HERMITIAN-EINSTEIN CONNECTIONS AND STABLE VECTOR BUNDLES OVER COMPACT COMPLEX SURFACES

by

N.P. Buchdahl

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

MPI 86-39

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N.P. Buchdahl
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3, West Germany

Summary.

A theorem of P.Gauduchon states that an arbitrary hermitian metric on complex surface has a conformal rescaling such that the associated Kähler form is then $\bar{\partial}\partial$-closed. Given such a form, the degree of a holomorphic line bundle can be defined in the usual way and with that, the notion of stability in the sense of Mumford and Takemoto for torsion-free sheaves. It is proved here that an indecomposable holomorphic vector bundle on the surface is stable iff it admits an irreducible Hermitian-Einstein connection, where "stable" and "Hermitian-Einstein" are both with respect to a given positive $\bar{\partial}\partial$-closed (1,1)-form. This generalizes a result of Donaldson, who proved this theorem in the case of algebraic surfaces in $\mathbb{P}_N$ equipped with a Kähler metric whose Kähler form is cohomologous to that of the Fubini-Study metric.

1. Introduction

Let X be a compact complex manifold of dimension n and E be a holomorphic vector bundle on X. It is well-known ([2]) that to each hermitian metric on E there is a unique hermitian connection inducing the $\bar{\partial}$-operator on E; the curvature F of this connection is an anti-self-adjoint section of $\Lambda^{1,1} \otimes \mathrm{End}E$. If h_0, h_1 are metrics on E, then the resulting curvatures are related by $F_1 = F_0 + \bar{\partial}_0(u^{-1}\partial_0 u)$, where u is the positive self-adjoint endomorphism $u = h_0^{-1}h_1$. Conversely, a unitary bundle with smooth unitary connection having curvature of type $(1,1)$ inherits a unique holomorphic structure by the Newlander-Nirenberg theorem.

If X has a Kähler metric and ω is the Kähler form, then the Yang-Mills equations for connections of this type reduce to $d\hat{F} = 0$, where $\hat{F} := * \frac{1}{(n-1)!}(F \wedge \omega^{n-1})$. In this case, the bundle and connection split up into the eigenspaces of the endomorphism $\hat{F}$, so if the connection is irreducible or if E is simple, then $\hat{F} = i\lambda 1$ for some $\lambda \in \mathbb{R}$. Such a connection, introduced by Kobayashi and by Hitchin, is called <u>Hermitian-Einstein</u> (H-E). The constant λ is determined by $c_1(E)$: $\lambda = \lambda_E = -\frac{2\pi}{(n-1)!V} \cdot \mu(E)$ where $V = \mathrm{Vol}(X)$ and $\mu(E) := (c_1(E) \cup \omega^{n-1})[X]/\mathrm{rank}(E)$.

The quantity $\mu(E)$ also features in the algebra-geometric notion of stability: E is <u>(semi-)stable</u> in the sense of Mumford and Takemoto if every coherent subsheaf $S \subset \mathcal{O}(E)$ with $0 < \mathrm{rank}\, S < \mathrm{rank}\, E$ satisfies $\mu(S) < \mu(E)$ $(\mu(S) \leq \mu(E))$. (The definition of μ for sheaves is given in section 3 below).

In [16], Narasimhan and Seshadri proved that an indecomposable holomorphic bundle on a Riemann surface is stable iff it admits an irreducible H-E connection. This result was later reproved by Donaldson [4] by a different method. About the same time, Kobayashi [13] and Lübke [15] showed that if a bundle on an arbitrary compact Kähler manifold admits an irreducible H-E connection, then it is stable. In [5], Donaldson showed in the case when X is an algebraic surface $X \hookrightarrow \mathbb{P}^N$ and ω is cohomologous to the restriction of the Fubini-Study form, the converse is also true. Recently, Uhlenbeck and Yau [22] have proved the general n-dimensional Kähler version of this theorem.

The case when X is a compact complex surface is perhaps the most interesting, for it is in this case that the differential topology of the underlying 4-manifold is intricately connected with this problem. For example, using a deep application of his results in [5], Donaldson has given a counterexample to the 5-dimensional h-cobordism conjecture [6]. The interaction between the complex and real analysis stems from the fact that H-E connections on bundles with $\mu = 0$ are precisely the anti-self-dual Yang-Mills connections.

In the case of complex surfaces, the notion of stability can be extended somewhat: given an arbitrary hermitian metric on X, a theorem of Gauduchon [7] states that there is a conformal rescaling of the metric, (unique up to a positive constant), such that the associated Kähler form ω satisfies $\bar{\partial}\partial\omega = 0$. If L is a holomorphic line bundle on X, the <u>degree</u> of L (with respect to ω) can then be defined by $\deg(L) = \deg(L,\omega) := \frac{i}{2\pi}\int_X f \wedge \omega$, where f is the curvature form of any hermitian connection on L compatible with $\bar{\partial}_L$. Since any two such curvature forms differ by a $\bar{\partial}\partial$-exact term, deg(L) is independent of the choice of connection. If $d\omega = 0$, then $\deg(L) = (c_1(L) \cup \omega)[X]$ as usual, but in general, deg(L) depends only on the image of $c_1(L)$ in $H^2(X,\mathbb{R})$ iff $b_1(X)$ is even; (see Proposition 2 below).

Having defined the degree of holomorphic line bundles, the definition of stability can be repeated verbatim, and the definition of H-E connections also remains unaltered, although it should be noted that when $d\omega \neq 0$, an H-E connection on E is a Yang-Mills connection compatible with $\bar{\partial}_E$ iff $\mu(E) = 0$. The main result to be proved here is (cf. [5]):

<u>Theorem 1</u>. Let X be a compact complex surface with a hermitian metric whose Kähler form is $\bar{\partial}\partial$-closed. Then an indecomposable holomorphic bundle on X is stable iff it admits an irreducible Hermitian-Einstein connection. This connection is unique.

("Stability" and "Hermitian-Einstein" are, of course, with respect to the given $\bar{\partial}\partial$-closed Kähler-form.)

The proof of Theorem 1 is by induction on the rank of the bundle, and is based on Donaldson's proof [4] of the theorem of Narasimhan and Seshadri. In brief outline this runs as follows: given the stable bundle E, a functional J(A) is constructed on the space of hermitian connections A on E compatible with $\bar{\partial}_E$, essentially equivalent to the L^2 norm of $\hat{F}(A) - i\lambda_E 1$. Choosing a minimizing sequence A_i for J and employing Uhlenbeck's weak compactness theorem [21] for connections on bundles, a limit connection A' is obtained with $J(A') \leq \inf J(A_i)$. Now A' might define a different holomorphic structure E' on the smooth underlying bundle, but in any case, by a semi-continuity of cohomology argument, Donaldson shows that there is a non-zero holomorphic map $\phi : E \longrightarrow E'$. If ϕ is not an isomorphism, he shows that $J(A') \geq 4\pi V^{-1/2} \nu_E(\ker\phi)$, where $\nu_E(S) := (\operatorname{rank} S)(\mu(E) - \mu(S))$ for $S \subset E$ and $V = \operatorname{Vol}(X)$. On the other hand, using the canonical filtrations of Harder and Narasimhan [11] and the inductive hypothesis, he can construct a connection A on E (compatible with $\bar{\partial}_E$) with $J(A) < 4\pi V^{-1/2} \nu_E(\ker\phi)$. This contradiction means that A' is compatible with $\bar{\partial}_E$ and minimizes J . A simple argument then shows that for A' to minimize J , necessarily $J(A') = 0$, giving $\hat{F}(A') = i\lambda_E 1$. The "only if" part of the argument is more straight-forward.

The main features of Donaldson's proof also appear here, the biggest strategic difference being that the Harder-Narasimhan filtrations are avoided by reversing the order of his arguments. However, the technical differences are somewhat more significant, owing to the appearance of singularities of one sort or another: torsion-free sheaves are no longer locally free, and sequences of connections only converge off finite sets of points. These difficulties are resolved generally by blowing-up and by appealing to the appropriate removability of singularities theorem of Hartogs, Serre or Uhlenbeck. Moreover, some of the techniques used by Donaldson in [5] can still be employed and indeed, these too play an essential rôle in the proof to be given here. The introduction and first section of [5] also contains more background material, and in particular, a clear description of the two equivalent formulations of the problem; namely, finding a certain connection on a fixed U(r)-bundle, or finding a certain hermitian metric on a fixed holomorphic bundle.

Acknowledgements. I am grateful to Prof.S.K.Donaldson for suggesting the generalization to $\bar{\partial}\partial$-closed Kähler forms, and to the Max-Planck-Institut für Mathematik in Bonn for its hospitality and support.

2. Hermitian Geometry

Let X be a compact complex surface and h be an hermitian metric on X. In local holomorphic coordinates z^a, the associated Kähler form is $\omega := \frac{i}{2} h_{a\bar{b}} dz^a \wedge dz^{\bar{b}}$; (all conventions here follow those in [10]). The volume form is $dV = \frac{1}{2}\omega\wedge\omega$, and if $* : \Lambda^{p,q} \longrightarrow \Lambda^{2-q,2-p}$ is the Hodge $*$-operator, then with respect to the inner product $(f,g) \longmapsto *(\bar{f}\wedge *g)$, the adjoint of $\Lambda^{p,q} \ni g \longmapsto g \wedge \omega \in \Lambda^{p+1,q+1}$ is denoted by $f \longmapsto \Lambda f$. On (1,1)-forms $f = f_{a\bar{b}}\, dz^a \wedge dz^{\bar{b}}$, $\Lambda f = -2ih^{a\bar{b}} f_{a\bar{b}}$, frequently denoted by $\hat{f}$. Note that $\Lambda\omega = 2$.

The $*$-operator on 2-forms satisfies $*^2 = 1$, and the decomposition into $\pm$ eigenspaces is $\Lambda^2_+ = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \mathrm{span}(\omega)$, $\Lambda^2_- = \ker\Lambda : \Lambda^{1,1} \longrightarrow \Lambda^0$.

With respect to the inner product $(f,g) \longmapsto \int_X \bar{f}\wedge *g$, a straightforward calculation gives

$$\partial^* g = -*\bar{\partial}*g = i\Lambda\bar{\partial}g + i*(\bar{\partial}\omega\wedge g) \;,\; g \in \Lambda^{1,0} \qquad \text{(a)} \qquad (2.1)$$

$$\partial^* f = -*\bar{\partial}*f = i(\Lambda\bar{\partial}-\bar{\partial}\Lambda)f + (*\bar{\partial}\omega)\Lambda f \;,\; f \in \Lambda^{1,1} \;. \qquad \text{(b)}$$

Let P be the second-order real elliptic operator on functions $P := i\Lambda\bar{\partial}\partial$, (so if h is flat, $P = \frac{1}{2}\Delta$ where Δ is the usual Laplacian having <u>negative</u> symbol). Then

$P^*f = *i\bar{\partial}\partial(\omega f) = i\Lambda\bar{\partial}\partial f + i*(\bar{\partial}\omega\wedge\partial f) - i*(\partial\omega\wedge\bar{\partial}f) + i(*\bar{\partial}\partial\omega)f$.

That is,

$$P^* = P + i*\bar{\partial}\omega\wedge\partial - i*\partial\omega\wedge\bar{\partial} + i*\bar{\partial}\partial\omega \quad . \qquad (2.2)$$

From (2.1)(a) and its complex conjugate, one easily obtains

$$\Delta' = \partial^*\partial = P + i{*}\bar\partial\omega\wedge\partial \quad (a) \qquad (2.3)$$

$$\Delta'' = \bar\partial^*\bar\partial = P - i\Lambda(\partial\bar\partial+\bar\partial\partial) - i{*}\partial\omega\wedge\bar\partial \quad (b)$$

$$\Delta = \Delta'+\Delta'' = 2\Delta'' + i\Lambda(\partial\bar\partial+\bar\partial\partial)+i{*}d\omega\wedge d \quad (c)$$

(Of course, $\bar\partial\partial+\partial\bar\partial = 0$ on functions, but (2.3) is valid for an arbitrary hermitian connection on a bundle, in which case $\partial\bar\partial+\bar\partial\partial$ is the (1,1) component of the curvature.) Adding (2.3) (a) and (b) and using (2.2) also gives

$$\Delta = P + P^* - i\Lambda(\partial\bar\partial+\bar\partial\partial) - i{*}\bar\partial\partial\omega \ . \qquad (2.4)$$

Now suppose that the metric h has been conformally scaled according to the theorem of Gauduchon [7] so that $\bar\partial\partial\omega = 0$. Then a number of easy but important consequences follow from these equations. The first of these is the existence of H-E connections on holomorphic line bundles. For if L is a line bundle with hermitian connection compatible with $\bar\partial_L$ and curvature $f \in \Lambda^{1,1}$ any other such curvature form has curvature $f+\bar\partial\partial \log u$ for some positive function u . Thus the equation to be solved is $P \log u = -i\hat{f}-\lambda$ where $\int_X (i\hat{f}+\lambda)dV = 0$. From (2.4), $\Delta = P+P^*$ on functions, so $\ker P = \ker P^* = \mathbb{R}$. By standard linear elliptic theory on compact manifolds, there exists a smooth solution u to $P \log u = -i\hat{f}-\lambda$, unique up to multiplication by a positive constant.

