$ on $P \times P$. A module is *self-dual* if it is a maximal isotropic module.

3.4. Almost self-dual modules. Let V and B be same as above. A submodule L in V is *almost self-dual* if it contains a self-dual module M and $B(v,w) \in p^{-1}\mathbb{O}_p$ for all v, $w \in L$ (see, e.g., [16], Section 10.6). Notice that $L/M \simeq \mathbb{Z}_p^k$ with $k = 0, 1, \dots, n$. Any almost self-dual module can be reduced by a symplectic transformation to the form

$$\begin{aligned}(\mathbb{O}_p e_1 \oplus \mathbb{O}_p e_{n+1}) \oplus \dots \oplus (\mathbb{O}_p e_k \oplus \mathbb{O}_p e_{n+k}) \oplus \\ \oplus (p^{-1}\mathbb{O}_p e_{k+1} \oplus \mathbb{O}_p e_{n+k+1}) \oplus \dots \oplus (p^{-1}\mathbb{O}_p e_m \oplus \mathbb{O}_p e_{n+m}) \oplus \\ \oplus \mathbb{Q}_p e_{m+1} \oplus \dots \oplus \mathbb{Q}_p e_n\end{aligned} \quad (3.1)$$

We draw an oriented graph $\Delta(V)$. Vertices are almost self-dual modules. If $R \supset R'$ then we draw an arrow from L to L'.

Note that If R, R' are connected by an arrow, then $R_\downarrow = (R')_\downarrow$ and $R^\uparrow = (R')^\uparrow$.

Any pair of lattices can be connected by an (non-oriented) way. Denote the subgraph whose vertices are lattices by $\Delta_0(V)$.

More generally, fix an isotropic subspace L and consider the subgraph $\Delta_L(V)$ whose vertices are almost self-dual lattices R such that $R_\downarrow = L$, $R^\uparrow = L^\perp$. We get a connected subgraph, moreover

$$\Delta_L(V) \simeq \Delta_0(L^\perp/L).$$

We get

$$\Delta(V) = \bigcup_{L \text{ is isotropic subspace}} \Delta_L(V).$$

If $L \subset M$, then Δ_M is contained in the closure of Δ_L in the sense of $\nearrow$-convergence.

3.5. Buildings, for details, see [3], [16]. Now we consider all k-plets of vertices of $\Delta(V)$ that are pairwise connected by edges.

First, consider the subgraph Δ_0. It can be shown that $k \leqslant n+1$ and each k-plet is contained in a (non-unique) $(n+1)$-plet. In this way we get a structure of an n-dimensional simplicial complex, it is called a *Bruhat–Tits building.* We denote it by $\mathrm{Bd}(V)$.

For a subgraph Δ_L we get a simplicial complex isomorphic $\mathrm{Bd}(L^\perp/L)$.

3.6. Relations. Let V, W be linear spaces. We say that a *relation* $P : V \rightrightarrows W$ is a submodule in $V \oplus W$.

EXAMPLE. Let $A : V \to W$ be a linear operator. Then its graph is a relation. $\square$

Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be relations. We define their product $S = QP : V \rightrightarrows Y$ as the set of all $v \oplus y \in V \oplus Y$ for which there exists $w \in W$ such that $v \oplus w \in P$, $w \oplus y \in Q$.

For a relation $P : V \to W$ we define its *kernel* as

$$\ker P = P \cap (V \oplus 0),$$

the *indefiniteness*

$$\mathrm{indef}\, P = P \cap (0 \oplus V),$$

the *domain of definiteness*

$$\mathrm{dom}\, P = \text{projection of } P \text{ to } V,$$

and the *image*

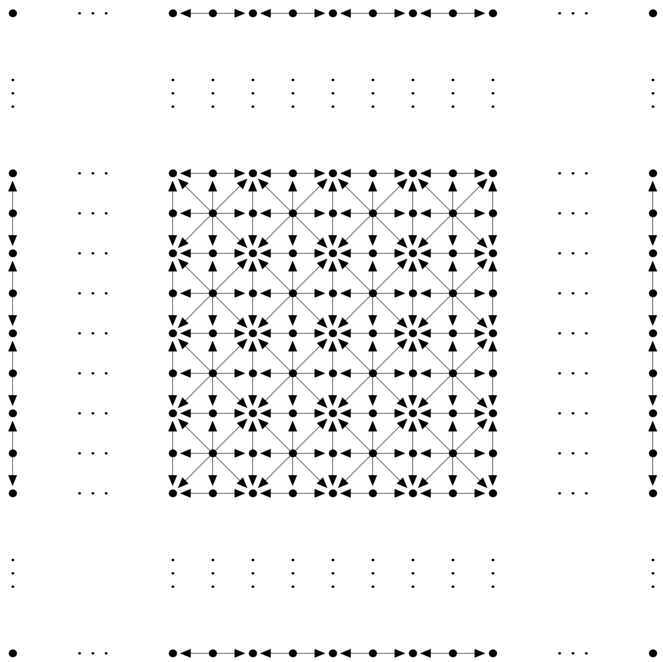
$$\mathrm{im}\, P = \text{projection of } P \text{ to } W.$$

We define the pseudo-inverse relation $P^\square : W \rightrightarrows V$ being the same submodule in $W \oplus V$. Evidentely,

$$(PQ)^\square = Q^\square P^\square.$$

3.7. The reformulation of the definition of product. We keep the same notation. Consider the space $\mathcal{Z} := V \oplus W \oplus W \oplus Y$ and submodules of $\mathcal{Z}$:

— the subspace $\mathcal{H}$ consisting of vectors $v \oplus w \oplus w \oplus y$;



Vertices of the central piece of the subcomplex are almost self-dual lattices of the form $L = p^{k_1}\mathbb{O}_p e_1 \oplus p^{k_2}\mathbb{O}_p e_2 \oplus p^{l_1}\mathbb{O}_p e_3 \oplus p^{l_2}\mathbb{O}_p e_4$. They are almost self-dual iff $k_1 + l_1$, $k_2 + l_2$ are 0 or -1. We also present limit points of this subcomplex.
1) Four boundary pieces correspond to almost self-dual submodules containing a line $\mathbb{Q}_p e_j$, e.g., $M = \mathbb{Q}_p e_1 \oplus p^{m_2}\mathbb{O}_p e_2 \oplus p^{m_2}\mathbb{O}_p e_4$. A sequence of lattices converges to M only if $k_1 \to -\infty$ and $k_2 = m_2$ starting some place.
2) Four extreme points correspond to Lagrangian planes, e.g., $N = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_4$. A sequence of lattices converges to N if $k_1 \to +\infty$, $k_2 \to -\infty$.

Figure 1: A reference to Subsections 3.2, 3.4. A subcomplex ('appartaments') of the building $\mathrm{Bd}(\mathbb{Q}_p^4)$.

— the subspace $\mathcal{A}$ consisting of vectors $0 \oplus w \oplus w \oplus 0$;

— the submodule $P \oplus Q \subset (V \oplus W) \oplus (W \oplus Y)$.

Then we do the following operations:

— consider the intersection $R = \mathcal{H} \cap (P \oplus Q)$

— consider the map $\theta : \mathcal{H} \to \mathcal{H}/\mathcal{A} \simeq V \oplus Y$

Then $QP = \theta(R)$.

3.8. Action on submodules. Let $P : V \rightrightarrows W$ be a relation, $T \subset V$. We define the submodule $PT \subset W$ as the set of $w \in W$ such that there is $v \in T$ satisfying $v \oplus w \in P$.

Remark. We can consider a submodule $T \subset V$ as a relation $0 \rightrightarrows V$. Therefore we can regard $PT : 0 \rightrightarrows W$ as the product of relations $T : 0 \rightrightarrows V$ and $Q : V \rightrightarrows W$. □

3.9. The Nazarov category. For a pair V, W of symplectic linear spaces we define a skew-symmetric bilinear form $B^{\ominus}$ on $V \oplus W$ by

$$B^{\ominus}(v \oplus w, v' \oplus w') = B_V(v, v') - B_W(w, w').$$

Denote by

— $\overline{\mathrm{Naz}}(V, W)$ the set of all self-dual submodules of $V \oplus W$;

— $\mathrm{Naz}(V, W)$ the set of $P \in \mathrm{Naz}(V, W)$ such that $\ker P$ and $\operatorname{indef} P$ are compact.

