ON AHLFORS' FINITENESS THEOREM

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SFB/MPI 84-67

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Contents:

+ Partially supported by NSF grants. The first author also acknowledges the support from the Max-Planck-Institut für Mathematik, Bonn, Germany.

Abstract

Ahlfors' finiteness theorem and its two complements namely the area-inequalities of Bers and the finiteness of the cusps due to Sullivan are some of the central results in the modern theory of Kleinian groups. Their proofs are analytic whereas their conclusions have a geometric flavor. In this paper we have attempted to explain the topological and group-theoretic genesis of these theorems. In case the domain of discontinuity is connected our approach is based on a structure theorem on planar regular coverings which is a partial extension of the Maskit's planarity theorem. In case the domain of discontinuity is not necessarily connected our approach is based on a relative version of the theorem "a finitely generated 3-manifold group is finitely presented" due to Scott and Shalen.

§1. Introduction (1.0) Let Γ be a Kleinian group, i.e. by definition, a discrete subgroup of Möbius transformations of the Riemann sphere S^2. Let Λ be its limit set which may be defined as the closure of the set of fixed points of elements of infinite order in Γ. The set $\Omega \stackrel{\text{def}}{=} S^2 - \Lambda$ is called the set of discontinuity of Γ, for, indeed it may be shown that Γ acts properly discontinuously on Ω - i.e. for every compact set $K \subseteq \Omega$ the set $\{\gamma \in \Gamma | \gamma K \cap K \neq \emptyset\}$ is finite. Classically one says that Γ is non-elementary if Λ has more than two points. This is equivalent to Γ being not virtually abelian. A central result in the modern theory of Kleinian groups is the following finiteness theorem of Ahlfors.

(1.1) Theorem (Ahlfors) If Γ is a finitely generated non-elementary Kleinian group then $\Gamma \backslash \Omega$ has finitely many components each of which, as an orbifold, is a hyperbolic Riemann surface of finite type.

(Recall that a Riemann surface is said to be of finite type if it is biholomorphic to a compact Riemann surface with at most finitely many points removed. Of course, (1.1) has content only when $\Omega \neq \emptyset$.)

Ahlfors' proof of this theorem in [1] with a gap filled by Greenberg, cf. [5], and in a different way by Bers, cf. [2], rests on showing finite-dimensionality of certain spaces of holomorphic q-differentials.

The theorem (1.1) has two major supplements. First, the area-inequality of Bers, cf. [3] asserts that

(1.2) <u>Theorem</u> (Bers) In the situation of (1.1),

(1.2.1) $\frac{1}{2\pi}$ {the hyperbolic area of $\Gamma\backslash\Omega$} $\leq 2(N-1)$

where N is the minimum number of generators of Γ.

Now S^2 may be regarded as the sphere at infinity of the hyperbolic 3-space H^3, which is the symmetric space for the Möbius group $\approx PSL_2(\mathbb{C})$. So the Möbius group, and in particular Γ, extend to $H^3 \cup S^2$; and Γ acts properly discontinuously on $\Omega \cup H^3$. Hence $M = \Gamma\backslash\{\Omega \cup H^3\}$ is a 3-manifold with boundary $\Gamma\backslash\Omega$, and int M has a structure of a hyperbolic 3-manifold, i.e. a complete Riemannian 3-manifold with constant curvature -1. The second supplement to (1.1) is due to Sullivan cf., [17].

(1.3) <u>Theorem</u> (Sullivan) Let Γ, Ω, be as in (1.1), and N as in (1.2), and M as defined above. Then the

(1.3.1) $\#\{\text{cusps of } M\} \leq 5N-4$.

The finiteness of the number of cusps, under some topological regularity hypothesis is due to Marden, cf. [2], theorem 6.4, and under a similar hypothesis Abikoff, cf. [17], p. 291, obtained the upper bound $3N-3$ for the same number.

The proofs of (1.1) - (1.3) use some deep analysis whereas the conclusions have a strong geometric flavor. The motivation of this paper is to understand the underlying topology and group theory of this theorem.

(1.4) Consider a finitely generated group Γ of homeomorphisms of S^2. We shall assume that Γ is orientation-preserving and torsion-free. Assume also that Γ leaves some open, non-empty subset $\Omega \subseteq S^2$ invariant on which it acts properly discontinuously. Later on we shall need to impose an appropriate "Kleinian condition" which ensures that the "limit set" $\Lambda \overset{\text{def}}{=\!=} S^2 - \Omega$ is minimal in some sense. We consider two cases:

i) Ω connected, ii) Ω not necessarily connected. We shall explain and partially extend (1.1) - (1.3), in case i) using only 2-dimensional topology whereas in case ii) we shall also use 3-dimensional topology.

(1.5) In the set-up of (1.4) assume that Ω is connected. Let Ω_e denote the end-compactification of Ω, and $e(\Omega) = \Omega_e - \Omega$ = the set of ends of Ω, cf. Freudenthal [4]. Then $e(\Omega)$ is a compact set, and Ω_e is homeomorphic to S^2. So Γ may as well be taken simply as a properly discontinuous group of homeomorphisms of a connected planar surface Ω. Since Γ acts properly discontinuously it extends continuously as a group of homeomorphisms of Ω_e. The <u>Kleinian condition</u> referred to above, in this case, is formulated as follows.

(1.5.1) There exists $\tilde{*} \in \Omega$ whose Γ-orbit accumulates at every end, i.e. the derived set $(\Gamma\tilde{*})' = e(\Omega)$.

Mimicking the proof of Hopf's theorem, cf. [7], [10],[11] on ends of groups it then follows from elementary topology of surfaces that we have the following four possibilities:

(1.5.2) i) $e(\Omega) = \emptyset$, $\Omega \approx S^2$, $\Gamma = \{e\}$,

ii) $e(\Omega) = \{\text{a point}\}$, $\Omega \approx \mathbb{R}^2$,

$\Gamma \approx \pi_1$ (a surface of finite type).

iii) $e(\Omega) = \{\text{two points}\}$, $\Omega \approx \mathbb{R}\times S^1$, $\Gamma \approx \mathbb{Z}$,

$\Gamma\backslash\Omega \approx$ a torus,

iv) $e(\Omega) \approx$ a Cantor set.

(Here a <u>surface of finite type</u> means a compact surface with at most finitely many points removed.)

To complete the picture, it remains to decide the structure of Γ and $\Gamma\backslash\Omega$ in case iv). It is proved in [11], using only the theory of ends of spaces and ends of groups, that in case iv), $\Gamma \approx \mathop{\Pi^*}_{i=1}^{n} \Gamma_i$ i.e. a finite free product where each $\Gamma_i \approx \pi_1$ (a surface of finite type). In this paper we show

(1.6) <u>Theorem A</u> Let Ω, Γ be as above, and assume that (1.5.1) holds. Then $\Gamma\backslash\Omega$ is a surface of finite type.