Next suppose that E is a holomorphic bundle with H-E

connection: $\hat{F} = \Lambda F = i\lambda 1$ for $\lambda = -2\pi V^{-1}\mu(E,\omega)$. If s is a global holomorphic section then from (2.3)(c),
$\| ds \|^2 = \langle s,\Delta s\rangle = -\lambda \| s \|^2 + \langle s, *id\omega\wedge ds\rangle$, (ds denoting the covariant derivature of s.). But $\langle s,*d\omega\wedge ds\rangle = \langle s,*\bar\partial\omega\wedge\partial s\rangle$
$= \langle s,*[-\partial(\bar\partial\omega s)+\partial\bar\partial\omega s]\rangle = -\langle *s,\partial(\bar\partial\omega s)\rangle = -\langle\partial^* *s, \bar\partial\omega s\rangle =$
$\langle *\bar\partial s, \bar\partial\omega s\rangle = 0$, so $\| ds\|^2 = -\lambda \| s\|^2$. Thus, just as in the Kähler case, one has the result of Kobayashi [12]:

Proposition 1. Let X be a compact surface with a metric whose Kähler form is $\bar\partial\partial$-closed. If E is a holomorphic bundle on X which admits an H-E connection, then if $\mu(E)<0$ it follows that $H^0(X,\mathcal{O}(E))=0$, and if $\mu(E)=0$, every holomorphic section is covariantly constant.

□

Corollary 1. If L is a holomorphic line bundle on the compact surface X such that $H^0(X,L) \neq 0$, then $\deg(L,\omega) \geq 0$ for any positive $\bar\partial\partial$-closed (1,1)-form ω , with equality iff L is trivial.

□

Corollary 2. Let ω be a positive $\bar\partial\partial$-closed (1,1)-form on the compact surface X , and let $\{e_1,\ldots,e_m\}$ be an integral basis for $H^2(X,\mathbb{Z})$/torsion. Then there exists $\varepsilon=\varepsilon(\omega)>0$ such that any holomorphic line bundle L on X with $c_1(L)\equiv\sum n^\alpha e_\alpha$ mod torsion and $H^0(X,L)\neq 0$ satisfies $\deg(L,\omega)\geq\varepsilon\sum|n^\alpha|$.

Proof. Let $e_\alpha \cdot e_\beta = q_{\alpha\beta}$ be the intersection matrix on $H^2(X,\mathbb{Z})$/torsion, $q^{\alpha\beta}$ the inverse. If f_α is a closed 2-form representing e_α, the (1,1)-component $\tilde{f}_\alpha$ of f_α is $\bar{\partial}\partial$-closed. If $\varepsilon>0$ is sufficiently small, $\omega \pm \varepsilon m \sum q^{\alpha\beta}\tilde{f}_\beta$ is positive for any $\alpha=1,\ldots,m$. By Corollary 1, $0 \leq \deg(L,\omega \pm \varepsilon m \sum q^{\alpha\beta}\tilde{f}_\beta) = \deg(L,\omega) \pm \varepsilon m n^\alpha$, (for if $f \in \Lambda^{1,1}$ represents $c_1(L)$, $\int f \wedge f_\beta = \int f \wedge \tilde{f}_\beta$). Thus $\deg(L,\omega) \geq \varepsilon m |n^\alpha|$ for all α, and summing over α gives the desired conclusion.

□

Corollary 3. An H-E connection on an indecomposable bundle is unique if one exists.

Proof. (cf.[4]). If E is a smooth unitary bundle with two integrable unitary connections A_0, A_1 inducing isomorphic holomorphic structures E_0, E_1 then, by definition, there is a complex automorphism g of E such that $\bar{\partial}_1 = g \circ \bar{\partial}_0 \circ g^{-1}$ and $\partial_1 = g^{*-1} \circ \partial_0 \circ g^*$. After a unitary change of gauge of one of them $[g(g^*g)^{-1/2}]$, g can be assumed positive self-adjoint. If A_0, A_1 are H-E connections, then the (holomorphic) isomorphism $g : E_0 \longrightarrow E_1$ is covariantly constant by Proposition 1, implying $0 = \partial_0(g^*g) = \partial_0(g^2)$ and $\bar{\partial}_0(g^2) = 0$. Since E_0 is indecomposable, $g^2 = \text{const}.1$ and since g is positive self-adjoint, $g = \text{const}.1$.

□

The next corollary is taken verbatim from [5]. For the proof (which is short), see that reference.

Corollary 4. Suppose that the main theorem has been proved for bundles of rank less than r. Then any r-bundle which admits an Hermitian-Einstein connection is a direct sum $\sum E_i$ of stable bundle E_i with $\mu(E_i)=\mu(E)$. In particular, it is semi-stable. If E admits an irreducible such connection, it is stable.

□

A slightly different version of (2.3)(c) will be of use subsequently. Suppose that E is a bundle with integrable hermitian connection having curvature F. Then (2.3)(c) gives $\Delta = 2\Delta'' + i\hat{F} + i{*}d\omega\wedge d$ for the full covariant Laplacian on sections. So if s is a local holomorphic section, $\Delta|s|^2=\Delta<s,s>=2<s,\Delta s>-2|ds|^2=2<s,i\hat{F}s>+2i<s,{*}d\omega\wedge ds>-2|ds|^2$. Using the same manipulations as before together with $\bar{\partial}s=0=\bar{\partial}\partial\omega$, one computes $<s,{*}d\omega\wedge ds> = -{*}\partial(|s|^2\bar{\partial}\omega)$. Thus $\Delta|s|^2+2i{*}\partial(|s|^2\bar{\partial}\omega) = 2<s,i\hat{F}s>-2|ds|^2$. Since $i\hat{F}$ is a real operator, taking the complex conjugate of this last equation and adding gives

$$\Delta|s|^2+i{*}\partial(|s|^2\bar{\partial}\omega)-i{*}\bar{\partial}(|s|^2\partial\omega)=2<s,i\hat{F}s>-2|ds|^2 , \qquad (2.5)$$

(s holomorphic) ,

which is the unintegrated version of the equation used for Proposition 1. Note that since $\bar{\partial}\partial\omega= 0$, the operator on the left of (2.5) satisfies the maximum principle, by theorem 3.1 of [8].

The last application of the equations (2.1)-(2.4) is the result promised in the introduction on the topological invariance of $\deg(-,\omega)$.

<u>Proposition 2</u>. If ω is a positive $\bar{\partial}\partial$-closed (1,1)-form on the compact surface X , then $\deg(L,\omega) = \frac{i}{2\pi}\int_X f_L \wedge \omega$ depends only on the image of $c_1(L)$ in $H^2(X,\mathbb{R})$ iff $b_1(X)$ is even.

<u>Remark</u>. $b_1(X)$ even is equivalent to the existence of a Kähler metric on X by results of Kodaira, Siu.

<u>Proof of proposition</u>. Suppose $b_1(X)$ is even. Under the map $H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^*)$ induced by $0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$, a representative $\bar{\partial}$-closed (0,1)-form g is mapped to $\frac{i}{2\pi}\int_X (\partial g - \bar{\partial}\bar{g}) \wedge \omega$ by $\deg(-,\omega)$, and of course, this map annihilate the image of $H^1(X,\mathbb{Z})$ in $H^1(X,\mathcal{O})$. Since b_1 is even, $H^1(X,\mathcal{O})$ has real dimension b_1 ([3]), and since $H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O})$ is always injective, $\deg: H^1(X,\mathcal{O}) \longrightarrow \mathbb{R}$ must be zero, otherwise the kernel would contain a lattice of rank greater than its dimension. Thus $\deg(L,\omega)$ depends only on $c_1(L) \in H^2(X,\mathbb{Z})$ in this case. Since $\int(\partial g - \bar{\partial}\bar{g}) \wedge \omega = 0$ for all $\bar{\partial}$-closed (0,1)-forms g , replacing g by ig shows that $\int \partial g \wedge \omega = 0$ for all such g , and similarly $\int \bar{\partial} h \wedge \omega = 0$ for all ∂-closed (1,0)-forms h . Thus if f_0, f_1 are (1,1)-forms such that $f_0 - f_1 = dh$ for some $h \in \Lambda^1$, then $\bar{\partial} h_{0,1} = 0 = \partial h_{1,0}$ giving $\int (f_0 - f_1) \wedge \omega = \int (\partial h_{0,1} + \bar{\partial} h_{1,0}) \wedge \omega = 0$. Thus $\deg(L,\omega)$ depends only on the image of $c_1(L)$ in $H^2(X,\mathbb{R})$

Now suppose that $\int(\partial g - \bar{\partial}\bar{g}) \wedge \omega = 0$ for all $\bar{\partial}$-closed (0,1)-forms g . Then as above, $\int \partial g \wedge \omega = 0 = \int \bar{\partial} h \wedge \omega$ for all $\bar{\partial}$-closed $g \in \Lambda^{0,1}$ and ∂-closed $h \in \Lambda^{1,0}$. Given such g , the equation

$Pu = i\Lambda\partial g$ has a solution u since $\int \Lambda\partial g dV = \int \partial g \wedge \omega = 0$, and moreover u is unique up to the addition of a constant. But this is just $\Lambda\partial\tilde{g} = 0$, where $\tilde{g} := g+\bar{\partial}u$. From (3.1)(b) it now follows that $\langle\partial\tilde{g},\partial\tilde{g}\rangle = \langle\tilde{g},\partial^*\partial\tilde{g}\rangle = \langle\tilde{g},[i(\Lambda\bar{\partial}-\bar{\partial}\Lambda)+*\bar{\partial}\omega\Lambda]\partial\tilde{g}\rangle = 0$, so g gives the unique $\bar{\partial}$-closed (1,0)-form $g' := \bar{\tilde{g}}$. Conversely, every holomorphic 1-form on a compact surface is closed ([3]), so that the map $H^1(X,\mathcal{O}) \longrightarrow H^0(X,\Omega^1)$ defined this way is invertible. Thus $h^{1,0}(X) = h^{0,1}(X)$ and $b_1(X) = h^{1,0}(X)+h^{0,1}(X)$ is even.

□

<u>Remark</u>. An easy continuation of this argument shows that when $b_1(X)$ is even, any real $\bar{\partial}\partial$-closed (1,1)-form ω is cohomologous mod im $\partial+\bar{\partial}$ to a d-closed real (1,1)-form, and any two such (cohomologous) d-closed (1,1)-forms differ by a d-exact term, so ω defines a unique element of $H^2(X,\mathbb{R})$.

In order to use the inductive hypothesis to prove Theorem 1, it is necessary to find sub-bundles of a given bundle. However, in general one can expect to find at most <u>subsheaves</u> which are sub-bundles off a finite set of points. To get sub-bundles therefore, these singular points have to be blown-up, and then appropriate metrics must be constructed on the blown-up space. For details of what follows, see [10], pp.182-187.

Let x be a point on the surface X and let $\tilde{X} \xrightarrow{\pi} X$ be the blow-up of X at x. Given the positive (1,1)-form ω on X, $\pi^*\omega$ is degenerate on the exceptional divisor $L = \pi^{-1}(x)$, but it can be modified as follows. If U is a sufficiently small neighbourhood of x and $\tilde{U} := \pi^{-1}(U)$, then there is a holomorphic projection $\pi_2 : \tilde{U} \longrightarrow \mathbb{P}_1$. Now L is the zero set of a section $s \in \Gamma(\tilde{X}, \mathcal{O}(-1))$, so let h_0 be the metric on $\mathcal{O}(-1)$ $(:= \mathcal{O}(L))$ over $\tilde{X} \setminus L$ such that $|s| \equiv 1$, and let h_1 be the standard metric on $\mathcal{O}(-1)$ over $\mathbb{P}_1$. Let ρ be any cut-off function with suppor in U such that $\rho = 1$ on a neighbourhood of x . Then $h := (1-\rho)h_0 + \rho\pi_2^* h_1$ is a metric on $\mathcal{O}(-1)$ and the resulting Cher form is $\sigma := \frac{i}{2\pi} \bar{\partial}\partial \log h \in \Lambda^{1,1}(\tilde{X})$. σ is identically zero outside of $\tilde{U}$ and is negative definite in directions tangent to L in a neighbourhood of L . Thus, for sufficiently small ε , $\tilde{\omega}_\varepsilon := \pi^*\omega - \varepsilon\sigma$ is positive.

If ω is $\bar{\partial}\partial$-closed, resp. d-closed then so too is $\tilde{\omega}_\varepsilon$, and if ω is rational ($d\omega = 0$ and $[\omega] \in H^2(X, \mathbb{Q})$) , so too is $\tilde{\omega}$ if ε is rational. These are the metrics used for the Kodaira embedding theorem.

If ω is $\bar{\partial}\partial$-closed, then in a neighbourhood W of x , $\omega = \partial u + \bar{\partial} v$ for some $u \in \Lambda^{0,1}$, $v \in \Lambda^{1,0}$. Since $\int_{\tilde{X}} \pi^*\omega \wedge \sigma$ does not depend on the choice of σ , it can be supposed that $\operatorname{supp}\sigma \subset\subset \tilde{W}$, from which it follows that $\int_{\tilde{X}} \pi^*\omega \wedge \sigma = 0$. Similarly, $\deg(-, \tilde{\omega}_\varepsilon)$ does not depend on the choice of σ , only on ε . Not also that since L has self-intersection -1 , $\int_{\tilde{X}} \sigma \wedge \sigma = -1$ and $\mathrm{Vol}(\tilde{X}, \tilde{\omega}_\varepsilon) = \frac{1}{2}\int_{\tilde{X}} \tilde{\omega}_\varepsilon^2 = \mathrm{Vol}(X) - \frac{1}{2}\varepsilon^2$.

3. Desingularization of sheaves

It is well-known that singularities on surfaces can be resolved by blowing-up [3], and the same is true for coherent analytic sheaves. This will be indicated shortly, but first a number of basic facts about sheaves will be recalled, taken directly from [17] pp.139-160. See also [9].