Theorem 3.3 a) *If $P \in \mathrm{Naz}(V, W)$, $Q \in \mathrm{Naz}(W, Y)$, then $QP \in \mathrm{Naz}(V, Y)$.*

b) *If $P \in \overline{\mathrm{Naz}}(V, W)$, $Q \in \overline{\mathrm{Naz}}(W, Y)$, then $QP \in \overline{\mathrm{Naz}}(V, Y)$.*

c) *If $P \in \mathrm{Naz}(V, W)$, $Q \in \mathrm{Naz}(W, Y)$ are lattices, then QP is a lattice.*

The statement a) was proved in Nazarov [12] (see also [22], Section 10.7), c) is obvious. Proof of b) is identical to proof of a).

Thus we get two similar categories[5], Naz and $\overline{\mathrm{Naz}}$. *The group of automorpisms of an object V is the symplectic group* $\mathrm{Sp}(V, \mathbb{Q}_p)$ (for both categories), an operator $V \to V$ is symplectic iff its graph is isotropic with respect to the form $B^{\ominus}$.

For $P \in \overline{\mathrm{Naz}}(V, W)$, we have

$$(\ker P)^{\perp\!\!\perp} = \operatorname{dom} P, \qquad (\operatorname{indef} P)^{\perp\!\!\perp} = \operatorname{im} P$$
$$\big((\ker P)_{\downarrow}\big)^{\perp} = (\operatorname{dom} P)^{\uparrow}, \qquad \big((\operatorname{indef} P)_{\downarrow}\big)^{\perp} = (\operatorname{im} P)^{\uparrow},$$

3.10. Extended Nazarov category. Now we add to the Nazarov category an infinite-dimensional object $V_{2\infty}$. This is the space of vectors

$$(x_1^+, x_2^+, \dots, x_1^-, x_2^-, \dots), \qquad \text{where } x_j^{\pm} \in \mathbb{Q}_p \text{ and } x_j^{\pm} \in \mathbb{O}_p \text{ for almost all } j.$$

[5]The Nazarov category is an analog of Krein–Shmulian type categories, see [16], [22]

Notice that $V_{2\infty}$ is not a linear space.

We introduce a bicharacter $\beta(\cdot,\cdot)$ on $V_{2\infty}\oplus V_{2\infty}$ by

$$\beta(x,y)=\exp\Bigl[2\pi i\sum_{j=1}^{\infty}(x_j^+y_j^- - x_j^-y_j^+)\Bigr]:=\prod_{j=1}^{\infty}\exp\bigl\{2\pi i(x_j^+y_j^- - x_j^-y_j^+)\bigr\}.$$

Notice that almost all factors of the product equal to 1. The sum in square brackets defining a symplectic form is not well defined, more precisely it is well defined modulo $\mathbb{O}_p$.

Objects of the *extended Nazarov category* **Naz** are finite-dimensional spaces V equipped with skew-symmetric non-degenerate bilinear forms B_V and with the corresponding bicharacters β_V and also the space $V_{2\infty}$.

Let V, W be two objects. We equip their direct sum with a bicharacter

$$\beta_{V\oplus W}(v\oplus w,v'\oplus w')=\frac{\beta_V(v,v')}{\beta_W(w,w')}.$$

A *morphism of the category* **Naz** is a self-dual submodule $P\subset V\oplus W$ such that $\ker P$ and $\operatorname{indef} P$ are compact.

Group $\mathbf{Sp}(2\infty,\mathbb{Q}_p)$ of automorphisms of $V_{2\infty}$ consists of $2\infty\times 2\infty$ matrices $r=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ such that

— all but a finite number of matrix elements are integer;

— matrix elements a_{ij}, b_{ij}, c_{ij}, d_{ij} tend to 0 as $i\to\infty$ for fixed j; also they tend to 0 as $j\to\infty$ for fixed i;

— matrices are symplectic in the usual sense,

$$r^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}r=\begin{pmatrix}0&1\\-1&0\end{pmatrix}=r\begin{pmatrix}0&1\\-1&0\end{pmatrix}r^t.$$

3.11. Heisenberg groups. For the sake of simplicity, set $p>2$. Denote by $\mathbb{T}_p\subset\mathbb{C}^\times$ the group of roots of unity of degrees p, p^2, p^3,… . Let V be an object of the Nazarov category. We define the Heisenberg group $\mathrm{Heis}(V)$ as a central extension of the Abelian group V by $\mathbb{T}_p$ in the following way. As a set, $\mathrm{Heis}(V)\simeq V\times\mathbb{T}_p$. The multiplication is given by

$$(v,\lambda)\cdot(w,\mu)=\bigl(v+w,\lambda\mu\cdot\beta_V(v,w)\bigr).$$

For a finite dimensional V we define a unitary representation Ψ of $\mathrm{Heis}(V)$ in $L^2(\mathbb{Q}_p^n)$ by the formula

$$\Psi(v^+\oplus v^-,\lambda)f(x)=\lambda f(x+v^+)\exp\Bigl\{2\pi i\Bigl(\sum v_j^+x_j+\frac12\sum v_j^+v_j^-\Bigr)\Bigr\}. \tag{3.2}$$

Next, consider the space $\mathcal{E}_\infty$ consisting of sequences $z=(z_1,z_2,\dots)$ such that $|z_j|\leqslant 1$ for all but a finite number of j. This space is an Abelian locally compact group, it admits a Haar measure. On the open subgroup $\mathbb{O}_p^\infty\subset\mathcal{E}_\infty$, the Haar measure is a product of probability Haar measures on $\mathbb{O}_p$. The whole space $\mathcal{E}_\infty$ is a countable disjoint union of sets $u + \mathbb{O}_p^\infty$.

We define the representation of the group $\mathrm{Heis}(V_{2\infty})$ in $L^2(\mathcal{E}_\infty)$ by the same formula (3.2).

3.12. The Weil representation of the Nazarov category. See [12], [13], for finite-dimensional case, see [22], Chapter 11.

Theorem 3.4 *For a $2n$-dimensional object of the category* **Naz** *we assign the Hilbert space $\mathcal{H}(V) := L^2(\mathbb{Q}_p^n)$. For the object $V_{2\infty}$, we assign the Hilbert space $\mathcal{H}(V_{2\infty}) := L^2(\mathcal{E}_\infty)$.*

a) *Let V, W be objects of* **Naz**. *Let P be a a morphism of category* **Naz**. *Then there is a unique up to a scalar factor bounded operator*

$$\mathrm{We}(P) : \mathcal{H}(V) \to \mathcal{H}(W)$$

such that

$$\Psi(w,1)\mathrm{We}(P) = \mathrm{We}(P)\Psi(v,1) \quad \textit{for all } v \oplus w \in P.$$

b) *Let V, W, Y be objects of* **Naz**. *Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be morphisms of* **Naz**. *Then*

$$\mathrm{We}(Q)\mathrm{We}(P) = s \cdot \mathrm{We}(QP),$$

where $s \in \mathbb{C}^\times$ is a nonzero scalar. In other words, we get a projective representation *of the category* **Naz**. *Also,*

$$\mathrm{We}(P^\square) = t \cdot \mathrm{We}(P)^*, \qquad t \in \mathbb{C}^\times.$$

For symplectic groups $\mathrm{Sp}(2n, \mathbb{Q}_p) = \mathrm{Aut}(\mathbb{Q}_p^{2n})$ the representation $\mathrm{We}(g)$ coincides with the Weil representation.

We present explicit formulas for some morphisms. First, let $V = W$ and P be a graph of a symplectic operator. There are simple formulas for special symplectic matrices:

$$\begin{aligned} \mathrm{We}\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} f(z) &= |\det A|^{1/2} f(zA); \\ \mathrm{We}\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(z) &= \exp\{\pi i z B z^t\}; \\ \mathrm{We}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(z) &= \int_{\mathbb{Q}_p^n} f(x) \exp\{2\pi i x z^t\}\, dx. \end{aligned} \tag{3.3}$$

Any element of $\mathrm{Sp}(2n, \mathbb{Q}_p)$ can be represented as a product of matrices of such forms, this allows to write an explicit formula for any element of $\mathrm{Sp}(2n, \mathbb{Q}_p)$.

