Good Reductions of Shimura Varieties of Hodge Type in Arbitrary Unramified Mixed Characteristic, Part I

Adrian Vasiu, Binghamton University

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ABSTRACT. We prove the existence of good smooth integral models of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic $(0, p)$. As a first application we solve a conjecture of Langlands for Shimura varieties of Hodge type. As a second application, for $p \geq 3$ (resp. for $p = 2$) we prove the existence in unramified mixed characteristic $(0, p)$ of integral canonical models of Shimura varieties of Hodge type that have compact factors (resp. that have compact factors and that pertain to abelian varieties in characteristic p which have zero p-ranks). Though the second application is new only for $p \leq 3$ and for non-unitary Shimura varieties, its proof is new, more direct, and more of a principle. The second application also represents progress toward the proof of a conjecture of Milne.

KEY WORDS: Shimura varieties, affine group schemes, abelian schemes, integral models, p-divisible groups, and F-crystals.

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1. Introduction

Let $p \in \mathbb{N}$ be a prime. Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at its prime ideal (p). Let $r \in \mathbb{N}$. Let $N \geq 3$ be a natural number relatively prime to p. Let $\mathcal{A}_{r,1,N}$ be the *Mumford moduli scheme* over $\mathbb{Z}_{(p)}$ that parameterizes isomorphism classes of principally polarized abelian schemes over $\mathbb{Z}_{(p)}$-schemes that are of relative dimension r and that are endowed with a symplectic similitude level-N structure (cf. [MFK, Thms. 7.9 and 7.10] applied to symplectic similitude level structures instead of only to level structures).

1.1. Basic properties. The $\mathbb{Z}_{(p)}$-schemes $\mathcal{A}_{r,1,N}$ have the following three properties:

(i) they are smooth and quasi-projective;

(ii) if $N_1 \in N\mathbb{N}$ is relatively prime to p, then the natural level-reduction $\mathbb{Z}_{(p)}$-morphism $\mathcal{A}_{r,1,N_1} \to \mathcal{A}_{r,1,N}$ is a pro-étale cover; thus the projective limit

$$\mathcal{M}_r := \text{proj.lim.}_{N \geq 3, (N,p)=1} \mathcal{A}_{r,1,N}$$

exists and is a regular, formally smooth $\mathbb{Z}_{(p)}$-scheme;

(iii) if Z is a regular, formally smooth scheme over $\mathbb{Z}_{(p)}$, then each morphism $Z_{\mathbb{Q}} \to \mathcal{M}_{r\mathbb{Q}}$ extends uniquely to a morphism $Z \to \mathcal{M}_r$ of $\mathbb{Z}_{(p)}$-schemes.

Property (i) is checked in [MFK, Thms. 7.9 and 7.10], cf. also Serre Lemma of [Mu, §21, Thm. 5]. Property (ii) is well known. Property (iii) is implied by the fact that each abelian scheme over $Z_{\mathbb{Q}}$ that has level-N structure for each natural number $N \geq 3$ relative prime to p, extends to an abelian scheme over Z (cf. the Néron–Ogg–Shafarevich criterion of good reduction and the purity result [Va2, Thm. 1.3]); such an extension is unique up to a unique isomorphism (cf. [FC, Ch. I, Prop. 2.7]). From Yoneda Lemma we get that the regular, formally smooth $\mathbb{Z}_{(p)}$-scheme $\mathcal{M}_r$ is uniquely determined by its generic fibre $\mathcal{M}_{r\mathbb{Q}}$ and by the *universal property* expressed by the property (iii). Thus one can view $\mathcal{A}_{r,1,N}$ as the *best* smooth integral model of $\mathcal{A}_{r,1,N\mathbb{Q}}$ over $\mathbb{Z}_{(p)}$. The main goal of this paper is to generalize properties (i) to (iii) to the context of Shimura varieties of Hodge type. Thus in this paper we prove the existence of *good* smooth integral models of Shimura varieties of Hodge type in unramified mixed characteristic and we list several main properties of them. We will begin with a list of notations and with a review on Shimura varieties.

