LINEAR DIFFERENTIAL EQUATIONS IN TWO VARIABLES OF RANK FOUR

Takeshi Sasaki*) and Masaaki Yoshida**)

*) Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

and

Kumamoto University
Department of Mathematics
Kumamomoto 860 Japan

**) Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

and

Kyushu University
Department of Mathematics
Fukuoka 812 Japan

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Introduction

We study systems of linear partial differential equations in two complex variables of rank (= complex dimension of the solution space) four. We state first our motivation and some backgrounds.

Let X be a Hermitian symmetric space, Γ be a properly discontinuous subgroup of the group $\mathrm{Aut}(X)$ of complex analytic automorphisms, M be the quotient variety $\Gamma\backslash X$ naturally equipped with the structure of an orbifold, $\pi : X \longrightarrow M$ be the natural projection and finally let $\psi = \pi^{-1} : M \longrightarrow X$ be the developing map of the orbifold M.

We think there should be a linear differential equation on M of which solution gives the developing map ψ. If such a differential equation exists, it is called the uniformizing equation of the orbifold M.

If X is a complex unit ball in $\mathbb{C}^n \subset \mathbb{P}^n$, then the uniformizing equation is a system of differential equations of the form

$$(0.1) \qquad \frac{\partial^2 z}{\partial x^i \partial x^j} = \sum p^k_{ij}(x)\frac{\partial z}{\partial x^k} + p^0_{ij}(x)z \qquad (i,j = 1,\dots,n)$$

where z is the unknown and $(x^1,\dots,x^n)$ is a system of local coordinates on M. The developing map $\psi : M \longrightarrow \mathbb{P}^n$ is given by the ratio of $n+1$ linearly independent solutions of (0.1). The system of coefficients $\{p^k_{ij}\}^n_{i,j,k=1}$ is the holomorphic projective structure on M naturally induced from $\pi : X \longrightarrow M$, and the coefficients $\{p^0_{ij}\}^n_{i,j=1}$ are determined by the projective structure. Namely, the integrability condition of the system (0.1) (the relation of the coefficients garanteeing (0.1) has $(n+1)$-dimensional solution space) says that each p^0_{ij} is a differential polynomial of $\{p^k_{ij}\}$. The differential equations of the form (0.1) are studied by many authors analytically and geometrically (see [Yos]).

We now turn to the case when $X = H \times H$ is the product of two upper half planes $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. Since X is a domain of the product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines, and since $\mathbb{P}^1 \times \mathbb{P}^1$ can be considered as a non-degenerate quadratic surface Q in the 3-dimensional projective space $\mathbb{P}^3$, the uniformizing equation should be a system of differential equations of rank four and the developing map $\psi : M \longrightarrow Q \subset \mathbb{P}^3$ is given by the ratio of four linearly independent solutions. In local coordinates (x,y) of M, such a differential equation can be written in the following form

$$(\mathrm{EQ})\begin{cases} \dfrac{\partial^2 z}{\partial x^2} = \ell \dfrac{\partial^2 z}{\partial x \partial y} + a\dfrac{\partial z}{\partial x} + b\dfrac{\partial z}{\partial y} + pz \\[2ex] \dfrac{\partial^2 z}{\partial y^2} = m\dfrac{\partial^2 z}{\partial x \partial y} + c\dfrac{\partial z}{\partial x} + d\dfrac{\partial z}{\partial y} + qz \end{cases} .$$

The coefficients ℓ and m gives the holomophic conformal structure $\ell dx^2 + 2dxdy + mdy^2$ on M naturally induced from the projection $\pi : X \longrightarrow M$ and the embeddings $X \hookrightarrow Q \hookrightarrow \mathbb{P}^3$.

In this way we encounter the differential equations of the form (EQ). Although some classical examples (so called hypergeometric differential equations in two variables) are known, this paper is the first systematic study of such differential equations, in which geometric as well as analytic studies are made.

We study the equation (EQ), especially its normalization and integrability condition, and establish some fundamental propositions and formulae. We make use of some differential geometric technique which is essential to endow the equation with a geometric meaning. It makes also possible to characterize (in terms of the coefficients) the property QR that "four linearly independent solutions are quadratically related".

To show that our study is effective, we construct the uniformizing equation on a Hilbert modular orbifold M found by F. Hirzebruch [Hir]. Recently R. Kobayashi and I. Naruki [K-N] succeeded to find the explicit conformal structure on M. Unlike the projective case (0.1), the conformal structure (i.e. ℓ, m) does not determine all the remaining coefficients. This phenomenon is characteristic in two dimensional conformal structure (see [Sas]). So we have to make a global consideration (the invariance under certain finite group) to find the differential equation . The equation thus obtained is the first non-trivial example of the equations of the form (EQ) which is non-hypergeometric. Its coefficients are given as follows:

$$\ell = -\frac{2-y^2-x^2y^2}{xy(1-x^2)} \qquad m = -\frac{2-x^2-x^2y^2}{xy(1-y^2)}$$

$$a = -\frac{3}{2}\frac{\partial}{\partial x}\log\frac{(1-x^2y^2)(2-x^2-y^2)}{1-x^2} + \frac{\ell}{2}\frac{\partial}{\partial y}\log\frac{(1-x^2y^2)^2(2-x^2-y^2)^2}{(1-y^2)^2(2-y^2-x^2y^2)}$$

$$c = \frac{m}{2}\frac{\partial}{\partial y}\log\frac{(2-x^2-x^2y^2)(1-x^2y^2)(2-x^2-y^2)}{(1-y^2)^2}$$

$$b = \frac{\ell}{2}\frac{\partial}{\partial x}\log\frac{(2-y^2-x^2y^2)(1-x^2y^2)(2-x^2-y^2)}{(1-x^2)^2}$$

$$d = -\frac{3}{2}\frac{\partial}{\partial y}\log\frac{(1-x^2y^2)(2-x^2-y^2)}{1-y^2} + \frac{m}{2}\frac{\partial}{\partial x}\log\frac{(1-x^2y^2)^2(2-x^2-y^2)^2}{(1-x^2)^2(2-x^2-x^2y^2)}$$

$$p = \frac{-2(x^2-y^2)}{(1-x^2)^2(1-y^2)} \qquad q = \frac{-2(y^2-x^2)}{(1-x^2)(1-y^2)^2}.$$

We also study the hypergeometric differential equations in two variables. Since those equations of rank four are studied very little, we make a large table of them, which will be a basic data in the future. We study the condition QR for them and express it in terms of their parameters. We further show that some of them (under QR) are transformed by an elementary (i.e. algebraic and logarithmic) change of variables into an equation of the following form:

$$(0.2) \quad \begin{cases} \dfrac{\partial^2 z}{\partial x^2} = p(x)z \\ \dfrac{\partial^2 z}{\partial y^2} = q(x)z \,. \end{cases}$$

This means that the monodromy group of the equation is the tensor product of those of the two ordinary differential equations (0.2).

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Contents

§ 1 Hypersurfaces in projective space

Let M be an n-dimensional complex local hypersurface in the complex projective space $\mathbb{P}^{n+1}$. We want to define a certain local projective invariant which is necessary to develop the theory of linear differential equations in the following sections. Let $i: M \longrightarrow \mathbb{P}^{n+1}$ be an immersion. We choose a lift e_0 of i to $\mathbb{C}^{n+2} - \{0\}$ which covers $\mathbb{P}^{n+1}$. The image $e_0(M)$ is locally a submanifold in $\mathbb{C}^{n+2}$. At each point $e_0(p)$, $p \in M$, we assicate a set of linearly independent vectors $e_1, \ldots, e_n, e_{n+1}$ such that the first n vectors $e_1, \ldots, e_n$ are tangent to $e_0(M)$. We call the set $e = \{e_0, \ldots, e_{n+1}\}$ a projective frame along M. We assume that $\det(e_0, \ldots, e_{n+1}) = 1$. The dependence of this frame on the point p is given by an infinitesimal equation

(1.1) $$de = \omega e,$$

where $\omega : p \longmapsto \omega(p)$ is a $sl(n+2, \mathbb{C})$ - valued holomorphic one form on M, which is called the Maurer- Cartan form. The integrability condition of (1.1) is given by

(1.2) $$d\omega = \omega \wedge \omega,$$

$$\text{i.e.}\quad d\omega_\beta^\alpha = \sum_{\gamma=0}^{n+1} \omega_\alpha^\gamma \wedge \omega_\gamma^\beta;\ 0 \le \alpha, \beta \le n+1.$$

Now the condition that $e_1, \ldots, e_n$ are tangent to $e_0(M)$ implies $\omega_0^{n+1} = 0$ and, by (1.2), we have

$$(1.3)\qquad \sum_{i=1}^{n} \omega_0^i \wedge \omega_i^{n+1} = 0.$$

Since $\omega^i := \omega_0^i$, $1 \leq i \leq n$, are linearly independent, Cartan's lemma shows the existence of a symmetric "tensor" h_{ij} such that

$$\omega_i^{n+1} = \sum_{j=1}^{n} h_{ij}\omega^j .$$

We define a symmetric 2-form φ_2 (the fundamental form) on M by

$$(1.4)\qquad \varphi_2 = \sum_{i,j=1}^{n} h_{ij}\omega^i\omega^j .$$

This form obviously depends on the choice of the frame e. Another possible frame $\tilde{e}$ is written as

$$\tilde{e} = ge \text{ , i.e. } \tilde{e}_\alpha = \sum_{\beta=0}^{n+1} g_\alpha^\beta e_\beta \quad \text{for} \quad g = (g_\alpha^\beta),$$

where the matrix g satisfies

$$g_0^i = g_0^{n+1} = g_i^{n+1} = 0, \quad 1 \leq i \leq n.$$

Let $\tilde{\omega}$ be the Maurer-Cartan form associated with $\tilde{e}$, then

$$\tilde{\omega} = dg\cdot g^{-1} + g\omega g^{-1}.$$

Using this identity we can see

$$\tilde{\varphi}_2 = \lambda^2\varphi_2, \quad \lambda = g_0^0 .$$

Namely the conformal class of φ_2 is independent of the choice of a frame. Now assume that this form is non-degenerate, equivalently that the matrix (h_{ij}) is non-degenerate. Then, by a frame change, we may assume

$$\det(h_{ij}) = 1, \quad \omega_0^0 + \omega_{n+1}^{n+1} = 0. \tag{1.5}$$

We next define a new quantity h_{ijk} by

$$\sum_{k=1}^n h_{ijk}\omega^k = dh_{ij} - \sum_{k=1}^n h_{ik}\omega_j^k - \sum_{k=1}^n h_{jk}\omega_i^k . \tag{1.6}$$

Under the condition (1.5), it is seen that h_{ijk} is symmetric with respect to subindices and behaves like a tensor. Namely under a frame change it varies as

$$\lambda\tilde{h}_{ijk} = \sum_{p,q,r=1}^n h_{pqr}g_i^p g_j^q g_k^r . \tag{1.7}$$

So we put

$$\varphi_3 = \sum_{i,j,k=1}^n h_{ijk}\omega^i\omega^j\omega^k . \tag{1.8}$$

This is called <u>the Fubini-Pick cubic form.</u> From (1.7) we can see

$$\tilde{\varphi}_3 = \lambda^2\varphi_3.$$

Hence, especially, the vanishing of φ_3 is independent of the choice of a frame satisfying (1.5). Moreover we have

Lemma (L. Berwald; see [Fla], § 12). The cubic form φ_3 vanishes if and only if the hypersurface is locally a quadric.

Let us recall the normalization (1.5) of the frame implies the so-called apolarity condition which is written as

$$(1.9) \qquad \sum_{i,j=1}^{n} h^{ij}h_{ijk} = 0, \text{ where } (h^{ij}) = (h_{ij})^{-1}.$$

For the detailed description refer [Sas].

§ 2 Linear differential equations in two variables of rank four

For notational simplicity, we denote by f_x (resp. f_y) the partial derivative of a function f with respect to x (resp. y). Consider a linear differential equation

$$
\text{(EQ)} \qquad \begin{cases} z_{xx} = \ell\, z_{xy} + a\, z_x + b\, z_y + pz \\ z_{yy} = m\, z_{xy} + c\, z_x + d\, z_y + pz \end{cases}
$$

where (x,y) are independent variables and z is the unknown. We assume throughout the paper that the rank (= dimension of the solution space) is four. Differentiate (EQ) to obtain

$$
\text{(2.1)} \qquad
\begin{aligned}
(1-\ell m)\, z_{xxy} &= \{\ell_y + a + bm + \ell(m_x + d + c\ell)\} z_{xy} \\
&\quad + \{a_y + bc + \ell(c_x + ca) + \ell q\} z_x \\
&\quad + \{b_y + bd + \ell(d_x + bc) + p\} z_y \\
&\quad + \{p_y + bq + \ell(q_x + cp)\} z \\
(1-\ell m)\, z_{xy} &= \{m_x + d + c\ell + m(\ell_y + \sigma + bm)\} z_{xy} \\
&\quad + \{c_x + ac + m(a_y + bc) + q\} z_x \\
&\quad + \{d_x + bc + m(b_y + bd) + mp\}\ z_y \\
&\quad + \{q_x + cp + m(p_y + bq)\} z.
\end{aligned}
$$

We have

$$
\text{(2.2)} \qquad 1 - \ell m \neq 0,
$$

otherwise the rank would be smaller than four. Let z^0,

z^1, z^2 and z^3 be linear independent solutions of (EQ) and put $z = (z^0, z^1, z^2, z^3)$. z defines a map from (x,y)-space into $\mathbb{P}^3$. The image is locally a surface S. The geometric treatment of this surface in which we are interested will be given in § 4. In this section we would like to present some basic formulae. Let us introduce a function θ by

$$(2.3) \qquad e^{2\theta} = \det(z, z_x, z_y, z_{xy}).$$

We call the function $e^{2\theta}$ <u>the normalization factor</u> of the equation (EQ). By differentiating (2.3) we have

$$2\theta_x = e^{-2\theta}\{\det(z, z_{xx}, z_y, z_{xy}) + \det(z, z_x, z_y, z_{xxy})\}$$

$$2\theta_y = e^{-2\theta}\{\det(z, z_x, z_{yy}, z_{xy}) + \det(z, z_x, z_y, z_{xyy})\}.$$