Classically, a Kleinian group Γ with $\Omega \neq \emptyset$ and connected, or more generally, leaving a component of Ω invariant is called a <u>function group</u>. Maskit, cf. [14],[15] has made a remarkable study of this interesting class of groups. From his arguments it is not difficult to see that given $\Omega \approx S^2 - \{\text{a Cantor set}\}$ and $\Gamma \approx$ a nontrivial free product of fundamental groups of surfaces, there are only finitely many (up to topological equivalence) properly discontinuous actions of Γ on Ω satisfying (1.5.1). This is a much more precise information than what can be obtained from (1.2) or (1.3). (In fact the equality in (1.2.1) is attained precisely when Γ is free and topologically equivalent to a Shottky group.)

(1.7) Now in the set-up of (1.4) consider the case where Ω is not necessarily connected. In this case we consider S^2 as the boundary of a closed ball D^3, and assume, first of all, that Γ extends continuously to D^3 and acts properly discontinuously on $\Omega \cup \text{int } D^3$. As noted previously, this condition is satisfied in the classical case. We now formulate "the Kleinian condition" as follows.

(1.7.1) There is no Γ-invariant open subset $\Omega_1 \supsetneq \Omega$ of S^2 such that the Γ-action on $\Omega_1 \cup (\text{int } D^3)$ is properly discontinuous.

Now $M = \Gamma\backslash\{\Omega \cup (\text{int } D^3)\}$ is a 3-manifold with boundary $\partial M = \Gamma\backslash\Omega$. In this paper we show

(1.8) <u>Theorem B</u> Let Ω, Γ, M be as above, and assume that (1.7.1) holds. Then ∂M has only finitely many components that are not open annuli or discs; moreover there are only finitely many homotopy classes (in M) of the annular components of ∂M.

The proof of theorem B provides a more precise information. It will be explained in §4 that the group $\Gamma \cong \pi_1(M)$ has finitely generated integer homology and well defined Betti numbers $b_i(\Gamma)$ and the Euler characteristic $\chi(\Gamma)$. If S is a surface of finite type, let us set

$$(1.8.1) \quad \chi_-(S) = \sum_C \max(0,-\chi(C))$$

where C ranges over the components of S. It will be shown in §5 that if we exclude the trivial case $\Gamma = \{e\}$ the following inequalities hold. Let α denote the number of homotopy classes of annular components (in M) of ∂M, and τ denote the number of toral components of ∂M. Then

$$(1.8.2) \quad \chi_-(\partial M) \leq -2\chi(\Gamma)$$

$$(1.8.3) \quad \alpha + \tau \leq -3\chi(\Gamma) + b_2(\Gamma) + 1$$

(1.9) Notice that in (1.2.1), since we are assuming Γ to be torsion free, the left-hand side, by the Gauss-Bonnet theorem and (1.1), is $|\chi(\Gamma\backslash\Omega)| = \chi_-(\partial M)$. Write b_i for $b_i(\Gamma)$. If N is the minimum number of generators of Γ then $b_1 \leq N$. So

$$-2\chi(\Gamma) = -2(1-b_1+b_2) \leq 2(b_1-1) \leq 2(N-1),$$

which is the right-hand side of (1.2.1). Thus, (1.8.2) explains and extends (1.2).

Next, in the classical case, a cusp of M corresponds to a conjugacy class of maximal parabolic subgroups, and a neighborhood of such a cusp is homeomorphic to (an annulus) $\times \mathbb{R}$ or (a torus) $\times \mathbb{R}$ according as the parabolic subgroup is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} + \mathbb{Z}$. Also the annuli corresponding to distinct rank-1 cusps are non-homotopic in M. Now (1.8.3) does not directly explain (1.3.1), since the cusps, by their very definition, do not appear in ∂M. Indeed, they would not show up even under quasi-conformal deformations of Γ. But it is possible to have <u>topological</u> deformations of Γ (in Homeo D^3) which do not change the topology of int M, but where the cusps would appear as annular or toral components of the boundary of the deformed M. This can be done by "blowing" of the cusp somewhat in the sense of real algebraic geometry. (More precisely, for each cusp of M, delete from D a Γ-invariant family of open horoballs which projects onto a neighborhood of the cusp in M. The closure of such a horoball meets ∂D^3 in a point, which is replaced by a circle which may be identified with the set of directions at the point tangential to ∂D^3. The resulting space D_O is clearly homeomorphic to D. The Γ-action extends to D_O since it was smooth on D.)[⊕] Now the right-hand side of (1.8.3) is

$$-3(1-b_1+b_2)+b_2+1 \leq 3b_1-4 \leq 3N-4$$

if $b_2 \neq 0$. (The proof actually shows that if $b_2 = 0$ then $\alpha+\tau \leq 3b_1-3$ unless $b_1 = 1$ in which case $\alpha+\tau \leq 1$.) So (1.8.3) explains and extends (1.3).

(1.10) In the situation of theorem B, with $\Gamma \neq \{e\}$, M is aspherical with infinite fundamental group. So no component

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⊕ The first author thanks J. McCarthy and J.-P. Otal for a useful conversation on this point.

of ∂M can be a sphere. But it is not difficult to construct examples of non-virtually-abelian Γ's so that some components of ∂M can be annuli or tori. In fact, infinitely many annuli can also occur. It remains undecided however whether disks can also occur as components of ∂M. A 2-dimensional analogue of theorem B is more precise: let Γ be a torsion-free, orientation-preserving, finitely generated group of homeomorphisms of D^2 which acts properly discontinuously on $\Omega \cup (\text{int } D^2)$ where Ω is a Γ-invariant open subset of ∂D^2 satisfying "the Kleinian condition" i.e. (1.7.1) with S^2 resp. D^3 replaced by S^1 resp. D^2. Then Γ is topologically conjugate to a fuchsian group (in Homeo D^2) - so $N = \Gamma \backslash \{\Omega \cup (\text{int } D^2)\}$ is a 2-manifold with compact boundary - in particular no component of N can be an arc. We see no reason however for this statement to carry over to dimension 3.

(1.11) Throughout this paper we shall assume that the group Γ under consideration is torsion-free. In case Γ is a finitely generated classical Kleinian group it is a relatively simple matter to pass to a torsion-free subgroup of finite index. In the topological case the existence of such a subgroup is a non-trivial question and probably cannot be settled by the elementary techniques of this paper.

§2. Structure of regular planar coverings

(2.1) Let Ω be a planar surface and Γ an orientation-preserving, properly discontinuous, torsion-free group of homeomorphisms of Ω. Then $p: \Omega \to \Gamma\backslash\Omega \overset{\text{def.}}{=} M$ is a regular planar covering. The following theorem describes $\pi_1(\Omega)$ as a normal subgroup of $\pi_1(M)$. Notice that we do not assume that Γ is finitely generated.

If G is any group and $A \leq G$ then $\langle\langle A\rangle\rangle$ denotes the smallest normal subgroup of G which contains A.

(2.2) Theorem Let $p: \Omega \to \Gamma\backslash\Omega = M$ be as in (2.1). Then there exists a family $\mathcal{S} = \{C_i\}_{i\in I}$ of mutually disjoint, non-nullhomotopic and pairwise non-homotopic simple closed curves in M such that

i) any compact set $K \subseteq M$ intersects only finitely many elements of $\mathcal{S}$, and

ii) $\pi_1(\Omega) = \langle\langle [C_i]_{i\in I}\rangle\rangle$, where $[C_i]$ denotes the conjugacy class in $\pi_1(M)$ defined by C_i.