Denote by $I(x)$ the function on $\mathbb{Q}_p$ defined by

$$I(x) = \begin{cases} 1, & |x| \leqslant 1; \\ 0, & \text{otherwise}. \end{cases}$$

Let $V=\mathbb{Q}_p^{2n}$, $W=V\oplus Y$, where $Y=\mathbb{Q}_p^{2n}$ or $V_{2\infty}$. Denote by $Y(\mathbb{O}_p)$ the lattice $\mathbb{O}_p^{2n}$ or $\mathbb{O}_p^{2\infty}$ respectively. Denote by

$$\lambda_W^V : V \rightrightarrows W$$

the direct sum of the graph of the unit operator $1_V : V \to V$ and the lattice $Y(\mathbb{O}_p) \subset Y$. Then

$$\mathrm{We}(\lambda_W^V)\, f(x_1,\dots,x_n,y_1,y_2,\dots) = f(x_1,\dots,x_n)\, I(y_1)I(y_2)\dots$$

Next, denote by

$$\theta_W^V : W \rightrightarrows W$$

the direct sum

$$1_V \oplus (Y(\mathbb{O}_p)\oplus Y(\mathbb{O}_p)) \,\subset\, (V\oplus V)\oplus(Y\oplus Y).$$

Then

$$\theta_W^V = \lambda_W^V\,\bigl(\lambda_W^V\bigr)^*, \qquad \bigl(\theta_W^V\bigr)^2=\theta_W^V, \qquad \bigl(\lambda_W^V\bigr)^*\,\lambda_W^V = 1_V. \tag{3.4}$$

The operator $\mathrm{We}(\theta_W^V)$ is the orthogonal projection to the space of functions of the form

$$f(x_1,\dots,x_n)\, I(y_1)I(y_2)\dots$$

4 Characteristic function

Here we define characteristic functions of double cosets $\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$ and formulate some theorems. Proofs are in the next section.

4.1. Construction. Consider the group

$$\mathrm{GL}(\alpha+k\infty,\mathbb{Q}_p) := \lim_{j\to\infty} \mathrm{GL}(\alpha+kj,\mathbb{Q}_p).$$

Let $g\in\mathrm{GL}(\alpha+k\infty,\mathbb{Q}_p)$ actually be contained in $\mathrm{GL}(\alpha+km,\mathbb{Q}_p)$,

$$g=\begin{pmatrix} a & b_1 & \dots & b_k\\ c_1 & d_{11} & \dots & d_{1k}\\ \vdots & \vdots & \ddots & \vdots\\ c_k & d_{k1} & \dots & d_{kk}\end{pmatrix} \in \mathrm{GL}(\alpha+km,\mathbb{Q}_p). \tag{4.1}$$

We write the following equation

$$\begin{pmatrix} v^+\\ y_1^+\\ \vdots\\ y_k^+\\ v^-\\ y_1^-\\ \vdots\\ y_k^-\end{pmatrix} = \left(\begin{matrix} a & b_1 & \dots & b_k\\ c_1 & d_{11} & \dots & d_{1k}\\ \vdots & \vdots & \ddots & \vdots\\ c_k & d_{k1} & \dots & d_{kk}\\ 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\end{matrix} \quad \begin{matrix} 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 0\\ \multicolumn{4}{c}{\begin{pmatrix} a & b_1 & \dots & b_k\\ c_1 & d_{11} & \dots & d_{1k}\\ \vdots & \vdots & \ddots & \vdots\\ c_k & d_{k1} & \dots & d_{kk}\end{pmatrix}^{t-1}}\end{matrix}\right) \begin{pmatrix} u^+\\ x_1^+\\ \vdots\\ x_k^+\\ u^-\\ x_1^-\\ \vdots\\ x_k^-\end{pmatrix}. \tag{4.2}$$

Here $u^\pm$, $v^\pm \in \mathbb{Q}_p^\alpha$ and $x_j^\pm$, $y_j^\pm \in \mathbb{Q}_p^m$.

Next, we define 3 spaces, V, H, ℓ_m:

1) Denote $V := \mathbb{Q}_p^\alpha \oplus \mathbb{Q}_p^\alpha$. We regard $u = u^+ \oplus u^-$, $v = v^+ \oplus v^-$ as elements of V. Equip V with the standard skew-symmetric bilinear form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2) Denote

$$H := H^+ \oplus H^- = \mathbb{Q}_p^k \oplus \mathbb{Q}_p^k \tag{4.3}$$

and equip this space with the standard skew-symmetric bilinear form.

3) Denote by ℓ_m the space $\mathbb{Q}_p^m$ equipped with the standard symmetric bilinear form $(z, w) = \sum z_j w_j$. We regard $x_j^\pm$, $y_j^\pm$ as elements of this space.

Consider the tensor product $H \otimes_{\mathbb{Q}_p} \ell_m$, vectors

$$\begin{pmatrix} x_1^+ & \dots & x_k^+ & x_1^- & \dots & x_k^- \end{pmatrix}, \quad \begin{pmatrix} y_1^+ & \dots & y_k^+ & y_1^- & \dots & y_k^- \end{pmatrix}$$

are regarded as elements of $H \otimes \ell_m$. We equip $H \otimes \ell_m$ with the tensor product of bilinear forms, this form is a skew-symmetric with matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1_m & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1_m \\ -1_m & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -1_m & 0 & \dots & 0 \end{pmatrix}.$$

Thus the operator in (4.2) is an operator

$$V \oplus (H \otimes \ell_m) \quad \to \quad V \oplus (H \otimes \ell_m)$$

preserving the skew-symmetric bilinear form.

For any self-dual module $Q \subset H$ we consider the self-dual module

$$Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^m \subset H \otimes \ell_m.$$

Notice, that $Q \otimes \mathbb{O}_p^m$ is a direct sum of m copies of Q.

Definition 4.1 *Fix self-dual submodules Q, $T \subset H$. We define a relation*

$$\chi_g(Q, T) : V \rightrightarrows V$$

as the set of all $u \oplus v \in V \oplus V$ for which there exist $x \in Q \otimes \mathbb{O}_p^m$, $y \in T \otimes \mathbb{O}_p^m$ such that (4.2) *holds.*

Definition 4.2 *We say that some property of a double coset* holds in a general position *if for every sufficiently large m the set of points $g \in \mathrm{GL}(\alpha + km, \mathbb{Q}_p)$, where the property does not hold, is a proper algebraic submanifold in* $\mathrm{GL}(\alpha + km, \mathbb{Q}_p)$.

4.2. Basic properties of characteristic functions.

Lemma 4.3 *$\chi_g(Q,T)$ does not depend on a choice of m.*

Theorem 4.4 *If g_1, g_2 are contained in the same double coset $\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$, then $\chi_{g_1}(Q,T)=\chi_{g_2}(Q,T)$.*

Thus, for any double coset $\mathfrak{g}\in\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$ we get a well-defined map

$$\chi_\mathfrak{g}:\ \mathrm{LMod}(H)\times\mathrm{LMod}(H)\quad\to\quad\Bigl\{\text{space of relations } V\rightrightarrows V\Bigr\}.$$

Therefore, we can write

$$\chi_\mathfrak{g}(Q,T),\qquad \text{where } \mathfrak{g}\in\mathbf{K}\setminus\mathbf{G}/\mathbf{K}.$$

We say that $\chi_\mathfrak{g}(\cdot,\cdot)$ is the *characteristic function* of the double coset $\mathfrak{g}$.

Theorem 4.5 $\chi_\mathfrak{g}(Q,T)\in\overline{\mathrm{Naz}}(V,V)$.

Theorem 4.6 *The following identity holds*

$$\chi_{\mathfrak{g}\star\mathfrak{h}}(Q,T)=\chi_\mathfrak{g}(Q,T)\,\chi_\mathfrak{h}(Q,T),$$

in the right-hand side we have a product of relations

4.3. Refinement of Theorem 4.5. Fix a double coset $\mathfrak{g}$. Substituting $x^\pm=0$, $y^\pm=0$ to the equation (4.2), we get an equation for $u\oplus v\in V\oplus V$. The explicit form (see equation (5.3)) is

$$\begin{cases} v^+ = au^+ \\ 0 \;\;\, = c_j u^+, & \text{for all } j\\ u^- = a^t v^- \\ 0 \;\;\, = b_j^t v^-, & \text{for all } j\end{cases}$$

Denote by $\Lambda(\mathfrak{g})\subset V\oplus V$ the linear subspace of solutions of this system. Notice that

$$\ker\Lambda(\mathfrak{g})=0,\qquad \operatorname{indef}\Lambda(\mathfrak{g})=0$$

(since g is an invertible matrix).

For $\mathfrak{g}$ being in a general position $\Lambda(\mathfrak{g})=0$.

Proposition 4.7 a) *For any self-dual Q, $T\in\mathrm{LMod}(H)$,*

$$\chi_\mathfrak{g}(Q,T)_\downarrow\supset\Lambda(\mathfrak{g}),\qquad \chi_\mathfrak{g}(Q,T)^\uparrow\subset\Lambda(\mathfrak{g})^\perp.$$

b) *If Q, T are self-dual lattices, then*

$$\chi_\mathfrak{g}(Q,T)_\downarrow=\Lambda(\mathfrak{g}),\qquad \chi_\mathfrak{g}(Q,T)^\uparrow=\Lambda(\mathfrak{g})^\perp.$$

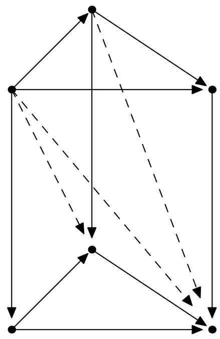


Figure 2: A reference to Subsection 4.5. A product of two simplices and additional arrows.