Then making use of (2.1) we get

$$(2.4) \qquad \begin{aligned} 2\theta_x &= a + \frac{1}{1-\ell m}\{\ell_y + a + bm + \ell(m_x + d + c\ell\} \\ 2\theta_y &= d + \frac{1}{1-\ell m}\{m_x + d + c\ell + m(\ell_y + a + bm)\}. \end{aligned}$$

For the sake of simplicity we put

$$(2.5) \qquad \begin{aligned} B^0 &= \{p_y + bq + \ell(q_x + cp)\}/(1-\ell m) & C^0 &= \{q_x + cp + m(p_y + bq)\}/(1-\ell m) \\ B^1 &= (A + \ell q)/(1-\ell m) & C^1 &= (C + q)/(1-\ell m) \\ B^2 &= (B + p)/(1-\ell m) & C^2 &= (D + mp)/(1-\ell m) \\ B^3 &= \{\ell_y + a + bm + \ell(m_x + d + c\ell)\}/(1-\ell m) & C_3 &= \{m_x + d + c\ell + m(\ell_y + a + bm)\}/(1-\ell m), \end{aligned}$$

where

$$A = a_y + bc + \ell(c_x + ac) \qquad B = b_y + bd + \ell(d_x + bc)$$
$$C = c_x + ac + m(a_y + bc) \qquad D = d_x + bc + m(b_y + bd).$$

In these abbreviations, (2.1) and (2.4) are written as

$$(2.1)' \quad \begin{cases} z_{xxy} = B^3 z_{xy} + B^1 z_x + B^2 z_y + B^0 z \\ z_{xyy} = C^3 z_{xy} + C^1 z_x + C^2 z_y + C^0 z \end{cases}$$

$$(2.4)' \quad B^3 = 2\theta_x - a, \quad C^3 = 2\theta_y - d.$$

We next choose a projective frame $e = \{e_0, e_1, e_2, e_3\}$ along the surface S as follows:

$$(2.7) \quad e_0 = z,\ e_1 = z_x,\ e_2 = z_y,\ e_3 = e^{-2\theta} z_{xy}\ .$$

Since

$$de_0 = e_1 dx + e_2 dy,$$

vectors e_1 and e_2 are tangent to S. The Maurer-Cartan form ω defined by $de = \omega e$ is equal to

$$(2.8) \quad \omega = \begin{pmatrix} 0 & dx & dy & 0 \\ p\,dx & a\,dx & b\,dx & e^{2\theta}(\ell dx + dy) \\ q\,dy & c\,dy & d\,dy & e^{2\theta}(m dy + dx) \\ e^{-2\theta}(B^0 dx + C^0 dy) & e^{-2\theta}(B^1 dx + C^1 dy) & e^{-2\theta}(B^2 dx + C^2 dy & -ad_x - ddy \end{pmatrix}$$

We assumed that (EQ) is of rank four. This condition is expressed by a system of algebraic relations of the differentials of the coefficients, which is called <u>the integrability condition</u> of (EQ). Let us define as usual the curvature form Ω by

$$(2.9) \qquad \Omega = d\omega - \omega \wedge \omega.$$

Then the integrability condition of (EQ) is nothing but the identity

$$\Omega = 0.$$

On the contrary this garantees that (EQ) is of rank four. We will rewrite this condition explicitly in terms of coefficients of (EQ).

<u>Lemma 2.1</u> Ω_0^α $(0 \leq \alpha \leq 3)$, Ω_1^i $(0 \leq i \leq 2)$, Ω_2^i $(0 \leq i \leq 2)$ and Ω_3^3 vanish identically.

<u>Proof.</u> We prove $\Omega_1^0 = 0$ as an example. By definition,

$$\Omega_1^0 = d(pdx) - bqdx \wedge dy - e^{2\theta}(\ell dx + dy) \wedge e^{-2\theta}(B^0dx + C^0dy)$$

$$= -(p_y + bq + \ell C^0 - B^0)\, dx \wedge dy.$$

Hence (2.5) shows $\Omega_1^0 = 0$. Other identities can be seen similarly from (2.5). □

The remaining components of Ω are calculated as follows:

$$\Omega_1^3 = \{(2\theta_x - a - B^3) - (2\theta_y - d - C^3)\ell\}\ dx \wedge dy$$

$$\Omega_2^3 = \{(2\theta_x - a - B^3)m - (2\theta_y - d - C^3)\ \}dx \wedge dy$$

$$\Omega_3^1 = e^{-2\theta}\{(2\theta_y - d)B^1 - B_y^1 - (2\theta_x - 2a)C^1 - cB^2 + C^0 + C_x^1\}dx \wedge dy$$

$$\Omega_3^2 = e^{-2\theta}\{-(2\theta_x - a)C^2 + C_x^2 + (2\theta_y - 2d)B^2 + bC^1 - B^0 - B_y^2\}\ dx \wedge dy$$

$$\Omega_3^0 = e^{-2\theta}\{(2\theta_y - d)B^0 - (2\theta_x - a)C^0 + pC^1 - qB^2 - B_y^0 + C_x^0\}\ dx \wedge dy.$$

It is now easy to rewrite the right handsides of these forms in terms of coefficients and as a result we have

<u>Proposition 2.2</u> The equation (EQ) is integrable if and only if the following conditions hold.

(IC0) $\quad (a + B^3)_y = (d + C^3)_x$

(IC1) $\quad \ell q_y - 2q_x - mp_y - (\ell\xi_y - \xi_x - 2\ell_y)q = R^1$

(IC2) $\quad mp_x - 2p_y - \ell q_x - (m\xi_x - \xi_y - 2m_x)p = R^2$

(IC3) $\quad p_{yy} - q_{xx} - mp_{xy} + \ell q_{xy}$

$$= cp_x - bq_y + (d + 2m_x + \xi_y - m\xi_x)p_y - (a + 2\ell_y + \xi_x - \ell\xi_y)q_x$$
$$+ (ma_y + 2c_x - 2c\ell_y - \ell c_y - c(\xi_x - \ell\xi_y))p$$
$$- (\ell d_x + 2b_y - 2bm_x - mb_x - b(\xi_y - m\xi_x))q,$$

where

$$(2.10)\qquad \begin{aligned} R^1 &= (d + 2C^3 + \xi_y)A - (2B^3 + \xi_x)C - cB + C_x - A_y \\ R^2 &= (a + 2B^3 + \xi_x)D - (2C^3 + \xi_y)B - bC + B_y - D_x \\ \xi &= \log(1 - \ell m). \end{aligned}$$

Proof. From the vanishing of Ω_1^3 and Ω_2^3 follows $2\theta_x = a + B^3$ and $2\theta_y = d + C^3$, namely (2.4)'. Hence (IC0). The conditions (IC1), (IC2) and (IC3) correspond to $\Omega_3^1 = 0$, $\Omega_3^2 = 0$ and $\Omega_3^0 = 0$ respectively, if we replace $2\theta_x$ and $2\theta_y$ by (2.4)'. Conversely, form (IC0) we can find θ up to an additive constant such that (2.4)' holds. This assures $\Omega_1^3 = \Omega_2^3 = 0$ for this choice of θ and furthermore (IC 1,2,3) implies the vanishing of Ω_3^1, Ω_3^2 and Ω_3^0 . □

§ 3 Transformation formulae

In this section we will prepare some transformation formulae of the differential equation (EQ). We first obtain the transformed equation when we perform a coordinate change $(x,y) \longrightarrow (u,v)$. Let

$$(3.1) \qquad \Delta = u_x v_y - u_y v_x$$

be the jacobian determinant of the coordinate change. We put

$$(3.2) \qquad \begin{aligned} \lambda &= \ell v_y^2 - 2v_x v_y + m v_x^2 \\ \mu &= \ell u_y^2 - 2u_x u_y + m u_x^2 \\ \nu &= \ell u_x v_y - u_x v_y - u_y v_x + m u_x v_x \end{aligned}$$

and define

$$(3.3) \qquad \begin{aligned} \alpha &= (v_x^2 - \ell v_x v_y)/\Delta, \quad \beta = (v_y^2 - m v_x v_y)/\Delta \\ \gamma &= (u_x^2 - \ell u_x u_y)/\Delta, \quad \delta = (u_y^2 - m u_x u_y)/\Delta \\ R(u) &= u_{xx} - (\ell u_{xy} + a u_x + b u_y) \\ S(u) &= u_{yy} - (m u_{xy} + c u_x + d u_y) \\ R(v) &= v_{xx} - (\ell v_{xy} + a v_x + b v_y) \\ S(v) &= v_{yy} - (m v_{xy} + c v_x + d v_y) \ . \end{aligned}$$

Then a calculation shows

<u>Proposition 3.1</u> Perform a coordinate change of the equation (EQ) from (x,y) to (u,v) and denote the coefficients of

the transformed equation by the same letter with bars. Then

(3.4)
$$\begin{aligned} \bar{\ell} &= -\frac{\lambda}{\nu} & \bar{m} &= -\frac{\mu}{\nu} \\ \bar{a} &= \frac{1}{\nu}(R(u)\beta - S(u)\alpha) & \bar{c} &= \frac{1}{\nu}(S(u)\gamma - R(u)\delta) \\ \bar{b} &= \frac{1}{\nu}(R(v)\beta - S(v)\alpha) & \bar{d} &= \frac{1}{\nu}(S(v)\gamma - R(v)\delta) \\ \bar{p} &= \frac{1}{\nu}(\alpha q - \beta p) & \bar{q} &= \frac{1}{\nu}(\delta p - \gamma q) \ . \end{aligned}$$

The normalization factor changes as

(3.5)
$$e^{2\bar{\theta}} = -\frac{\nu}{\Delta^3} e^{2\theta}.$$

We next derive a formula when the unknown function z is multiplied by a factor e^{ρ}. If we put $z = e^{-\rho}w$, then it is easy to see that

(3.6)
$$\begin{aligned} z_x &= e^{-\rho}(w_x - \rho_x w) \qquad\qquad z_y = e^{-\rho}(w_y - \rho_y w) \\ z_{xx} &= e^{-\rho}\{w_{xx} - 2\rho_x w_x - (\rho_{xx} - \rho_x^2)w\} \\ z_{xy} &= e^{-\rho}\{w_{xy} - \rho_x w_y - \rho_y w_x - (\rho_{xy} - \rho_x\rho_y)w\} \\ z_{yy} &= e^{-\rho}\{w_{yy} - 2\rho_y w_y - (\rho_{yy} - \rho_y^2)w\} . \end{aligned}$$

From these identities we have

Proposition 3.2 Perform a change of the unknown z by multiplying a factor e^{ρ}. Then the coefficients of the transformed equation, which are denoted by the same letter with primes, are gvien as follows

$$\ell' = \ell \qquad m' = m$$

$$a' = a + 2\rho_x - \ell\rho_y \qquad c' = c - m\rho_y$$

$$b' = b - \ell\rho_x \qquad d' = d + 2\rho_y - m\rho_x$$

$$p' = p - a\rho_x - b\rho_y + (\rho_{xx} - \rho_x^2) - \ell(\rho_{xy} - \rho_x\rho_y) \qquad q' = q - c\rho_x - d\rho_y + (\rho_{yy} - \rho_y^2) - m(\rho_{xy} - \rho_x\rho_y)$$

The normalization factor changes as follows

(3.8) $$e^{2\theta'} = e^{4\rho+2\theta} \quad .$$

§ 4 Linear differential equations defining maps into quadrics

§§ 4.1 Conformal structures

We have denoted by S the surface defined by the equation (EQ) by use of their independent solutions. To this surface we have associated a projective frame defined by (2.7). Then, from the expression of the corresponding Maurer-Cartan form in (2.8) and by the definition (1.4), the conformal structure φ of S is given by

$$\varphi = \ell dx^2 + 2dxdy + mdy^2 . \tag{4.1}$$

§§ 4.2 Fubini-Pick cubic form

We want to calculate the Fubini-Pick cubic form i.e. its coefficients h_{ijk} of S. For this purpose it is necessary to modify the frame e so that it satisfies (1.5). Put

$$\bar{e}_0 = e_0, \bar{e}_1 = \lambda e_1,\ \bar{e}_2 = \lambda e_2,\ \bar{e}_3 = \lambda^{-2} e_3 , \tag{4.2}$$

λ being a function to be determined. With respect to this frame

$$\bar{\omega}^1 = \lambda^{-1} dx , \qquad \bar{\omega}^2 = \lambda^{-1} dy ,$$
$$\bar{\omega}_1^3 = \lambda^4 e^{2\theta}(\ell\omega^1 + \omega^2),\quad \bar{\omega}_2^3 = \lambda^4 e^{2\theta}(m\omega^2 + \omega^1).$$

Hence the tensor $\bar{h}_{ij}$ is given by

$$(4.3) \qquad \bar{h}_{ij} = \lambda^4 e^{2\theta}\begin{pmatrix} \ell & 1 \\ 1 & m \end{pmatrix}$$

and, if we take

$$(4.4) \qquad \lambda = e^{-\frac{1}{2}\theta}(\ell m - 1)^{-\frac{1}{8}} \quad ,$$

then $\det \bar{h}_{ij} = 1$. In this choice of λ

$$\begin{aligned} d\bar{e}_3 &= -2\, d \log \lambda\; \bar{e}_3 + \lambda^{-2} d\, e_3 \\ &\equiv -(2\, d \log \lambda + a\, dx + d\, dy)\, \bar{e}_3 \quad \mathrm{mod}\ (\bar{e}_0, \bar{e}_1, \bar{e}_2). \end{aligned}$$