(For any planar covering a family $\mathcal{S}$ of the type described in (2.2) will be said to be admissible.)

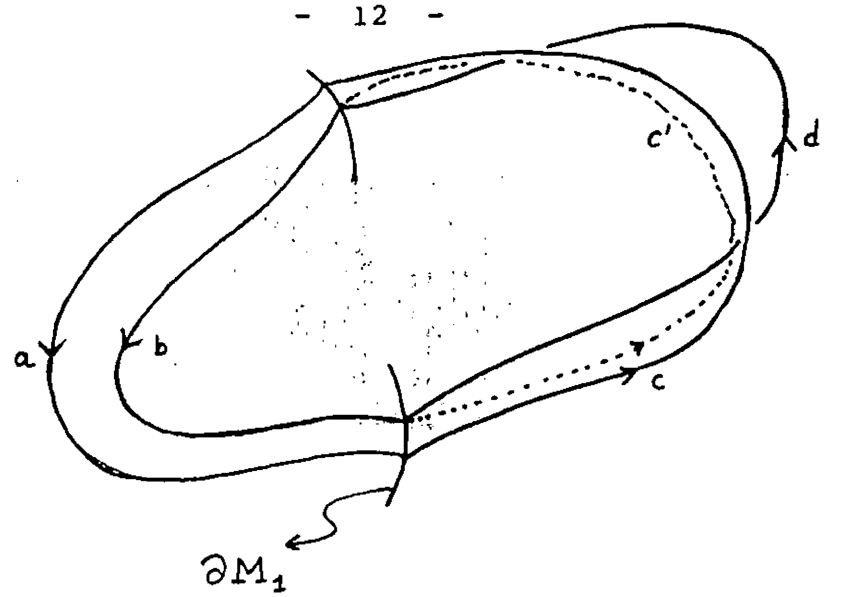
Proof: In case M is a surface of finite type then the family $\mathcal{S}$ is necessarily finite and the theorem in this case is due to Maskit, cf. [13] theorem 3.

Call a connected sub-surface S of M <u>incompressible</u> if, choosing a base-point $*$ in S, the induced map $\pi_1(S,*) \to \pi_1(M,*)$ is injective. Let S be a compact subsurface with boundary. If no component of $M - \text{int } S$ is $\approx$ a closed disk it is well-known that S is incompressible. So if S itself is not incompressible then we can attach to it the closed-disk-components of $M - \text{int } S$ and obtain a new compact surface $S_1 \supseteq S$ which, moreover, is incompressible.

Now let M_1 M_2 ... be an exhaustion of M by compact sub-surfaces with boundary. Here $M_i \subseteq \text{int } M_{i+1}$, $i = 1,2\ldots$. (One can obtain such exhaustions either from a triangulation of M or by a proper smooth Morse function.) Moreover applying the process described in the above paragraph, and changing notation if necessary, we may also assume that M_i's are incompressible for $i = 1,2,\ldots$. Choosing a base-point $*$ in M_1 we thus obtain an increasing sequence $\pi_1(M_1,*) \subseteq \pi_1(M_2,*)$ ... whose union is clearly $\pi_1(M,*)$. Let $\tilde{*}$ be a base-point in Ω lying over $*$ and write $N = p_*\pi_1(\Omega,\tilde{*})$, and $N_i = N \cap \pi_1(M_i,*)$. So N is the union of the increasing sequence $N_1 \subseteq N_2$ ... . If $\tilde{M}_i$ is the component of $p^{-1}(M_i)$ containing $\tilde{*}$, it is clear that $p|_{\tilde{M}_i}: \tilde{M}_i \to M_i$ is a regular planar covering with the group of covering transformations isomorphic to N_i, $i = 1,2,\ldots$.

We now start constructing an admissible family $\mathcal{S}$ of simple closed curves as asserted in the theorem. By [13], cf. lemma 5 and theorem 3, there exists a family $\mathcal{S}' = \{C'_{ij}\}$ $i = 1,2,\ldots,$ $j = 1,2,\ldots,n_i < \infty$, of disjoint simple closed curved such that $\{C'_{11}, C'_{12}, \ldots, C'_{k\,n_k}\}$ form an admissible family for $p|_{\tilde{M}_k}: \tilde{M}_k \to M_k$. (This $\mathcal{S}'$ may not be admissible for $p: \Omega \to M$ since the condition i) of admissibility may fail.) We shall replace the C'_{ij}'s by other mutually disjoint simple closed curves C_{ij}'s such that only finitely many C_{ij}'s will intersect any given M_k, and $\{C_{11}, \ldots, C_{k,n_k}\}$ is still an admissible family for $p|_{\tilde{M}_k}: \tilde{M}_k \to M_k$. Then clearly the new family will be admissible for $p: \Omega \to M$.

First we may assume that C'_{ij}, if it intersects M_k for $k < i$, then it intersects ∂M_k transversely. Now let $C_{1j} = C'_{1j}$, $j = 1,2,\ldots,n_1$. Among the C'_{2j}'s suppose there is a pair, say C'_{21} and C'_{22} such that $C'_{21} \cap M_1$ has an arc-component which is parallel to an arc-component of $C'_{22} \cap M_1$. Choose orientations so that, say, $C'_{21} = a*c$, $C'_{22} = b*d$ where a and b are a pair of parallel arc-components lying in M_1. Let c' be an arc parallel to (but disjoint from) c and having the same end-points as b. Let $C''_{22} = c'*d^{-1}$. Then clearly $\langle\langle[C'_{21}],[C'_{22}]\rangle\rangle = \langle\langle[C'_{21}],[C''_{22}]\rangle\rangle$, and $C''_{22} \cap M_1$ has one <u>less</u> arc-component lying in M_1 than $C'_{22} \cap M_1$. Continuing this process we replace the C'_{2j}'s by C_{2j}'s, $j = 1,2,\ldots,n_2$ such that for no two distinct values j_1, j_2 of j, an arc-component of $C_{2j_1} \cap M_1$ is parallel to that of $C_{2j_2} \cap M_1$.



Proceeding further inductively we construct a family $\mathcal{S} = \{C_{ij}\}$ of disjoint simple closed curves such that no arc-component of $C_{i_1 j_1} \cap M_k$ is parallel to an arc-component of $C_{i_2 j_2} \cap M_k$ if $i_1, i_2 > k$ and $(i_1, j_1) \neq (i_2, j_2)$; and such that

$$<<[C_{11}], \ldots, [C_{kn_k}]>> = <<[C'_{11}], \ldots, [C'_{kn_k}]>>.$$

Since each M_i is compact a family of mutually non-parallel and disjoint simple closed curves and properly embedded arcs with end-points on ∂M_i is finite.⊕ So it follows that at most finitely many C_{ij}'s can intersect M_k if $k < i$. This proves that $\mathcal{S}$ is an admissible family. q.e.d.

2.3 Remark: The theorem (2.2) is not valid if Γ is not

⊕ cf. (4.4) for a precise statement. This fact is crucial also in the proof of theorem B.

finitely generated and contains torsion. It is easy to construct examples when (2.2) fails in such cases.