1.2. Notations. Let $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m\mathbb{C}}$ be the two dimensional torus over $\mathbb{R}$ such that we have identifications $\mathbb{S}(\mathbb{R}) = \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C}) = \mathbb{G}_{m\mathbb{C}}(\mathbb{C}) \times \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ with the property that the monomorphism $\mathbb{R} \hookrightarrow \mathbb{C}$ induces the map $z \to (z, \bar{z})$; here $z \in \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$.

Let O be a commutative $\mathbb{Z}$-algebra. We recall that a group scheme F over O is called *reductive* if it is smooth and affine and its fibres are connected and have trivial unipotent radicals. Let $\text{Lie}(F)$ be the Lie algebra over O of F. The group schemes $\mathbb{G}_{mO}$ and $\mathbb{G}_{aO}$ are over O. For a free module M of finite rank over O, let $M^* := \text{Hom}_O(M, O)$ and let $\mathbf{GL}_M$ be the reductive group scheme over O of linear automorphisms of M. A bilinear form ψ_M on M is called perfect if it defines naturally an isomorphism $M \tilde{\to} M^*$. If M has even rank and if ψ_M is a perfect, alternating form on M, then $\mathbf{Sp}(M, \psi_M)$ and $\mathbf{GSp}(M, \psi_M)$ are viewed as reductive group schemes over O.

Let k be a perfect field of characteristic p. Let $W(k)$ be the ring of Witt vectors with coefficients in k. Always $n \in \mathbb{N}$. Let $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles of $\mathbb{Q}$. Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of $\mathbb{Q}$ with the p-component omitted; we have $\mathbb{A}_f = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$. If $O \in \{\mathbb{A}_f, \mathbb{A}_f^{(p)}, \mathbb{Q}_p\}$, then the group $F(O)$ is endowed with the coarsest topology that makes all maps $O = \mathbb{G}_{aO}(O) \to F(O)$ associated to morphisms $\mathbb{G}_{aO} \to F$ of O-schemes to be continuous; thus $F(O)$ is a totally discontinuous locally compact group. Each continuous action of a totally discontinuous locally compact group on a scheme will be in the sense of [De2, Subsubsection 2.7.1] and it will be a right action.

1.3. Shimura varieties. A *Shimura pair* $(G, \mathcal{X})$ consists from a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class $\mathcal{X}$ of homomorphisms $\mathbb{S} \to G_{\mathbb{R}}$ that satisfy Deligne's axioms of [De2, Subsubsection 2.1.1]: the Hodge $\mathbb{Q}$–structure on $\mathrm{Lie}(G)$ defined by any $h \in \mathcal{X}$ is of type $\{(-1,1),(0,0),(1,-1)\}$, no simple factor of the adjoint group G^{ad} of G becomes compact over $\mathbb{R}$, and $\mathrm{Ad}(h(i))$ is a Cartan involution of $\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})$. Here $\mathrm{Ad} : G_{\mathbb{R}} \to \mathbf{GL}_{\mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{R}})}$ is the adjoint representation. These axioms imply that $\mathcal{X}$ has a natural structure of a hermitian symmetric domain, cf. [De2, Cor. 1.1.17]. For $h \in \mathcal{X}$ we consider the Hodge cocharacter

$$\mu_h : \mathbb{G}_{m\mathbb{C}} \to G_{\mathbb{C}}$$

defined on complex points by the rule: $z \in \mathbb{G}_{m\mathbb{C}}(\mathbb{C})$ is mapped to $h_{\mathbb{C}}(z,1) \in G_{\mathbb{C}}(\mathbb{C})$.

The most studied Shimura pairs are constructed as follows. Let W be a vector space over $\mathbb{Q}$ of even dimension $2r$. Let ψ be a non-degenerate alternative form on W. Let $\mathcal{Y}$ be the set of all monomorphisms $\mathbb{S} \hookrightarrow \mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$ that define Hodge $\mathbb{Q}$–structures on W of type $\{(-1,0),(0,-1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. The pair $(\mathbf{GSp}(W,\psi), \mathcal{Y})$ is a Shimura pair that defines a *Siegel modular variety*. Let L be a $\mathbb{Z}$-lattice of W such that ψ induces a perfect form $\psi : L \otimes_{\mathbb{Z}} L \to \mathbb{Z}$. Let

$$K(N) := \{g \in \mathbf{GSp}(L,\psi)(\widehat{\mathbb{Z}}) | g \bmod N \text{ is identity}\} \text{ and } K_p := \mathbf{GSp}(L,\psi)(\mathbb{Z}_p).$$