Again modify $\bar{e}$ defining $\tilde{e}$ by

$$(4.5) \qquad \tilde{e}_0 = \bar{e}_0, \tilde{e}_1 = \bar{e}_1,\ \tilde{e}_2 = \bar{e}_2,\ \tilde{e}_3 = \bar{e}_3 + \mu\bar{e}_1 + \nu\bar{e}_2$$

With respect to this frame we have

$$\begin{aligned} \tilde{\omega}_3^3 &= -2\, d \log \lambda - a\, dx - d dy + \\ &\quad + \lambda^3 e^{2\theta}\{\mu(\ell\, dx + dy) + \nu(m\, dy + dx)\}. \end{aligned}$$

In order that $\tilde{\omega}_3^3 = 0$ it is sufficient to choose μ and ν so that they satisfy the following equalities.

$$\begin{aligned} \lambda^3 e^{2\theta}(\mu\ell + \nu) &= 2\lambda_x/\lambda + a \\ \lambda^3 e^{2\theta}(\mu + \nu m) &= 2\lambda_y/\lambda + d\ . \end{aligned}$$

Namely,

$$(4.6)\qquad \begin{cases} \mu = \lambda^{-3}e^{-2\theta}(\ell m-1)^{-1}\{(2\lambda_x/\lambda+a)m-(2\lambda_y/\lambda+d)\} \\ \nu = \lambda^{-3}e^{-2\theta}(\ell m-1)^{-1}\{(2\lambda_y/\lambda+d)\ell-(2\lambda_x/\lambda+a)\}. \end{cases}$$

Now that we have chosen a projective frame $\tilde{e}$ which satisfies the condition (1.5), we can compute h_{ijk}. We now drop $\sim$'s for simplicitiy. A computation shows

$$\begin{pmatrix} \omega_1^1 & \omega_1^2 \\ \omega_2^1 & \omega_2^2 \end{pmatrix} = \begin{pmatrix} adx+d\log\lambda-\mu\lambda^3e^{2\theta}(\ell dx+dy) & bdx-\nu\lambda^3e^{2\theta}(\ell dx+dy) \\ cdy-\mu\lambda^3e^{2\theta}(dx+mdy) & ddy+d\log\lambda-\nu\lambda^3e^{2\theta}(dx+mdy) \end{pmatrix}.$$

Hence by defintion (1.6) we have

$$\begin{aligned} \Sigma h_{11i}\omega^i &= dh_{11} - 2\Sigma h_{1i}\omega_1^i \\ &= d(\lambda^4e^{2\theta}\ell) - 2\lambda^4e^{2\theta}\ell\{adx+d\log\lambda - \mu\lambda^3e^{2\theta}(\ell dx+dy)\} \\ &\qquad - 2\lambda^4e^{2\theta}\{bdx-\nu\lambda^3 2^{2\theta}(\ell dx+dy)\}. \end{aligned}$$

By (4.6),

$$(4.7.1)\qquad \begin{aligned} h_{111} &= \lambda^5e^{2\theta}\{6\ell\lambda_x/\lambda + 2\ell\theta_x + \ell_x - 2b\} \\ h_{112} &= \lambda^5e^{2\theta}\{4\lambda_x/\lambda + 2\ell\lambda_y/\lambda + 2\ell\theta_y + \ell_y + 2a\}. \end{aligned}$$

Similarly,

$$\begin{cases} h_{122} = \lambda^5 e^{2\theta}\{4\lambda_y/\lambda + 2m\lambda_x/\lambda + 2m\theta_x + m_x + 2d\} \\ h_{222} = \lambda^5 e^{2\theta}\{6m\lambda_y/\lambda + 2m\theta_y + m_x - 2c\} \end{cases} .$$

Proposition 4.1 The cubic form of the surface S associated with the differential equation (EQ) vanishes identically if and only if

$$(4.8.1) \quad \begin{cases} b = \frac{\ell}{2}(\frac{\ell_x}{\ell} - \frac{3}{4}\xi_x - \theta_x) \\ c = \frac{m}{2}(\frac{m_y}{m} - \frac{3}{4}\xi_y - \theta_y) \end{cases}$$

$$(4.8.2) \quad \begin{cases} a = \frac{1}{4}\xi_x + \theta_x - \frac{\ell}{2}(\frac{\ell_y}{\ell} - \frac{1}{4}\xi_y + \theta_y) \\ d = \frac{1}{4}\xi_y + \theta_y - \frac{m}{2}(\frac{m_x}{m} - \frac{1}{4}\xi_x + \theta_x) \end{cases} .$$

Proof. Recall $\xi = \log(1 - \ell m)$. Then it is enough to use (4.4). ∎

Remark 4.2 The tensor $\{h_{ijk}\}$ satisfies the apolarity condition (1.9). In this instance

$$mh_{111} - 2h_{112} + \ell h_{122} = 0$$

$$mh_{112} - 2h_{122} + \ell h_{222} = 0.$$

So, if $\ell m - 4 \neq 0$, then the vanishing of h_{ijk} is equivalent to $h_{111} = h_{222} = 0$, i.e. the condition (4.8.1). The condition (4.8.2) follows from (4.8.1).

§§ 4.3 Equations with the property QR

The equation (EQ) is said to have property QR if four linearly independent solutions are quadratically related, namely if its coefficients satisfy (4.8) (cf. Lemma in § 1). Now assume that (EQ) has this property. Since $\varphi = \ell dx^2 + 2dxdy + mdy^2$ is non-degenerate, we can find new coordinates such that $\ell = m = 0$. (Such a coordinate system is called asymptotic). Then (4.8) implies $b = c = 0$ and $a = \theta_x$, $d = \theta_y$. Perform a transformation of the unknown multiplying $e^{-\theta/2}$. Then the equation changes as

$$(4.9.1) \qquad z_{xx} = p'z, \qquad p' = p + \frac{\theta_x^2 - \theta_{xx}}{4}$$

$$(4.9.2) \qquad z_{yy} = q'z, \qquad q' = q + \frac{\theta_y^2 - \theta_{yy}}{4}.$$

The integrability condition of (2.13) is easily seen to be

$$(4.10) \qquad q'_x = p'_y = 0.$$

Let z^1, z^2 (resp. z^3, z^4) be solutions of (4.9.1) (resp. 4.9.2). Then z^1z^3, z^2z^3, z^1z^4 and z^2z^4 are four linearly independent solutions. So we have proved

Proposition 4.3 Assume that (EQ) has the property QR. Then it is (locally) reduced to the pair of ordinary differential equations (4.9) by a coordinate change and by a renormalization.

This reduction corresponds to the fact that a quadric in $\mathbb{P}^3$

is ruled and decomposed into $\mathbb{P}^1 \times \mathbb{P}^1$.

Let us examine the integrability condition. The condition (4.10) corresponds to (IC1) and (IC2) in Proposition 2.2. The condition (IC3) is $p_{yy} - q_{xx} = 0$ and follows from (4.10). More precisely we have

Proposition 4.4 Given functions $\ell, m, a, b, c, d, p, q$ and θ assume that these functions satisfy the condition (4.8) and identities (IC1) and (IC2). Then the differential equation (EQ) with $\ell, m, \ldots, p, q$ as its coefficients is integrable with $e^{2\theta}$ as the normalization factor and has the property QR.

Proof. Substituting the expression of a, b, c and d in (4.8) into the right handsides of (2.4), we see the identities (2.4), namely the vanishing $\Omega_1^3 = \Omega_2^3 = 0$, and hence (IC0). We will examine (IC3). For this purpose we follow again the reduction process stated above in terms of frames. If we perform a coordinate change from (x,y) to (u,v), then the corresponding frame e defined in (2.7) and $\tilde{e}$ defined analogously with respect to the coordinate (u,v) are related as follows.

$$\tilde{e} = g_1 e ,$$

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_u & y_u & 0 \\ 0 & x_v & y_v & 0 \\ * & * & * & 1/\delta \end{pmatrix} \quad \delta = x_u y_v - x_u y_u \ .$$

We next renormalize the unknown z by multiplying e^{ρ}. Denote by $\bar{e}$ the frame associated with $e^{\rho}z$. Then

$$\bar{e} = g_2\tilde{e} \ ,$$

$$g_2 = \begin{pmatrix} e^{\rho} & & & \\ \rho_u & e^{\rho} & & \\ \rho_v & 0 & e^{\rho} & \\ * & * & * & e^{-3\rho} \end{pmatrix} .$$

Hence $\bar{e} = g_2 g_1 e$. Now let $\bar{\Omega}$ be the curvature form with respect to $\bar{e}$. Then, since $\bar{\Omega}$ is a tensor,

$$\bar{\Omega} = (g_2 g_1)^{-1}\Omega(g_1 g_2) \ .$$

From this relation, using also (IC0), we see

$$\bar{\Omega}_3^1 = e^{4\rho}\delta(x_u\Omega_3^1 + x_u\Omega_3^2)$$
$$\bar{\Omega}_3^2 = e^{4\rho}\delta(y_u\Omega_3^1 + y_u\Omega_3^2)$$
$$\bar{\Omega}_3^0 \equiv e^{4\rho}\delta\Omega_3^0 \mod (\Omega_3^1,\Omega_3^2) .$$

We have seen that the identity $\bar{\Omega}_3^0 = 0$ follows from $\bar{\Omega}_3^1 = \bar{\Omega}_3^2 = 0$ provided that the coordinates (u,v) is choosen so that $\ell = m = 0$ and that $\rho = -\frac{1}{2}\theta$. So if we assume $\Omega_3^1 = \Omega_3^2 = 0$ then from the above three identities follows $\Omega_3^0 = 0$ which is the condition (IC3). Hence in view of

Proposition 2.2, we have proved the assertion. ■

§§ 4.4 Moduli of equations with a fixed conformal structure

Assume further that we are given two equations with the same conformal structure and the same normalization factor. Let p_i, q_i $(i = 1,2)$ be the respective coefficients. Put $P = p_1 - p_2$ and $Q = q_1 - q_2$. Then they satisfy linear differential equation

$$(4.11) \qquad \begin{cases} \ell Q_y - 2Q_x - mP_y - (\ell\xi_y - \xi_x - 2\ell_y)Q = 0 \\ mP_x - 2P_y - \ell Q_x - (m\xi_x - \xi_y - 2m_x)P = 0. \end{cases}$$

This gives the moduli of (EQ) with the property QR. When we are concerned with global equations (e.g. the Fuchsian differential equations on $\mathbb{P}^2$), to determine the solution space of (4.11) turns out to be a very important problem. Here we give an example.

Example 4.5 . Let us consider the equation F_2 defined by P. Appell (see §§ 6.1). The corresponding equation (4.11) which is defined on $\mathbb{P}^2(x,y)$, (x,y) being the inhomogeneous coordinate, is

$$(4.12) \qquad \begin{cases} y(1-y)Q_y - 2(1-x)(1-y)Q_x - x(1-x)P_y - (3y-2)Q = 0 \\ x(1-x)P_x - 2(1-x)(1-y)P_y - y(1-y)Q_x - (3x-2)P = 0. \end{cases}$$

The following are special solutions of this equation:

$$(1) \quad \begin{cases} p = \dfrac{\beta}{x^2(1-x)} \\[2ex] q = \dfrac{\beta'}{y^2(1-y)} \end{cases} \qquad\qquad (2) \begin{cases} p = \dfrac{1}{(1-x)(2-x-y)} \\[2ex] q = \dfrac{1}{(1-y)(2-x-y)} \end{cases} .$$

The first one appears in F_2. The second solution is related to the fact that the equation F_4 (see §§ 6.1) can be transformed to an equation which has the same conformal structure as that of F_2 (see §§ 6.4). It is interesting to find all rational solutions of (4.12).

§5 Uniformizing differential equation of a Hilbert modular orbifold

§§ 5.1 Hilbert modular orbifold M on $\mathbb{P}^2$ ([Hir])

Let $\mathcal{O}$ be the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{2})$ and let $\Gamma(2)$ be the principal congruence subgroup of $SL(2,\mathcal{O})$ associated with the ideal (2) of $\mathcal{O}$:

$$\Gamma(2) = \{g \in SL(2,\mathcal{O}) \mid g \equiv \text{identity} \bmod (2)\}.$$

Let further $\Gamma(2) \subset \Gamma' \subset SL(2,\mathcal{O})$ be the group such that $\Gamma'/\Gamma(2)$ is the center of $SL(2,\mathcal{O})/\Gamma(2)$. We note that $[\Gamma':\Gamma(2)] = 2$ and $SL(2,\mathcal{O})/\Gamma'$ is isomorphic to the symmetric group of degree four. The group $SL(2,\mathcal{O})$ acts on $H \times H$ as follows

$$g:(z_1,z_2) \longmapsto (g'z_1,g''z_2) \qquad g \in SL(2,\mathcal{O})$$

where g' and g'' are two embeddings of $SL(2,\mathcal{O})$ into $SL(2,\mathbb{R})$. Let finally Γ be the group generated by Γ' and the involution

$$\tau:(z_1,z_2) \longmapsto (z_2,z_1) .$$

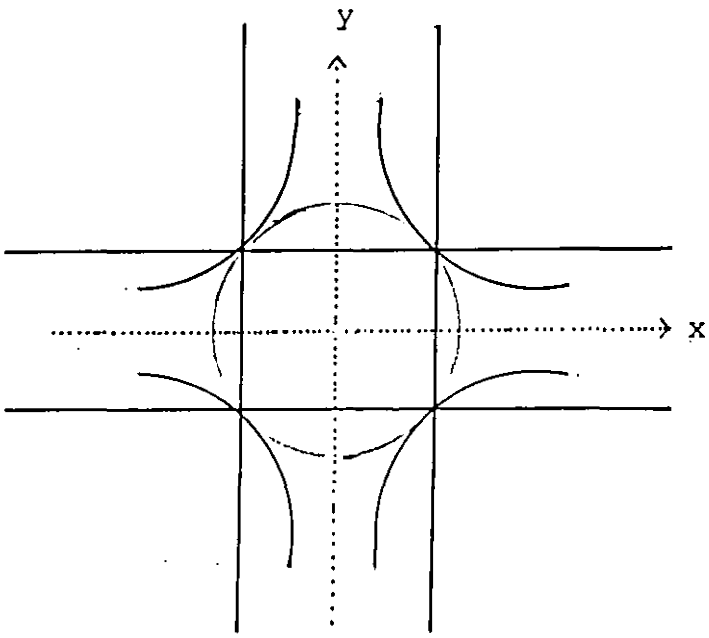
The factor space $H \times H/\Gamma$ is isomorphic to $\mathbb{P}^2$ minus six points. The ramification locus of the natural projection

$$\pi:H \times H \longrightarrow \mathbb{P}^2$$

is given by $D = \{D = 0\}$ where

$$D = (1-x^2)(1-y^2)(1-x^2y^2)(2-x^2-y^2) \tag{5.1}$$

provided that the affine coordinates x,y are suitably chosen. The above six points are exactly six multiple points of D. The branching of π is caused by the involution τ and its conjugates, and so the branching index is two. The projective plane $\mathbb{P}^2$ equipped with the ramification locus D and the index 2 will be refered to as the orbifold M



Orbifold M

The group G of projective transformations which leave M invariant is order 48 which is generated by three reflections:

$$T_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

with respect to the associated homogeneous coordinates.