§3. Proof of theorem A

(3.1) Lemma. Let Ω be a planar surface and Γ a finitely generated, orientation-preserving torsion free, properly discontinuous group of homeomorphisms of Ω. Then $M = \Gamma\backslash\Omega$ is a surface of finite type iff $\pi_1(\Omega)$ regarded as a subgroup of $\pi_1(M)$ is normally generated by finitely many elements.

Proof. The "only if" part follows from Maskit's theorem noted in (2.2). The "if" part is trivial if M is compact. So we may assume that M is noncompact, hence $\pi_1(M) \cong$ a free group. Now M is of finite type $\Longleftrightarrow$ $\pi_1(M)$ is finitely generated $\Longleftrightarrow$ $\pi_1(M)/[\pi_1(M),\pi_1(M)]$ is finitely generated. We also have a short exact sequence $\pi_1(\Omega) \hookrightarrow \pi_1(M) \twoheadrightarrow \Gamma$. Now if $\{x_1,\ldots,x_n\} \subseteq \pi_1(M)$ is such that its image in Γ generates Γ, and $\{y_1,\ldots,y_n\} \subseteq \pi_1(M)$ is such that $\pi_1(\Omega) = \langle\langle y_1,\ldots,y_n\rangle\rangle$, it is easy to see that the image of $\{x_1,\ldots,x_m, y_1,\ldots,y_n\}$ in $\pi_1(M)/[\pi_2(M),\pi_1(M)]$ is a system of generators. This finishes the proof. q.e.d.

(3.2) Proof of (1.4) Let $p: \Omega \to \Gamma\backslash\Omega \overset{def}{=} M$ be the canonical projection and regard $* = p(\tilde{*})$ as a base-point in M. Let $\{x_1,\ldots,x_n\} \subseteq \pi_1(M,*)$ be such that its image in Γ generates Γ. Let $A_1,\ldots,A_n$ be closed curves based at $*$ representing $x_1,\ldots,x_n$ resp. Let $A = \bigcup_{i=1}^{n} A_i$ and $B = p^{-1}(A)$. Then B is a closed connected subset of Ω. Now the "Kleinian condition" (1.5.1) implies that the closure of B in Ω_e contains $e(\Omega)$.

We claim that each component of $\Omega - B$ is simply connected. For indeed $\Omega_e \approx S^2$, so a Jordan curve C lying in a component α of $\Omega - B$ is a common boundary of two disks D_1, D_2 whose union is Ω_e. If B intersected both D_1 and D_2 it would also intersect C since B is connected. But this is not possible by construction. So suppose B does not intersect D_1. But then $D_1 \cap e(\Omega) = \emptyset$ since B accumulates at every end. So $D_1 \subseteq \Omega$ and again since $B \cap D_1 = \emptyset$ we must have $D_1 \subseteq \alpha$ i.e. α is simply connected.

Now let $\mathcal{S}$ be an admissible family (§2) for the regular planar covering $p: \Omega \to M$. Since A is compact there are only finitely many elements of $\mathcal{S}$, say $C_1,\ldots,C_n$, which intersect A. We claim that $\mathcal{S} = \{C_1,\ldots,C_n\}$. For indeed if C is any simple closed curve in M such that $C \cap A = \emptyset$ and which lifts to a simple closed curve $\tilde{C}$ in Ω then $\tilde{C} \subsetneq \Omega - B$. Since each component of $\Omega - B$ is simply connected $\tilde{C}$ is null-homotopic.

So C is also null-homotopic, hence $C \notin \mathcal{S}$.

By Lemma 3.1, it now follows that M is of finite type.

q.e.d.

§4. Preliminaries for the proof of theorem B

(4.0) From this section on we shall view Ahlfors' theorem from the viewpoint of 3-dimensional topology. We shall try to separate the homological parts from those which depend on the homotopy considerations and then also point out the special features of the low-dimensional topology. It is amazing to see how these different features are intricately intertwined in the original analytic proof of Ahlfors.

(4.1) A group G is said to have <u>finitely generated integer homology</u> if $H_i(G;\mathbb{Z})$ is finitely generated for all $i \geq 0$ and $= 0$ for sufficiently large i. In this case define <u>the Euler characteristic of</u> G, denoted $\chi(G)$, to be

$$\sum_{i=0}^{\infty} (-1)^i \dim H_i(G;\mathbb{Q}).$$

It is easy to see that if two groups G_1 and G_2 have finitely generated integer homology then so does their free product $G_1 * G_2$. In fact, $H_0(G_1*G_2;\mathbb{Z}) \simeq \mathbb{Z}$, $H_i(G_1*G_2;\mathbb{Z}) \simeq H_i(G_1;\mathbb{Z}) + H_i(G_2,\mathbb{Z})$ for $i \geq 1$, and $\chi(G_1*G_2) = \chi(G_1) + \chi(G_2)-1$.

Recall also that if G is finitely generated (resp. finitely presented) then $H_i(G;\mathbb{Z})$ is finitely generated for $i \leq 1$, (resp. $i \leq 2$).

(4.2) In the sequel we shall often use the following fact from the homotopy theory. Let X,Y be two connected CW-complexes and let $\pi_1(X) \xrightarrow{\varphi} \pi_1(Y)$ be a homomorphism. Then there exists a cellular map $X_2 \xrightarrow{f} Y$ from the 2-skeletan X_2 of X into Y such that $f_* = \varphi$ (defined w.r.t. a choice of a base-point.) If in addition Y is aspherical, i.e. $\pi_i(Y) = 0$ for $i > 1$, then f extends to a cellular map from X to Y.

(4.3) Let $\Gamma \simeq \pi_1(M^3)$ be finitely generated. Then $H_1(M;\mathbb{Z}) \simeq H_1(\Gamma;\mathbb{Z}) \simeq \Gamma/[\Gamma,\Gamma]$ is finitely generated also. It is a well-known fact that if ∂M contains a handle i.e. equivalently two simple closed curves intersecting transversely at exactly one point then at least one of the curves is non-homologous to zero in M; and if there are r disjoint handles then $\dim H_1(M;\mathbb{Q}) \geq r$. This implies that if $\pi_1(M)$ is generated by n elements then ∂M cannot contain more than n handles. Thus the real difficulty in the proof of theorem B is to control the ends of ∂M.

If M^3 is compact then the homology-sequence of the pair $(M, \partial M)$ and Lefschetz duality shows $\chi(\partial M) = 2\chi(M)$. If no component of ∂M is a sphere then each component of ∂M has $\chi \leq 0$, so one has a bound for the number of components with $\chi < 0$ in terms of $\chi(M)$.

We shall be extending partially these considerations when M is non-compact which would "explain" the finiteness in Ahlfors' theorem.

We first recall some terminology and facts from 2- and 3-dimensional topology.

(4.4) Let S be a compact orientable surface of genus g with $b \geq 0$ boundary components. Then the number of non-nullhomotopic, pairwise-non-homotopic disjoint simple closed curves and non-boundary-parallel properly embedded arcs[+] is at most $3g - 3 + 2b$ if this number is > 0, and 1 if $g = 1, b = 0$, and 0 if $g = 0, b = 0$ or 1.