Corollary 4.8 *For $\mathfrak{g}$ being in a general position, we get a map*

$$\mathrm{LLat}(H) \times \mathrm{LLat}(H) \to \mathrm{LLat}(V \oplus V).$$

4.4. Values of characteristic functions on the distinguished boundary.

Theorem 4.9 *Let Q, T range in the Lagrangian Grassmannian* $\mathrm{LGr}(H)$*. Then*

a) $\chi_{\mathfrak{g}}(Q,T)$ *is a Lagrangian subspace in $V \oplus V$.*

b) *The map*

$$\chi_{\mathfrak{g}} : \mathrm{LGr}(H) \times \mathrm{LGr}(H) \to \mathrm{LGr}(V \oplus V)$$

is rational.

c) *For $\mathfrak{g}$ being in a general position, $\chi_{\mathfrak{g}}(Q,T) \in \mathrm{Sp}(V, \mathbb{Q}_p)$ a.s. on* $\mathrm{LGr}(H) \times \mathrm{LGr}(H)$.

A precise description of the subset of $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$, where the last property holds, is given below in Subsection 5.8.

There is a more exotic statement in the same spirit.

Proposition 4.10 *For all $\mathfrak{g}$ for almost all $(Q,T) \in \mathrm{LGr}(H) \times \mathrm{LGr}(H)$, the condition $(u^+ \oplus u^-) \oplus (v^+ \oplus v^-) \in \chi_{\mathfrak{g}}(Q,T)$ can be written as an equation*

$$\begin{pmatrix} v^+ \\ u^- \end{pmatrix} = Z(Q,T) \begin{pmatrix} v^- \\ u^+ \end{pmatrix}$$

there $Z(Q,T)$ is a symmetric matrix.

4.5. Extension of characteristic function to buildings. Next, we can apply the definition of characteristic function to almost self-dual lattices Q, T.

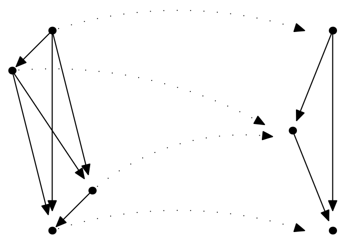


Figure 3: A reference to Subsection 4.5. A morphism of oriented graphs

Proposition 4.11 *If Q, T are self-dual modules, then $\chi_{\mathfrak{g}}(Q,T)$ is almost self-dual.*

Now we construct an oriented graph $\Delta(H \bowtie H)$. Vertices are ordered pairs (Q,T) of almost self-dual lattices in H. We draw an arrow from (Q,T) to (Q',T') if $Q \supset Q'$, $T \supset T'$.

Consider the product of simplicial complexes $\mathrm{Bd}(H) \times \mathrm{Bd}(H)$. It is polyhedral complex, whose strata are products of simplices. Two vertices (Q,T) and (Q',T') are connected by an arrow if $Q \supset Q'$ and $T = T'$ or $Q = Q'$ and $T \supset T'$. However, we have draw more arrows, this provides a simplicial partition of each product of simplices (see, e.g., [5]). Finally, we get a $2k$-dimensional simplicial complex $\mathrm{Bd}(H \bowtie H)$ (it also is a subcomplex of the complex $\mathrm{Bd}(H \oplus H)$).

Let Φ, Ψ be two oriented graphs, assume that number of edges connecting any pair of vertices is $\leqslant 1$. We say that a map $\sigma : \mathrm{Vert}(\Phi) \to \mathrm{Vert}(\Psi)$ is a *morphism of graphs* if for any arrow $a \to b$ in Φ we have $\sigma(a) = \sigma(b)$ or there is an arrow $\sigma(a) \to \sigma(b)$.

Theorem 4.12 *A characteristic function $\chi_{\mathfrak{g}}$ is a morphism of oriented graphs*

$$\Delta(H \bowtie H) \to \Delta(V \oplus V). \tag{4.4}$$

Remark. Let $a_1, \dots a_l$ be vertices of a simplex of the complex $\mathrm{Bd}(H \bowtie H)$. Therefore their images $c_1, \dots c_l$ are contained in one simplex of the complex $\mathrm{Bd}(V \oplus V)$ (some vertices can coincide). Therefore we can extend our map to affine map of simplices. In this way we get a piece-wise affine map of simplicial complexes

$$\mathrm{Bd}(H \bowtie H) \to \mathrm{Bd}(V \oplus V).$$

4.6. Continuity.

Theorem 4.13 *Let Q_j, Q, T_j, T be almost self-dual modules. If $Q_j \nearrow Q$, $T_j \nearrow T$, then*

$$\chi_{\mathfrak{g}}(Q_j, T_j) \nearrow \chi_{\mathfrak{g}}(Q,T).$$

Notice that characteristic function can be discontinuous with respect to the Hausdorff convergence. Moreover, the restriction of $\chi_{\mathfrak{g}}$ to $\mathrm{LGr}(H) \times \mathrm{LGr}(H)$ can be discontinuous in the topology of Grassmannian.

4.7. Involution.

Proposition 4.14 *If $u \oplus v \in \chi_{\mathfrak{g}}(Q,T)$, then $v \oplus u \in \chi_{\mathfrak{g}^*}(T,Q)$.*

4.8. Additional symmetry. For a nonzero $\lambda \in \mathbb{Q}_p^\times = \mathbb{Q}_p \setminus 0$, we define an operator $M(\lambda)$ in H given by $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$, by the same symbol we denote the operator $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$ in the space V.

Theorem 4.15

$$\chi_{\mathfrak{g}}\bigl(M(\lambda)Q, M(\lambda)T\bigr) = M(\lambda^{-1})\chi_{\mathfrak{g}}(Q,T)M(\lambda).$$

4.9. Another semigroup of double cosets. Now consider the group $\widetilde{\mathbf{G}} = \mathrm{Sp}(2\alpha + 2k\infty, \mathbb{Q}_p)$ of symplectic matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, consider its subgroup $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ consisting of matrices $\begin{pmatrix}g & 0\\ 0 & g^{t-1}\end{pmatrix}$, consider the same $\mathbf{K} = \mathrm{O}(\infty, \mathbb{O}_p) \subset \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$. Consider the semigroup of double cosets $\mathbf{K}\backslash\widetilde{\mathbf{G}}/\mathbf{K}$. We define characteristic function $\chi_{\widetilde{\mathfrak{g}}}(Q,T)$ in the same way, in formula (4.2) instead the matrix $\begin{pmatrix}g & 0\\ 0 & g^{t-1}\end{pmatrix}$ we write a symplectic matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{Sp}(2\alpha + 2k\infty, \mathbb{Q}_p)$.

Theorem 4.16 *All the statements of this section hold for $\chi_{\widetilde{\mathfrak{g}}}(Q,T)$ except Theorem* 4.15.

5 Proofs

5.1. Independence of representatives. To shorten expressions, set $k = 2$. Let $h \in \mathrm{O}(m, \mathbb{O}_p)$, let $\mathfrak{I}(h)$ be given by (2.1). Then characteristic function of $g\mathfrak{I}(h)$ is determined by

$$\begin{pmatrix} v^+\\ y_1^+\\ y_2^+\\ v^-\\ y_1^-\\ y_2^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b_1h & b_2h\\ c_1 & d_{11}h & d_{12}h\\ c_2 & d_{21}h & d_{22}h \end{matrix} & \begin{matrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{matrix}\\ \begin{matrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} a & b_1h & b_2h\\ c_1 & d_{11}h & d_{12}h\\ c_2 & d_{21}h & d_{22}h \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} u^+\\ x_1^+\\ x_2^+\\ u^-\\ x_1^-\\ x_2^- \end{pmatrix}.$$

or

$$\begin{pmatrix} v^+\\ y_1^+\\ y_2^+\\ v^-\\ y_1^-\\ y_2^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b_1 & b_2\\ c_1 & d_{11} & d_{12}\\ c_2 & d_{21} & d_{22} \end{matrix} & \begin{matrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{matrix}\\ \begin{matrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{matrix} & \begin{pmatrix} a & b_1 & b_2\\ c_1 & d_{11} & d_{12}\\ c_2 & d_{21} & d_{22} \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} u^+\\ hx_1^+\\ hx_2^+\\ u^-\\ hx_1^-\\ hx_2^- \end{pmatrix}.$$

We introduce new variables $\widetilde{x}_1^\pm = hx_1^\pm$, $\widetilde{x}_2^\pm = hx_2^\pm$ and come to the equation for χ_g. Notice that modules $Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^m$ are invariant with respect to $\mathrm{O}(m, \mathbb{O}_p)$.