Let $E(G,\mathcal{X}) \hookrightarrow \mathbb{C}$ be the number subfield of $\mathbb{C}$ that is the field of definition of the $G(\mathbb{C})$-conjugacy class of the cocharacters μ_h's of $G_{\mathbb{C}}$, cf. [Mi2, p. 163]. We recall that $E(G,\mathcal{X})$ is called the reflex field of $(G,\mathcal{X})$. The *Shimura variety* $\mathrm{Sh}(G,\mathcal{X})$ is identified with the canonical model over $E(G,\mathcal{X})$ of the complex Shimura variety

$$\mathrm{Sh}(G,\mathcal{X})_{\mathbb{C}} := \mathrm{proj.lim.}_{K \in \Sigma(G)} G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f)/K),$$

where $\Sigma(G)$ is the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation (see [De1], [De2], and [Mi1] to [Mi4]). Thus $\mathrm{Sh}(G,\mathcal{X})$ is an $E(G,\mathcal{X})$-scheme together with a continuous $G(\mathbb{A}_f)$-action. For C a compact subgroup of $G(\mathbb{A}_f)$ let

$$\mathrm{Sh}_C(G,\mathcal{X}) := \mathrm{Sh}(G,\mathcal{X})/C.$$

Let $K \in \Sigma(G)$. A classical result of Baily and Borel allows us to view $\mathrm{Sh}_K(G,\mathcal{X})_{\mathbb{C}} = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f)/K)$ as a finite, disjoint union of normal, quasi-projective varieties over $\mathbb{C}$ and not only of complex space (see [BB, Thm. 10.11]). Thus $\mathrm{Sh}_K(G,\mathcal{X})$ is a normal, quasi-projective $E(G,\mathcal{X})$-scheme. If K is small enough, then $\mathrm{Sh}_K(G,\mathcal{X})$ is in fact a smooth, quasi-projective $E(G,\mathcal{X})$-scheme. Let H be a compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

We recall that the group $G_{\mathbb{Q}_p}$ is called *unramified* if and only if it has a Borel subgroup and splits over an unramified, finite field extension of $\mathbb{Q}_p$. See [Ti] for *hyperspecial subgroups* of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. In what follows we will only need the following three things: (i) the group

$G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ has hyperspecial subgroups if and only if $G_{\mathbb{Q}_p}$ is unramified, (ii) the subgroup H of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ is hyperspecial if and only if it is the group of $\mathbb{Z}_p$-valued points of a reductive group scheme over $\mathbb{Z}_p$ whose generic fibre is $G_{\mathbb{Q}_p}$, and (iii) each hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ is a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

Let v be a prime of $E(G,\mathcal{X})$ that divides p. Let $k(v)$ be the residue field of v. Let $e(v)$ be the index of ramification of v. Let $O_{(v)}$ be the localization of the ring of integers of $E(G,\mathcal{X})$ with respect to v.

1.3.1. Definitions. (a) By an *integral model* of $\mathrm{Sh}_K(G,\mathcal{X})$ over $O_{(v)}$ we mean a faithfully flat $O_{(v)}$-scheme whose generic fibre is $\mathrm{Sh}_K(G,\mathcal{X})$.

(b) By an *integral model* of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$ we mean a faithfully flat $O_{(v)}$-scheme equipped with a continuous $G(\mathbb{A}_f^{(p)})$-action whose generic fibre is the $E(G,\mathcal{X})$-scheme $\mathrm{Sh}_H(G,\mathcal{X})$ equipped with its natural continuous $G(\mathbb{A}_f^{(p)})$-action.

In this paper we study integral models of $\mathrm{Sh}_K(G,\mathcal{X})$ and $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$. The subject has a long history, the first main result being the existence of the moduli schemes $\mathcal{A}_{r,1,N}$ and $\mathcal{M}_r$. This is so as we have natural identifications

$$\mathcal{A}_{r,1,N\mathbb{Q}} = \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W,\psi),\mathcal{Y}) \text{ and } \mathcal{M}_{r\mathbb{Q}} = \mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi),\mathcal{Y})$$

(see [De1], [Mi2], [Va1], etc.). In particular, see [Va1, Ex. 3.2.9 and Subsection 4.1] and [De1, Thm. 4.21] for the natural continuous action of $\mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$ on $\mathcal{M}_r$.