§§ 5.2 Kobayashi-Naruki's result ([K-N])

Since a quadratic surface in $\mathbb{P}^3$ has a natural holomorphic conformal structure (see § 1), the factor space M has a holomorphic conformal structure outside D. The conformal structure on M is explicitly known. In terms of the inhomogeneous coordinates (x,y), it is given by

$$\varphi = \ell dx^2 + 2dxdy + mdy^2$$

where

$$(5.3) \qquad \ell = -\frac{2-y^2-x^2y^2}{xy(1-x^2)}, \quad m = -\frac{2-x^2-x^2y^2}{xy(1-y^2)} .$$

§§ 5.3 Differential equation giving the developing map

We would like to find the differential equation of the form (EQ) which gives the inverse map of π :

$$\psi : \mathbb{P}^2 \longrightarrow H \times H$$

The space $H \times H$ can be considered as a domain in the non-degenerate quadratic surface Q (which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$) in $\mathbb{P}^3$:

$$\begin{array}{ccccccc} H \times H & \subset & \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow[\sim]{i} & Q & \hookrightarrow & \mathbb{P}^3 \\ (z^1, z^2) & \mapsto & ([1,z^1],[1,z^2]) & \mapsto & [1,z^1,z^2,z^1z^2] & & \end{array}$$

where [ ] stands for the ratio. Accordingly we have

$$\mathrm{Aut}(H\times H) \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow[\sim]{i_*} \mathrm{Aut}(\mathbb{P}^3, Q)$$

where $\mathrm{Aut}(X)$ stands for the group of complex analytic automorphisms of a manifold X and $\mathrm{Aut}(\mathbb{P}^3,Q)$ stands for the group of projective transformations leaving Q invariant. This implies that the multi-valued map

$$z := i\circ\psi : \mathbb{P}^2 \longrightarrow Q \subset \mathbb{P}^3$$

has the monodromy $i_*\Gamma \subset PGL(4,\mathbb{C})$. If we fix inhomogeneous coordinates (x,y) on $\mathbb{P}^2$ and homogeneous coordinates $(z^0,\dots,z^3)$ on $\mathbb{P}^3$, then the 4-vector function $(z^0(x,y),\dots,z^3(x,y))$ has the monodromy in $GL(4,\mathbb{C})$.

Now consider the linear differential equation with the unknown z and the independent variables (x,y):

$$(5.4)\qquad \begin{vmatrix} z & z^0 \dots & z^3 \\ z_x & z_x^0 \dots & z_x^3 \\ z_y & z_y^0 \dots & z_y^3 \\ z_{xy} & z_{xy}^0 \dots & z_{xy}^3 \\ z_{xx} & z_{xx}^0 \dots & z_{xx}^3 \end{vmatrix} = 0, \qquad \begin{vmatrix} z & z^0 \dots & z^3 \\ z_x & z_x^0 \dots & z_x^3 \\ z_y & z_y^0 \dots & z_y^3 \\ z_{xy} & z_{xy}^0 \dots & z_{xy}^3 \\ z_{yy} & z_{yy}^0 \dots & z_{yy}^3 \end{vmatrix} = 0$$

then the ratios of the coefficients are single valued (because $z^0,\dots,z^3$ change only linearly). Studying local properties of the coefficients in Lemma 5.5 and 5.7 we will know that the coefficients are rational functions. Without loosing generality we can assume

$$\begin{vmatrix} z^0 & \dots & z^3 \\ z_x^0 & \dots & z_x^3 \\ z_y^0 & \dots & z_y^3 \\ z_{xy}^0 & \dots & z_{xy}^3 \end{vmatrix} \neq 0$$

so that the equation (5.4) is of the form (EQ).

On the other hand paragraph two tells us the coefficients ℓ and m in (5.4) give the conformal structure on M. Since Kobayashi-Naruki determined ℓ and m as in §§ 5.2, we shall determine the remaining coefficients. The equation (5.4) is uniquely determined by the group Γ up to the normalization (multiplication of any non-zero functions to $z^0,\dots,z^3$). Normalize the equation as follows

$$(5.5) \qquad e^{2\theta} = (1 - \ell m)^{-\frac{7}{2}} (xy)^{-6} .$$

<u>Lemma 5.1</u>' The normalization factor is G-invariant. Namely, when the coordinate transformation $(x,y) \longmapsto (u,v)$ induced by $g \in G$ changes (see Proposition 3.1) $\ell \longrightarrow \bar{\ell},\dots,q \longrightarrow \bar{q}$, we have

$$e^{2\bar{\theta}} = -\frac{\nu}{\Delta^3} e^{2\theta} = (1 - \bar{\ell}\,\bar{m})^{-\frac{7}{2}} (uv)^{-6}.$$

<u>Proof</u> Straightforward calculation. ∎

Then by proposition 4.1 the four coefficients are

$$a = -\frac{3}{2}\left(\log\frac{(1-x^2y^2)(2-x^2-y^2)}{(1-x^2)(1-y^2)}\right)_x \qquad c = \frac{m}{2}\left(\log\frac{(2-x^2-x^2y^2)(1-x^2y^2)(2-x^2-y^2)}{(1-y^2)^2}\right)$$

$$(5.6) \qquad + \frac{\ell}{2}\left(\log\frac{(1-x^2y^2)^2(2-x^2-y^2)^2}{(1-y^2)^2(2-y^2-x^2y^2)}\right)_y$$

$$b = \frac{\ell}{2}\left(\log\frac{(2-y^2-x^2y^2)(1-x^2y^2)(2-x^2-y^2)}{(1-x^2)^2}\right)_y \qquad d = -\frac{3}{2}\left(\log\frac{(1-x^2y^2)(2-x^2-y^2)}{(1-x^2)(1-y^2)}\right)$$

$$+ \frac{m}{2}\left(\log\frac{(1-x^2y^2)^2(2-x^2-y^2)^2}{(1-x^2)^2(2-x^2-x^2y^2)}\right)_x$$

Remark 5.2. The normalization is unique if your require that it is invariant under the action of G and that the four coefficients a,b,c and d have poles only along D and $\{xy = 0\}$.

Once the normalization is fixed the equation (5.4) is uniquely determined by the group Γ. Let us call the equation (UEQ).

Lemma 5.3. The equation (UEQ) is G-invariant.

Proof. It follows from Lemma 5.1 and the fact that the orbifold M is G-invariant. ∎

Remark 5.4 The equation (EQ) is invariant under the coordinate change $(x,y) \longrightarrow (u,v)$ if and only if (by the notations in Proposition 3.1)

$$\overline{\ell}(u,v) = \ell(u,v), \ldots, \overline{q}(u,v) = q(u,v).$$

The conformal structure and the normalization do not determine all the coefficients of (EQ). This is a characteristic feature of two dimensional conformal structures of hypersurfaces (see [Sas]). We shall use Lemma 5.3 to determine the remaining coefficients p and q in the following subsections and obtain the explicit form of the equation (UEQ).

Theorem The developing map $\psi : M \longrightarrow Q \subset \mathbb{P}^2$ of the orbifold M is given by the ratios of the four linearly independent solutions of the equation (UEQ) of which coefficients are given by (5.3), (5.6) and

$$p = \frac{2(y^2-x^2)}{(1-x^2)^2(1-y^2)}, \qquad q = \frac{2(x^2-y^2)}{(1-x^2)(1-y^2)^2}.$$

The remaining part of this section is devoted to the proof of the theorem.

§§ 5.4 Determination of the coefficients p and q

Lemma 5.5 Outside the curve D the coefficients p and q have at worst simple poles along $\{xy = 0\}$.

Lemma 5.6 The coefficients p and q are even functions with respect to x and y and satisfies $p(x,y) = q(y,x)$. In particular they are holomorphic outside the curve D.

Lemma 5.7 The coefficients p and q have at worst double poles along the curve D.

Lemma 5.8 The coefficients p and q can be written in the following form:

$$p = \frac{P(x,y)}{D^2} , \qquad q = \frac{Q(x,y)}{D^2}$$

where P and Q are even polynomials of degree ≤ 18 such that

$$P(x,y) = Q(y,x).$$

Proof of Lemma 5.5 Outside the ramification locus D, the developing map ψ is locally biholomorphic. Choosing local coordinates (u,v) and a normalization suitably, (UEQ) is locally equivalent to the system with coefficients $\ell_0, m_0, a_0, b_0, c_0, d_0, p_0$ and q_0 which are zero, namely

$$(5.7) \qquad \begin{cases} z_{uu} = 0 \\ z_{vv} = 0 \end{cases}$$

which has the normalization factor $e^{2\theta_0} = 1$. We shall trace the procedure in the opposite direction starting from (5.7) and see local properties of (UEQ).

A coordinate change $(u,v) \longrightarrow (x,y)$ transforms (5.7) into an equation $\overline{(5.7)}$ with the coefficients $\bar{\ell}, \bar{m}, \ldots, \bar{p}, \bar{q}$ (see Proposition 3.1):

$$(5.8) \qquad \begin{aligned} &\nu = -x_u y_u - x_v y_u , \qquad \Delta = x_u y_v - x_v y_u \\ &\bar{\ell} = -2y_u y_v/\nu , \qquad \bar{m} = -2x_u x_v/\nu \\ &\bar{a}, \bar{b}, \bar{c}, \bar{d} = \frac{1}{\nu} \times \text{(holomorphic function)} \\ &\bar{p}, \bar{q} = 0 \end{aligned}$$

with the normalization

$$e^{2\overline{\theta}} = -\nu/\Delta^3 \ .$$

We note that ν and $y_u y_v$ (as well as ν and $x_u x_v$) have no divisor in common, otherwise the Jacobian Δ of the biholomorphic map has a divisor.

A change of the unkown $z \longrightarrow e^{\rho} z$ transforms $(\overline{5.7})$ into (UEQ) with the coefficients $\ell,\ldots,q$ (see Proposition 3.2):

$$(5.9) \qquad \begin{array}{ll} \ell = \overline{\ell} & m = \overline{m} \\ a = \overline{a} + 2\rho_x - \ell\rho_y & c = \overline{c} - m\rho_y \\ b = \overline{b} - \ell\rho_x & d = \overline{d} + 2\rho_y - m\rho_x \\ p = \rho_{xx} - \rho_x^2 - \ell(\rho_{xy} - \rho_x\rho_y) & q = \rho_{yy} - \rho_y^2 - m(\rho_{xy} - \rho_x\rho_y) \\ \quad -\overline{a}\rho_x - \overline{b}\rho_y & \quad -\overline{c}\rho_x - \overline{d}\rho_y \end{array}$$

with the normalization factor

$$(5.10) \qquad e^{2\theta} = -e^{4\rho}\nu/\Delta^3 \ .$$

Since we have

$$-\frac{2-y^2-x^2y^2}{xy(1-x^2)} = \ell = \overline{\ell} = -2y_u y_v/\nu \ ,$$

outside D, ν has simple zeros along $\{xy = 0\}$. Since we have

$$e^{2\theta} = (1 - \ell m)^{-\frac{7}{2}}(xy)^{-6}$$

$$= xy\left(-2\frac{(1-x^2y^2)(2-x^2-y^2)}{(1-x^2)(1-y^2)}\right)^{-\frac{7}{2}} ,$$

(5.10) implies that ρ has no poles along $\{xy = 0\}$. Thus (5.9) shows that p and q have at worst simple poles along $\{xy = 0\}$ outside D. ∎

Proof of Lemma 5.6 We make use of the G-invariance of (UEQ). Change the coordinates by $T_2 \in G$ (resp. $T_1T_2T_1 \in G$), namely $(-x,y)$ (resp. $\longrightarrow (x,-y)$). By the transformation formula Proposition 3.1 we have $\overline{p}(u,v) = p(x,y)$ and $\overline{q}(u,v) = q(x,y)$. On the other hand, G-invariance means $\overline{p}(u,v) = p(u,v)$ and $\overline{q}(u,v) = q(u,v)$. Thus we have $p(u,v) = p(x,y)$ and $q(u,v) = q(x,y)$, which tells us p and q are even functions with respect to x and y. This implies in particular that simple poles of p and q do not occur. Analogously T_1-invariance leads to $p(x,y) = q(y,x)$. ∎

Proof of Lemma 5.7 Along the ramification locus the map π: $H\times H \longrightarrow M$ is locally equivalent to the quotient map with respect to the involution $\tau:(z^1,z^2) \longrightarrow (z^2,z^1)$. Choosing local coordinates (u,v) suitably, π is given by

$$H \times H \ni (z^1,z^2) \longrightarrow (u,v) = (z^1 + z^2 - (z^1 - z^2)^2, (z^1-z^2)^2)$$

and $\{v = 0\}$ is the ramification locus. The differential equation in (u,v)-coordinates with the four linearly independent solutions 1, z^1, z^2 and z^1z^2 is given by the following coefficients:

$$\ell_0 = 1 \qquad m_0 = 1 - \frac{1}{4v}$$
$$a_0 = b_0 = 0 \qquad c_0 = -d_0 = \frac{1}{2v}$$
$$p_0 = 0 \qquad q_0 = 0$$

namely

$$(5.11) \qquad \begin{aligned} z_{uu} &= z_{uv} \\ z_{vv} &= (1 - \frac{1}{4v}) z_{uv} + \frac{1}{2v} z_u - \frac{1}{2v} z_v \end{aligned}$$

with the normalization factor

$$e^{2\theta_0} \doteq \frac{1}{2} v^{-\frac{1}{2}} \quad .$$

It is obtained by $\det(w, w_u, w_v, w_{uv}, w_{uu}) = 0$ and $\det(w, w_u, w_v, w_{uv}, w_{vv}) = 0$, where w is the transposed vector of $(1, z^1, z^2, z^1 z^2, z)$. We shall do the same thing as in the proof of Lemma 5.5.