Let E be a coherent analytic sheaf on a complex manifold X . The singularity set of B is $S(B) = \{x \in X : B_x$ is not a free $\mathcal{O}_x$-module$\}$ and is an analytic set in X of codimension ≥ 1 . Thus B has a well-defined rank, b say. The torsion subsheaf $\tau(B)$ is defined by $\tau(B)_x$ = torsion submodule of B_x , and $\tau(B)$ is coherent. If $\tau(B)=0$, then B is <u>torsion-free</u> and codim $S(B) \geq 2$. Thus if X is compact and B is torsion-free, B has a well-defined first Chern class. An equivalent definition of torsion-free is that the canonical homomorphism $B \longrightarrow B^{**}$ is injective, where $B^* := \mathrm{Hom}(B,\mathcal{O})$. If $B = B^{**}$, then B is <u>reflexive</u> and codim $S(B) \geq 3$. In general, B is reflexive iff it is torsion-free and <u>normal</u>, where normal means that $\Gamma(U,B) \longrightarrow \Gamma(U \setminus A,B)$ is injective for any analytic set A of codim ≥ 2 in an open set $U \subset X$. Thus for arbitrary B , it follows B^* is reflexive. In general, a reflexive sheaf of rank 1 is a line bundle, so the determinant of a coherent analytic sheaf B of rank b is $\det B := (\Lambda^b B)^{**}$. If $B \longrightarrow C$ is a monomorphism of torsion-free sheaves of ranks $b \leq c$, then $\Lambda^b B \longrightarrow \Lambda^b C$ is also a monomorphism since the kernel is a torsion subsheaf; thus if $b = c$, $\det B \longrightarrow \det C$ is also a monomorphism.

If $0 \to A \to B \to C \to 0$ is an exact sequence of sheaves with

B reflexive, then lemma 1.1.16 of [17] states that A is normal if C is torsion-free. If C is not torsion-free, then the <u>maximal normal extension</u> $\hat{A}_B$ of A in B is given by $\hat{A}_B := \ker[B \twoheadrightarrow C/\tau(C)]$; thus there is a monomorphism $A \to \hat{A}_B$ and in this way it generally suffices to deal with reflexive subsheaves of bundles in questions related to stability.

In the case when X is a compact surface, torsion-free sheaves are singular only at finitely many points and reflexive sheaves are locally free. If ω is a positive $\bar{\partial}\partial$-closed (1,1)-fo on X, the degree of a coherent analytic sheaf B of rank b o X is $\deg(B) = \deg(B,\omega) := \deg(\det B,\omega)$, and $\mu(B) = \mu(B,\omega) :=$ $\deg(B,\omega)/b$. It follows from Corollary 1 that if $B \to C$ is a monomorphism of torsion-free sheaves of the same rank, then $\mu(B) \leq \mu(C)$. Also, despite its possibly non-topological nature, $\deg(-,\omega)$ behaves well with respect to exact sequences $0 \to A \to B \to C \to 0$ of torsion-free sheaves, for since $\det B \simeq (\det A)\otimes(\det C)$ off a finite set of points, this isomorphism extends by Hartogs' theorem to all of X , giving $\deg(B) = \deg(A) + \deg(C)$.

With these preliminaries out of the way, the desingularizati of torsion-free sheaves on surfaces can now be described.

Let B be a torsion-free sheaf in a neighbourhood of $0 \in \mathbb{C}^2$ singular only at 0. Then in a neighbourhood of 0, B is given by an exact sequence $0 \to \mathcal{O}^m \xrightarrow{f} \mathcal{O}^n \to B \to 0$, where $f(x)$ is an $n\times m$ matrix of holomorphic functions which has rank m for $x \neq 0$. A measure of the degree of the singularity at 0 is given by $\operatorname{rank} f(0)$. If this is zero, a second measure is given by the smallest integer p such that $\mathfrak{m}_0^p$ is contained in the ideal $I(f)_0$ generated by the germs of the $m\times m$ subdeterminants of f, where $\mathfrak{m}_0$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$.

By elementary row and column operations, f is equivalent to a matrix of the form $\begin{pmatrix}1 & 0\\ 0 & g\end{pmatrix}$ where 1 is the unit $k\times k$ matrix ($k=\operatorname{rank} f(0)$) and $g(0)=0$. Blowing-up the origin gives $\pi^*g=\tilde{g}s$ where $s=\operatorname{diag}(t^{a_1},\dots t^{a_{m-k}})$, $a_i>0$, $t\in\Gamma(\mathcal{O}(-1))$ defining the exceptional divisor L, with $\tilde{g}$ non-singular and having a non-zero entry in each column. In terms of diagrams, this is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \begin{matrix}\mathcal{O}^k\\ \oplus\\ \mathcal{O}^{m-k}\end{matrix} & \xrightarrow[\pi^*g]{1} & \begin{matrix}\mathcal{O}^k\\ \oplus\\ \mathcal{O}^{n-k}\end{matrix} & \longrightarrow & \pi^*B & \longrightarrow & 0\\
 & & \Big\downarrow{\scriptstyle 1\oplus s} & & \Vert & & \Big\downarrow & & \\
0 & \longrightarrow & \begin{matrix}\mathcal{O}^k\\ \oplus\\ \sum\mathcal{O}(-a_i)\end{matrix} & \xrightarrow[\tilde{g}]{1} & \begin{matrix}\mathcal{O}^k\\ \oplus\\ \mathcal{O}^{n-k}\end{matrix} & \longrightarrow & \tilde{B} & \longrightarrow & 0
\end{array}.$$

Here $\tilde{B}$ is defined by the lower row.

Now let $\tilde{B}_1 := \tilde{B}/\tau(\tilde{B})$, $\tilde{A} := \ker[\mathcal{O}^n \longrightarrow \tilde{B}_1]$, so $\tilde{A}$ is locally free and the map $\tilde{f}:\tilde{A}\longrightarrow\mathcal{O}^n$ is of rank $\geq k+\operatorname{rank}\tilde{g}$ at each point. In particular, $\tilde{f}$ has rank m off L and rank $>k$ at generic points of L. If $k=0$, then at every point $x\in L$, the smallest p such that $\mathfrak{m}_x^p\subset I(\tilde{f})_x$ is clearly less than that for $I(f)_0$. In this case, the procedure can be repeated at each of the singular points of $\tilde{B}$ until eventually the rank of the derived map $\tilde{f}$ is positive at every point. Thus in either case, the rank of $\tilde{f}$ can be increased by blowing-up, and after finitely many such blow-ups a diagram of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^n & \xrightarrow{\pi^*f} & \mathcal{O}^n & \longrightarrow & \pi^*B & \longrightarrow & 0 \\
 & & \downarrow & & \wr\downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{A} & \longrightarrow & \mathcal{O}^n & \longrightarrow & \tilde{B} & \longrightarrow & 0
\end{array}$$

is arrived at, where the lower row is an exact sequence of bundles.

It follows from the above that if $0 \to A \to E \to B \to 0$ is an exact sequence of sheaves on a compact surface X with E locally free and B torsion-free, then there is a modification $\tilde{X} \xrightarrow{\pi}$ consisting of finitely many blow-ups and vector bundles $\tilde{A},\tilde{B}$ on $\tilde{X}$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*A & \longrightarrow & \pi^*E & \longrightarrow & \pi^*B & \longrightarrow & 0 \\
 & & \cap\downarrow & & \wr\downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{A} & \longrightarrow & \pi^*E & \longrightarrow & \tilde{B} & \longrightarrow & 0
\end{array} \qquad (3.1)$$

has exact rows, commutes, and has the lower row an exact sequence of bundles. Moreover, off the exceptional divisor, the vertical arrows are isomorphisms. This will referred to as a <u>desingularization</u> of B .

<u>Remarks</u>. (a) Since A is locally free, so too is π^*A , so $\pi^*A \longrightarrow \pi^*E$ is a monomorphism of sheaves even though π is not flat. Moreover, since $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ and $\pi^1_*\mathcal{O}_{\tilde{X}} = 0$ (Thm.I.9.1 [3]) it follows $\pi_*\pi^*A = A$ and $\pi^1_*\pi^*A = 0$. Applying π_* to the top row of (3.1) then gives $\pi_*\pi^*B = B$ and since $\ker(\pi_*\pi^*B \longrightarrow \pi_*\tilde{B})$ is a torsion sheaf and B is torsion-free

it follows $B \longrightarrow \pi_*\tilde{B}$ is injective; this implies $\pi_*\tilde{A} = A$.

(b) In general, if $0 \to A' \to \pi^*E \to B' \to 0$ is exact with B' torsion-free, then π_*B' is torsion-free so $K := \ker[\pi_*B' \longrightarrow \pi_*^1 A']$ is also; this implies π_*A' is locally-free. If L is any component of the exception divisor and $A'|_L = \Sigma\, \mathcal{O}(a_i)$, then necessarily $a_i \leq 0$ for all i because $A'|_L \longrightarrow \pi^*E|_L$ is injective off a finite set and $\pi^*E|_L$ is trivial. (If all a_i vanish it is easy to show $A' = \hat{\pi}^*\hat{\pi}_*A'$, where $\hat{\pi}$ is the blowing-down map for L .)

(c) If X is compact with positive $\bar{\partial}\partial$-closed (1,1)-form ω and $\tilde{X} \xrightarrow{\pi} X$ is the blow-up of X at $x \in X$, let $\tilde{\omega}_\varepsilon = \pi^*\omega - \varepsilon\sigma$ be one of the forms constructed in Section 2. If $\tilde{C}$ is a line bundle on $\tilde{X}$, then by Theorem I.9.1 [3], $\tilde{C} = \pi^*C \otimes \mathcal{O}(k)$ for some $C \in \mathrm{Pic}(X)$. Since $\pi_*\mathcal{O}(k) = \mathcal{O}_X$ if $k \leq 0$ and $\pi_*\mathcal{O}(k) = m_x^k$ for $k > 0$, $\pi_*\tilde{C} = C$ or $C \otimes m_x^k$. In either case, $\det(\pi_*\tilde{C}) = C$, so it follows that $\deg(\tilde{C},\tilde{\omega}_\varepsilon) = \deg(C,\omega) - \varepsilon\sigma \cdot c_1(\tilde{C}) = \deg(\pi_*\tilde{C},\omega) - \varepsilon\sigma \cdot c_1(\tilde{C})$. If now $\tilde{C}$ is an arbitrary torsion-free sheaf on $\tilde{X}$, then $\pi_*\tilde{C}$ is a torsion-free sheaf on X and the isomorphism $\det\pi_*\tilde{C} = \pi_*\det\tilde{C}$ off a finite subset extends to an isomorphism $\det\pi_*\tilde{C} = \det[\pi_*\det\tilde{C}]$ over X by Hartogs' theorem. Thus $\deg(\tilde{C},\tilde{\omega}_\varepsilon) = \deg(\det\tilde{C},\tilde{\omega}_\varepsilon) = \deg(\pi_*\det\tilde{C},\omega) - \varepsilon\sigma \cdot c_1(\det\tilde{C}) = \deg(\det\pi_*\tilde{C},\omega) - \varepsilon\sigma \cdot c_1(\det\tilde{C}) = \deg(\pi_*\tilde{C},\omega) - \varepsilon\sigma \cdot c_1(\tilde{C})$.

(d) With $X,\tilde{X}$ as in (c), suppose that $L = \pi^{-1}(x)$ is the exceptional line and $\tilde{C}$ on $\tilde{X}$ is locally free of rank n . Suppose moreover that $\tilde{C}|_L = \sum \mathcal{O}(-a_i)$ for some $a_i \geq 0$ and that $C := \pi_*\tilde{C}$ is locally free.

By the Riemann-Roch theorem, the holomorphic Euler characteristic for $\tilde{C}$ is given by $\chi(\tilde{C}) = \frac{1}{2}p_1(\tilde{C})+\frac{1}{2}c_1(\tilde{X})\cdot c_1(\tilde{C})+n\chi(\mathcal{O}_{\tilde{X}})$, where $p_1 = c_1^2-2c_2$ and $\chi(\mathcal{O}_{\tilde{X}})$ is the birational invariant $\frac{1}{12}(c_1(\tilde{X})^2+c_2(\tilde{X})) = \frac{1}{12}(c_1(X)^2+c_2(X))$. Moreover, $c_1(\tilde{X}) = c_1(X)+t$ where $t = c_1(\mathcal{O}(1))$. On the other hand, using the Leray spectral sequence, $\chi(\tilde{C}) = \chi(C)-\chi(\pi_*^1\tilde{C})$. Now, $\pi_*^1\tilde{C}$ is supported at x and thus is annihilated by m_x^{p+1} for sufficiently large p (by the Rückert Nullstellensatz [9]), and it follows that $\pi_*^1\tilde{C} = \pi_*^1\tilde{C}|_{\tilde{L}^{(p)}}$ where $L^{(p)}$ is the p-th formal neighbourhood of L in $\tilde{X}$. From the exact sequence $0 \to \mathcal{O}_L(q) \to \mathcal{O}_{L^{(q)}} \to \mathcal{O}_{L^{(q-1)}} \to 0$, it follows that if a_1, say, is the largest a_i, then $\tilde{C}(a_1)|_L$ ha[s] a non-vanishing section extending to all orders. By induction, $\tilde{C}|_{L^{(q)}}$ can be expressed in terms of extensions by line bundles $\mathcal{O}(-a_i)$, so for purposes of computing $\chi(\pi_*^1\tilde{C})$ it can be supposed that $\tilde{C} = \sum\mathcal{O}(-a_i)$. Since $\pi_*\mathcal{O}(-a_i) = \mathcal{O}_{\tilde{X}}$, the Riemann-Roch formu[la] gives $\chi(\pi_*^1\tilde{C}) = \chi(\pi_*^1\sum\mathcal{O}(-a_i)) = \chi(\sum\mathcal{O}_X)-\chi(\sum\mathcal{O}(-a_i)) = n\chi(\mathcal{O}_X) - \sum\chi(\mathcal{O}(-a_i)) = n\chi(\mathcal{O}_X)-[\sum\frac{1}{2}a_i(1-a_i)+\chi(\mathcal{O}_{\tilde{X}})] = \frac{1}{2}\sum a_i(a_i-1)$. Substitutin[g] this into $\chi(\tilde{C}) = \chi(C) - \chi(\pi_*^1\tilde{C})$ and using $c_1(\tilde{C}) = c_1(C) - at$ for $a = \sum a_i$ gives $p_1(C) = p_1(\tilde{C}) + \sum a_i^2$. In particular, $p_1(C) \geq p_1(\tilde{C})$.