In 1976 Langlands conjectured the existence of a good integral model of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$, provided H is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (see [La, p. 411]); unfortunately, Langlands did not explain what good is supposed to stand for. Only in 1992, an idea of Milne made it significantly clearer how to characterize and identify the good integral models. Milne's philosophy can be roughly summarized as follows (cf. [Mi2]): under certain conditions, the good regular, formally smooth integral models should be uniquely determined by (Néron type) universal properties that are similar to the property 1.1 (iii).

1.3.2. Definitions. (a) We say $(G,\mathcal{X})$ has *compact factors*, if for each simple factor F of the adjoint group G^{ad} of G there exists a simple factor of $F_{\mathbb{R}}$ which is compact.

(b) Suppose that $e(v)=1$. An affine, flat group scheme $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ that extends G (i.e., whose generic fibre is G) is called a *quasi-reductive group scheme for* $(G,\mathcal{X},v)$, if there exists a reductive, normal, closed subgroup scheme $G^{\mathrm{r}}_{\mathbb{Z}_p}$ of $G_{\mathbb{Z}_p}$ and a cocharacter $\mu_v : \mathbb{G}_{mW(k(v))} \to G^{\mathrm{r}}_{\mathbb{Z}_p} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(W(k(v)))$, such that the extension of μ_v to $\mathbb{C}$ via an (any) $O_{(v)}$-monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ defines a cocharacter of $G_{\mathbb{C}}$ that is $G(\mathbb{C})$-conjugate to the cocharacters μ_h of $G_{\mathbb{C}}$ introduced above ($h \in \mathcal{X}$).

(c) Let Y be a smooth $O_{(v)}$-scheme of finite type. We say that Y is a *Néron model* of its generic fibre $Y_{E(G,\mathcal{X})}$ over $O_{(v)}$, if for each smooth $O_{(v)}$-scheme Z, every morphism $Z_{E(G,\mathcal{X})} \to Y_{E(G,\mathcal{X})}$ extends uniquely to a morphism $Z \to Y$ of $O_{(v)}$-schemes.

Definition (a) was introduced in [Va3, Def. 1.1]. Definition (b) is a variation of [Re2, Def. 1.5]; more precisely, the group $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ is an *h–hyperspecial* subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ in the sense of loc. cit. Definition (c) is well known, cf. [BLR, Ch. 1, 1.2, Def. 1].

1.4. Constructing integral models. Until the end we will assume that the Shimura pair $(G, \mathcal{X})$ is of *Hodge type* i.e., there exists an injective map

$$f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$$

for some symplectic space (W, ψ) over $\mathbb{Q}$; thus $f : G \hookrightarrow \mathbf{GSp}(W, \psi)$ is a monomorphism such that we have $f_{\mathbb{R}} \circ h \in \mathcal{Y}$ for all elements $h \in \mathcal{X}$.

We recall that we identity $\mathcal{M}_{r\mathbb{Q}} = \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})$. Let $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let $G_{\mathbb{Z}_{(p)}}$ be the Zariski closure of G in $\mathbf{GL}_{L_{(p)}}$. Until the end we will also assume that we have an identity $H = K_p \cap G_{\mathbb{Q}_p}(\mathbb{Q}_p)$; thus $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$.

The functorial morphism $f_0 : \mathrm{Sh}(G, \mathcal{X}) \to \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G,\mathcal{X})}$ defined by f (see [De1, Cor. 5.4]) is a closed embedding as it is so over $\mathbb{C}$ (cf. [De1, Prop. 1.15]). The morphism f_0 gives birth naturally to a morphism of $E(G, \mathcal{X})$-schemes

$$f_p : \mathrm{Sh}_H(G, \mathcal{X}) \to \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G,\mathcal{X})}$$

which is finite (cf. Proposition 2.2.1 (b)). Thus we can speak about the normalization