A coordinate change $(u,v) \longrightarrow (x,y)$ transforms (5.11) into an equation $\overline{(5.11)}$ with the coefficients $\overline{\ell}, \ldots, \overline{q}$:

$$(5.12) \qquad \begin{aligned} &\nu = x_u y_v + (1 - \frac{1}{4v}) x_u y_u - x_u y_v - x_v y_u \\ &\overline{\ell}, \overline{m} = \frac{1}{\nu v} \times \text{(holomorphic function)} \\ &\overline{a}, \overline{b}, \overline{c}, \overline{d} = \frac{1}{\nu v} \times \text{(holomorphic function)} \quad \text{(see (3.4) carefully)} \\ &\overline{p} = \overline{q} = 0 \end{aligned}$$

with the normalization factor

$$e^{2\overline{\theta}} = -\frac{\nu}{\Delta^3} \frac{1}{2} v^{-\frac{1}{2}} \quad .$$

A change of the unknown $z \longrightarrow e^{\rho} z$ transforms $(\overline{5.11})$ into (UEQ) with coefficients $\ell,\ldots,q$ satisfying (5.9) with the normalization factor this time

$$(5.13) \qquad e^{2\theta} = -e^{4\rho}\left(\frac{\nu}{\Delta^3}\,\frac{1}{2}\,v^{-\frac{1}{2}}\right).$$

In case $v = (1 - x^2) \times (\text{unit})$, by (5.13), we have

$$\rho = \frac{1}{4}\log(\nu v^4) \times (\text{unit}),$$

where (unit) stands for a non-vanishing holomorphic function. So either v divides ν or not, ρ_x has at most simple pole along $v = 0$ and ρ_y is holomorphic. Thus (5.9) tells that $\bar{a}$ (resp. $\bar{b}$) has at most simple (resp. double) poles along $\{v = 0\}$ and that p and q have at most double poles along $\{v = 0\}$. The case when $v = (1 - y^2)\times(\text{unit})$ the argument is exactly the same and when $v = (2 - x^2 - y^2)\times(\text{unit})$ the argument is simpler. ∎

Proof of Lemma 5.8 Apply Lemma 5.6 near the line at infinity. Then we see that p and q are rational functions of degree -2. Since D is an even symmetric polynomial of degree 20 the lemma is derived by Lemma 5.5 ~ 5.7. ∎

We now study the effect of T_3-invariance. The transformation T_3 in inhomogeneous coordinates is given by

$$u = \frac{2-x+y}{x+y}, \qquad v = \frac{2-x-y}{x+y}$$

The transformation formula (Propositon 3.1) tells that

$$\bar{p}(u,v) = \frac{1}{\nu}(\alpha q - \beta p), \quad \bar{q} = \frac{1}{\nu}(\delta p - \gamma q)$$

where

$$\nu = \frac{uv}{xy} \quad \frac{4}{(x+y)^2}$$

$$\alpha = \frac{(1-y)(2-y^2+xy-xy^2-x^2y)}{2xy(x+y)(1-x)}, \quad \beta = \frac{(1+x)(2-x^2+xy+xy^2+x^2y)}{2xy(x+y)(1+y)}$$

$$\gamma = \frac{(1+y)(2-y^2+xy+x^2y+xy^2)}{2xy(x+y)(1+x)}, \quad \delta = \frac{(1-x)(2-x^2+xy-x^2y-xy^2)}{2xy(x+y)(1-y)}.$$

The T_3-invariance $\bar{p}(u,v) = p(u,v)$, $\bar{q}(u,v) = q(u,v)$ is stated by

$$(5.14) \qquad \frac{P(u,v)}{D^2(u,v)} = \frac{1}{2uv(u+v)D^2(x,y)}\left\{\frac{1+v}{1+u}\, BP(x,y) - \frac{1-v}{1-u}\, AQ(x,y)\right\}$$

where

$$A = A(x,y) = 2 - y^2 + xy - xy^2 - x^2y$$

$$B = B(x,y) = A(-y,-x).$$

The corresponding formula for Q is derived by $P(x,y) = Q(y,x)$ (Lemma 5.6, from (5.14). The following formulae are useful:

$$x + y = \frac{4}{u+v}, \quad x - y = \frac{-2(u-v)}{u+v}$$

$$2 + x + y = \frac{2(2+u+v)}{u+v}, \quad 2 - x - y = -\frac{2(2-u-v)}{u+v}$$

$$2 + x - y = \frac{4v}{u+v}, \quad 2 - x + y = \frac{4u}{u+v}$$

$$1 - x = -\frac{2(1-u)}{u+v}, \quad 1 + x = \frac{2(1+v)}{u+v}$$

$$1 - y = -\frac{2(1-v)}{u+v}, \quad 1 + y = \frac{2(1+u)}{u+v}$$

$$1 + xy = \frac{4(1+uv)}{(u+v)^2}, \quad 1 - xy = -\frac{2(2-u^2-v^2)}{(u+v)^2}$$

$$2 - x^2 - y^2 = -\frac{8(1-uv)}{(u+v)^2}$$

Since we have

$$D(x,y) = 2^{10}(u+v)^{-10}D(u,v)$$

(5.14) is equivalent to

$$(5.16) \qquad P(u,v) = \frac{2^{-20}(u+v)^{20}}{2uv(u+v)}\left\{\frac{1+v}{1+u}\,BP(x,y) - \frac{1-v}{1-u}\,AQ(x,y)\right\}$$

Since we have $1 + u = (1+y)(u+v)/2$. BP must be a multiple of $1+y$. Since B is prime and P is even, we can put

$$P(x,y) = (1-y^2)P_1(x,y) \quad \text{and} \quad Q(x,y) = (1-x^2)Q_1(x,y)$$

where P_1 and Q_1 are even polynomials of degree ≤ 16 such that $P_1(y,x) = Q_1(x,y)$. Then the equality (5.16) is equivalent to

$$(5.17) \qquad P_1(u,v) = -\frac{2^{-19}(u+v)^{17}}{uv}\{BP_1(x,y) - AQ_1(x,y)\}.$$

Put

$$R(x,y) = (P_1(x,y) - P_1(y,x))/2$$
$$S(x,y) = (P_1(x,y) + P_1(y,x))/2$$

then (5.17) is equivalent to

$$(5.18)\quad \begin{aligned} &4^{-2}(u+v)^2(2+x-y)(2-x+y)(R(u,v)+S(u,v)) \\ &= -2^{19}(u+v)^{17}\{(4-(x-y)^2)R(x,y) + (x+y)(2xy-x+y)S(x,y)\}. \end{aligned}$$

This tells that S has $(2+x-y)(2-x+y)$ as a factor. Since S is even it is divisible by

$$(2 \pm x \pm y) := (2+x+y)(2+x-y)(2-x+y)(2-x-y).$$

Put

$$S = (2 \pm x \pm y)\ S_1,\ R = (x^2 - y^2)R_1$$

where S_1 and R_1 are even symmetric polynomials of degree ≤ 12 and ≤ 14, respectively. Then (5.18) is equivalent to

$$(5.19)\quad \begin{aligned} &(u^2-v^2)\ R_1(u,v) + (2 \pm u \pm v)S_1(u,v) \\ &= -2^{-15}(u+v)^{15}\{(x^2-y^2)R_1(x,y) + \\ &\quad +(x+y)(2xy-x+y)(2+x+y)(2-x-y)S_1(x,y)\}. \end{aligned}$$

Exchange in (5.19) x and y as well as u and v and we have an equality (5.19)'.

Add (5.19) and (5.19)' then we have

(5.20) $$S_1(u,v) = 2^{-10}(u+v)^{10}S_1(x,y).$$

Substract (5.19) from (5.19)' and put

$$R_2 = R_1 - 4S_1$$

then we have

(5.21) $$R_2(u,v) = 2^{-12}(u+v)^{12}R_2(x,y).$$

By the equalities (5.20) and (5.21) we know that the even symmetric polynomials R_2 and S_1 are of degree ≤ 12 and ≤ 10, respectively.

Lemma 5.9 Let $f(x,y)$ be a polynomial of degree $\leq d$ such that

$$f(u,v) = 2^{-d}(u+v)^d f(x,y).$$

Then the homogeneous polynomial (of degree d) $\bar{f}(x,y,z) = z^d f(x/z,y/z)$ is T_3-invariant, namely $\bar{f}(T_3(x,y,z)) = \bar{f}(x,y,z)$.

Proof Easy. ∎

Lemma 5.10 Any polynomial in (x,y,z) which is invariant under the action of the group $G = \langle T_1,T_2,T_3\rangle$ is a polynomial in

$$\bar{A} = z^2 + x^2 + y^2$$
$$\bar{B} = (x^2 - y^2)^2 + 8(x^2 + y^2)z^2$$
$$\bar{C} = (x^2 - y^2)^2 z^2$$

Proof Change the coordinates by $(X,Y,Z) = (x-y, x+y, 2z)$ then the matrices T_1, T_2 and T_3 are transformed into

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} & -1 & \\ -1 & & \\ & & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

respectively. The fundamental invariant of this group are easily seen to be $X^2+Y^2+Z^2$, $X^2Y^2+Y^2Z^2+Z^2X^2$ and $X^2Y^2Z^2$, which are (in terms of x,y,z) $\overline{A}, \overline{B}$ and $\overline{C}$, respectively. ∎

Note 5.11 The group is the so-called imprimitive reflection group sometimes denoted by $G(2,1,3)$.

Lemma 5.12 There are constants $a_i (1 \le i \le 5)$ and $b_j (1 \le j \le 7)$ such that

$$S_1 = a_1 A^5 + a_2 A^3 B + a_3 AB^2 + a_4 A^2 C + a_5 BC$$
$$R_2 = b_1 A^6 + b_2 A^4 B + b_3 A^2 B^2 + b_4 B^3 + b_5 A^3 C + b_6 ABC + b_7 C^6$$

where

$$A = \overline{A}(x,y,1) = 2 + x^2 + y^2$$
$$B = \overline{B}(x,y,1) = (x^2 - y^2)^2 + 8(x^2 + y^2)$$
$$C = \overline{C}(x,y,1) = (x^2 - y^2)^2 .$$

Proof It follows from Lemmas 5.9 and 5.10. ∎

Summing up, we now know that the coefficients p and q are expressed as follows:

$$(5.22)\qquad \begin{aligned} p &= \frac{1-y^2}{D^2}[(x^2-y^2)R_2 + \{(2 \pm x \pm y)+4(x^2-y^2)\}S_1] \\ q &= \frac{1-x^2}{D^2}[-(x^2-y^2)R_2 + \{(2 \pm x \pm y) - 4(x^2-y^2)\}S_1] \end{aligned}$$

we shall determine the twelve constants a_i and b_j by making use of (IC1) and (IC2) in §2 (see also Proposition 4.4) of which left hand sides we shall call L^1 and L^2:

$$(5.23)\qquad \begin{aligned} L^1 &:= \ell q_y - 2q_x - mp_y - (\ell\xi_y - \xi_x - 2\ell_y)q = R^1 \\ L^2 &:= mp_y - 2p_y - \ell q_x - (m\xi_x - \xi_y - 2m_x)p = R^2 \end{aligned}$$

<u>Lemma 5.13</u> Compare the coefficients of maximal poles along $\{1 - x^2 = 0\}$ in the both sides of (5.23) then we get

$$(5.24)\qquad a_1 + a_2 + a_3 = 0 , \qquad a_4 + a_5 = 0$$

$$(5.25)\qquad b_1 + b_2 + b_3 + b_4 = 0, \quad b_5+b_6 = 0, \quad b_7 = -2.$$

<u>Proof</u> Put $X = 1 - x^2$ then we have

$$p = \frac{U(y)}{X^2} + 0(X^{-1}), \quad U(y) = \left.\frac{R_2+(12+(1-y^2)S_1)}{(1-y^2)^4}\right|_{x^2=1}$$

$$q = \frac{V(y)}{X} + 0(1) \quad , \quad V(y) = \left.\frac{-R_2+(4+(1-y^2)S_1)}{(1-y^2)^5}\right|_{x^2=1}$$

and

$$L^1 = \frac{1}{xX^2}\{2V - \frac{2(1-y^2)}{y} V_y + \frac{U_y}{y}\} + 0(X^{-1})$$

$$L^2 = \frac{1}{X^3}\{- \frac{2U}{y} + \frac{4(1-y^2)V}{y}\} + 0(X^{-2}) \quad .$$

On the other hand one can show by (long) computation that

$$R^1 = - \frac{4}{(1-y^2)xX^2} + 0(X^{-1})$$

$$R^2 = \frac{12}{yX^3} + 0(X^{-2}) \qquad .$$

The identity $L^1 = R^1$ leads to $S_1|_{X=0} = 0$ which implies (5.24). The identity $L^2 = R^2$ leads to $R_2|_{X=0} = -2(1-y^2)^4$ which implies (5.25). ∎