5.2. Reformulation of definition. The equation (4.2) determines a linear subspace in

$$\Big(V \oplus (H \otimes \ell_m)\Big) \oplus \Big(V \oplus (H \otimes \ell_m)\Big).$$

We regard it as a linear relation

$$\xi : \Big((H \otimes \ell_m) \oplus (H \otimes \ell_m)\Big) \rightrightarrows \big(V \oplus V\big).$$

Then $\chi_{\mathfrak{g}}$ is the image of the submodule

$$\eta_{Q,T} = (Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^m) \oplus (T \otimes_{\mathbb{O}_p} \mathbb{O}_p^m)$$

under ξ.

5.3. Immediate corollaries. The relation ξ is a morphism of the category $\overline{\mathrm{Naz}}$. A module $\eta_{Q,T}$ is self-dual. Therefore $\xi\,\eta_{Q,T}$ is almost self-dual Theorem 4.5 is proved.

The same argument implies Theorem 4.9.a and Proposition 4.11.

Also Lemma 4.3 became obvious.

5.4. Continuity (Theorem 4.13). We calculate the product $\xi\,\eta_{Q,T}$ according Subsection 3.7. By Lemma 3.2 both steps of calculation are continuous.

5.5. Products. Proof of Theorem 4.6. To shorten notation, set $k = 2$. Let

$$g = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} \in \mathrm{GL}(\alpha+2l, \mathbb{Q}_p), \qquad h = \begin{pmatrix} a' & b_1' & b_2' \\ c_1' & d_{11}' & d_{12}' \\ c_2' & d_{21}' & d_{22}' \end{pmatrix} \in \mathrm{GL}(\alpha+2m, \mathbb{Q}_p).$$

Let $v \oplus w \in \chi_{\mathfrak{g}}(Q,T)$, $u \oplus v \in \chi_{\mathfrak{h}}(Q,T)$. Then there are $x \in Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^m$, $y \in T \otimes_{\mathbb{O}_p} \mathbb{O}_p^m$ such that

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ y_1^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} a' & b_1' & b_2' & 0 & 0 & 0 \\ c_1' & d_{11}' & d_{12}' & 0 & 0 & 0 \\ c_2' & d_{21}' & d_{22}' & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{pmatrix} a' & b_1' & b_2' \\ c_1' & d_{11}' & d_{12}' \\ c_2' & d_{21}' & d_{22}' \end{pmatrix}^{t-1}$}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ u^- \\ x_1^- \\ x_2^- \end{pmatrix}. \tag{5.1}$$

Also there are $X \in Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^l$, $Y \in T \otimes_{\mathbb{O}_p} \mathbb{O}_p^l$ such that

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ Y_2^+ \\ w^- \\ Y_1^- \\ Y_2^- \end{pmatrix} = \begin{pmatrix} a & b_1 & b_2 & 0 & 0 & 0 \\ c_1 & d_{11} & d_{12} & 0 & 0 & 0 \\ c_2 & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix}^{t-1}$}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v^+ \\ X_1^+ \\ X_2^+ \\ v^- \\ X_1^- \\ X_2^- \end{pmatrix}. \tag{5.2}$$

We write (5.2) as

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ y_1^+ \\ Y_2^+ \\ y_2^+ \\ w^- \\ Y_1^- \\ y_1^- \\ Y_2^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} v^+ \\ X_1^+ \\ y_1^+ \\ X_2^+ \\ y_2^+ \\ v^- \\ X_1^- \\ y_1^+ \\ X_2^- \\ y_2^- \end{pmatrix}.$$

Applying (5.1) we come to

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ y_1^+ \\ Y_2^+ \\ y_2^+ \\ w^- \\ Y_1^- \\ y_1^- \\ Y_2^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{t-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \begin{matrix} a' & 0 & b_1' & 0 & b_2' \\ 0 & 1 & 0 & 0 & 0 \\ c_1' & 0 & d_{11}' & 0 & d_{12}' \\ 0 & 0 & 0 & 1 & 0 \\ c_2' & 0 & d_{21}' & 0 & d_{22}' \end{matrix} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} & \begin{pmatrix} a' & 0 & b_1' & 0 & b_2' \\ 0 & 1 & 0 & 0 & 0 \\ c_1' & 0 & d_{11}' & 0 & d_{12}' \\ 0 & 0 & 0 & 1 & 0 \\ c_2' & 0 & d_{21}' & 0 & d_{22}' \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} u^+ \\ X_1^+ \\ x_1^+ \\ X_2^+ \\ x_2^+ \\ u^- \\ X_1^- \\ x_1^+ \\ X_2^- \\ x_2^- \end{pmatrix}$$

Now

$$X \oplus x \in Q \otimes (\mathbb{O}_p^l \oplus \mathbb{O}_p^m), \qquad Y \oplus y \in T \otimes (\mathbb{O}_p^l \oplus \mathbb{O}_p^m),$$

and we get $u \oplus w \in \chi_{\mathfrak{g}\star\mathfrak{h}}(Q,T)$. Thus,

$$\chi_{\mathfrak{g}\star\mathfrak{h}}(Q,T) \supset \chi_{\mathfrak{g}}(Q,T)\chi_{\mathfrak{h}}(Q,T).$$

But both sides are self-dual, therefore they coincide.

5.6. Morphism of graphs (Theorem 4.12). Consider the map

$$\mathrm{LMod}(H) \times \mathrm{LMod}(H) \to \mathrm{LMod}(H \otimes \ell_m) \times \mathrm{LMod}(H \otimes \ell_m)$$

given by $(Q,T) \mapsto (Q \otimes_{\mathbb{O}_p} \mathbb{O}_p^m, P \otimes_{\mathbb{O}_p} \mathbb{O}_p^m)$.

Lemma 5.1 *This map is a morphism of graphs*

$$\Delta(H \bowtie H) \to \Delta\big((H \otimes \ell_m) \bowtie (H \otimes \ell_m)\big).$$

This statement is obvious.

Next, we have an embedding of complexes

$$\mathrm{Bd}\big((H\otimes\ell_m)\bowtie(H\otimes\ell_m)\big)\to\mathrm{Bd}\big((H\otimes\ell_m)\oplus(H\otimes\ell_m)\big).$$

On the other hand, the linear relation ξ is a morphism of the category Naz. Therefore it induces a morphism of graphs $\Delta\big((H\otimes\ell_m)\oplus(H\otimes\ell_m)\big)\to\Delta(V\oplus V)$, see [22], Proposition 10.7.6.

5.7. Proof of Proposition 4.7. We have

$$\operatorname{indef}\xi=\Lambda(\mathfrak{g}).$$

Therefore $\Lambda(\mathfrak{g})\subset\xi\,\eta_{Q,T}\subset\Lambda(\mathfrak{g})^{\perp}$. This is the statement a) of Proposition 4.7.

Also, if R is a relation $V\rightrightarrows W$, $Y\subset V$ is a lattice, then $(RY)_{\downarrow}=(\operatorname{indef}R)_{\downarrow}$. This implies b).