$$\mathcal{N}$$

of $\mathcal{M}_{rO_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_H(G, \mathcal{X})$. If $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, then f_p is a closed embedding (for instance, see [Va1, Rm. 3.2.14]) and thus $\mathcal{N}$ is in fact the normalization of the Zariski closure of $\mathrm{Sh}_H(G, \mathcal{X})$ in $\mathcal{M}_{rO_{(v)}}$. As $G(\mathbb{A}_f^{(p)})$ acts on $\mathrm{Sh}_H(G, \mathcal{X})$ and $\mathcal{M}_r$, we get a natural continuous action of $G(\mathbb{A}_f^{(p)})$ on $\mathcal{N}$. Let

$$\mathcal{N}^{\mathrm{s}}$$

be the formally smooth locus of $\mathcal{N}$ over $O_{(v)}$; it is a $G(\mathbb{A}_f^{(p)})$-invariant, open subscheme of $\mathcal{N}$ such that we have identities $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})} = \mathrm{Sh}_H(G, \mathcal{X})$ (cf. Lemma 2.2.2). If $p > 2$ let $\mathcal{N}^{\mathrm{m}} := \mathcal{N}^{\mathrm{s}}$. If $p = 2$ let $\mathcal{N}^{\mathrm{m}}$ be the open subscheme of $\mathcal{N}^{\mathrm{s}}$ that is defined in Subsubsection 3.5.1. In this paper we study the following sequence

$$\mathcal{N}^{\mathrm{m}} \hookrightarrow \mathcal{N}^{\mathrm{s}} \hookrightarrow \mathcal{N} \to \mathcal{M}_{rO_{(v)}}$$

of morphisms of $O_{(v)}$-schemes in order to prove the following three basic results.

1.5. Basic Theorem. *We assume that $e(v) = 1$ (i.e., v is unramified over p) and that the $k(v)$-scheme $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty. Then we have:*

(a) *The $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{s}}$ is the unique regular, formally smooth integral model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of index of ramification 1 and is a faithfully flat $O_{(v)}$-algebra, then each morphism $Z_{E(G,\mathcal{X})} \to \mathrm{Sh}_H(G, \mathcal{X})$ extends uniquely to a morphism $Z \to \mathcal{N}^{\mathrm{s}}$ of $O_{(v)}$-schemes.*

(b) *For each algebraically closed field k of characteristic p, the natural morphism $\mathcal{N}^{\mathrm{s}}_{W(k)} \to \mathcal{M}_{rW(k)}$ induces $W(k)$-epimorphisms at the level of complete, local rings of residue field k (i.e., it is a formally closed embedding at all k-valued point of $\mathcal{N}^{\mathrm{s}}_{W(k)}$).*

(c) *Suppose that $(G, \mathcal{X})$ has compact factors. Let $H^{(p)}$ be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $\mathcal{N}$ is a pro-étale cover of $\mathcal{N}/H^{(p)}$. Then $\mathcal{N}^{\mathrm{s}}/H^{(p)}$ is a Néron model of its generic fibre $\mathrm{Sh}_{H\times H^{(p)}}(G, \mathcal{X})$ over $O_{(v)}$.*

1.6. Main Theorem. *Suppose that $e(v) = 1$ and $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for $(G, \mathcal{X}, v)$. Let $\mathcal{N}^{\mathrm{m}}_{k(v)\mathrm{red}}$ be the reduced scheme of $\mathcal{N}^{\mathrm{m}}_{k(v)}$. Let $\mathcal{P}^{\mathrm{m}}$ be the normalization of the Zariski closure of $\mathcal{N}^{\mathrm{m}}_{k(v)\mathrm{red}}$ in $\mathcal{N}_{k(v)}$. If $p = 2$ we consider the following condition:*

(*) each abelian variety that is the pull back of the universal abelian scheme over $\mathcal{M}_r$ via a geometric point of $\mathcal{N}^{\mathrm{m}}_{k(v)}$, has p-rank 0.