(4.5) Let T be a compact, orientable surface in a connected orientable 3-manifold M which is properly embedded, i.e. $T \cap \partial M = \partial T$. We say that T is <u>incompressible</u> <u>in</u> M if for every disk $D \subset M$ with $D \cap \partial T = \partial D$, ∂D is the boundary

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+ i.e. an arc α such that $\partial\alpha = \alpha \cap \partial S$ and there does not exist an arc $\beta \subseteq \partial S$ with $\partial\alpha = \partial\beta$ such that $\alpha \cup \beta$ bounds a disk.

of a disk in T. It is a standard consequence of Dehn's lemma and the loop theorem cf. [6] or [9] that T is incompressible iff for each component T_i of T, the canonical map $\pi_1(T_i) \to \pi_1(T)$ is injective. It follows easily (for example by van-Kampen's theorem and the theory of generalized free products) that if N is a compact, connected submanifold of M whose frontier+ is an incompressible surface then $\pi_1(N) \to \pi_1(M)$ is injective.

(4.6) A connected, oriented 3-manifold M is said to be <u>boundary-irreducible</u> if for every properly embedded disk $D \subseteq M$ (i.e. $\partial D = D \cap \partial M$), ∂D bounds a disk in ∂M. Again by Dehn's lemma and the loop theorem this is equivalent to the fact that $\pi_1(S) \to \pi_1(M)$ is injective for every component S of ∂M. From this algebraic characterization it follows that if M is boundary-irreducible, so is any of its covering space.

(4.7) It is a standard consequence of the sphere theorem cf. [6],[9] that a connected 3-manifold M is aspherical iff

(4.7.1)

i) M is not closed, or else $\pi_1(M)$ is infinite,

and ii) every embedded 2-sphere in M bounds a compact simply connected submanifold.

+ The frontier of N means the "boundary" in the sense of general topology; it will be denoted by FrN.

It follows that a connected 3-dimensional, proper submanifold of an aspherical 3-manifold is aspherical iff it satisfies ii).

(4.8) It was shown in [16] that if the fundamental group of a connected, orientable 3-manifold M is finitely generated then it is finitely presented. It had been shown previously, cf. [8], that if $\pi_1(M)$ is finitely presented, $\not\approx \mathbb{Z}$, and admits no non-trivial free-product decomposition then there is a compact 3-manifold $N \subseteq \text{int } M$ such that ∂N is incompressible in M and $\pi_1(N) \to \pi_1(M)$ is an isomorphism. The main result of this section is essentially a relative version of the latter result.

(4.9) If $\Gamma \approx \pi_1(M^3)$ is finitely generated then by the remarks in (4.2) and (4.8) $H_i(\Gamma;\mathbb{Z})$ is finitely generated for $i \leq 2$. Moreover, suppose that M^3 is a connected aspherical manifold. Then $H_i(\Gamma;\mathbb{Z}) \approx H_i(M;\mathbb{Z})$ for all i, and of course $H_i(M;\mathbb{Z}) = 0$ for $i \geq 4$ and $H_3(M,\mathbb{Z}) \approx \mathbb{Z}$ (resp. 0) if M is closed and orientable (resp. otherwise). So in this case Γ has finitely generated integer homology.

(4.10) Definition. Let G be a group, and $G_1,\ldots,G_k$, $k \geq 0$ its subgroups $\not\approx e$. ($k = 0$ means the collection of subgroups is empty.) We say that G is decomposable relative to $G_1,\ldots,G_k$ if either i) $G \approx \mathbb{Z}$ and $k = 0$ or ii) $G = H_1 * H_2$, a free product with $H_1 \neq \{e\} \neq H_2$ such that each G_i is contained in a conjugate of H_1 or H_2.

Evidently, "decomposability rel. to $G_1,\ldots,G_k$" depends only on the conjugacy classes of G_i's.

Also "indecomposable rel. to $G_1,\ldots,G_k$" will mean "not decomposable rel. to $G_1,\ldots,G_k$".

(4.11) Lemma. Let G be a subgroup of a free product $A*B$, $A \neq \{e\} \neq B$. Let $G_1,\ldots,G_k$, $k \geq 0$ be subgroups of G such that each G_i is conjugate to a subgroup of A or of B. Suppose that G is indecomposable relative to $G_1,\ldots,G_k$ then G is contained in a conjugate of A or B and, in fact, all G_i's are conjugates of subgroups of A or all of them conjugates of subgroups of B.

Proof. This is immediate from the Kurosh subgroup theorem.

q.e.d.

(4.12) The rest of this section is devoted to proving the following.

Proposition. Let M be a connected, aspherical 3-manifold such that $\Gamma = \pi_1(M)$ is finitely generated and $\neq \{e\}$. Let $T_1,\ldots,T_k$, $k \geq 0$ be compact, connected, mutually disjoint surfaces contained in ∂M. Let $\varphi_i\colon \pi_1(T_i) \to \Gamma$ be the maps induced by the inclusion and $\Gamma_i = \operatorname{im} \varphi_i$ (which are well-defined up to conjugacy). Suppose $\Gamma_i \neq \{e\}$ and Γ is indecomposable rel. to Γ_i. Then there is a compact, connected, aspherical 3-manifold $N \subseteq M$ such that $T_1 \cup \ldots \cup T_k \subseteq N$ and the canonical map $\pi_1(N) \overset{\alpha}{\to} \Gamma$ is an isomorphism.

The proof extends over (4.13) - (4.18).

(4.13) A submanifold N of M is called <u>ample</u> if

$$(4.13.1) \quad \begin{cases} \text{i)} & N \text{ is compact and connected,} \\ \text{ii)} & N \cap \partial M = T_1 \cup T_2 \cup \dots \cup T_k \\ \text{iii)} & \text{There is a homomorphism } \Gamma \overset{\beta}{\to} \pi_1(N) \end{cases}$$

which is a right-inverse to the canonical homomorphism $\pi_1(N) \overset{\alpha}{\to} \Gamma$, i.e. $\alpha \circ \beta = 1$, such that for $i = 1,2,\dots,k$ the diagram

$$\begin{array}{ccc} & \pi_1(T_i) & \\ \swarrow & & \searrow \\ \Gamma & \xrightarrow{\beta} & \pi_1(N) \end{array}$$

commutes modulo inner automorphism of Γ. (The slanted arrows are induced by inclusion and are defined modulo inner automorphisms of Γ.)

If N is ample, clearly by iii) the map α is surjective. Now if FrN is incompressible then by (4.5) α is also injective. Thus to prove (4.12) we need to produce an aspherical ample submanifold with incompressible frontier.

Notice that if N, N_1 are submanifolds of M with N ample, $N \subset N_1$, and N_1 satisfies i) and ii) then N_1 is ample also.

(4.14) Lemma. There exists an ample submanifold of M.