5.8. Values on the distinguished boundary. Now let Q, T be Lagrangian subspaces in H.

First, we prove Proposition 4.10. A Lagrangian subspace $Q\subset H$ of general position is a graph of an operator $H^+\to H^-$, and matrix of this operator is symmetric (see, e.g., [22], Theorem 3.1.4). To shorten notation, set $k=2$. The equation (4.2) can be written in the form

$$\begin{pmatrix}v^+\\ y_1^+\\ y_2^+\\ u^-\\ t_{11}x_1^++t_{12}x_2^+\\ t_{12}x_1^++t_{22}x_2^+\end{pmatrix}\begin{pmatrix}a&b_1&b_2&0&0&0\\ c_1&d_{11}&d_{12}&0&0&0\\ c_2&d_{21}&d_{22}&0&0&0\\ 0&0&0&a^t&c_1^t&c_2^t\\ 0&0&0&b_1^t&d_{11}^t&d_{21}^t\\ 0&0&0&b_2^t&d_{12}^t&d_{22}^t\end{pmatrix}\begin{pmatrix}u^+\\ x_1^+\\ x_2^+\\ v^-\\ q_{11}y_1^++q_{12}y_2^+\\ q_{12}y_1^++q_{22}y_2^+\end{pmatrix},\tag{5.3}$$

We denote

$$\varkappa:=\begin{pmatrix}q_{11}&q_{12}\\ q_{12}&q_{22}\end{pmatrix},\qquad \tau:=\begin{pmatrix}t_{11}&t_{12}\\ t_{12}&t_{22}\end{pmatrix}$$

and write (5.3) as

$$v^+=au^++bx^+\tag{5.4}$$
$$y^+=cu^++dx^+\tag{5.5}$$
$$u^-=a^tv^-+c^t\varkappa y^+\tag{5.6}$$
$$\tau x^+=b^tv^-+d^t\varkappa y^+.\tag{5.7}$$

We regard lines (5.5),(5.7) as a system of equations for x^+, y^+. The matrix of the system is

$$\Omega(\varkappa,\tau)=\begin{pmatrix}-d&1\\ \tau&-d^t\varkappa\end{pmatrix}.$$

Evidently, the polynomial $\det\Omega(\varkappa,\tau)$ is not zero. Indeed, fix $\varkappa$ and take $\tau=p^{-N}\cdot 1$. If N is sufficiently large, then the determinant is $\neq 0$. Thus, outside the hypersurface

$$\det\Omega(\varkappa,\tau)=0$$

we can express x^+ and y^+ as functions of u^+, v^-. After substitution of x^+, y^+ to (5.4),(5.6), we get a dependence of u^-, v^+ in u^+, v^-.

This also proves Theorem 4.9.b (rationality of characteristic function).

Next, we prove Theorem 4.9.c. Denote

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and write the equation (4.2) in the form

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ q_{11}y_1^+ + q_{12}y_2^+ \\ q_{12}y_1^+ + q_{22}y_2^+ \end{pmatrix} \begin{pmatrix} a & b_1 & b_2 & 0 & 0 & 0 \\ c_1 & d_{11} & d_{12} & 0 & 0 & 0 \\ c_2 & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & A^t & C_1^t & C_2^t \\ 0 & 0 & 0 & B_1^t & D_{11}^t & D_{21}^t \\ 0 & 0 & 0 & B_2^t & D_{12}^t & D_{22}^t \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ u^- \\ t_{11}x_1^+ + t_{12}x_2^+ \\ t_{12}x_1^+ + t_{22}x_2^+ \end{pmatrix},$$

or

$$v^+ = au^+ + bx^+ \tag{5.8}$$

$$y^+ = cu^+ + dx^+ \tag{5.9}$$

$$v^- = A^t u^- + C^t \tau x^+ \tag{5.10}$$

$$y_+ = B^t u^- + D^t \tau x^+. \tag{5.11}$$

We consider lines (5.9), (5.11) as equations for y^+, x^+. The matrix of the system is

$$\Xi(\varkappa, \tau) = \begin{pmatrix} 1 & -d \\ \varkappa & -D^t\tau \end{pmatrix}.$$

Its determinant equals

$$\det \Xi(\varkappa, \tau) = \det(-D^t\tau + \varkappa d)$$

If it is nonzero, we get a linear operator $u \mapsto v$. We come to the following statement

Proposition 5.2 *If there exists a pair of symmetric matrices $\varkappa$, τ such that* $\det(-D^t\tau + \varkappa d) \neq 0$, *then* $\chi_{\mathfrak{g}}(Q, T) \in \mathrm{Sp}(V, \mathbb{Q}_p)$ *a.s. on* $\mathrm{LGr}(H) \times \mathrm{LGr}(H)$.

5.9. Involution. Proof of Proposition 4.14. We write the defining relation for $\chi_{g^{-1}}$,

$$\begin{pmatrix} v^+ \\ y^+ \\ v^- \\ y^- \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \end{pmatrix} \begin{pmatrix} u^+ \\ x^+ \\ u^- \\ x^- \end{pmatrix},$$

represent this in the form

$$\begin{pmatrix} u^+ \\ x^+ \\ u^- \\ x^- \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1}} \\ 0 & 0 & & \end{pmatrix} \begin{pmatrix} v^+ \\ y^+ \\ v^- \\ y^- \end{pmatrix}$$

and come to desired statement.

5.10. Proof of Theorem 4.15. We write (4.2) as

$$\begin{pmatrix} u^+ \\ x^+ \\ u^- \\ x^- \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & & & \\ & \lambda^{-1} & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix} \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1}} \\ 0 & 0 & & \end{pmatrix} \begin{pmatrix} \lambda & & & \\ & \lambda & & \\ & & \lambda^{-1} & \\ & & & \lambda^{-1} \end{pmatrix} \begin{pmatrix} v^+ \\ y^+ \\ v^- \\ y^- \end{pmatrix}$$

or

$$\begin{pmatrix} \lambda u^+ \\ \lambda x^+ \\ \lambda^{-1} u^- \\ \lambda^{-1} x^- \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1}} \\ 0 & 0 & & \end{pmatrix} \begin{pmatrix} \lambda v^+ \\ \lambda y^+ \\ \lambda^{-1} v^- \\ \lambda^{-1} y^- \end{pmatrix}$$

5.11. Another reformulation of the definition. Consider the space $W = V \oplus (H \otimes \ell_m)$. For any self-dual submodule $Q \subset H$, consider the linear relation $\Lambda : V \rightrightarrows W$ defined by

$$\Lambda_Q = 1_V \oplus (Q \otimes \mathbb{O}_m) \subset (V \oplus V) \oplus (Q \otimes \ell_m).$$

Then $\chi_{\mathfrak{g}}$ is a product of linear relations

$$\chi_{\mathfrak{g}}(Q, T) = (\Lambda_T)^{\square} \begin{pmatrix} g & 0 \\ 0 & g^{t-1} \end{pmatrix} \Lambda_Q.$$

6 Multiplicativity theorem

Theorem 2.2 (multiplicativity theorem) formulated above is a representative of wide class of theorems, their proofs are standard, below we refer to proofs [16], Chapter VIII.

6.1. Corners of orthogonal matrices.

Lemma 6.1 *Let A be a $m \times m$ matrix with elements $\in \mathbb{O}_p$. Then there exists N and a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{O}(m+N, \mathbb{O}_p)$.*

Proof. Denote by $\mathbf{B}_m$ the set of all possible $m \times m$ left upper corners of matrices $g \in \mathrm{O}(\infty, \mathbb{O}_p)$.

1) The set $\mathbf{B}_m$ is closed with respect to matrix products. Indeed, let

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{O}(m+N, \mathbb{O}_p), \qquad \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \mathrm{O}(m+N', \mathbb{O}_p).$$

Then

$$\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A' & 0 & B' \\ 0 & 1 & 0 \\ C' & 0 & D' \end{pmatrix} = \begin{pmatrix} AA' & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \in \mathrm{O}(m+N+N', \mathbb{O}_p).$$

2) If $A \in \mathbf{B}_m$, $A' \in \mathbf{B}_n$, then $\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \in \mathbf{B}_{m+n}$.

3) It is more-or-less clear that for any $z \in \mathbb{O}_p$ we have

$$\begin{pmatrix} z \end{pmatrix} \in \mathbf{B}_1, \qquad \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in \mathbf{B}_2.$$

4) $\mathbf{B}_m$ contains matrices of permutations.

Now we can produce any matrix with integer elements. □

6.2. Admissible representations. Denote by $\mathbf{K}_m$ the subgroup in $\mathbf{K}$ consisting of matrices of the form $\begin{pmatrix} 1_m & 0 \\ 0 & * \end{pmatrix}$.

Let τ be a unitary representation of $\mathbf{K}$ in a Hilbert space H. Denote by $H(m)$ the subspace of $\mathbf{K}_m$-fixed vectors. Denote by $P(m)$ the operator of orthogonal projection to $H(m)$. We say, that τ is *admissible* if $\cup_m H(m)$ is dense in H.

We say, that a representation of $\mathbf{G}$ is $\mathbf{K}$-*admissible* if its restriction to $\mathbf{K}$ is admissible.

6.3. Continuation of representations. Denote by $\mathbf{B}_\infty$ the semigroup of all infinite matrices A such that:

a) $a_{ij} \in \mathbb{O}_p$;

b) for each i the sequence a_{ij} tends to 0 as $j \to \infty$; for each j the sequence a_{ij} tends to 0 as $i \to \infty$.

We say that a sequence of matrices $A^{(j)} \in \mathbf{B}_\infty$ weakly converges to A if we have convergence of each matrix element, $a_{kl}^{(j)} \to a_{kl}$.

Denote by $\mathbf{O}(\infty, \mathbb{O}_p)$ the group of all orthogonal matrices $\in \mathbf{B}_\infty$.