We do the same thing along $2 - x^2 - y^2 = 0$. Put $X = 2 - x^2 - y^2$ and define r_0, r_1, s_0 and s_1 by

$$R_2|_{x^2=2-y^2-X} = r_0(y) + r_1(y)X + 0(X^2)$$

$$S_1|_{x^2=2-y^2-X} = s_0(y) + s_1(y)X + 0(X^2)$$

then we have

$$p = \frac{U^1(y)}{X^2} + \frac{U^2(y)}{X} + 0(1)$$

$$q = \frac{W^1(y)}{X^2} + \frac{W^2(y)}{X} + 0(1)$$

where

$$U^1(y) = \frac{2r_0+\{4(1-y^2)+8\}s_0}{(1-y^2)^6}$$

$$U^2(y) = \frac{2(1-2y^2)(2r_0+(4(1-y^2)+8)s_0)}{(1-y^2)^8}$$

$$+ \frac{2(1-y^2)r_1-r_0+4y^2s_0+(4(1-y^2)+8)(1-y^2)s_1}{(1-y^2)^7} \tag{5.26}$$

$$W^1(y) = \frac{2r_0+\{-4(1-y^2)+8\}s_0}{(1-y^2)^6}$$

$$W^2(y) = \frac{(1-3y^2)(-2r_0+(4(1-y2)-8)s_0)}{(1-y^2)^8}$$

$$+ \frac{2(1-y^2)r_1-r_0-(8+4y^2)s_0-(4(1-y^2)-8)(1-y^2)s_1}{(1-y^2)^7} .$$

So we have

$$L^1 = \frac{-4y^2U^1-4(2-y^2)W^1}{xX^3} +$$

$$\frac{1}{X^2}\left\{\frac{4W^1}{x} - \frac{4xW^1}{1-y^2} - 4xW^2 - \frac{y(1-y^2)(2yU^2+U^1_y)-4(1+y^2)U^1}{x(1-y^2)}\right.$$

$$\left. + \frac{4y(3-y^2)W^1 + x^2(1-y^2)(2yW^2+W^1_y)}{xy(1-y^2)}\right\}$$

$$L^2 = \frac{-4y^2U^1 - 4(2-y^2)W^1}{yX^3} +$$

$$+ \frac{1}{X^2}\left\{\frac{4W^1}{y} + \frac{2y^2(1-y^2)U^2-4(1+y^2)U^1}{y(1-y^2)} - 2(2yU^2+U^1_y) + \frac{4yU^1}{1-y^2}\right.$$

$$\left. - \frac{4(3-y^2)W^1+2x^2(1-y^2)W^2}{y(1-y^2)}\right\} .$$

On the other hand, one can show by (long) computation that R^1 and R^2 have no pole along $X = 0$. The (IC) leads to

$$y^2U^1 + (2-y^2)W^1 = 0 \tag{5.27}$$

$$4W^1 + 2U^1 + y^2(1-y^2)U^2 + (2-y^2)(1-y^2)W^2 + y(1-y^2)U^1_y = 0 \tag{5.28}$$

$$4y(2-y^2)W^1 - 2y(2-y^2)(1-y^2)W^2 + 4y(1+y^2)U^1$$
$$-2y^3(1-y^2)U^2 - y^2(1-y^2)U^1_y + (1-y^2)(2-y^2)W^1_y = 0. \tag{5.29}$$

From (5.28) and (5.29), using the differential of (5.27), we can eliminate U^1_y and W^1_y and obtain

$$(3 - y^2)W^1 + (1 + y^2)U^1 = 0$$

which together with (5.27) and (5.28) implies

$$U^1(y) = W^1(y) = 0 \tag{5.30}$$

$$y^2U^2 + (2 - y^2)W^2 = 0. \tag{5.31}$$

The definition of U^1 and W^1 ((5.26) and (5.30)) imply

$$s_0 = r_0 = 0. \tag{5.32}$$

Since we know by Lemma 5.13 that

$$S_1 = (A^2 - B)\{-a_2A^3 - a_3A(A^2 + B) + a_4C\},$$

$s_0 = 0$ implies

$$S_1|_{X=0} = -4^3a_2 - 4a_3(4^2+16+4(1-y^2)^2)+a_4\cdot 4(1-y)^2 = 0$$

and so

$$a_2 = -2a_3, \qquad a_4 = 4a_3.$$

Thus we have

$$\begin{aligned} S_1 &= (A^2 - B)(2A^3 - A(A^2 + B) + 4C)a_3 \\ &= 4^2a_3(2 - x^2 - y^2)(1 - x)^2(1 - y^2)(1 - x^2y^2) = 4^2a_3D. \end{aligned}$$

Analogously by Lemma 5.13 and $r_0 = 0$ imply

$$b_4 = 0, \quad b_2 = -2b_3 \ , \quad b_5 - \frac{b_7}{4} = 4b_3 \ .$$

Thus we have

$$\begin{aligned} (5.34) \qquad R_2 &= 4^2b_3A(2 - x^2 - y^2)(1 - x^2y^2)(1 - x^2)(1 - y^2) \\ &\quad + b_7C(2 - x^2 - y^2)(1 - x^2y^2). \end{aligned}$$

The equalities (5.33) and (5.34) together with (5.31) lead to

$$a_3 = 0, \qquad b_3 = \frac{b_7}{16} = -\frac{1}{8} \ .$$

Therefore we have

$$S_1 = 0$$

$$R_2 = -2(2-x^2-y^2)(1-x^2y^2)\{(2+x^2+y^2)(1-x^2)(1-y^2)+(x^2-y^2)^2\}$$

$$= -2(2-x^2-y^2)^2(1-x^2y^2)^2$$

and finally

$$p = \frac{(1-y^2)(x^2-y^2)R_2}{D^2} = \frac{-2(x^2-y^2)}{(1-x^2)^2(1-y^2)}$$

$$q = \frac{2(x^2-y^2)}{(1-x^2)(1-y^2)^2} \quad ,$$

ending the proof of the theorem.

§ 6 Hypergeometric differential equations in two variables of rank four

§§ 6.1 Table of coefficients

As generalizations of the Gauss hypergeometric differential equation, several hypergeometric differential equation (HGDE for short) in two variables are known ([Erd]). They are denoted $F_1,\dots,F_4$ (Appell's HGDE), $G_1,G_2,G_3,\mathit{H}_1,\dots,\mathit{H}_7$ and confluent HGDE's $\Phi_1,\Phi_2,\Phi_3,\Psi_1,\Psi_2,\Xi_1,\Xi_2,\Gamma_1,\Gamma_2,\mathrm{H}_1,\dots,\mathrm{H}_{11}$. These HGDE's are systems of linear partial differential equations of rank three or four. We are interested in those of rank four. They are $F_2,F_3,F_4,G_3,\mathit{H}_1,\dots,\mathit{H}_7;\Psi_1,\Psi_2,\Xi_1,\Xi_2,\Gamma_1,\mathrm{H}_1,\dots,\mathrm{H}_{11}$. Since we intend this paper to be a basic data of differential equations of rank 4, we tabulate their coefficients:

<table>
<tr><td rowspan="5">name</td><td>ℓ</td><td>m</td></tr>
<tr><td>a</td><td>c</td></tr>
<tr><td>b</td><td>d</td></tr>
<tr><td>p</td><td>q</td></tr>
<tr><td>parameters included</td><td>$1 - \ell m$</td></tr>
</table>

	$\frac{y}{1-x}$	$\frac{x}{1-y}$
	$\frac{(\alpha+\beta+1)x-y}{x(1-x)}$	$\frac{\beta' x}{y(1-y)}$
F_2	$\frac{\beta y}{x(1-x)}$	$\frac{(\alpha+\beta'+1)y-\gamma'}{y(1-y)}$
	$\frac{\alpha\beta}{x(1-x)}$	$\frac{\alpha\beta'}{y(1-y)}$
	$\alpha,\beta,\beta':\ \gamma,\gamma'$	$\frac{1-x-y}{(1-x)(1-y)}$
	$\frac{-y}{x(1-x)}$	$\frac{-x}{y(1-y)}$
	$\frac{(\alpha+\beta+1)x-\gamma}{x(1-x)}$	0
F_3	0	$\frac{(\alpha'+\beta'+1)y-\gamma}{y(1-y)}$
	$\frac{\alpha\beta}{x(1-x)}$	$\frac{\alpha'\beta'}{y(1-y)}$
	$\alpha,\alpha',\beta,\beta':\gamma$	$\frac{xy-x-y}{(1-x)(1-y}$

	$\dfrac{2y}{1-x-y}$	$\dfrac{2x}{1-x-y}$
	$\dfrac{(\alpha+\beta+1)x-\gamma(1-y)}{x(1-x-y)}$	$\dfrac{(\alpha+\beta+1)x-\gamma x}{y(1-x-y)}$
F_4	$\dfrac{(\alpha+\beta+1)y-\gamma' y}{x(1-x-y)}$	$\dfrac{(\alpha+\beta+1)y-\gamma'(1-x)}{y(1-x-y)}$
	$\dfrac{\alpha\beta}{x(1-x-y)}$	$\dfrac{\alpha\beta}{y(1-x-y)}$
	$\alpha,\beta:\gamma,\gamma'$	$\dfrac{(1-x-y)^2-4xy}{(1-x-y)^2}$
	$\dfrac{y\{(2+4x)(1+4y)-(2+4y)x\}}{x\{(1+4x)(1+4y)-xy\}}$	$\dfrac{x\{(2+4y)(1+4x)-(2+4x)y\}}{y\{(1+4x)(1+4y)-xy\}}$
	$-\dfrac{(1+4y)\{1-\alpha+(4\beta+6)x\}+2\alpha xy}{x\{(1+4x)(1+4y)-xy\}}$	$\dfrac{x\{2\alpha(1+4x)+(1-\alpha+(4\beta+6)x)\}}{y\{(1+4x)(1+4y)-xy\}}$
G_3	$\dfrac{y\{(2\beta(1+4y)+(1-\beta+(4\alpha+6)y)\}}{x\{(1+4x)(1+4y)-xy\}}$	$-\dfrac{(1+4x)\{1-\beta+(4\alpha+6)y\}+2\beta xy}{y\{(1+4x)(1+4y)-xy\}}$
	$\dfrac{-\beta(\beta+1)(1+4y)+\alpha(\alpha+1)}{x\{(1+4x)(1+4y)-xy\}}$	$\dfrac{-\alpha(\alpha+1)(1+4x)+\beta(\beta+1)}{y\{(1+4x)(1+4y)-xy\}}$
	α,α'	$\dfrac{-3-12x-12y-54xy+81x^2y^2}{(1+4x+4y+15xy)^2}$

	$\frac{-y(1-y)}{(1-x)(1+y)}$	$\frac{x(1-y)}{y(1+y)}$
	$\frac{\gamma xy-(1+y)(\delta-(\alpha+\beta+1)x)}{x(1-x)(1+y)}$	$\frac{-\gamma x}{y(1+y)}$
H_1	$\frac{y(2\alpha y-\beta)}{x(1-x)(1+y)}$	$\frac{\alpha-1-(\beta+\gamma+1)y}{y(1+y)}$
	$\frac{\alpha\beta(1+y)+\beta\gamma y}{x(1-x)(1+y)}$	$\frac{-\beta\gamma}{y(1+y)}$
	$\alpha,\beta,\gamma;\delta$	$\frac{(1+y)^2-4xy}{(1-x)(1+y)^2}$
	$\frac{-xy}{x(1-x)}$	$\frac{x}{y(1+y)}$
	$\frac{-\varepsilon+(\alpha+\beta+1)x}{x(1-x)}$	0
	$\frac{-\beta y}{x(1-x)}$	$\frac{\alpha-1-(\gamma+\delta+1)y}{y(1+y)}$
H_2	$\frac{\alpha\beta}{x(1-x)}$	$\frac{-\delta\gamma}{y(1+y)}$
	$\alpha,\beta,\gamma,\delta;\varepsilon$	$\frac{1+y-xy}{(1-x)(1+y)}$

	$-\dfrac{y\{(1-y)(1-4x)+x(1-2y)\}}{x(1-4x)(1-y)}$	$-\dfrac{x(1-2y)}{y(1-y)}$
	$-\dfrac{(1-y)\{\gamma-(4\alpha+6)x\}-2\beta xy}{x(1-4x)(1-y)}$	$\dfrac{2\beta x}{y(1-y)}$
H_3	$\dfrac{y\{2(\alpha+1)(1-y)-\gamma+(\alpha+\beta+1)y\}}{x(1-4x)(1-y)}$	$\dfrac{(\alpha+\beta+1)\ y-\gamma}{y(1-y)}$
	$\dfrac{\alpha(\alpha+1)(1-y)\ +\ \alpha\beta y}{x(1-4x)(1-y)}$	$\dfrac{\alpha\beta}{y(1-y)}$
	$\alpha,\beta;\gamma$	$\dfrac{-y^2+y-x}{(1-y)^2(1-4x)}$
	$\dfrac{4y\ -\ 2y^2}{(1-y)(1-4x)}$	$\dfrac{2x}{1-y}$
	$\dfrac{2\beta xy\ -\ (1-y)\{\gamma-(4\alpha+4)x\}}{x(1-y)(1-4x)}$	$\dfrac{2\beta x}{y(1-y)}$
H_4	$\dfrac{y\{(1-y)(3\alpha+2)-(\delta-(\alpha+\beta)y)\}}{x(1-y)(1-4x)}$	$\dfrac{(\alpha+\beta)y-\delta}{y(1-y)}$
	$\dfrac{\alpha(\alpha+1)(1-y)+\alpha\beta y}{x(1-y)(1-4x)}$	$\dfrac{\alpha\beta}{y(1-y)}$
	$\alpha,\beta;\gamma,\delta$	$\dfrac{(1-y)^2\ -\ 4x}{(1-y)^2(1-4x)}$