(e) If E is a holomorphic bundle on the compact surface X then the Chern classes of holomorphic subbundles $E' \subset E$ must satisfy certain restrictions. To see this, <u>fix</u> an hermitian metri[c] on E, so E' and the quotient E'' have induced hermitian me[trics]. In a unitary frame, the induced connection A on E has the f[orm]

$$A = \begin{pmatrix} A' & \beta \\ -\beta^* & A'' \end{pmatrix}$$

where A', A'' are the induced connections on E', E'' and $\beta \in \Lambda^{0,1}(\mathrm{Hom}(E'', E'))$ is a $\bar\partial$-closed form representing the extension $0 \to E' \to E \to E'' \to 0$, (cf.e.g.[4]). (Conversely, A', A'', β gives E as smooth bundle a holomorphic structure, and any $\tilde\beta$ of the form $t\beta + \bar\partial\gamma$ for $t \in \mathbb{C}\setminus 0$ gives an isomorphic structure.) The curvature of this connection is

$$F = F(A) = \begin{pmatrix} F' - \beta\wedge\beta^* & \nabla\beta \\ -\nabla\beta^* & F'' - \beta^*\wedge\beta \end{pmatrix} . \qquad (3.2)$$

The characteristic class $p_1(E) = (c_1^2 - 2c_2)(E)$ is given by $p_1(E) = \frac{1}{4\pi^2}\int_X \mathrm{tr} F\wedge F$, so if ω is a positive (1,1)-form on X,

$$p_1(E) = \frac{1}{4\pi^2}(\|F_+\|^2 - \|F_-\|^2) = \frac{1}{4\pi^2}(\tfrac{1}{2}\|\hat F\|^2 - \|F_-\|^2) . \qquad (3.3)$$

The first and second Chern forms are $c_1 = \frac{i}{2\pi}\,\mathrm{tr}\, F$ and $c_2 = \frac{1}{8\pi^2}[\mathrm{tr}\, F^2 - (\mathrm{tr}\, F)^2]$, (where $F^2 := F\wedge F$). With $G := F' - \beta\wedge\beta^*$ and $B := \beta\wedge\beta^*$, one calculates $(c_2 - c_1^2)(F') = \frac{1}{8\pi^2}[\mathrm{tr}\, G^2 + (\mathrm{tr}\, G)^2 + 2\mathrm{tr}(G\wedge B) + 2(\mathrm{tr}\, G\wedge \mathrm{tr}\, B)] - (2\pi)^{-2}\,\mathrm{tr}\,\gamma\wedge\gamma^*$, where γ is the component of $\beta\otimes\beta$ in $\Lambda^{0,2}\otimes S^2E'\otimes\Lambda^2E''^*$; (cf.[10] pp.416-418 for similar calculations). It follows that there are constants $C_1, C_2 > 0$ depending only on the sup norm of $F(A)$, and thus only on E and ω, such that $*(c_2 - c_1^2)(F') \le C_1 + C_2|\beta|^2$. Furthermore, since β is a (0,1)-form, $|\beta|^2 = -i\,\mathrm{tr}\Lambda\beta\wedge\beta^* = i\,\mathrm{tr}\hat G - i\,\mathrm{tr}\hat F'$, so if ω is $\bar\partial\partial$-closed, it follows that $\int|\beta|^2 dV \le -2\pi\deg(E', \omega) + \text{const}$. Thus there are constants $C_4, C_5 > 0$ depending only on E and ω such that $(c_2 - c_1^2)(E') \le C_4 - C_5\deg(E', \omega)$.

Now suppose that $A \subset E$ only has torsion-free quotient. Let $\tilde{X} \xrightarrow{\pi} X$ be a desingularizating space for E/A and $\tilde{A}$ be the "desingularization" of A. For the metrics $\tilde{\omega}_\varepsilon$ on $\tilde{X}$ constructed as in Section 2, $|\pi^*f|$ compares uniformly with $|f|$ for a two-form f on X by choosing the scaling factors ε appropriately. By remarks (b),(c) above, $(c_2-\frac{1}{2}c_1^2)(A) \leq (c_2-\frac{1}{2}c_1^2)(\tilde{A}) \leq C_4-C_5 \deg(\tilde{A},\tilde{\omega}) + \frac{1}{2}c_1(\tilde{A})^2 \leq C_4-C_5 \deg(A,\omega) + \frac{1}{2}c_1(A)^2$, so the inequality

$$(c_2-c_1^2)(A) \leq C_4-C_5 \deg(A,\omega) \tag{3.4}$$

is valid for any $A \subset E$ with torsion-free quotient, with $C_4,C_5>0$ constants depending only on E,ω.

(f) The last observation is the following: by definition, $\deg(-,\omega)$ ignores the singularities of torsion-free sheaves. However this is also true on the level of forms in the following sense: if Q is a torsion-free quotient of a bundle E and the latter is given an hermitian connection as above, then off $S(Q)$ the bundle Q inherits an hermitian connection and thus gives a curvature form F_Q on $X\setminus S(Q)$. The claim is that $\mathrm{tr}\,\hat{F}_Q$ is integrable and indeed $\frac{i}{2\pi}\int_X \mathrm{tr}\,\hat{F}_Q\, dV = \deg(Q,\omega)$, where the right hand side is defined in the usual way. To see this, it suffices to assume that $\mathrm{rank}Q = 1$ (otherwise replace E,Q by $\Lambda^q E,\Lambda^q Q$), and then Q is the image in $\det Q$ of a holomorphic map $E \to \det Q$ which is surjective outside $S(Q)$. Locally, the singular part of F_Q is then $\bar{\partial}\partial\log|f|^2$, where f is a rankE-tuple of holomorphic functions whose only common zero is the singular point. Pulling back to the desingularization space $\tilde{X} \xrightarrow{\pi} X$,

$\pi^*\log|f|^2 = \log|\tilde{f}|^2 + \sum a_j \log|s_j|^2$ where $\tilde{f}$ is non-vanishing, s_j is the holomorphic function defining the exceptional line L_j, and $a_j \in \mathbb{Z}$. By the Poincaré-Lelong lemma ([10] p.388), $\log|s_j|^2$ is integrable and $\pi^* F_Q = F_{\tilde{Q}} + 2\pi i \sum a_j T_{L_j}$ in the sense of currents. Since $\int_{L_j} \pi^*\omega = 0$, this gives $\int_X F_Q \wedge \omega = \int_{\tilde{X}} \pi^*(F_Q \wedge \omega) = \int_{\tilde{X}} F_{\tilde{Q}} \wedge \pi^*\omega$, and since $\tilde{Q} = (\pi^*\det Q) \otimes K$ for some line bundle K with curvature $\sum b_j \sigma_j$, it follows that $\frac{i}{2\pi} \int_X F_Q \wedge \omega = \deg(Q,\omega)$, as claimed.

4. Construction of subsheaves

Let X be a compact surface and ω be a fixed positive $\bar{\partial}\partial$-closed (1,1)-form on X. If B is a torsion-free sheaf on X, a subsheaf $A \subset B$ will be called admissable if A is coherent and $0 < \text{rank } A < \text{rank } B$. Then B can be one of two types; namely, B has an admissable subsheaf (type I) or, B has no admissable subsheaves (type II). All of the analysis in this section will deal excusively with a bundle E of type I.

The following fact will be used frequently (cf.[5] p.3): if E is a bundle which is not stable, then there exists a stable admissable $A \subset E$ with E/A torsion-free and $\mu(A) \geq \mu(E)$.

Lemma 1. If E is a bundle on X, then $\{\deg(A) : A \subset E$ is admissable$\}$ is bounded above.

Proof. If not, there exists a sequence $A_i \subset E$ with $\mu(A_i) \uparrow \infty$. Without loss of generality, E/A_i is torsion-free, and passing to a subsequence, rank $A_i = a$ is constant. Then $\det A_i \to \Lambda^a E$ is injective, and $\deg(\det A_i) \uparrow \infty$. Fix a connection on $\Lambda^a E$, and on $(\det A_i)^*$ put the H-E connection. Then (2.5) applied to the non-zero section of $(\det A_i)^* \otimes \Lambda^a E_i$ yields a contradiction for i large enough.

□

If $A \subset E$ is admissable of rank a, let $\nu_E(A) := a(\mu(E) - \mu(A))$. By Lemma 1, the possible values of ν_E

are bounded below, and indeed, if E is stable, then $\nu_E(A)>0$ for all admissable A.

<u>Lemma 2</u>. If E is a stable bundle on X and if there exists an admissable $A \subset E$ of rank a such that $\nu_E(A) = \inf\{\nu_E(A'):A'\subset E$ is admissable$\}$, then

(a) A is stable; and

(b) $B := E/A$ is torsion-free and stable.

<u>Proof</u>. (a) If $C \subset A$ is admissable of rank c, $a(\mu(E)-\mu(A)) \leq c(\mu(E)-\mu(C)) < a(\mu(E)-\mu(C))$ since $c < a$ and $\mu(E) > \mu(C)$.

(b) If $\hat{A}$ is the maximal normal extension of A in E, then $a(\mu(E)-\mu(A)) \leq a(\mu(E)-\mu(\hat{A}))$, so $\mu(\hat{A}) \leq \mu(A)$. On the other hand, $A \to \hat{A}$ is a monomorphism so $\mu(A) \leq \mu(\hat{A})$. Thus $\mu(A) = \mu(\hat{A})$, giving $\nu_E(\hat{A}) = \nu_E(A)$. By (a), $\hat{A}$ is stable, so $A \to \hat{A}$ must be an isomorphism. Thus $B = E/A$ is torsion-free.

If $C \subset B$ is admissable with torsion-free quotient, let $K := \ker(E \to B/C)$. A quick calculation gives $\mu(C) = \mu(E) - \frac{1}{c}(\nu_E(K)-\nu_E(A)) \leq \mu(E) < \mu(B)$, $c = \text{rank } C$.

□

The strategy of this section is to produce subsheaves $A \subset E$ with this infimum property, to desingularize these, and show that (eventually) such A can be assumed to be subbundles; this process commences with the next lemma.

Lemma 3. Let S be a torsion-free sheaf on X and let $\{L_i\}_{i=1}^{\infty}$ be a sequence of line bundles such that $|\mu(L_i)| \leq$ Const. and $\Gamma(X, L_i^* \otimes S) \neq 0$. Then there is a subsequence with $c_1(L_i)$ constant.

Proof. By replacing S with S^{**} if necessary, it can be assumed that S is locally free. If $\operatorname{rank} S = 1$, the result follows from Corollary 2. If $\operatorname{rank} S > 1$, pick a non-zero homomorphism $L_1 \to S$ and let $S_1 := S/L_1$, $S_1' := S_1/\tau(S_1)$, $\hat{L}_1 := \ker S \to S_1'$. From the exact sequence $0 \to L_i^* \otimes \hat{L}_1 \to L_i^* \otimes S \to L_i^* \otimes S_1' \to 0$ it follows that the sequences $\Gamma(X, L_i^* \otimes \hat{L}_1)$ and $\Gamma(X, L_i^* \otimes S_1')$ cannot both be almost always zero, so the result follows by induction on $\operatorname{rank} S$.

□

The next lemma is the key lemma of this section even though its proof is trivial when (X,ω) is algebraic and straightforward when X is Kähler.

Lemma 4. Let E be a bundle of rank r on X and suppose that the main theorem has been proved for bundles of rank less than r . Then

(a) If E is of type I, then there exists a stable admissable $A \subset E$ with torsion-free quotient such that $\mu(A) = \sup\{\mu(A') : A' \subset E \text{ is admissable}\}$.

(b) If, moreover, E is semi-stable, then there exists an admissable $B \subset E$ such that $\nu_E(B) = \inf\{\nu_E(B') : B' \subset E$ is admissable} .

<u>Proof</u>. (a) Choose a sequence of admissable $A_i \subset E$ with $\mu(A_i) \uparrow M := \sup\{\mu(A') : A' \subset E\}$, and without loss of generality, each A_i is stable and has torsion-free quotient. If $\mu(A_i)$ is eventually constant, then A_i satisfies the requirements of the lemma for large enough i, so suppose that this is not the case. By passing to a subsequence it can be supposed that rank $A_i = a$ is constant and $\mu(A_i)$ is strictly increasing.

Since $\mu(\det A_i) = a\mu(A_i)$ and $\det A_i \to \Lambda^a E$ is non-zero, Lemma 3 implies that there is a subsequence with $c_1(A_i)$ constant. By Proposition 2 therefore, it must be the case that $b_1(X)$ is odd. Since each A_i is stable, it admits an H-E connection by the inductive hypothesis, so by (3.3), $\{(c_1^2-2c_2)(A_i)\}$ is bounded above. On the other hand, by (3.4), $\{(c_1^2-c_2)(A_i)\}$ is bounded below, so it follows that a subsequence has $c_2(A_i)$ constant. By passing to yet another subsequence, it can be assumed that $\{A_i\}$ is topologically constant.