Lemma 6.2 *The group* $\mathrm{O}(\infty, \mathbb{O}_p)$ *is dense in* $\mathbf{O}(\infty, \mathbb{O}_p)$ *and in* $\mathbf{B}_\infty$.

PROOF. Let $S \in \mathbf{B}_\infty$. Consider its left upper corner of size $m \times m$. Consider $g_m \in \mathbb{O}(\infty, \mathbb{O}_p)$ having the same left upper corner. Then g_m weakly converges to S, □

Theorem 6.3 a) *Let τ be a unitary representation of* $\mathbf{K} = \mathrm{O}(\infty, \mathbb{O}_p)$. *The following conditions are equivalent:*

— *τ is admissible;*

— *τ admits a weakly continuous extension to the group* $\mathbf{O}(\infty, \mathbb{O}_p)$*;*

— *τ admits a weakly continuous extension to a representation $\widetilde{\tau}$ of the semigroup $\mathbf{B}_\infty$ such that $\widetilde{\tau}(A^t) = \widetilde{\tau}(A)^*$, $\|\widetilde{\tau}(A)\| \leqslant 1$ for all A.*

b) *For an admissible representation τ,*

$$P(m) = \widetilde{\tau}\begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a statement in the spirit of [24]. We omit a proof, since it is a one-to-one repetition of proof of [16], Theorem VIII.1.4 about symmetric groups (admissibility implies semigroup continuation), the only new detail is Lemma 6.1). Admissibility follows from continuity by [16], Proposition VIII.1.3.

Corollary 6.4 *Denote*

$$\Theta_N^{(m)} = \begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$

The projector $P(m)$ is a weak limit of the sequence

$$P(m) = \lim_{N\to\infty} \tau(\Theta_N^{(m)}). \tag{6.1}$$

Proof. The sequence $\Theta_N^{(m)} \in \mathbb{O}(\infty, \mathbb{O}_p)$ weakly converges to the matrix $\begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{B}_\infty$. We refer to the statement b) of the theorem. $\square$

6.4. Proof of Theorem 2.2. We keep the notation of Subsection 2.3. Let $v \in H^{\mathbf{K}}$, $g \in G_j = \mathrm{GL}(\alpha + km, \mathbb{Q}_p)$, let $q \in \mathbf{K}_j$. Then

$$\rho(q)\rho(g)v = \rho(g)\rho(q)h = \rho(g)h,$$

i.e., $v \in H(j)$. Thus the subspace $\cup_j H(j)$ is $\mathbf{G}$-invariant. Its closure is an admissible representation of $\mathbf{G}$. In $\bigl(\cup_j H(j)\bigr)^\perp$ Theorem 2.2 holds by a trivial reason (the space of fixed vectors $\mathbf{K}$ is zero).

Thus, without loss of generality we can assume that ρ is admissible.

Now let g, $h \in \mathbf{G}$, let $\mathfrak{g}$, $\mathfrak{h} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ be the corresponding double cosets. Let $P = P(0)$ be the projector to $\mathbf{K}$-fixed vectors. Applying Corollary 6.4, we obtain

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(h) = P\rho(g)P\rho(h) = \lim_{N\to\infty} P\rho(g)\rho(\mathfrak{J}(\Theta_N^{(0)}))\rho(h) = \lim_{N\to\infty} P\rho(g\mathfrak{J}(\Theta_N)h),$$

here $\mathfrak{J} : \mathbf{K} \to \mathbf{G}$ is the embedding (2.1). By the definition ($\Theta_N^{(0)}$ is Θ_N from Subsection 2.3), we get $\overline{\rho}(\mathfrak{g} \star \mathfrak{h})$.

6.5. Variation of construction. Train. We can define multiplication of double cosets

$$\mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q \times \mathbf{K}_q \setminus \mathbf{G}/\mathbf{K}_r \to \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_r,$$

in definition of product of double cosets (Subsection 2.2), we simply change Θ_N by $\Theta_N^{(q)}$. An explicit formula of the product is the same (2.3). Thus we get a category (*train $\mathcal{T}(\mathbf{G}, \mathbf{K})$ of the pair* $(\mathbf{G}, \mathbf{K})$).

Next, for any unitary representation ρ of the group $\mathfrak{G}$, a double coset $\mathfrak{g} \in \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q$ determines an operator $\overline{\rho}(\mathfrak{g}) : H(r) \to H(q)$ by the formula

$$\overline{\rho}(g) := P(q)\rho(g), \qquad g \in \mathfrak{g}.$$

For any

$$\mathfrak{g} \in \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q \qquad \mathfrak{h} \in \mathbf{K}_q \setminus \mathbf{G}/\mathbf{K}_r,$$

the following identity holds

$$\rho(\mathfrak{g})\rho(\mathfrak{h}) = \rho(\mathfrak{g} \star \mathfrak{h}),$$

i.e., we get a representation of the category $\mathcal{T}(\mathbf{G}, \mathbf{K})$.

Also it can be shown that this construction is a bijection between the set of $\mathbf{K}$-admissible $*$-representations of $\mathbf{G}$, the proof is the same as in [20].

Also a construction of characteristic functions and their properties survive for double cosets $\mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q$.

7 Representations of G

7.1. Existence of representations. Let

$$\begin{pmatrix} a & b_1 & \dots & b_k \\ c_1 & d_{11} & \dots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{k1} & \dots & d_{kk} \end{pmatrix} \in \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p).$$

Consider embedding $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p) \to \mathrm{Sp}(2(\alpha + k\infty), \mathbb{Q}_p)$ given by

$$\iota : g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{t-1} \end{pmatrix}.$$

For any

$$r = \begin{pmatrix} r_{11} & \dots & r_{1\,2n} \\ \vdots & \ddots & \vdots \\ r_{2n\,1} & \dots & r_{2n\,2n} \end{pmatrix} \in \mathrm{Sp}(2k, \mathbb{Q}_p)$$

consider the matrix $\sigma(r) = 1_{2\alpha} \oplus (r \otimes 1_\infty)$,

$$\sigma(r) := \begin{pmatrix} 1_\alpha & 0 & \dots & 0 & 0 \\ 0 & r_{11} \cdot 1_\infty & \dots & 0 & r_{1k} \cdot 1_\infty \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1_\alpha & 0 \\ 0 & r_{11} \cdot 1_\infty & \dots & 0 & r_{1k} \cdot 1_\infty \end{pmatrix}$$

This matrix is not contained in $\mathrm{Sp}(2(\alpha + k\infty), \mathbb{Q}_p)$, because it is not finite. However, the map

$$q \mapsto \sigma(r^{-1})\, q\, \sigma(r) \tag{7.1}$$

is an automorphism of $\mathrm{Sp}(2(\alpha+k\infty),\mathbb{Q}_p)$. Emphasize that this automorphism fixes the subgroup $\mathbf{K}=\mathrm{O}(\infty,\mathbb{O}_p)$.

We consider the representation $\rho(r)$ of $\mathrm{GL}(\alpha+k\infty,\mathbb{Q}_p)$ given by the formula

$$\rho_r(g)=\mathrm{We}\bigl(\sigma(r^{-1})\iota(g)\sigma(r)\bigr),$$

where $\mathrm{We}(\cdot)$ is the Weil representation, see Subsection 3.12.

Recall that the Weil representation is projective.

Lemma 7.1 *The representation ρ_r is equivalent to a linear representation, i.e., there is a function* (a trivializer) $\gamma:\mathbf{G}\to\mathbb{C}^\times$ *such that $\gamma(g)\rho_r(g)$ is a linear representation. Moreover, we can choose $\gamma(g)=1$ on* $\mathrm{O}(\infty,\mathbb{O}_p)$.

PROOF. First, the restriction of the Weil representation of $\mathrm{Sp}(2n,\mathbb{Q}_p)$ to $\mathrm{GL}(n,\mathbb{Q}_p)$ is linear, see (3.3). Therefore, restricting the Weil representation to each finite-dimensional group $G_j=\mathrm{GL}(\alpha+kj,\mathbb{Q}_p)$ we get a representation equivalent to a linear representation (for finite-dimensional groups the automorphism (7.1) is inner). Denote by $\gamma_j(g)$ the trivializer for G_j. Ratio $\gamma(g)_j/\gamma(g)_{j+1}$ of two trivializers is a character $G_j\to\mathbb{C}^\times$. All characters of $G_j\to\mathbb{C}^\times$ has the form $\varphi(\det h)$, where φ is a character $\mathbb{Q}^\times\to\mathbb{C}^\times$. Correcting $\gamma_{j+1}(g)\mapsto\gamma_{j+1}(g)\psi(\det g)$, we can assume that $\gamma_{j+1}(g)=\gamma_j(g)$ on G_j.