	$\dfrac{y\{(1-y)(1-4x)-xy\}}{x\{(1+4x)(1-y)-2xy\}}$	$\dfrac{x(12x-1)}{(1+4x)(1-y)-2xy}$
	$-\dfrac{(1-y)\{1-\gamma+4(\alpha+1)x\}-(2+\alpha-2\beta)xy}{x\{(1+4x)(1-y)-2xy\}}$	$\dfrac{x\{2(1-\gamma+4(\alpha+1)x)-(2+\alpha-2\beta)(1-4x)\}}{y\{(1+4x)(1-y)-2xy\}}$
H_5	$\dfrac{y\{\gamma-(\alpha+\beta+1)y-(3\alpha+2)(1-y)}{x\{(1+4x)(1-y)-2xy\}}$	$\dfrac{2(3\alpha+2)xy-(1+4x)\{\gamma-(\alpha+\beta+1)y\}}{y\{(1+4x)(1-y)-2xy\}}$
	$\dfrac{\alpha(\alpha+1)(1-y)-\alpha\beta y}{x\{(1+4x)(1-y)-2xy\}}$	$\dfrac{2\alpha(\alpha+1)x-\alpha\beta(1+4x)}{y\{(1+4x)(1-y)-2xy\}}$
	$\alpha,\beta;\gamma$	$\dfrac{1+8x-y+16x^2-36xy+27xy^2}{\{(1+4x)(1-y)-2y\}^2}$
	$\dfrac{y\{(1+4x)(1+y)-(2+y)x\}}{x(1+4x)(1+y)}$	$\dfrac{x(2+y)}{y(1+y)}$
	$-\dfrac{(1+y)\{1-\beta+(4\alpha+6)x\}+\gamma xy}{x(1+4x)(1+y)}$	$\dfrac{\gamma x}{y(1+y)}$
H_6	$\dfrac{y\{2\alpha(1+y)+(1-\alpha+(\beta+\gamma+1)y)}{x(1+4x)(1+y)}$	$\underline{\alpha-1-(\beta+\gamma+1)y}$
	$\dfrac{-\alpha(\alpha+1)(1+y)+\beta\gamma y}{x(1+4x)(1+y)}$	$\dfrac{-\beta\gamma}{y(1+y)}$
	α,β,γ	$\dfrac{(1+4x)(1+y)^2-(2+y)\{1+4x)(1+y)-x(2+y)\}}{(1+4x)(1+y)^2}$

	$\frac{-y^2 - 4y}{(1-4x)(1+y)}$	$\frac{3x}{1+y}$
	$\frac{\gamma xy - \{\delta - (4\alpha + 6)x\}(1+y)}{x(1-4x)(1+y)}$	$\frac{\gamma x}{y(1+y)}$
H_7	$-\frac{y\{1+\alpha+(\beta+\gamma+2\alpha+1)y\}}{x(1-4x)(1+y)}$	$\frac{\alpha - 1 - (\beta + \gamma + 1)\,y}{y(1+y)}$
	$\frac{-\beta\gamma y + \alpha(\alpha+1)(1+y)}{x(1-4x)(1+y)}$	$\frac{-\beta\gamma}{y(1+y)}$
	$\alpha,\beta,\gamma;\delta$	$\frac{(1-4x)(1+y)^2 + 3xy(4+y)}{(1-4x)(1+y)^2}$
	$\frac{y}{1-x}$	0
	$\frac{-\gamma + (\alpha+\beta+1)x}{x(x-1)}$	$\frac{x}{y}$
Ψ_1	$\frac{\beta y}{x(1-x)}$	$\frac{-\delta + y}{y}$
	$\frac{\alpha\beta}{x(1-x)}$	$\frac{\alpha}{y}$
	$\alpha,\beta;\gamma,\gamma'$	1

Ψ_2	0	0
	$\frac{-\gamma + x}{x}$	$\frac{x}{y}$
	$\frac{y}{x}$	$\frac{-\gamma' + y}{y}$
	$\frac{\alpha}{x}$	$\frac{\alpha}{y}$
	$\alpha;\gamma,\gamma'$	1
Ξ_1	$\frac{-y}{x(1-x)}$	$\frac{-x}{y}$
	$\frac{-\gamma + (\alpha+\beta+1)x}{x(1-x)}$	0
	0	$\frac{-\gamma + y}{y}$
	$\frac{\alpha\beta}{x(1-x)}$	$\frac{\alpha'}{y}$
	$\alpha,\alpha',\beta;\gamma$	$\frac{-x}{1-x}$

Ξ_2	$\frac{-y}{x(1-x)}$	$\frac{-x}{y}$
	$\frac{-\gamma+(\alpha+\beta+1)x}{x(1-x)}$	0
	0	$\frac{-\gamma}{y}$
	$-\frac{\alpha\beta}{x(1-x)}$	$\frac{1}{y}$
	$\alpha,\beta;\gamma$	$\frac{-x}{1-x}$
Γ_1	$\frac{y}{x}$	$\frac{\alpha}{y}$
	$\frac{\beta-1-(\alpha+\beta'+1)x}{x(1+x)}$	$\frac{x}{y}$
	$\frac{\alpha y}{x(1+x)}$	$\frac{\beta'-1-y}{y}$
	$\frac{-\alpha\beta'}{x(1+x)}$	$\frac{-\beta}{y}$
	α,β,β'	$\frac{-\alpha+x}{x}$

	$\frac{-y}{1-x}$	$\frac{x}{y}$
	$\frac{-\delta + (1+\alpha+\beta+y)x}{x(1-x)}$	$\frac{-x}{y}$
H_1	$\frac{-(\beta-y)y}{x(1-x)}$	$\frac{\alpha-1-y}{y}$
	$\frac{(\alpha+y)\beta}{x(1-x)}$	$\frac{-\beta}{y}$
	$\alpha, \beta; \delta$	$\frac{1}{1-x}$
	$\frac{-x}{1-x}$	$\frac{x}{y}$
	$\frac{-\delta + (\alpha+\beta+1)x}{x(1-x)}$	0
H_2	$\frac{(1-\alpha+y)x - \beta y}{x(1-x)}$	$\frac{\alpha-1-y}{y}$
	$\frac{\alpha\beta + \gamma x}{x(1-x)}$	$\frac{-\gamma}{y}$
	$\alpha, \beta, \gamma; \delta$	$\frac{y - xy + x^2}{y(1-x)}$

H_3	$\frac{x}{1-x}$	$\frac{x}{y}$
	$\frac{-\delta + (\alpha+\beta+1)x}{x(1-x)}$	0
	$\frac{(1-\alpha)x - \beta y}{x(1-x)}$	$\frac{\alpha - 1}{y}$
	$\frac{\alpha\beta + x}{x(1-x)}$	$\frac{-1}{y}$
	$\alpha,\beta;\delta$	$\frac{y - xy + x^2}{(1-x)y}$
H_4	0	$\frac{x}{y}$
	$\frac{-\delta + x}{x}$	0
	$\frac{-y}{x}$	$\frac{\alpha - 1 - y}{y}$
	$\frac{\alpha}{x}$	$\frac{-\gamma}{y}$
	$\alpha,\gamma;\delta$	1

	0	$\frac{x}{y}$
	$\frac{-\delta + x}{x}$	0
H_5	$\frac{-y}{x}$	$\frac{\alpha - 1}{y}$
	$\frac{\alpha}{x}$	$\frac{-1}{y}$
	$\alpha ; \delta$	1
	$-\frac{y(1 - 3x)}{x(1 - 4x)}$	$-\frac{x}{y}$
	$\frac{2xy + (4\alpha + 6)x - \gamma}{x(1 - 4x)}$	$\frac{2x}{y}$
H_6	$\frac{(2\alpha + 2 - \gamma + y)y}{x(1 - 4x)}$	$\frac{-\gamma + y}{y}$
	$\frac{\alpha(\alpha + 1 + y)}{x(1 - 4x)}$	$\frac{\alpha}{y}$
	$\alpha ; \gamma$	$\frac{-x}{1 - 4x}$

	$\frac{4y}{1-4x}$	0
	$\frac{-\gamma + (1 + 4\alpha + 2y)x}{x(1-4x)}$	$\frac{2x}{y}$
H_7	$\frac{(3\alpha + 2 - \delta + y)y}{x(1-4x)}$	$\frac{-\delta + y}{y}$
	$\frac{(1 + \alpha + y)\alpha}{x(1-4x)}$	$\frac{\alpha}{y}$
	$\alpha;\gamma,\delta$	1
	$\frac{y(1+2x)}{x(1+4x)}$	$\frac{2x}{y}$
	$\frac{\beta - 1 - (6 + 4\alpha + y)x}{x(1+4x)}$	$\frac{x}{y}$
H_8	$\frac{y(1 + \alpha + y)}{x(1+4x)}$	$\frac{\alpha - 1 - y}{y}$
	$\frac{-\alpha(1+\alpha) + \beta y}{x(1+4x)}$	$\frac{-\beta}{y}$
	α,β	$\frac{-1}{1+4x}$

	$\frac{-2y}{1-4x}$	$\frac{2x}{y}$
	$\frac{-\delta + (4\alpha + 6)x}{x(1-4x)}$	0
H_9	$\frac{-y(1+\alpha+y)}{x(1-4x)}$	$\frac{\alpha - 1 - y}{y}$
	$\frac{\alpha(1+\alpha) - \beta y}{x(1-4x)}$	$\frac{-\beta}{y}$
	$\alpha, \beta; \delta$	$\frac{1}{1-4x}$
	$\frac{-2y}{1-4x}$	$\frac{2x}{y}$
	$\frac{-\delta + (4\alpha + 6)x}{x(1-4x)}$	0
H_{10}	$\frac{-(1+\alpha)y}{x(1-4x)}$	$\frac{\alpha - 1}{y}$
	$\frac{\alpha(1+\alpha) - y}{x(1-4x)}$	$\frac{-1}{y}$
	$\alpha; \delta$	$\frac{1}{1-4x}$

H_{11}

0	$\frac{x}{y(1+y)}$
$\frac{-\delta + x}{x}$	0
$\frac{-y}{x}$	$\frac{\alpha - 1 - (\beta + \gamma + 1)y}{y(1+y)}$
$\frac{\alpha}{x}$	$\frac{-\beta\gamma}{y(1+y)}$
$\alpha, \beta, \gamma;\ \delta$	1

§§ 6.2 Normalization factors

It is important to know the normalization factors of the equations. Since (2.4) gives the expressions of θ_x and θ_y in terms of the coefficients, we integrate them to know θ, up to additive constants. The following is the normalization factor $e^{2\theta}$ (up to multiplicative constants) of each Appell's HGDE's F_2, F_3, F_4. We omit the others.

$$(F_2):\quad x^{\varepsilon_0} y^{\varepsilon_0'} (1-x)^{\varepsilon_1} (1-y)^{\varepsilon_1'} (1-x-y)^{-\delta}$$

$$\varepsilon_0 = -2\gamma \qquad \varepsilon_0' = -2\gamma'$$

$$\varepsilon_1 = \alpha+\beta+1-\gamma-\beta', \qquad \varepsilon_1' = \alpha+\beta'+1-\gamma'-\beta$$

$$\delta = \alpha+\beta+\beta'+2-\gamma-\gamma'$$

$$(F_3):\quad x^{\varepsilon_0} y^{\varepsilon_0'} (1-x)^{\varepsilon_1} (1-y)^{\varepsilon_1'} (xy-x-y)^{\delta}$$

$$\varepsilon_0 = \alpha'+\beta'-2\gamma, \qquad \varepsilon_0' = \alpha+\beta-2\gamma.$$

$$\varepsilon_1 = \gamma-\alpha-\beta-1, \qquad \varepsilon_1' = \gamma-\alpha'-\beta'-1$$

$$\delta = \gamma-\alpha-\alpha'-\beta-\beta'-1$$

$$(F_4):\quad x^{\varepsilon_0} y^{\varepsilon_0'} (1-x-y)\{(1-x-y)^2-4xy\}^{\delta}$$

$$\varepsilon_0 = -2\gamma, \qquad \varepsilon_0' = -2\gamma'$$

$$\delta = \gamma+\gamma'-\alpha-\beta-5/2$$

§§ 6.3 Equivalence between some equations

By changing coordinates and normalization factors F_3 and H_2 are transformed into F_2, and H_6 is transformed into H_3. The following shows the explicit transformation. The equations in the left hand sides have (x,y) and z as independent variables and the unknown, and the ones in the right hand sides have (u,v) and w as independent variables and the unknown

$$F_3(\alpha,\alpha',\beta,\beta';\gamma) \longrightarrow F_2(\beta+\beta'+1-\gamma,\beta,\beta';\beta+1-\alpha,\beta'+1-\alpha')$$

$$(x,y) \longrightarrow (u,v) = (1/x,1/y)$$

$$z \longrightarrow w = u^{-\beta}v^{-\beta'}z$$

$$H_2(\alpha,\beta,\gamma,\delta;\varepsilon) \longrightarrow F_2(\alpha+\beta,\beta,\gamma;\varepsilon\,,\gamma-\delta+1)$$

$$(x,y) \longrightarrow (u,v) = (x,-1/y)$$

$$z \longrightarrow w = v^{\gamma}z$$

$$H_6(\alpha,\beta,\gamma) \longrightarrow H_3(\alpha+\gamma,\gamma;\gamma-\beta+1)$$

$$(x,y) \longrightarrow (u,v) = (-x,-1/y)$$

$$z \longrightarrow w = v^{-\gamma}z \quad .$$

§§ 6.4 Relation between F_2 and F_4

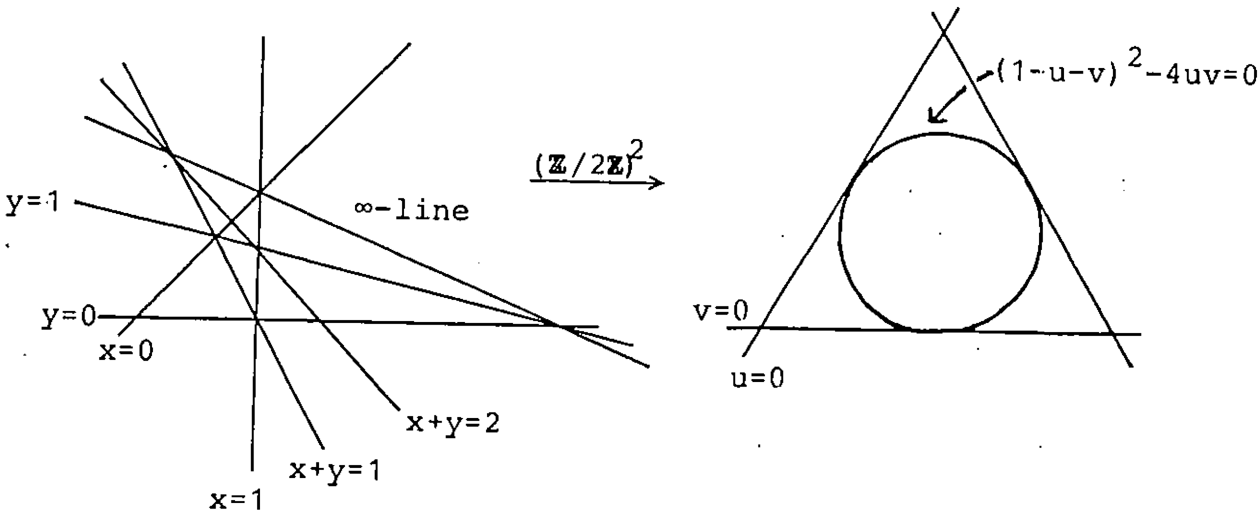
Lift the F_4 to the four sheeted covering of the (u,v)-space branching along the two lines $u = 0$ and $v = 0$ with indices two. In terms of coordinates

$$u = \left(\frac{x}{x+y-2}\right)^2 , \qquad v = \left(\frac{y}{x+y-2}\right)^2 .$$