Now recall that $\deg : \mathrm{Pic}(X) \to \mathbb{R}$ induces $\deg : H^0(X,\mathcal{O}) \to \mathbb{R}$ and this annihilates the rank $b_1(X)$ lattice $H^1(X,\mathbb{Z}) \hookrightarrow H^0(X,\mathcal{O})$. Since $b_1(X)$ is odd by assumption, Proposition 2 implies that $\deg : H^0(X,\mathcal{O}) \to \mathbb{R}$ is not identically zero, so $\ker(\deg)/H^1(X,\mathbb{Z}) = T$, a torus, and $\mathrm{Pic}_0(X) = H^1(X,\mathcal{O})/H^1(X,\mathbb{Z}) = T \times \mathbb{R}$. After picking a basis for $H^1(X,\mathcal{O})$ as $\mathbb{R}$-vector space*, the component of $L_1^* \otimes L_i$ in T can be assumed to converge to some element of T, and on the other hand, the component in $\mathbb{R}$ also converges since it is measured by deg and $\deg(L_i) \uparrow aM$. Thus (a subsequence of the) L_i converges to some $L_\infty \in \mathrm{Pic}(X)$ with $\mu(L_\infty) = aM$.

Now let $L \in \mathrm{Pic}_0(X)$ be a line bundle with $\mu(L) = 1$, and

* and setting $L_i := \det A_i$

set $\tilde{A}_i := A_i \otimes L^{-\mu(A_i)}$, so $\mu(\tilde{A}_i) = 0$, $\{\tilde{A}_i\}$ is topologically constant, and of course, $\tilde{A}_i$ is stable. By the inductive hypothesi $\tilde{A}_i$ admits a (unique) H-E connection, and this is moreover an anti-self-dual Yang-Mills connection. The curvatures F_i of these connections satisfy $\| F_i \|^2_{L^2} = 4\pi^2 p_1(\tilde{A}_i)$ = constant, so by Uhlenbeck's weak compactness theorem [21], ([18, 5]), there is a finite set $S = \{x_1, \ldots, x_N\} \subset X$ such that a subsequence of these connections (on the same underlying smooth bundle) converges weakly in $L^p_{1,loc}(X \setminus S)$ for any p to an anti-self-dual connection over $X \setminus S$. By the removable singularities theorem [20], this connectio extends across S to a smooth ASD connection on a (possibly topologically different) bundle $\tilde{A}_\infty$. This ASD connection gives $\tilde{A}_\infty$ a unique holomorphic structure.

Since $\det \tilde{A}_i = L_i \otimes L^{-a\mu(A_i)}$ and this converges to $L_\infty \otimes L^{-aM}$ it follows that $\det \tilde{A}_\infty = L_\infty \otimes L^{-aM}$ and $\mu(\tilde{A}_\infty) = 0$. Setting $A_\infty := \tilde{A}_\infty \otimes L^M$, it follows that $\mu(A_\infty) = M$ and $A_i \to A_\infty$ weakly in $L^p_{1,loc}(X \setminus S)$ for any p (in the sense of connections).

It suffices now to produce a non-zero holomorphic map $A_\infty \to E$, for if A'_∞ is one of the stable components of A_∞ whose existence is asserted by Corollary 4, and if $A'_\infty \to E$ is non-zero, then $A'_\infty \to E$ must be a sheaf inclusion else the image I satisfies $M = \mu(A'_\infty) < \mu(I)$. Moreover, A'_∞ must be equal to its maximal normal extension $\hat{A}'_\infty$ in E (since the latter must have $\mu = M$ and is therefore semi-stable), so A'_∞ has torsion-free quotient.

The existence of a non-zero holomorphic map $A_\infty \to E$ is proved by repitition of Donaldson's argument [5] pp.22-23, and will be an argument appearing here subsequently also.

For each j, there is a non-zero holomorphic map $s_j : A_j \to E$. Fix an hermitian connection on E compatible with $\bar{\partial}_E$ and, as before, A_j is equipped with its H-E connection. From (2.5),
$\Delta|s_j|^2 + i{*}\partial(|s_j|^2\bar{\partial}\omega) - i{*}\bar{\partial}(|s_j|^2\partial\omega) \le (|\hat{F}_j| + |\hat{F}_E|)|s_j|^2 \le \text{Const.}|s_j|^2$,
so by Theorem 9.20 [8] it follows that $\sup_X |s_j|^2 \le C\| s_j\|^2_{L^8(X)}$.
Choose balls B_α about the points $x_\alpha \in S$ such that A_∞, E are holomorphically trivial on them and such that $C^4\sum \text{Vol}(B_\alpha) = \frac{1}{2}$, and normalize s_j so that $\| s_j\|_{L^8(X)} = 1$. Since the connections converge weakly in $L^p_{1,loc}(X \setminus S)$ for any p and $\bar{\partial}_j s_j = 0$, it follows that $\| s_j\|_{L^8_2(K)} \le \text{Const.}(\| s_j\|_{L^8(K)} + 1) \le \text{Const.}$ for $K := X \setminus \cup B_\alpha$, (using also the C^0 bound on s_j). Thus $\{s_j\}$ has a subsequence converging weakly in $L^8_2(K)$ and strongly in $C^0(K)$ to a limit s_∞ which satisfies $\bar{\partial}_\infty s_\infty = 0$. Since $\| s_j\|^8_{L^8(K)} \ge \frac{1}{2}$ for all j, the limit is non-zero, and by Hartog's theorem, it extends to X to give a non-zero holomorphic map $A_\infty \to E$. This completes the proof of (a).

The proof of (b) is essentially identical. If $B \subset E$ is not stable, then there exists stable $B' \subset B$ which has E/B' torsion-free and $\nu_E(B') \le \nu_E(B)$. The proof of (a) can then be repeated by choosing a minimizing sequence for ν_E and passing to a subsequence of constant rank.

□

Let $\tilde{X} \xrightarrow{\pi} X$ be a modification of X consisting of N blow-ups, and let ω be a positive $\bar{\partial}\partial$-closed (1,1)-form on X. Let $\sigma_1, \ldots, \sigma_N$ be forms constructed as in Section 2, one for each component of the exceptional divisor and all pulled-back to $\tilde{X}$.

Suppose that $\alpha_1,\ldots,\alpha_N > 0$ are such that, if $\rho := \sum\alpha_i\sigma_i$, then $\pi^*\omega - \rho$ is positive. Then $\tilde{\omega}_\varepsilon := \pi^*\omega - \varepsilon\rho$ is positive for any $\varepsilon \in (0,1]$ since $\pi^*\omega$ is positive semi-definite. If E is an r-bundle on X, then by Lemma 4 (a), there is for each ε subsheaf $A(\varepsilon) \subset \pi^*E$ maximizing $\mu(A,\tilde{\omega}_\varepsilon)$ over all admissable $A \subset \pi^*E$. This can be strengthened as follows:

<u>Lemma 5</u>. There exists $\varepsilon_0 > 0$ and a stable admissable $A_0 \subset \pi^*E$ such that $\mu(A_0,\tilde{\omega}_\varepsilon) = \sup\{\mu(A,\tilde{\omega}_\varepsilon) : A \subset \pi^*E$ is admissable$\}$ for all $\varepsilon \in (0,\varepsilon_0]$.

<u>Proof</u>. Take $\varepsilon_1 = 1$ and choose $A_1 \subset \pi^*E$ according to Lemma 4(
Suppose that there exists $\varepsilon_2 < \varepsilon_1$ and $A_2 \subset \pi^*E$ with $\mu(A_2,\tilde{\omega}_{\varepsilon_2}) > \mu(A_1,\tilde{\omega}_{\varepsilon_2})$. Without loss of generality, A_2 has torsion-free quotient so by remark (b) of Section 3, $\rho\cdot c_1(A_2) \leq 0$ Moreover, using remark (c); $\mu(A_1,\tilde{\omega}_{\varepsilon_1}) = \mu(\pi_*A_1)-\varepsilon_1\rho\cdot c_1(A_1)/a_1 \geq \mu(\pi_*A_2)-\varepsilon_1\rho\cdot c_1(A_2)/a_2 = \mu(A_2,\tilde{\omega}_{\varepsilon_1})$ and $\mu(A_1,\tilde{\omega}_{\varepsilon_2}) = \mu(\pi_*A)-\varepsilon_2\rho\cdot c_1(A_1)/a_1 < \mu(\pi_*A_2)-\varepsilon_2\rho\cdot c_1(A_2)/a_2 = \mu(A_2,\tilde{\omega}_{\varepsilon_2})$. These imply $(\varepsilon_1-\varepsilon_2)[\rho\cdot c_1(A_1)/a_1-\rho\cdot c_1(A_2)/a_2] < 0$, so $\rho\cdot c_1(A_1)/a_1 < \rho\cdot c_1(A_2)/a_2$. Here $a_i = \text{rank } A_i$.

Now replace (ε_1,A_1) by (ε_2,A_2). This process must terminate after finitely many steps because $\rho\cdot c_1(A_j)$ is bounded above by zero, all the α_i's are positive, and the coefficients of the σ_i's in $c_1(A_j)$ are all non-negative integers.

□

<u>Corollary 5</u>. If E is ω-stable, then

(a) π^*E is $\tilde{\omega}_\varepsilon$-stable for all ε sufficiently small, and

(b) there exists $\varepsilon_0 > 0$ and admissable $B_0 \subset \pi^*E$ such that $\nu_{\pi^*E}(B_0, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^*E}(B, \tilde{\omega}_\varepsilon) : B \subset \pi^*E$ is admissable$\}$ for all $\varepsilon \in (0, \varepsilon_0]$.

Proof. (a) Let $M := \sup\{\mu(A,\omega) : A \subset E$ is admissable$\}$. Since M is realized by some $A \subset E$ by lemma 4(a) and E is stable, it follows $M < \mu(E)$. Let $A_0 \subset \pi^*E$ be the A_0 given by Lemma 5. Then $\mu(A_0, \tilde{\omega}_\varepsilon) = \mu(\pi_* A_0, \omega) - \varepsilon\rho \cdot c_1(A_0)/a_0 \leq M - \varepsilon\rho \cdot c_1(A_0)/a_0 < \mu(E,\omega) = \mu(\pi^*E, \tilde{\omega}_\varepsilon)$ if ε is small enough.

(b) Take ε_1 small enough so that π^*E is $\tilde{\omega}_{\varepsilon_1}$-stable for $\varepsilon \leq \varepsilon_1$. Choose $B_1 \subset \pi^*E$ according to Lemma 4(b) and repeat the argument of Lemma 5.

□

Thus stability is preserved under pull-backs to blow-ups (in the above sense). [Semi-stability is not preserved!]. The following lemma shows that this is also true of the desingularization process:

Lemma 6. With $X, \tilde{X}, \omega, \rho$ as in Lemma 5, let B be a torsion-free sheaf of rank $\leq r$ on X and suppose that $\tilde{B}$ on $\tilde{X}$ is a desingularization of B according to Section 3. Then

(a) If B is ω-stable, it follows that $\tilde{B}$ is $\tilde{\omega}_\varepsilon$-stable for $\varepsilon > 0$ sufficiently small;

(b) If B is given by an exact sequence $0 \to A \to E \to B \to 0$

with rank $A \leq r$ and $0 \to \tilde{A} \to \pi^*E \to \tilde{B} \to 0$ is the desingularized sequence, it follows that $\tilde{A}$ is $\tilde{\omega}_\varepsilon$-stable for sufficiently small $\varepsilon > 0$ if A is ω-stable.

Proof. (a) There is nothing to prove if $\tilde{B}$ has no admissable subsheaves, so suppose that it has such subsheaves. By the remark (a) of Section 3, there is an exact sequence $0 \to B \to \pi_*\tilde{B} \to Q \to 0$ where Q := quotient is supported on $S(B)$. It follows that $\det B = \det(\pi_*\tilde{B})$, so $\mu(B) = \mu(\pi_*\tilde{B})$. Now, $\pi_*\tilde{B}$ is also stable: if $A \subset \pi_*\tilde{B}$ is admissable, let I be the image of A in Q under the composition $A \hookrightarrow \pi_*\tilde{B} \twoheadrightarrow Q$. Then $A' := \ker(A \to I$ is an admissable subsheaf of B , and since B is stable it follows $\mu(A') < \mu(B)$. But as above, $A' = A$ off a finite subset so $\mu(A) = \mu(A') < \mu(B) = \mu(\pi_*\tilde{B})$.

By Lemma 5, there exists $A_0 \subset \tilde{B}$ such that $\mu(A_0,\tilde{\omega}_\varepsilon) = \sup\{\mu(A,\tilde{\omega}_\varepsilon) : A \subset \tilde{B}\}$ for all ε small enough. So if $a = \operatorname{rank} A_0$, $b = \operatorname{rank} B$ and $\delta := \mu(\pi_*\tilde{B}) - \mu(\pi_*A_0)$, then $\delta > 0$ and $\mu(A_0,\tilde{\omega}_\varepsilon) = \mu(\pi_*A_0,\omega) - \varepsilon\rho\cdot c_1(A_0)/a = \mu(\pi_*\tilde{B},\omega) - \varepsilon\rho\cdot c_1(A_0)/a = \mu(\tilde{B},\tilde{\omega}_\varepsilon) - \delta + \varepsilon(\rho\cdot c_1(\tilde{B})/b - \rho\cdot c_1(A_0)/a) < \mu(\tilde{B},\tilde{\omega}_\varepsilon)$ if ε is small enough.

(b) The same proof as (a) works (and is simpler since $\pi_*\tilde{A} = A$ is stable by hypothesis).

□

The next lemma is somewhat technical and is required for the proof of the main result of this section which follows it.

Lemma 6. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be an element of ℓ_2 all of whose entries α_i are positive, and let $\{a^j\}_{j=1}^{\infty}$ be a sequence in ℓ_2 such that all entries a_i^j in $a^j = (a_1^j, a_2^j, \ldots)$ are non-negative integers (so almost all a_i^j are zero for fixed j.) Suppose that $A_j := \langle \alpha, a^j \rangle = \sum_{i=1}^{\infty} \alpha_i a_i^j$ is strictly increasing. Then $\{\| a^j \|_{\ell_2}\}$ is unbounded.