Proof. We know $\Gamma = \pi_1(M)$ is finitely presented. Choose a finite 2-complex K and an isomorphism $J: \pi_1(K) \to \Gamma$. Let $\alpha_i = J^{-1} \circ \varphi_i$ where $\varphi_i: \pi_1(T_i) \to \Gamma$ are the canonical maps. By (4.2) α_i is induced by a map $f_i: T_i \to K$. Let Z_i be the mapping cylinder$^+$ of f_i, and L, the complex obtained from the disjoint union of K and Z_i's by identifying K with its image in each Z_i. So L is a 3-dimensional complex containing K as a deformation retract and T_i's are naturally identified with disjoint subcomplexes of L. Again by (4.2) J is induced by a map $K \overset{f}{\to} M$, which extends to a map $L \overset{g}{\to} M$ (since there exists a deformation-retraction $L \to K$). By construction, $g_{|T_i}$ induces φ_i, so $g_{|T_i}$ is homotopic to the inclusion map $T_i \hookrightarrow M$. By the homotopy extension property for polyhedra, we may assume that after modifying g by a homotopy if necessary (and still calling it g), we have $g_{|T_i}$ is the inclusion map $T_i \hookrightarrow M$. By a further general-position homotopy we may assume that $g(L) \cap \partial M = \bigcup_{i=1}^{k} T_i$.

Now let N be a regular neighborhood of $g(L)$ such that $N \cap \partial M = \bigcup_{i=1}^{k} T_i$. Then N is ample: indeed set $\beta = g_* \circ J^{-1}$ where we consider g_* as the map $\pi_1(L) \to \pi_1(N)$. It is easy

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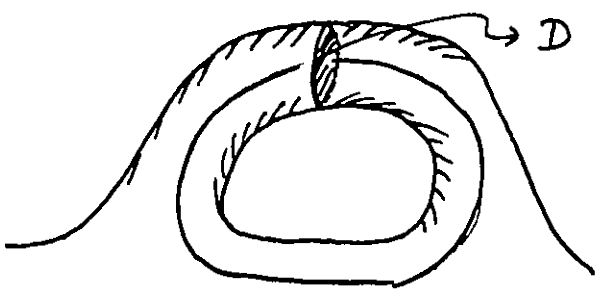
+ Recall that $Z_i = \{(T_i \times [0,1]) \cup K\}/\sim$ where for $x \in T_i$ one identifies $(x,1)$ with $f_i(x)$.

to see that the conditions in (4.13.1) hold. q.e.d.

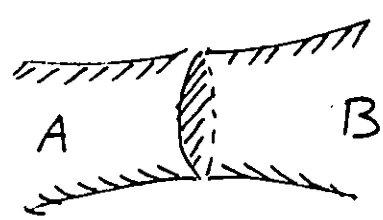
(4.15) Cutting or thickening along a compressing disk: Let N be a 3-dimensional submanifold of a 3-manifold M such that $\mathrm{Fr}N$ is a compressible surface, i.e. by definition, there exists a 2-disk $D \subset \mathrm{int}\, M$ such that $D \cap \mathrm{Fr}N = \partial D$ and ∂D does not bound a disk in $\mathrm{Fr}N$. Such a disk is called a compressing disk. A compressing disk D is contained either in N or in $M - \mathrm{int}\, N$. Let $E \subset \mathrm{int}\, M$ be a regular neighborhood of D in N or $M - \mathrm{int}\, N$, and set $N_1 = \overline{N-E}$ or $N \cup E$. In the first (resp. second) case we shall say that N_1 is obtained from N by cutting along D, (resp. by thickening along D).

(4.16) Lemma. Let N be an ample submanifold of M with compressible frontier. Suppose $D \subseteq N$ is a compressing disk, and N_1 is obtained from N by cutting along D. Then N_1 itself or a component of N_1 is ample.

Proof. There are two possibilities: either N_1 is connected, or it has two components say A and B. In the first (resp. second) case $\pi_1(N) = \pi_1(N_1) * \mathbb{Z}$ (resp. $\pi_1(N) = \pi_1(A) * \pi_1(B)$), cf. figure i) (resp. ii)) below .



(i)



(ii)

Case 1. ($\pi_1(N) = \pi_1(N_1) * \mathbb{Z}$). Since D is disjoint from $T_1 \cup T_2 \cup \ldots \cup T_k$ it is clear that $\Gamma_i = \text{im}\, \varphi_i$ are contained in the conjugates of $\pi_1(N_1)$ in $\pi_1(N)$. So since Γ is, by hypothesis, indecomposable w.r.t. Γ_i, it follows by (4.9) that Γ is conjugate to a subgroup of $\pi_1(N_1)$. This provides the required homomorphism $\beta: \Gamma \to \pi_1(N_1)$ making N_1 ample.

Case 2. ($\pi_1(N) = \pi_1(A) * \pi_1(B)$). By the argument as in case 1, now Γ_i are conjugates of subgroups of $\pi_1(A)$ or of $\pi_1(B)$. So by (4.11) Γ is conjugate to a subgroup of $\pi_1(A)$ or $\pi_1(B)$ - say of $\pi_1(A)$. Then as above A is ample. q.e.d.

(4.17) We note one more property of an ample submanifold: let D be a compressing disk for an ample submanifold N, then ∂D does not bound a disk in ∂N. Indeed ∂D lies in FrN and does not bound a disk in FrN. So if it bounds a disk Δ in ∂N then $\Delta \subseteq N \cap \partial M$, so Δ contains one of the T_i's, say T_1. But then $\Gamma_1 = \{e\}$, contrary to the hypothesis.

(4.18) Proof of (4.12): As noted in (4.13), we need to produce an aspherical ample submanifold with incompressible frontier. By (4.14) we know that ample submanifolds exist. To an ample submanifold N attach its complexity: $C(N) \stackrel{\text{def}}{=} \sum_B \{1+(\text{genus } B)^2\}$ where B runs over the components of ∂N. Let N_0 be an ample submanifold with the least complexity. We show that N_0 is aspherical and has incompressible frontier.

Let $\Sigma \subseteq \text{int } N_0$ be a 2-sphere. By (4.7) we need to show that Σ bounds a compact, simply connected submanifold in N_0. Since M is aspherical, there exists such a submanifold $E \subseteq M$. If $E \not\subseteq N_0$ then $N_1 = N_0 \cup E$ is a compact connected submanifold. Clearly $N_1 \cap \partial M = N \cap \partial M = T_1 \cup \ldots \cup T_k$. So by the remark in (4.13), N_1 is ample. Now the components of ∂N_1 are among those of ∂N_0 and clearly ∂N_1 has at least one component less than those in ∂N_0. So $C(N_1) < C(N_0)$ contradicting the definition of N_0. So N_0 is aspherical. In particular no component of ∂N_0 is a sphere. (For otherwise N_0 would be simply connected, and so $\Gamma = \{e\}$, contrary to our hypothesis).

Next suppose that FrN_0 is compressible. Let D be a compressing disk, and $\partial D \subseteq$ the component B of ∂N_0.

<u>Case 1</u>. ($D \not\subseteq N_0$). Let N_1 be obtained from N by thickening along D. Again as above N_1 is ample. If $B-\partial D$ is disconnected then the components of ∂N_1 are those of ∂N except that B is replaced by two components B_1, B_2 such that genus B = genus B_1 + genus B_2. Also genus $B_i > 0$, $i = 1,2$, for otherwise D would not be a compressing disk by (4.17). It is now clear that $C(N_1) < C(N_0)$ - a contradiction. If $B-\partial D$ is connected then the components of N_1 are those of N except that B is replaced by a component B_1 with genus B_1 = genus $B-1$. So again $C(N_1) < C(N_0)$ - a contradiction.