Looking from upstairs, the projection is the quotient map by the group $G(\cong (\mathbb{Z}/2\mathbb{Z})^2)$ generated by

$$g_1 : (x,y) \longrightarrow \left(\frac{-x}{1-x} , \frac{y}{1-x}\right) ,$$

$$g_2 : (x,y) \longrightarrow \left(\frac{x}{1-y} , \frac{-y}{1-y}\right) .$$



$\mathbb{P}^2(x,y)$ $\qquad\qquad$ $\mathbb{P}^2(u,v)$

Then we have the lift $\widetilde{F}_4(\alpha_1,\beta_1,\beta_1',\gamma_1)$ of $F_4(\alpha,\beta;\gamma,\gamma')$. The coefficients are given as follows:

$$\tilde{F}_4 \qquad \begin{aligned} \ell &= \frac{y}{1-x} & m &= \frac{x}{1-y} \\ a &= \frac{(\alpha_1+\beta_1+1)x-2\beta_1}{x(1-x)} & c &= \frac{\beta_1' x}{y(1-y)} \\ b &= \frac{\beta_1 y}{x(1-y)} & d &= \frac{(\alpha_1+\beta_1'+1)y-2\beta_1'}{y(1-y)} \\ p &= \frac{\alpha_1\beta_1}{x(1-x)} + \frac{\gamma_1}{(1-x)(x+y-2)^2} & q &= \frac{\alpha_1\beta_1'}{y(1-y)} + \frac{\gamma_1}{(1-y)(x+y-2)^2}. \end{aligned}$$

The equation $\tilde{F}_4$ includes F_2 with restricted parameters:

$$\tilde{F}_4(\alpha_1,\beta_1,\beta_1',0) = F_2(\alpha_1,\beta_1,\beta_1';2\beta_1,2\beta_1')$$

which are exactly those F_2 invariant under G. The condition that F_2 is invariant under G (i.e. $\gamma = 2\beta$, $\gamma' = 2\beta'$) happens to be a part of the condition QR for F_2 (see §§ 6.5). Thus under the condition QR, $\tilde{F}_4$ coincides with F_2.

<u>Note</u> S. Nishiyama ([Nis]) studied $F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1)$ algebro-geometrically in detail.

§§ 6.5 <u>Condition QR for hypergeometric differential equations</u>

Let us study the condition QR ((4.8)) for HGDE's. We first examine (4.8.1) for F_2 as follows:

$$2\theta_2 = \frac{\alpha+\beta+1-\gamma-\beta' x}{1-x} - \frac{\gamma(2-x)}{x} + \frac{\alpha+\beta+2-2\gamma-\gamma'+(\gamma+\beta')y+(\gamma+\beta')x}{1+x-y}$$

$$\frac{(1-\ell m)_x}{1-\ell m} = \frac{1}{1-x-y} - \frac{1}{1-x}$$

$$\frac{\ell_x - 2b}{\ell} = \frac{1}{1-x} - \frac{2\beta}{x} \quad .$$

The equality (4.8.1) is

$$0 = \frac{2b-\ell_x}{\ell} + \theta_x + \frac{3}{4}\,\frac{(1-\ell m)_x}{1-\ell m}$$

$$= -\frac{\alpha+\beta+\frac{1}{2}-\gamma-\beta' x}{1-x} + \frac{\gamma(2-x)-4\beta}{x}$$

$$- \frac{\alpha+\beta+\frac{1}{2}-2\gamma-\gamma'+(\gamma+\beta')y+(\gamma+\beta')x}{1-x-y} \quad .$$

This implies

$$\gamma = 2\beta, \quad \gamma' = 2\beta', \quad \beta+\beta' = \alpha+\frac{1}{2} \; .$$

The equality (4.8.2) produces the same condition.

For other HGDE's we make similar computations and obtain the following result.

HGDE	QR
$F_2(\alpha,\beta,\beta';\gamma,\gamma')$	$\gamma = 2\beta,\ \gamma' = 2\beta',\ \beta+\beta' = \alpha+1/2$
$F_3(\alpha,\alpha',\beta,\beta';\gamma)$	$\alpha+\beta = \alpha'+\beta' = 1,\quad \gamma = 3/2$
$F_4(\alpha,\beta;\gamma,\gamma')$	$\gamma+\gamma' = \alpha+\beta+1$
$H_2(\alpha,\beta,\gamma,\delta;\varepsilon)$	$\gamma+\delta = 1,\ \varepsilon = 2\beta,\ \beta = \alpha+1/2$
$H_3(\alpha,\beta;\gamma)$	$\alpha = \beta = 1/2,\quad \gamma = 3/2$
$H_6(\alpha,\beta,\gamma)$	$\alpha = \beta = 0,\quad \gamma = 1/2$
$\Xi_2(\alpha,\beta;\gamma)$	$\alpha+\beta = 1,\quad \gamma = 3/2$
$H_{10}(\alpha;\delta)$	$\delta = \alpha+1$
other HGDE	No parameter satisfies the condition QR

§§ 6.6 F_4 under QR

Change the normalization of F_4 by $z \longrightarrow x^{\frac{\gamma}{2}} y^{\frac{\gamma'}{2}} z$ to get an equation F_4' with the following coefficients:

$$F_4' \qquad \ell = \frac{2y}{1-x-y} \qquad m = \frac{2x}{1-x-y}$$

$$a = \frac{\delta}{1-x-y} \qquad c = \frac{\delta x}{y(1-x-y)}$$

$$b = \frac{\delta y}{x(1-x-y)} \qquad d = \frac{\delta}{1-x-y}$$

$$p = \frac{\varepsilon-(\gamma+\gamma')\delta}{2x(1-x-y)} + \frac{\lambda}{4x^2} \qquad q = \frac{\varepsilon-(\gamma+\gamma')\delta}{2y(1-x-y)} + \frac{\lambda'}{4y^2}$$

where

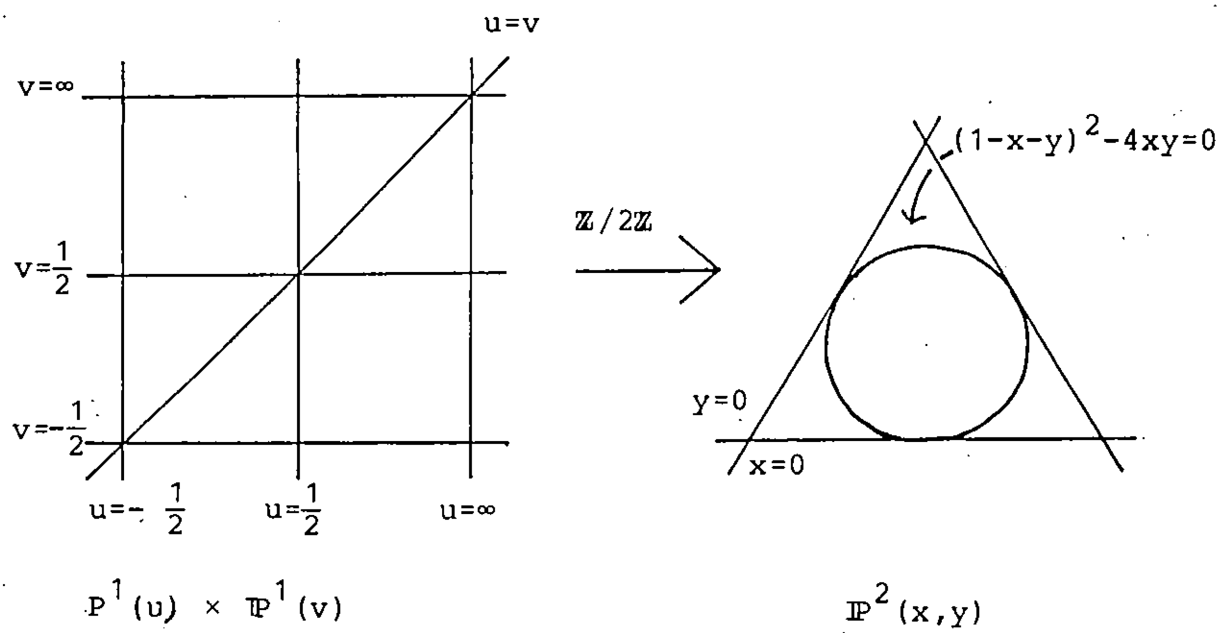
$$\delta = \alpha+\beta+1-\gamma-\gamma', \qquad \varepsilon = 2\alpha\beta-\gamma\gamma'$$

$$\lambda = \gamma^2-2\gamma, \qquad \lambda' = \gamma'^2-2\gamma'$$

The condition QR is "$\delta = 0$". Lift the F_4' to the double covering of the (x,y) - space branching along the curve $(1-x-y)^2 - 4xy = 0$. In terms of coordinates, perform the coordinate transformation

$$x = (1+2u)(1+2v)/4, \quad y = (1-2u)(1-2v)/4.$$

Looking from upstairs the projection is the quotient map under the involution $(u,v) \longrightarrow (v,u)$.



Under the condition QR the equation thus obtained is

$$z_{uu} = p(u)z, \qquad z_{vv} = q(v)z$$

where

$$p(u) = \frac{2\varepsilon}{1-4u^2} + \frac{\lambda}{(1+2u)^2} + \frac{\lambda'}{(1-2u)^2}$$

$$q(v) = \frac{2\varepsilon}{1-4v^2} + \frac{\lambda'}{(1-2v)^2} + \frac{\lambda}{(1+2v)^2} \quad .$$

Each of which is expressed by Riemann's P-function

$$\left\{ \begin{array}{lll} u(\text{resp. } v) = -\frac{1}{2} & u(\text{resp. } v) = \frac{1}{2} & u(\text{resp. } v) = \infty \\ \frac{1}{2} + \sqrt{1+\lambda} & \frac{1}{2} + \sqrt{1+\lambda'} & -\frac{1}{2} + \sqrt{\lambda+\lambda'-2\varepsilon+1} \\ \frac{1}{2} - \sqrt{1+\lambda} & \frac{1}{2} - \sqrt{1+\lambda'} & -\frac{1}{2} - \sqrt{\lambda+\lambda'-2\varepsilon+1} \end{array} \right\}$$

§§ 6.7 Ξ_2 and H_{10} under QR

Let us define an equation $\Xi(\lambda,\mu,\nu)$ with the parameters λ,μ and ν by

$$\ell = \frac{y}{2(1-x)} \qquad m = \frac{2x}{y}$$

$$a = \frac{1}{2(1-x)} \qquad c = 0$$

$$b = 0 \qquad d = \frac{-2}{y}$$

$$p = \frac{\lambda y}{4x(1-x)} + \frac{\mu}{x^2(1-x)} \qquad q = \frac{\lambda}{y} + \nu \ .$$

This equation has the property QR and includes both Ξ_2 and H_{10} under QR as follows (using the convention in §§ 6.3):

$$H_{10}(\alpha, 1+\alpha) \longrightarrow \Xi(-1, \frac{\alpha^2-1}{4}, 0)$$
$$(x,y) \longrightarrow (u,v) = (4x,y)$$
$$z \longrightarrow w = u^{\frac{\alpha+1}{2}} z$$

$$\Xi_2(\alpha, 1-\alpha, \tfrac{3}{2}) \longrightarrow \Xi(0, -\alpha(1-\alpha), 4)$$
$$(x,y) \longrightarrow (u,v) = (1/x, \sqrt{y})$$
$$z \longrightarrow w = z \quad .$$

The equation $\Xi(\lambda,\mu,\nu)$ is decomposed into two ordinary differential equations by the elementary change of variables. Indeed put

$$u = \frac{1}{2} \int \frac{1}{x}\sqrt{\frac{2x-1}{1-x}}\, dx + i \log (y\sqrt{x})$$

$$v = \frac{1}{2} \int \frac{1}{x}\sqrt{\frac{2x-1}{1-x}}\, dx - i \log (y\sqrt{x})$$

then the fundamental form

$$\frac{y}{2(1-x)}\, dx^2 + 2dxdy + \frac{2x}{y}\, dy^2$$

is conformally equivalent to the form $dudv$.

§§ 6.8 H_3 and H_6 under QR

We do not know whether the equations H_3 and H_6 under QR are decomposed into two ordinary differential equations or not.

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