Proof. Suppose on the contrary that $\| a^j \| \leq B$ for all j. If, for each i, $\{a_i^j\}_{j=1}^{\infty}$ is almost always zero, choose k_0 such that $\sum_{i \geq k_0} \alpha_i^2 < (A_2/B)^2$, and choose N so large that $a_i^j = 0$ for all $i \leq k_0$ if $j \geq N$. Then for $j \geq N$, $A_2 < A_j = \sum_{i \geq k_0} \alpha_i a_i^j \leq (\sum_{i \geq k_0} \alpha_i^2)^{1/2} (\sum (a_i^j)^2)^{1/2} < (A_2/B) \cdot B = A_2$, a contradiction.

So there exists k such that $\{a_k^j\}_{j=1}^{\infty}$ is not almost zero, and let k_0 be the first such k. Since $\| a^j \| \leq B$, $\{a_{k_0}^j\}$ is bounded, so there is a subsequence which has $a_{k_0}^j = a_{k_0} \neq 0$ constant, with $a_1^j, \ldots, a_{k_0-1}^j = 0$ for all j.

Since $\{A_j\}$ is strictly increasing, there exists M such that $A_M > \alpha_{k_0} a_{k_0}$. If every entry after the k_0-th in the subsequence is almost always zero, choose k_1 so that $\sum_{i \geq k_1} \alpha_i^2 < (A_M - \alpha_{k_0} a_{k_0})^2 B^{-2}$ and $N > M$ so large that $a_i^j = 0$ for all i with $k_0 < i \leq k_1$ if $j \geq N$. Then for $j \geq N$, the same contradiction as above ensues, giving another entry which is not almost always zero. Repeating this argument B^2+1 times gives the desired conclusion.

□

Proposition 3. Let X be a compact surface with positive $\bar{\partial}\partial$-closed (1,1)-form ω, and suppose that the main theorem has been proved for bundles of rank less that r. If E is an ω-stable r-bundle on X which has an admissable subsheaf, then there exist

(i) a modification $\tilde{X} \xrightarrow{\pi} X$ consisting of N blow-ups;

(ii) $\alpha_1, \ldots, \alpha_N > 0$ such that, if $\sigma_1, \ldots, \sigma_N$ are forms constructed as in Section 2 and $\rho := \sum \alpha_i \sigma_i$, the form $\pi^*\omega - \rho$ is positive;

(iii) $\varepsilon_0 > 0$ and a <u>subbundle</u> $A \subset \pi^*E$ such that $\nu_{\pi^*E}(A, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^*E}(A', \tilde{\omega}_\varepsilon) : A' \subset \pi^*E$ is admissable$\}$ for all $\varepsilon \in (0, \varepsilon_0]$, where $\tilde{\omega}_\varepsilon := \pi^*\omega - \varepsilon\rho$.

Proof. By Lemma 4(b) there exists $A_0 \subset E$ satisfying $\nu_E(A_0) = \inf\{\nu_E(A') : A' \subset E$ is admissable$\}$, and the quotient $B_0 := E/A_0$ is automatically torsion-free and stable by Lemma 2.

If B_0 is locally free, then there is nothing more to do, so suppose this is not the case. Desingularize B_0 to get $\tilde{X}_0 \xrightarrow{\pi} X$ together with $\pi^*B_0 \twoheadrightarrow \tilde{B}_0$, $\pi^*A_0 \hookrightarrow \tilde{A}_0$. Let $\{\sigma_i\}$ by any of the forms of Section 2 (one for each exceptional line), and choose $\alpha_i > 0$ so that $\rho_0 := \sum \alpha_i \sigma_i$ has $\pi^*\omega - \rho_0$ positive.

By Corollary 5(b), there exists $A_1 \subset \pi^*E$ satisfying (iii), except that it may not be a sub-bundle. If not, for any positive ε sufficiently small one has

$\nu_E(\pi_* A_1) + \varepsilon\rho_0 \cdot c_1(A_1) = \nu_{\pi^*E}(A_1, \tilde{\omega}) \leq \nu_{\pi^*E}(\tilde{A}_0, \tilde{\omega}_\varepsilon) = \nu_E(\pi_*\tilde{A}_0) + \varepsilon\rho_0 \cdot c_1(\tilde{A}$

Since $\pi_*\tilde{A}_0 = A_0$ letting $\varepsilon \to 0$ gives $\nu_E(\pi_*A_1) \leq \nu_E(A_0)$, and by definition of A_0 , the reverse inequality holds also. So $\nu_E(\pi_* A_1) = \nu_E(A_0)$, giving $\rho_0 \cdot c_1(A_1) \leq \rho_0 \cdot c_1(\tilde{A}_0)$. If equality holds here, then $\tilde{A}_0$ satisfies the requirements of the proposition.

Suppose then that $\rho_0 \cdot c_1(A_1) < \rho_0 \cdot c_1(\tilde{A}_0)$. Desingularize the torsion-free sheaf $B_1 := \pi^* E/A_1$ to get $\tilde{X}_1 \xrightarrow{\pi_1} \tilde{X}_0$, $\pi_1^* B_1 \longrightarrow\!\!\!\!\to \tilde{B}_1$, $\pi_1^* A_1 \hookrightarrow \tilde{A}_1$. Choose more σ's and α's so that $\rho_1 := \pi_1^* \rho_0 + \sum \alpha_i \sigma_i$ has $\pi^* \omega - \rho_1$ positive, where π denotes $\tilde{X}_1 \longrightarrow X$. Now choose A_2 according to Corollary 5(b) so that $\nu_{\pi^* E}(A_2, \tilde{\omega}_\varepsilon) = \inf\{\nu_{\pi^* E}(A', \tilde{\omega}_\varepsilon) : A' \subset \pi^* E\}$, where $\tilde{\omega}_\varepsilon = \pi^* \omega - \varepsilon \rho_1$. [It is important to use $\pi : \tilde{X}_1 \to X$ rather than $\pi_1 : \tilde{X}_1 \to \tilde{X}_0$ at this point.] Again one obtains $\nu_E(\pi_* A_2) \leq \nu_E(\pi_* \tilde{A}_1)$, and since $\pi_* \tilde{A}_1 = \pi_{0*} \pi_{1*} \tilde{A}_1 = \pi_{0*} A_1$, it follows as before that $\nu_E(\pi_* A_2) = \nu_E(A_0)$ and $\rho_1 \cdot c_1(A_2) \leq \rho_1 \cdot c_1(\tilde{A}_1)$. If equality holds here then $\tilde{A}_1$ satisfies the requirements of the proposition; otherwise, repeat the process again.

If this procedure fails to terminate, then there is an infinite sequence of modifications $\ldots \to \tilde{X}_{j+1} \to \tilde{X}_j \to \ldots \to X$ with $A_{j+1}, \tilde{A}_j \subset \pi^* E$ on $\tilde{X}_{j+1}$ satisfying $\nu_E(\pi_* A_{j+1}) = \nu_E(\pi_* \tilde{A}_j) = \nu_E(A_0)$ and $\rho_{j+1} \cdot c_1(A_{j+1}) < \rho_{j+1} \cdot c_1(\tilde{A}_j)$, where π denotes $\tilde{X}_{j+1} \to X$. Here $\rho_{j+1} = \pi_{j+1}^* \rho_j + \sum \alpha_i \sigma_i$ for some $\alpha_i > 0$ and σ_i belonging to the modification $\tilde{X}_{j+1} \to \tilde{X}_j$.

Since $\tilde{A}_j$ results from the desingularization of the torsion-free sheaf $B_j = \pi^* E/A_j$ on $\tilde{X}_j$, $\rho_{j+1} \cdot c_1(\tilde{A}_j) \leq \rho_j \cdot c_1(A_j)$; (indeed, this is strict). Thus $\{\rho_{j+1} \cdot c_1(\tilde{A}_j)\}$ is a strictly decreasing sequence. By passing to a subsequence, it can be assumed that $\operatorname{rank}(\tilde{A}_j) = a$ is constant, and then the equation $\nu_E(\pi_* \tilde{A}_j) = \nu_E(A_0)$ implies $\mu(\pi_* \tilde{A}_j)$ is constant. Since $\pi_* \tilde{A}_j$

is contained in E and has torsion-free quotient, it follows from Lemma 3 that there is a subsequence with $c_1(\pi_*\tilde{A}_j)$ constant. Since $0 \to \pi_*\tilde{A}_j \to E \to \pi_*\tilde{B}_j \to 0$ is exact off a finite subset, $c_1(\pi_*\tilde{B}_j)$ is also constant. Thus if $c_1(\pi_*\tilde{A}_j) = \beta \in H^2(X,\mathbb{Z})$ and $c_1(\pi_*\tilde{B}_j) = \gamma \in H^2(X,\mathbb{Z})$, then it follows that $c_1(\tilde{A}_j) = \beta + \sum a_i^j\sigma_i$ and $c_1(\tilde{B}_j) = \gamma - \sum a_i^j\sigma_i$ for some non-negative integers a_i^j. If $\rho_{j+1} = \sum \alpha_i\sigma_i$, then $\rho_{j+1}\cdot c_1(\tilde{A}_j) = -\sum a_i^j\alpha_i$ is strictly decreasing with j, and since $\mathrm{Vol}(\tilde{X}_{j+1}, \pi^*\omega - \rho_{j+1}) = \mathrm{Vol}(X) - \frac{1}{2}\sum \alpha_i^2$, the infinite sequence of α's is in ℓ_2. By Lemma 7, $\| a^j \|^2 := \sum_i (a_i^j)^2$ is an unbounded sequence.

Now, by Lemma 2, A_j and B_j on $\tilde{X}_j$ are stable with respect to $\pi^*\omega - \varepsilon\rho_j$ for ε sufficiently small. So by Lemma 6, $\tilde{A}_j$ and $\tilde{B}_j$ on $\tilde{X}_{j+1}$ are stable with respect to some positive $\bar{\partial}\partial$-closed $(1,1)$-form on $\tilde{X}_{j+1}$ (not necessarily $\pi^*\omega - \varepsilon\rho_{j+1}$). By the inductive hypothesis, they admit H-E connections and therefore satisfy Lübke's inequality ([14]): with $A = \tilde{A}_j$, $B = \tilde{B}_j$, rank $A = a$, rank $B = b$, this states $(\frac{a-1}{2a} c_1^2 - c_2)(A) \le 0$ and $(\frac{b-1}{2b} c_1^2 - c_2)(B) \le 0$. Adding these together and substituting $c_1(A) = \beta + \sum a_i^j\sigma_i$, $c_1(B) = \gamma - \sum a_i^j\sigma_i$, $c_2(E) = c_2(A) + c_2(B) + c_1(A)\cdot c$ gives $0 \ge \frac{a-1}{2a}\beta\cdot\beta + \frac{b-1}{2b}\gamma\cdot\gamma + \beta\cdot\gamma - c_2(E) + \frac{r}{2ab}\| a^j \|^2$ after a short calculation with some fortuitous cancellations; ($r = a+b$ of course). Since all terms except the last on the right are independent of j in this inequality, the desired contradiction has been achieved because $\| a^j \|$ is unbounded.

□

5. Proof of Theorem 1

In order to prove the main theorem, a certain functional, to be given shortly, must be minimized. The set over which this minimization is performed is the set of all integrable L_1^p connections on a fixed U(r)-bundle, each connection inducing the same holomorphic structure. By the Newlander-Nirenberg theorem, a smooth integrable connection induces a holomorphic structure, but it is not immediately clear that the same is true of general L_1^p connections. However, the following result shows that if p is large enough, this is indeed the case. The proof was suggested by the proof for the case $n = 1$ in [1].

Lemma 8. Let B_1 denote the open unit polydisc in $\mathbb{C}^n$ centred at the origin. Let A be an $r \times r$ matrix of (0,1)-forms with coefficients in $L^p_{1,loc}(B_1)$ satisfying $\bar{\partial}A+A\wedge A = 0$, where $p \geq 2n$. Then $A = u^{-1}\bar{\partial}u$ for some $u \in L^p_{2,loc}(B_1)$.

Proof. Consider first the following: Let $\mathcal{U}$ denote the Banach manifold of invertible $r \times r$ matrices on $\mathbb{P}_n$ with coefficients in L_2^p, $\mathcal{M}$ denote the Banach space of $r \times r$ matrices on $\mathbb{P}_n$ with coefficients in L^p, and $\mathcal{A}$ denote the Banach space of $r \times r$ matrices of (0,1)-forms on $\mathbb{P}_n$ with coefficients in L_1^p. Let $\mathcal{M}^{\perp}$ be the subspace of $\mathcal{M}$ perpendicular in L^2 to the constant matrices.

Since $p > n$, the Sobolev embedding theorem shows that the map ϕ given by

$$\mathcal{U}\times\mathcal{A} \ni (u,A) \mapsto \bar\partial^*(u^{-1}\bar\partial u+u^{-1}Au) = -i\Lambda\partial(u^{-1}\bar\partial u+u^{-1}Au)$$

is a smooth map of Banach manifolds $\mathcal{U}\times\mathcal{A} \to \mathcal{M}^\perp$, where the adjoint is with respect to the Fubini-Study metric on $\mathbb{P}_n$. The partial derivative of ϕ in the $\mathcal{U}$-direction at $(1,0)$ is $T\mathcal{U} \ni v \mapsto \Delta'' v \in \mathcal{M}^\perp$, which is surjective with kernel the constants. By the implicit function theorem, the equation $\bar\partial^*(u^{-1}\bar\partial u+u^{-1}Au) = 0$ has a solution $u \in \mathcal{U}$ for all $A \in \mathcal{A}$ sufficiently small.