<u>Case 2</u>. ($D \subseteq N_0$) Let N_1 be obtained from N_0 by cutting

along D. By (4.14) N_1 itself or a component of N_1, say A, is ample. If N_1 is connected then as in case 1, $C(N_1) < C(N_0)$ - a contradiction. If N_1 is not connected, clearly the components of ∂A are among the components of ∂N_1 except that B is replaced by a component of lower genus. So again $C(N_1) < C(N_0)$ - a contradiction.

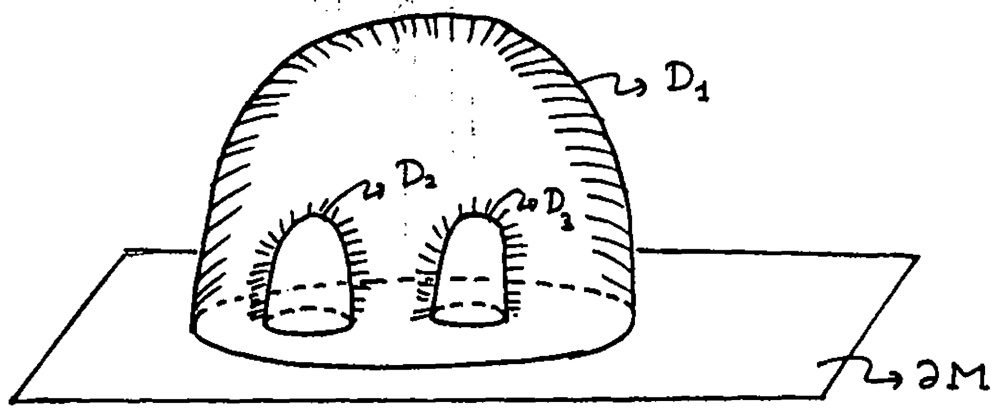
These contradictions show that N_0 has incompressible frontier. This finishes the proof of (4.12). q.e.d.

§5. Proof of theorem B

(5.0) Recall that Γ is a torsion-free, properly discontinuous, orientation-preserving group of homeomorphisms of D^3 which acts properly discontinuously on $\Omega \cup (\text{int } D^3)$ where Ω is an open subset of ∂D^3 and the Kleinian condition (1.7.1) holds. We set $\Lambda = \partial D^3 - \Omega$ and $M = \Gamma \backslash D^3 - \Lambda$. Since $D^3 - \Lambda$ is contractible, M is aspherical.

(5.1) Lemma. A connected, simply connected 3-dimensional submanifold K which is a closed subset of M and whose frontier is a finite union $D_1 \cup D_2 \cup \dots \cup D_n$ of properly embedded disks i.e. $D_i \cap \partial M = \partial D_i$. Then K is compact.

(K should be thought of as in the following picture.



).

Proof. Lift K to a connected simply connected submanifold $\tilde{K}$ of $D^3 - \Lambda$. Let $\bar{\tilde{K}}$ be its closure in D^3. It is easy to see that for all $\gamma \in \Gamma - \{e\}$, we have $\gamma(\bar{\tilde{K}}) \cap (\bar{\tilde{K}}) = \emptyset$. Also $\bar{\tilde{K}} - \tilde{K} \subseteq \partial D^3$. If $\bar{\tilde{K}} - \tilde{K} \not\subseteq \Omega$ then clearly it will have a non-empty (2-dimensional) interior, say L and Γ will act properly discontinuously on $(D^3 - \Lambda) \cup \{\bigcup_{\gamma \in \Gamma} \gamma L\}$ which contradicts (1.7.1). So $\bar{\tilde{K}} = \tilde{K}$ i.e. $\tilde{K}$ and hence K is compact. q.e.d.

(5.2) First reduction: We can write $\partial M = \bigcup_{i=1}^{\infty} S_i$, where S_i is an open subsurface of finite type. $\bar{S}_i \subseteq S_{i+1}$, and each component of S_i is incompressible (in the sense of (2.2)) in the component of ∂M in which it lies. Set $M_i = S_i \cup \{\text{int } M\}$. Note that together with M, each M_i satisfies the property mentioned in (5.1), namely, (5.2.1) A connected, simply connected, 3-dimensional submanifold K which is a closed subset of M_i and whose frontier is a union of finitely many properly embedded disks is compact.

(5.2.2) We shall call K as in (5.2.1) a test-submanifold.

(5.2.3) Now the right-hand sides of (1.8.2) and (1.8.3) depend only in $\operatorname{int} M$, and are finite. So if we establish (1.8.2) and (1.8.3) for M_i, $i = 1,2,3,\ldots$ then they clearly hold for M and theorem B would also be proved. In other words, it suffices to show

Proposition Let M be an orientable aspherical 3-manifold such that ∂M is of finite type, $\Gamma \cong \pi_1(M)$ is finitely generated and $\neq \{e\}$. Suppose that M satisfies (5.2.1). Then (1.8.2) and (1.8.3) hold.

(5.3) Remarks i) Let D be a properly embedded disk in M, U = a regular neighborhood of D in M, and $N = M - \operatorname{int} U$. If M satisfies (5.2.1) so does N.

(Indeed, let K be a test-submanifold of N. By an ambient isotopy (in N) we may remove the intersection of K with $\operatorname{Fr} U$ (in M). Now the $\operatorname{Fr} K$ (in M) is an union of finitely many embedded disks. Since M satisfies (5.2.1), K is compact.)

ii) Let M be an orientable, connected, aspherical 3-manifold. Assume $\pi_1(M) \neq \{e\}$ and M is boundary-irreducible (cf. 4.6).

Then M satisfies (5.2.1).

(Indeed let K be a test-submanifold and $\mathrm{Fr}K$ is an union of properly embedded disks D_i, $i = 1,2,\ldots,r$. By boundary-irreducibility ∂D_i bounds a disk D_i' in ∂M. So by (4.7) $D_i \cup D_i'$ bounds a simply connected compact submanifold E_i of M. For each i, either $K \subseteq E_i$ or $K \cap E_i = D_i$. If $K \subseteq E_i$ for some i then K is clearly compact. But otherwise $M = K \cup E_1 \cup \ldots \cup E_r$ and so $\pi_1(M) = \{e\}$, a contradiction.)

(5.4) <u>Second reduction</u> Suppose M, Γ are as in (5.2.3) and M is boundary-reducible cf. (4.6). Let D be a properly embedded disk in M such that ∂D does not bound a disk in ∂M. Let U be a regular neighborhood of D and $N = M - \mathrm{int}\, U$.