In this way we choose a trivializer γ on the whole group $\mathbf{G}$. Restriction of γ to $\mathrm{O}(\infty,\mathbb{O}_p)$ must be a character on $\mathrm{O}(\infty,\mathbb{O}_p)\to\mathbb{C}^\times$. The only non-trivial character is $\det(u)$. We change the trivializer $\gamma(g)$ to $\det(g)\gamma(g)$. □

Lemma 7.2 *In the model of Subsection* 3.12, *the subspace $L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}}$ of $\mathbf{K}$-fixed vectors of ρ_r coincides with the space of functions of the form*

$$f(z_1,\dots,z_\alpha)I(z_{\alpha+1})I(z_{\alpha+2})\dots$$

PROOF. Without loss of generality, we can set $\alpha=0$. We regard $\mathcal{E}_{k\infty}$ as the space of $\infty\times k$ matrices $Z=\{z_{ij}\}$ with elements in $\mathbb{Q}_p$ (all but a finite number of matrix elements are in $\mathbb{O}_p$). The group $\mathbf{K}=\mathrm{O}(\infty,\mathbb{O}_p)$ acts by left multiplications

$$\mathrm{We}(u)f(Z)=f(Zu).$$

We must show that $\prod_{ij}I(z_{ij})$ is a unique $\mathrm{O}(\infty,\mathbb{O}_p)$-invariant function in $L^2(\mathcal{E}_{k\infty})$.

The group $\mathrm{O}(\infty,\mathbb{O}_p)$ contains the group $S(\infty)$ of finite permutations of the set $\mathbb{N}$. According zero-one law (see, e.g., [31], §4.1), the action of $S(\infty)$ on the set $\mathbb{O}_p^{k\infty}\subset\mathcal{E}_{k\infty}$ is ergodic. Let $\Omega\subset\mathcal{E}_{k\infty}$ be an invariant set. Let $\xi\in\mathcal{E}_{k\infty}\setminus\mathbb{O}_p^{k\infty}$. Assume that the measure of the set $\Omega\cap\mathbb{O}_p^{k\infty}$ is non-zero, say ν_0. Since Ω is $S(\infty)$-invariant, for any $\mathfrak{s}\in S(\infty)$, the set $\Omega\cap(\xi\mathfrak{s}+\mathbb{O}_p^{k\infty})$ has the same measure ν_0. However there is a countable number of disjoint sets of the form $\xi\mathfrak{s}+\mathbb{O}_p^{k\infty}$, therefore the measure of Ω is infinite. □

Corollary 7.3 *Let $\alpha=0$. Then the representation ρ_r contains a unique irreducible $\mathbf{K}$-spherical representation of $\mathbf{G}$.*

PROOF. We take the $\mathbf{G}$-cyclic span of the unique $\mathbf{K}$-fixed vector. $\square$

Next, consider the subgroup $\mathrm{GL}(1,\mathbb{Q}_p) \subset \mathrm{Sp}(2k,\mathbb{Q}_p)$ consisting of matrices $\begin{pmatrix} \lambda\cdot 1_k & 0 \\ 0 & \lambda^{-1}\cdot 1_k \end{pmatrix}$, where $\lambda \in \mathbb{Q}_p^\times$.

Lemma 7.4 *If r, r' are contained in the same double coset*

$$\mathrm{GL}(1,\mathbb{Q}_p) \setminus \mathrm{Sp}(2k,\mathbb{Q}_p)/\mathrm{Sp}(2k,\mathbb{O}_p),$$

then $\rho_r \simeq \rho_{r'}$.

PROOF. First, if $q \in \mathrm{GL}(1,\mathbb{Q}_p)$, then the automorphism (7.1) fixes the subgroup $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$.

Second, if $t \in \mathrm{Sp}(2k,\mathbb{O}_p)$, then $\sigma(t)$ is contained in the group $\mathbf{Sp}$ of automorphisms of the infinite object of the Nazarov category. Therefore the operator $\mathrm{We}(\sigma(t))$ is well-defined, it intertwines ρ_r and ρ_{rt}. $\square$

7.2. Characteristic functions and representations. By Lemma 7.2, we can identify the space of $\mathbf{K}$-fixed vectors of ρ_r and the space of the Weil representation of $\mathrm{Sp}(2\alpha,\mathbb{Q}_p)$.

Theorem 7.5 *The representation of the semigroup $\mathbf{K}\setminus\mathbf{G}/\mathbf{K}$ in the space of $\mathbf{K}$-fixed vectors of ρ_r is given by the formula*

$$\overline{\rho}_r(\mathfrak{g}) = s\cdot \mathrm{We}\big(\chi_{\mathfrak{g}}(r\mathbb{O}_p^{2k}, r\mathbb{O}_p^{2k})\big), \qquad s\in\mathbb{C}^\times.$$

PROOF. We use the notation and statements of Subsection 3.12. Let V and H be the same as in Section 4. Let $Y = V_{2k\infty}$, $W = V\oplus Y$ The operator of projection $\mathcal{H}(V\oplus Y)$ to $\mathcal{H}(V\oplus Y)^{\mathbf{K}} \simeq \mathcal{H}(V)$ is $\mathrm{We}(\theta_W^V)$. Therefore

$$\overline{\rho}(\mathfrak{g}) = s'\cdot \mathrm{We}(\theta_W^V)\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\theta_W^V)$$

as an operator $L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}} \to L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}}$. The operator

$$\mathrm{We}(\lambda_W^V) : L^2(\mathbb{Q}_p^\alpha) \to L^2(\mathcal{E}_{\alpha+k\infty})$$

is an operator of isometric embedding, the image is $\mathcal{H}(V\oplus V_{2k\infty})^{\mathbf{K}}$. Therefore we can write $\overline{\rho}(\mathfrak{g})$ as

$$\begin{aligned}\overline{\rho}(\mathfrak{g}) = s^{''}\cdot \mathrm{We}(\lambda_W^V)^*\mathrm{We}(\theta_W^V)\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\theta_W^V)\mathrm{We}(\lambda_W^V) =\\ = s^{'''}\cdot \mathrm{We}(\lambda_W^V)^*\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\lambda_W^V) =\\ = s^{''''}\cdot \mathrm{We}\Big[(\lambda_W^V)^*\sigma(r^{-1})\iota(g)\sigma(r)\lambda_W^V\Big]. \quad (7.2)\end{aligned}$$

Next, $\sigma(r)\lambda_W^V : V \rightrightarrows V\oplus Y$ is a direct sum of $1_V \subset V\oplus V$ and the lattice in Y given by

$$\sigma(r)Y(\mathbb{O}) = \sigma(r)(H(\mathbb{O})\otimes\mathbb{O}^\infty) = (rH(\mathbb{O}))\otimes\mathbb{O}^\infty).$$

We apply Subsection 5.11 for the expression in square brackets in (7.2).

7.3. Constructions of representations. Now we extend the previous construction. Consider the embedding

$$\iota_l : \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p) \to \mathrm{Sp}(2l\alpha + 2lk\infty, \mathbb{Q}_p)$$

given by

$$g \mapsto \begin{pmatrix} g & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & g & 0 & \dots & 0 \\ 0 & \dots & 0 & g^{t-1} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & g^{t-1} \end{pmatrix}.$$

This is a $2l \times 2l$ block matrix, each block of this matrix has size $(\alpha + k\infty) \times (\alpha + k\infty)$.

Next, for a matrix $r \in \mathrm{Sp}(2kl, \mathbb{Q}_p)$ we take

$$\sigma(r) := 1_{2\alpha l} \oplus (r \otimes 1_\infty)$$

and consider the representation of $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ given by

$$\rho_r(g) = \mathrm{We}(\sigma(r)^{-1} \iota_l(g) \sigma(r)).$$

Set $\alpha = 0$. As above, each representation ρ_r of $\mathbf{G} = \mathrm{GL}(k\infty, \mathbb{Q}_p)$ contains a unique $\mathbf{K}$-spherical subrepresentation.

Conjecture 7.6 *Any* $\mathbf{K}$*-spherical representation of* $\mathrm{GL}(k\infty, \mathbb{Q}_p)$ *is a subrepresentation in* $\varphi(\det(g))\, \rho_r(g)$*, where* $\varphi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ *is a character. Representations* ρ_r *are parametrized by the set*

$$\bigcup_l \mathrm{GL}(l, \mathbb{Q}_p) \setminus \mathrm{Sp}(2kl, \mathbb{Q}_p)/\mathrm{Sp}(2kl, \mathbb{O}_p).$$

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