Now suppose that A is simply a matrix of $(0,1)$-forms with coefficients in $L^p_{1,loc}(B_1)$ satisfying $\bar\partial A+A\wedge A = 0$. Pull-back $A|_{B_r}$ to B_1 by the holomorphic map $B_1 \ni z \mapsto rz \in B_r$ to give $\tilde A_r \in L^p_1(B_1)$. Then $\|\tilde A_r\|_{L^p_1(B_1)} \le \text{Const.}\ r^{1-2n/p}\|A\|_{L^p_1(B_r)}$. Let η be a cutoff function with support in B_1 and with $\eta = 1$ on $B_{1/2}$. Then if $A_r := \eta\tilde A_r$, $\|A_r\|_{L^p_1} \le \text{Const.}\ \|\tilde A_r\|_{L^p_1(B_1)} \le$ Const. $r^{1-2n/p}\|A\|_{L^p_1(B_r)}$, and the last term on the right can be made arbitrarily small by shrinking r since $p \ge 2n$ and $A \in L^p_1(B_{1/2})$.

The matrices A_r can now be regarded as defined on $\mathbb{P}_n$, so if r is small enough, there exists u such that $\bar\partial^*(u^{-1}\bar\partial u+u^{-1}A_r u) = 0$. If $A'_r := u^{-1}\bar\partial u+u^{-1}A_r u$, then $\bar\partial A'_r + A'_r\wedge A'_r = u^{-1}(\bar\partial A_r + A_r\wedge A_r)u = u^{-1}[\bar\partial(\eta\tilde A_r)+(\eta\tilde A_r)\wedge(\eta\tilde A_r)]u$. Thu near 0, A'_r satisfies the (overdetermined in general) elliptic system $\bar\partial^* A'_r = 0$, $\bar\partial A'_r = -A'_r\wedge A'_r$ and is therefore smooth there. By the usual Newlander-Nirenberg theorem $A'_r = v^{-1}\bar\partial v$ for some smooth v defined near 0, and if $\tilde w := vu^{-1} \in L^p_2$ then $\tilde w^{-1}\bar\partial \tilde w = \tilde A_r$ near 0. Reverting to the original coordinat gives $A = w^{-1}\bar{\partial}w$ for some $w \in L_2^p$ defined near 0, and the conclusion of the lemma follows by applying this result at each point of B_1 and using the triviality of all holomorphic vector bundles on B_1.

□

Remark. With simple alterations the above proof can be sharpened to $p > n$.

The functional to be minimized can now be given - it is almost identical to Donaldson's [4], so the same notation will be used.

For hermitian $r \times r$ matrices M, the trace norm is $\nu(M) := \operatorname{tr}(M^*M)^{1/2} = \sum_{i=1}^{r} |\lambda_i|$ where $\{\lambda_i\}$ are the eigenvalues of M repeated according to multiplicity. As explained in [4], it defines a norm, and if $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ then $\nu(M) \geq |\operatorname{tr}A| + |\operatorname{tr}D|$. If s is a section of the endomorphisms of a $U(r)$-bundle E on the compact surface X, set $N(s) := \| \nu(s) \|_{L^p(X)}$, and for a connection A on E with curvature F in $\Lambda^{1,1}(\operatorname{End}E)$, the functional is $J(A) := N(i\hat{F} + \lambda 1)$, where $\lambda = \lambda_E = \frac{1}{irV} \int_X \operatorname{tr}\hat{F}\, dV$. Here p will be some fixed number greater than 4.

The following lemma corresponds to Lemma 3 of [4].

Lemma 9. Suppose that Theorem 1 has been proved for bundles of rank less than r. If E is a stable holomorphic r-bundle on X which can be expressed as an extension $0 \to B \to E \to C \to 0$ with B,C stable, then there is a smooth hermitian connection A on E compatible with $\bar{\partial}_E$ such that $J(A) < 4\pi V^{1/p-1}\nu_E(B)$.

<u>Proof</u>. On B, C , fix the H-E connections which exist by the inductive hypothesis, and let $\beta \in \Lambda^{0,1}(\mathrm{Hom}(C,B))$ be a $\bar{\partial}$-closed (0,1)-form representing the extension $0 \to B \to E \to C \to 0$.

If Q is the operator $Q := -i\Lambda\partial\bar{\partial}$, then $Q = i\Lambda\bar{\partial}\partial - i\Lambda(\partial\bar{\partial}+\bar{\partial}\partial) = P-i\hat{F}$ (cf.(2.2), (2.3)), so from (2.4) it follows that $Q+Q^* = P+P^*-2i\hat{F} = \Delta-i\hat{F}$. For the induced H-E connection on $\mathrm{Hom}(C,B)$, $\hat{F} = i(\lambda_B-\lambda_C)1$, and since E is stable, $\lambda_B > \lambda_C$. Thus Q^* has no kernel and Q is surjective; in particular, there exists $\gamma \in \mathrm{Hom}(C,B)$ such that $\Lambda\partial(\beta+\bar{\partial}\gamma) = 0$.

If β is thus modified so that $\Lambda\partial\beta = 0$, now rescale it so that $\sup_X|\beta| = 1$; ($\beta \neq 0$ since E is stable). Using $t\beta$ in place of β for $t = \bar{t} \neq 0$, (3.2) shows that the curvature of the induced connection on E has

$$i\hat{F}_E(t)+\lambda_E 1 = \begin{pmatrix} (\lambda_E-\lambda_B)1-it^2\Lambda\beta\wedge\beta^* & 0 \\ 0 & (\lambda_E-\lambda_C)1-it^2\Lambda\beta^*\wedge\beta \end{pmatrix} .$$

Since $\lambda_B > \lambda_E > \lambda_C$, when t is small enough all of the eigenvalues of the top term are negative and all those of the bottom are positive. For such such t , it follows that $\nu(i\hat{F}_E(t)+\lambda_E 1) = -\mathrm{tr}[(\lambda_E-\lambda_B)1-it^2\Lambda\beta\wedge\beta^*]+\mathrm{tr}[(\lambda_E-\lambda_C)1-it^2\Lambda\beta^*\wedge\beta] = 4\pi V^{-1}\nu_E(B)-2t^2|\beta|^2$. Since $|\beta|^2 \leq 1$, taking t sufficiently small gives $N(i\hat{F}_E(t)+\lambda_E 1) < 4\pi V^{1/p-1}\nu_E(B)$.

□

The next step is the equivalent of Lemma 1 of [4], but in the current setting, it is made considerably more complicated by the presence of singularities of one sort or another.

Suppose, as usual, that E is a stable r-bundle on the compact surface X, where stability is with respect to a fixed positive $\bar{\partial}\partial$-closed (1,1)-form ω. If E has an admissable subsheaf, pull-back E to the modification $\tilde{X} \xrightarrow{\pi} X$ given by Proposition 3 and fix one of the forms $\tilde{\omega}_\varepsilon$ described there. By Proposition 3 and Lemma 9, π^*E admits a smooth connection A with $J(A) < 4\pi\tilde{V}^{1/p-1}m$, where $\tilde{V} = \mathrm{Vol}(\tilde{X},\tilde{\omega}_\varepsilon)$ and $m := \inf\{\nu_{\pi^*E}(S,\tilde{\omega}_\varepsilon) : S \subset \pi^*E$ is admissable$\}$. If E has no admissable subsheaves, no blowing-up is required for what follows. To simplify notation, $(\tilde{X},\pi^*E,\tilde{\omega}_\varepsilon)$ will temporarily be denoted by (X,E,ω) when E is of type I.

Now choose a sequence A_i of smooth connections on E which minimize the functional J. Since line bundles admit H-E connections, it can be assumed that the induced connections on $\det E$ are all the same; namely, the H-E connection.

Since $J(A_i)$ is comparable with the usual L^p norm of the self-dual component of the curvature $F(A_i)$, $\| F(A_i) \|_{L^2}$ is bounded. By the weak compactness theorem of Uhlenbeck [21], ([18,S]), there is a finite subset $S = \{x_1,\ldots,x_N\} \subset X$ and local gauge transformations such that the gauge-transformed connections converge weakly in $L^2_{1,loc}(X\setminus S)$. In fact, an inspection of the proof of Corollary 23 [5] shows that the sequence can be assumed to converge weakly in $L^p_{1,loc}(X\setminus S)$, for all that is required in the proof of that corollary is a uniform bound on the L^p norm of the self-dual component of the curvatures. The transition functions of the resulting "bundle" on $X \setminus S$ are then continuous, and (as in [5]), Section 3 of [21] applies to construct global gauge transformations from the local ones. Thus, after suitable bundle automorphisms of the underlying U(r)-bundle, (a subsequence of) the gauge-transformed sequence, also denoted by A_i, converges weakly in $L^p_{1,loc}(X \setminus S)$ to a connection A' with $F(A') \in L^2(X)$ and $\hat{F}(A') \in L^p(X)$. By semi-continuity, $J(A') \leq \inf J(A_i)$.

The connection A' has curvature of type $(1,1)$, so by Lemma 8 it induces a holomorphic structure; denote this holomorphic bundle on $X \setminus S$ by E'. Since the connections on $\det E$ do not change in the sequence, $\det E' = \det E$ and $\mathrm{tr}F(A') = \mathrm{tr}F(A_0)$.

Following Donaldson [5] again, a non-zero holomorphic map $E \to E'$ will now be constructed, as in the proof of Lemma 4. Let g_j be the complex automorphism intertwining A_0 and A_j, with $\det g_j = 1$ for all j; (that is, g_j is the map which gives the isomorphism between the holomorphic structure E_0 defined by A_0 and that which is defined by A_j.) By (2.5), $\Delta|g_j|^2 + i*\partial(|g_j|^2\bar{\partial}\omega) - i*\bar{\partial}(|g_j|^2\partial\omega) \leq 2(|\hat{F}|+|\hat{F}_j|)|g_j|^2$, so by Theorem 9.20 [8] there is a constant C, independent of j, such that $\sup_X |g_j|^2 \leq C[\|g_j\|^2_{L^2(X)} + \|(|\hat{F}_0|+|\hat{F}_j|)|g_j|^2\|_{L^4(X)}]$. By Hölder's inequality, it follows that $\sup_X |g_j|^2 \leq C\|g_j\|^2_{L^q(X)}$ for $q = 8p/p-4$ and some new constant C, using the uniform bound on $\|\hat{F}_j\|_{L^p}$. Since $\{A_j\}$ converges weakly in $L^p_{1,loc}(X \setminus S)$ and $p > 4$, the A_j's are bounded in $C^0(K)$ for any compact $K \subset X \setminus S$. Repeating the argument of Lemma 4, after rescaling g_j to $\tilde{g}_j$ satisfying $\|\tilde{g}_j\|_{L^q(X)} = 1$ and choosing small balls B_α about the points $x_\alpha \in S$, a subsequence of the $\tilde{g}_j$'s can be found which converges weakly in $L^p_2(K_0)$ and strongly in $L^q(K_0)$ to a non-zero limit $\tilde{g}$ representing a holomorphic map $E_0 \to E'$, where $K_0 = X \setminus \cup B_\alpha$. Since ∂K_0 is pseudo-concave, $\tilde{g}$ extends to $X \setminus S$, and by diagonalization ([18]) it can be assumed that $\tilde{g}_j$ is converging weakly to $\tilde{g}$ in $L^p_{2,loc}(X \setminus S)$.

Since the connections on $\det E$, $\det E'$ are the same, $\det \tilde{g}$ is a holomorphic function on $X \setminus S$, and therefore constant by Hartog's theorem. Suppose that $\det \tilde{g} = 0$. Then $\tilde{g}$ has non-zero kernel at every point, giving a diagram on $X \setminus S$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \tilde{g}\downarrow & & \| & & \\
 & & & & & \tilde{g} & & & \\
0 & \longleftarrow & C & \longleftarrow & E' & \longleftarrow & Q & \longleftarrow & 0 ,
\end{array}
\qquad (5.1)
$$

where K = kernel, Q = quotient, C = cokernel. If $\mathcal{O}(E)_x$ is generated by sections $e_1, \ldots e_r \in \Gamma(B_\alpha, \mathcal{O}(E))$ as $\mathcal{O}_x$-module for each $x \in B_\alpha$, then the images of $e_1, \ldots, e_r$ in $\Gamma(B_\alpha \setminus \{x_\alpha\}, Q)$ generate Q_y as $\mathcal{O}_y$-module for each $y \in B_\alpha \setminus \{x_\alpha\}$. By a theorem of Serre [19], $i_* Q$ is a coherent analytic sheaf on X, where $i : X \setminus S \to X$ is inclusion. [Indeed $i_* Q$ is locally free in a neighbourhood of $x_\alpha \in S$, being torsion-free and normal there; $E \to i_* Q$ need not be surjective at x_α though.] It follows that $i_* K$ is coherent, so in particular, E has an admissable subsheaf.

Now, off a codimension ≥ 1 analytic subset of $X \setminus S$, (5.1) is a diagram of bundles. By definition, the $\bar{\partial}$-operator induced on Q as a quotient of E and the $\bar{\partial}$-operator induced on Q as a sub-bundle of E' via $\tilde{g} : Q \to E'$ are the same on this complement. Since E and E' have the same unitary structures (where the latter is defined), the induced connections on Q are the same; in particular, they have the same curvatures F_Q, so by the remark (e) at the end of section 3, $\operatorname{tr} \hat{F}_Q$ is integrable and $\frac{i}{2\pi} \int_X \operatorname{tr} \hat{F}_Q dV = \deg(Q,\omega)$. [If $\tilde{Q} := E/i_*K$, then $\tilde{Q} = i_*Q$ off S, so $\det\tilde{Q} = \det(i_*Q)$. For simplicity, the symbol Q is being used in place of $\tilde{Q}$ here.]