<u>Case 1</u> (D separates M). Let N_1, N_2 be two components of N. By (5.3) each of N_1, N_2 satisfies (5.2.1). We note that $\pi_1(N_1) \neq \{e\} \neq \pi_1(N_2)$. For otherwise, say $\pi_1(N_1) = \{e\}$. Since N_1 satisfies (5.2.1), we see that N_1 must be compact and $\partial N_1 \approx S^2$ but then ∂D would bound a disk in ∂M - a contradiction. Also both N_1, N_2 are orientable and aspherical and their boundaries are of finite type. Moreover

$$\Gamma \approx \pi_1(N_1) * \pi_1(N_2)$$

is a nontrivial free product. By Grushko's theorem each of $\pi_1(N_i)$, $i = 1,2$ has fewer minimum number of generators than Γ, and so by induction on the minimum number of generators we may assume that (1.8.2) and (1.8.3) hold for N_1 and N_2. But then

$$\begin{aligned}\chi_-(\partial M) &= \chi_-(\partial N_1) + \chi_-(\partial N_2) + 2 \\ &\leq -2\chi(\pi_1(N_1)) - 2\chi(\pi_1(N_2)) + 2 \\ &= -2\{\chi(\pi_1(N_1) * \pi_1(N_2)) \\ &= -2\chi(\Gamma)\end{aligned}$$

i.e. (1.8.2) holds for M. Secondly the toral and annular components of ∂M are clearly disjoint from D. So if α_i, τ_i denote the number of homotopy classes (in ∂N_i) of the annular components and the number of toral components of ∂N_i, $i = 1,2$ and α, τ denote the number of homotopy classes (in ∂M) of the annular components and the number of toral components of ∂M then

$$\alpha \leq \alpha_1 + \alpha_2,^{\oplus} \qquad \tau = \tau_1 + \tau_2$$

So

$$\begin{aligned}\alpha+\tau = \alpha_1+\tau_1+\alpha_2+\tau_2 \leq &-3\chi(\pi_1(N_1))+b_2(\pi_1(N_1)) + 1 \\ &-3\chi(\pi_1(N_2))+b_2(\pi_1(N_2) + 1\end{aligned}$$

⊕ In fact $\alpha = \alpha_1 + \alpha_2$ also.

$$= -3\{\chi(\Gamma)+1\} + b_2(\Gamma) + 2$$

$$\leq -3\chi(\Gamma)+b_2(\Gamma) + 1$$

So (1.8.3) also holds for M.

Case 2 (D does not separate M). If $\pi_1(N) \approx \{e\}$ then clearly $\pi_1(M) \approx Z$ and ∂M is a torus. So (1.82) and (1.8.3) are valid for M. So assume $\pi_1(N) \not\approx \{e\}$. Again N is orientable, aspherical with ∂N of finite type, and

$$\Gamma \approx \pi_1(N) * \mathbb{Z} .$$

As before, by induction on the minimum number of generators we may assume that (1.8.2) and (1.8.3) hold for N. By a calculation as above we see that they hold for M also.

In other words we have shown

(5.4.1) Proposition It suffices to prove (5.2.3) under the additional assumption that M is boundary-irreducible.

(5.5) Third reduction Now suppose M, Γ are as in (5.2.3), M is boundary-irreducible but Γ is decomposable relative to $G_i = \mathrm{im}(\pi_1(T_i) \to \Gamma)$ where T_i, $i = 1,2,\ldots,k$ are the non-simply connected components of ∂M. (Note that $G_i \not\approx \{e\}$ by the boundary-irreducibility.) If $\Gamma \approx \mathbb{Z}$ and $k = 0$, clearly

(1.8.2) and (1.8.3) hold for M. So suppose

$$\Gamma \cong \Gamma_1 * \Gamma_2 \qquad \text{(a nontrivial product)}$$

and each G_i is conjugate to a subgroup of Γ_1 or Γ_2 - say, by reindexing if necessary, G_i's, $1 \leq i \leq \ell$ are conjugate to subgroups of Γ_1 and G_i's $\ell+1 \leq i \leq k$ are conjugate to subgroups of Γ_2. Let $\tilde{M}_j$ be the covering of M w.r.t. Γ_j, $j = 1,2$. There exists components $\tilde{T}_i$ $1 \leq i \leq \ell$ (resp. $\ell+1 \leq i \leq k$) of $\partial\tilde{M}_1$ (resp. $\partial\tilde{M}_2$) which are mapped homeomorphically onto T_i. Set

$$N_1 = (\text{int } \tilde{M}_1) \cup \tilde{T}_1 \cup \tilde{T}_2 \cup \ldots \cup \tilde{T}_\ell,$$

$$N_2 = (\text{int } \tilde{M}_2) \cup \tilde{T}_{\ell+1} \cup \ldots \cup \tilde{T}_k.$$

By (4.6) $\tilde{M}_j$ and N_j are boundary-irreducible. By induction on the minimum number of generators we may assume that (1.8.2), (1.8.3) hold for N_j, $j = 1,2$. As in (5.4) one sees that (1.8.2), (1.8.3) also hold for M.

In other words,

(5.5.1) <u>Proposition</u> It suffices to prove (5.2.3) under the assumptions that M is boundary-irreducible and Γ is indecomposable relative to $G_i = \text{im}(\pi_1(T_i) \to \Gamma)$ where T_i are the non-simply connected components of ∂M.

(5.6) Proof of (5.2.3) We make in addition the assumptions stated above. Let T_i' be compact subsurfaces which are deformation-retracts of T_i. By (4.12) there exists a compact connected aspherical 3-manifold $N \subseteq M$ such that $T_1' \cup \ldots \cup T_k' \subseteq N$ and the canonical map $\pi_1(N) \overset{\alpha}{\to} \Gamma$ is an isomorphism. Also by boundary-irreducibility $\pi_1(T_i') \overset{\simeq}{\to} G_i$. In particular no component of $\partial T_i'$ is contractible in M. So no component of $\partial N - \text{int}\{T_1' \cup \ldots \cup T_k'\}$ is a disk. Also since N is aspherical and $\Gamma \neq \{e\}$, no component of ∂N is a sphere. So every component of $\partial N - \text{int}\{T_1' \cup \ldots \cup T_k'\}$ has Euler characteristic ≤ 0. Hence

$$\chi_-(\partial M) = -\sum_{i=1}^{k} \chi(T_i') \leq -\chi(\partial N) = -2\chi(N) = -2\chi(\pi_1(N)) = -2\chi(\Gamma)$$

which proves (1.8.2).

Next let U be the union of the toral components of ∂M, so U has τ components. Clearly $U \subseteq \partial N$. Let V denote the toral components of $\partial N - U$, and W the remaining components of ∂N. Then the number α of homotopy classes (in ∂M) of the annular components of ∂M is at most the maximum number of disjoint, non-nullhomotopic and pairwise nonhomotopic simple closed curves in $V \cup W$. The maximum number of such curves in V are just the number v of components of V, and this number in W is

$$-\frac{3}{2}\chi(W) = -\frac{3}{2}\chi(\partial N) = -3\chi(N) = -3\chi(\pi_1(N)) = -3\chi(\Gamma)$$

cf. (4.4). On the other hand,

$$\begin{aligned} v + \tau &\leq \#\text{ components of } \partial N \\ &\leq b_2(N) + 1 = b_2(\Gamma) + 1 \end{aligned}$$

Thus

$$\alpha + \tau \leq v - \frac{3}{2}\chi(W) + \tau \leq -3\chi(\Gamma) + b_2(\Gamma) + 1$$

which proves (1.8.3).

This finishes the proof of (5.2.3) and hence of theorem B.

q.e.d.

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