Off the codim ≥ 1 subset of $X \setminus S$, (3.2) gives

$$F(A') = \begin{pmatrix} F_Q - \beta\wedge\beta^* & \nabla\beta \\ -\nabla\beta^* & F_C - \beta^*\wedge\beta \end{pmatrix},$$

and moreover, $\operatorname{tr} \hat{F}(A') = \operatorname{tr} \hat{F}_Q + \operatorname{tr} \hat{F}_C = \operatorname{tr} \hat{F}(A_0) = ir\lambda_E$. Using the property of ν stated earlier in this section, $\nu(i\hat{F}(A') + \lambda_E 1) \geq |\operatorname{tr}(i\hat{F}_Q - i\Lambda\beta\wedge\beta^* + \lambda_E 1)| + |\operatorname{tr}(i\hat{F}_C - i\Lambda\beta^*\wedge\beta + \lambda_E 1)|$ $2|\operatorname{tr} i\hat{F}_Q + |\beta|^2 + \lambda_E q|$, where $q = \operatorname{rank} Q$ and $|\beta|^2 = -i\operatorname{tr}\Lambda\beta\wedge\beta^*$. Thus $J(A') = N(i\hat{F}(A') + \lambda_E 1) = \|\nu(i\hat{F}(A') + \lambda_E 1)\|_{L^p(X)} \geq$ $V^{1/p-1}\|\nu(i\hat{F}(A') + \lambda_E 1)\|_{L^1(X)} \geq 2V^{1/p-1}\|\operatorname{tr} i\hat{F}_Q + |\beta|^2 + \lambda_E q\|_{L^1(X)} \geq 2V^{1/p-1}|\int_X(\operatorname{tr} i\hat{F}_Q + |\beta|^2 + \lambda_E q)dV| =$ $2V^{1/p-1}|\int_X |\beta|^2 dV + (\lambda_E - \lambda_Q)qV|$. Since $\lambda_E = -2\pi V^{-1}\mu(E)$ and E is stable, $\mu(E) < \mu(Q)$ and $\lambda_E > \lambda_Q$; thus $J(A') \geq 2V^{1/p}q(\lambda_E - \lambda_Q) = 4\pi V^{1/p-1}\nu_E(i_*K)$, with equality only if $\beta = 0$. This, however contradicts $J(A') \leq \inf J(A_i) < 4\pi^{1/p-1}\nu_E(i_*K)$; (recall that since E has admissable subsheaves, we are actually working here on the modification given by Proposition 3).

Thus when E is of either type, $\tilde{g} : E \to E'$ is an isomorphism. Unfortunately, a priori this is only an isomorphism outside S , so it must be shown that $\tilde{g} \in L^p_2(X)$.

Recall that the unscaled g_j's had $\det g_j = 1$. Since the unscaled endomorphisms $\tilde{g}_j$ converge in C^0 off a neighbourhood of S to $\tilde{g}$ with $\det \tilde{g} \neq 0$, the scaling factors must be bounded above and below, and since $\{\tilde{g}_j\}$ is bounded in $C^0(X)$, so too is $\{\tilde{g}_j^{-1}\}$. Thus $\tilde{g}, \tilde{g}^{-1} \in L^p_{2,loc}(X \setminus S) \cap L^\infty(X)$. Now, there is no loss of generality in assuming that $\tilde{g}_j$ and $\tilde{g}$ are positive and self-adjoint, for the replacement of g_j by $(g_j^* g_j)^{1/2}$ amounts to a unitary gauge transformation. If u_j is the positive self-adjoint endomorphism $u_j = \tilde{g}_j^* \tilde{g}_j = \tilde{g}_j^2$, then $\{u_j\}$ is converging weakly in $L^p_{2,loc}(X \setminus S)$ to $u = \tilde{g}^* \tilde{g} = \tilde{g}^2$. In a fixed <u>holomorphic</u> frame defined by A_0 , the curvature forms are then $F(A_j) = F(A_0) + \bar{\partial}(u_j^{-1}\partial_0 u_j)$. Since $\{F(A_j)\}$ is converging weakly in $L^2(X)$ and $u_j^{-1}\partial_0 u_j$ is a sequence of (1,0)-forms, it follows that $u_j^{-1}\partial_0 u_j$ converges weakly in $L^2_1(X)$ and (without loss of generality), $u^{-1}\partial_0 u \in L^2_1(X)$. Since $u, u^{-1} \in L^\infty(X)$, it follows easily that $u \in L^2_2(X)$. Since $\tilde{g}, \tilde{g}^{-1}$ are positive, self-adjoint and bounded, it follows easily that if A is a matrix such that $A + \tilde{g}^{-1}A\tilde{g} \in L^q$, then $A \in L^q$. For $A = \tilde{g}^{-1}D\tilde{g}$, one has $A + \tilde{g}^{-1}A\tilde{g} = u^{-1}Du \in L^2_1 \subset L^4$, so $A \in L^4$. Then $DA + \tilde{g}^{-1}DA\,\tilde{g} =$

$D(u^{-1}Du) + A\tilde{g}^{-1}A\tilde{g} - \tilde{g}^{-1}A\tilde{g}A \in L^2$, implying $DA \in L^2$, and $A \in L^2_1$. Thus in a unitary frame, the connection forms for the limit connection A' lie in $L^2_1(X)$. But now Theorem 1.3 of [21] can be applied. Since $\int_B \|F\|^2 dV$ is invariant under dilations of the ball B (using a flat metric, but in the non-flat case the difference can be neglected), by choosing a sufficiently small neighbourhood B of a bad point $x_\alpha \in S$, the connection can be gauge-transformed as in Proposition 22 of [5]. Arguing then as in Corollary 23 of [5] using the fact that $\hat{F}(A') \in L^p(X)$, it follows that the gauge-transformed connection form actually lies in $L^p_1(\frac{1}{2}B)$ (provided B is sufficiently small). It follows that $F(A') \in L^p(X)$, and by repeating the earlier argument, u and then $\tilde{g}$ are in $L^p_2(X)$.

The connection $A' \in L^p_1$ now minimizes the functional J , and it must be shown that this minimum is 0.

Recall the operators $P = i\Lambda\bar{\partial}\partial$ and $Q = -i\Lambda\bar{\partial}\partial$. Since $P + P^* = \Delta + i\hat{F}$ and $Q + Q^* = \Delta - i\hat{F}$, $R := P + Q$ satisfies $R + R^* = 2\Delta$. Any solution $s \in L^p_2(\mathrm{End}E)$ of $Rs = 0$ is necessary of the form $s = \mathrm{const}.1$; this is true even though R may not have smooth coefficients, because a sequence of smooth connections A'_j can be chosen converging strongly in L^p_1 to A' and the corresponding operators R_j have the same second order term, first order terms converging in L^p_1 and zeroth order terms converging in L^p . Thus

$0 = \langle s,Rs\rangle = \lim\langle s,R_j s\rangle = \lim\langle d_{A'_j}s, d_{A'_j}s\rangle = \langle d_A s, d_A s\rangle$, implying $s = \mathrm{const}.1$.

The same type of elementary approximation argument shows that there is a unique solution $s \in L^p_2(\mathrm{End}E) \cap (\ker R)^\perp$ to

$Rs = i\hat{F}(A') + \lambda_E 1$ (since $i\hat{F}(A') + \lambda_E 1$ lies in $(\ker R^*)^{\perp}$) . Exactly the same argument as in [4] now shows that in order for A' to minimize J , it must be the case that $J(A') = 0$, otherwise $g_t := 1+ts$ gives a connection A_t with $J(A_t) < J(A')$ for small enough t .

In the case when E has no admissable subsheaves, it has now been shown that E admits an H-E connection. In the case that E does have admissable subsheaves, it has been shown that π^*E admits an H-E connection for each of the forms $\tilde{\omega}_\varepsilon$ of Proposition 3, where $\tilde{X} \xrightarrow{\pi} X$ is the modification described in that proposition. The final task is to push these down to X .

Recall that the forms σ_i of Proposition 3 could have support in arbitrarily small neighbourhoods of the exceptional lines they represent, so $\tilde{\omega}_\varepsilon - \pi^*\omega$ can have support in an arbitrarily small neighbourhood of the exceptional divisor D . Shrinking these supports (and necessarily, the coefficients α_i at the same time) gives a sequence of forms $\{\tilde{\omega}_j\}$, say, and corresponding connections $\tilde{A}_j$ on π^*E such that $\tilde{A}_j$ is an H-E connection for $\tilde{\omega}_j$. Thus if $\{x_1,\ldots,x_M\} = \pi(D)$, then off each fixed (but arbitrarily small neighbourhood) of $\pi(D)$ the sequence $\tilde{A}_j$ can be viewed as a sequence of connections A_j on E , which for j large enough, are all H-E connections for ω . The constants λ_E in this sequence are of course changing: $(\lambda_E)_j = -2\pi\mu(E)/\mathrm{Vol}(\tilde{X},\tilde{\omega}_j)$, with $\mathrm{Vol}(\tilde{X},\tilde{\omega}_j) \to \mathrm{Vol}(X)$.

Applying the argument of Uhlenbeck-Sedlacek-Donaldson once again, there exist $x_{M+1},\ldots,x_N \in X$ such that, if $S := \{x_1,\ldots,x_N\}$, then after suitable gauge transformations the A_j converge weakly in $L^p_{1,loc}(X\setminus S)$ to an H-E conection A with finite Yang-Mills action over $X\setminus S$. (The U-S-D argument is still applicable even though it is being applied over $X\setminus\bigcup_\alpha B^j_\alpha$ with $B^j_\alpha \to \{x_\alpha\}$, as an inspection of [18] quickly shows.) By ellipticity, A is smooth, and since, in a neighbourhood of any point of X the connection A can be twisted by an H-E connection on a trivial line bundle so that the resulting connection has $\lambda = 0$, it follows from the removable singularities theorem [20] that A extends across S to an H-E connection on a (possibly topologically different) bundle E'. The new holomorphic bundle E' is automatically semi-stable by Corollary 4. If U is any neighbourhood of S, then for sufficiently large j,

$\int_{X\setminus U} \mathrm{tr}\hat{F}(A_j)\,dV = ir(\lambda_E)_j \mathrm{Vol}(X\setminus U)$, so $\mu(E') = \mu(E)$.

It remains therefore to construct a non-zero holomorphic map $E \to E'$ or $E' \to E$. Choose a small ball B_α about x_α and set $U := \bigcup_\alpha B_\alpha$, $\tilde{U} := \pi^{-1}(U)$. The balls B_α are chosen small enough that E has a connection A_0 (compatible with $\bar{\partial}_E$) which is smooth and moreover is flat in all B_α. Pull A_0 back to $\tilde{X}$ and let g_j be the endomorphism intertwining π^*A_0 with $\tilde{A}_j$. Using the Laplacian Δ_j on $\tilde{X}$ determined by $\tilde{\omega}_j$, as well as the $*$ and Λ operators for $\tilde{\omega}_j$, (2.3) gives

$$\Delta_j|g_j|^2 + i*\partial(|g_j|^2\bar{\partial}\tilde{\omega}_j) - i*\bar{\partial}(|g_j|^2\partial\tilde{\omega}_j) \le 2\langle g_j, i\hat{F}(\tilde{A}_j)g_j - g_j i\hat{F}_0\rangle, \tag{5.2}$$

where $i\hat{F}(\tilde{A}_j) = 2\pi\mu(E)/\mathrm{Vol}(\tilde{X},\tilde{\omega}_j)$. If $\mu(E) > 0$, replace g_j by g_j^{-1}; otherwise leave g_j as it is. Then in $\tilde{U}$, $\hat{F}_0 = 0$

and the right-side of (5.2) is ≤ 0 . Since $\bar{\partial}\partial\tilde{\omega}_j = 0$, Theorem 3.1 of [8] (the maximum principle) gives $\sup_{\tilde{U}}|g_j|^2 \leq \sup_{\partial\tilde{U}}|g_j|^2$. On the other hand, outside $\tilde{U}$ the forms $\tilde{\omega}_j$ all agree for large enough j , and in $\tilde{X}\setminus\tilde{U}$ one has the usual bound $\Delta|g_j|^2 \leq \text{Const.}|g_j|^2$, where Δ is simply determined by ω . By Theorem 9.20 of [8] it now follows that $\sup_{\tilde{X}}|g_j|^2 \leq C\|g_j\|^2_{L^8(\tilde{X}\setminus\tilde{U}',\pi^*\omega)}$, where $U' \subset\subset U$ is slightly smaller.

Now choose $U'' \subset U'$ such that $C^4\text{Vol}(U'') = \frac{1}{2}$ and fix a non-singular metric $\tilde{\omega}$ on $\tilde{X}$ such that $\text{supp}(\tilde{\omega}-\pi^*\omega) \subset \tilde{U}''$. Normalize g_j so that $\|g_j\|_{L^8(\tilde{X},\tilde{\omega})} = 1$; (here it is assumed $\mu(E) \leq 0$, otherwise use g_j^{-1} as above). Then since $\text{Vol}(\tilde{U}'',\tilde{\omega}) \leq \text{Vol}(U'',\omega)$, the usual calculation gives $\|g_j\|^8_{L^8(\tilde{X}\setminus\tilde{U}'',\pi^*\omega)} \geq \frac{1}{2}$.

Now regard g_j as defined on $X\setminus S$. Then $\|g_j\|^8_{L^8(X\setminus U'')} \geq \frac{1}{2}$, and exactly the same argument as in the proof of Lemma 4 (i.e.[5] p.23) shows that the g_j's have a subsequence weakly convergent in $L^8_2(X\setminus U'')$ and strongly convergent in $C^0(X\setminus U'')$ to a limit g representing a non-zero holomorphic map $E \to E'$ (or $E' \to E$) over $X\setminus U''$, and by Hartogs' theorem, this extends to X . Since $\mu(E) = \mu(E')$, E is stable and E' is semi-stable, this map must be an isomorphism. This completes the proof of the theorem.

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