(a) *If $p = 2$ we assume that the condition (*) holds. Then $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is a non-empty, open closed subscheme of $\mathcal{N}_{k(v)}$.*

(b) *Suppose that $\mathcal{N}_{E(G,\mathcal{X})} = \mathrm{Sh}_H(G, \mathcal{X})$ is a closed subscheme of $\mathcal{M}_{rE(G,\mathcal{X})}$. Then the $k(v)$-scheme $\mathcal{P}^{\mathrm{m}}$ is regular and formally smooth. Moreover, for each algebraically closed field k of characteristic p, the natural morphism $\mathcal{P}^{\mathrm{m}}_k \to \mathcal{M}_{rk}$ induces k-epimorphisms at the level of complete, local rings of residue field k (i.e., it is a formally closed embedding at all k-valued point of $\mathcal{P}^{\mathrm{m}}_k$).*

(c) *Suppose that $(G, \mathcal{X})$ has compact factors. If $p = 2$, we also assume that the condition (*) holds. Then $\mathcal{N}^{\mathrm{m}}$ is the disjoint union of an open closed subscheme $\mathcal{N}^{\mathrm{p}}$ of $\mathcal{N}$ and of the $E(G, \mathcal{X})$-scheme $\mathcal{N}_{E(G,\mathcal{X})} \setminus \mathcal{N}^{\mathrm{p}}_{E(G,\mathcal{X})}$ which is an open closed subscheme of $\mathcal{N}_{E(G,\mathcal{X})}$. Moreover $\mathcal{N}^{\mathrm{p}}$ is a pro-étale cover of a smooth, projective $O_{(v)}$-scheme.*

1.7. Main Corollary. *We assume that $e(v) = 1$, that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, and that $(G, \mathcal{X})$ has compact factors. If $p = 2$ we also assume that the condition 1.6 (*) holds.*

(a) *Then we have $\mathcal{N}^{\mathrm{m}} = \mathcal{N}^{\mathrm{s}} = \mathcal{N}$ and moreover $\mathcal{N}$ is the integral canonical model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ as defined in [Va1, Def. 3.2.3 6)].*

(b) *Let $H^{(p)}$ be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $K := H \times H^{(p)}$ is contained in $K(N)$ for some natural number N that is at least 3 and that is relatively prime to p; thus we have a natural finite morphism*

$$f(N) : \mathrm{Sh}_K(G, \mathcal{X}) \to \mathcal{A}_{r,1,NE(G,\mathcal{X})} = \mathrm{Sh}_{K(N)}(\boldsymbol{GSp}(W, \psi), \mathcal{Y})_{E(G,\mathcal{X})}.$$

Then the normalization $\mathcal{Q}$ of $\mathcal{A}_{r,1,NO_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_K(G, \mathcal{X})$ is a smooth, projective $O_{(v)}$-scheme that can be identified with $\mathcal{N}/H^{(p)}$ and that is the Néron model of $\mathrm{Sh}_K(G, \mathcal{X})$ over $O_{(v)}$.

1.8. On contents. We detail on the contents of this Part I. In Section 2 we list conventions, notations, and few basic properties that pertain to the injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$ and to Hodge cycles.

In Section 3 we include crystalline applications. In Subsections 3.1 to 3.3 we introduce basic notations and review three recent results that pertain to p-divisible groups and that play a central role in Sections 4 and 5. The results are: (i) *de Jong extension theorem* (see [dJ2]), (ii) *a motivic conjecture of Milne* proved in [Va4, Thm. 1.2], and (iii) a variant of *Faltings deformation theory*. In Subsection 3.4 we prove the Basic Theorem 1.5. Extra crystalline properties needed in Sections 4 and 5 are gathered in Subsection 3.5.

See Lemma 4.1 for a simple criterion on when the $k(v)$-scheme $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty. In Subsection 4.2 we apply Theorem 1.5 (a) and Lemma 4.1 to prove the existence of good regular, formally smooth integral models of $\mathrm{Sh}_{\tilde{H}}(G,\mathcal{X})$ over $O_{(v)}$ for a large class of maximal compact, open subgroups $\tilde{H}$ of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (the class includes all hyperspecial subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$). Corollary 4.2.3 can be viewed as a complete solution of Langlands' conjecture (see paragraph before Definitions 1.3.2) for Shimura varieties of Hodge type.

In Section 5 we use Lemma 2.2.4 (i.e., [Va3, Cor. 4.3]), [Va5], Theorem 3.1, and Subsection 3.4 to prove the Main Theorem 1.6 (see Subsections 5.1 to 5.7) and the Main Corollary 1.7 (see Subsection 5.8).

Appendices A and B review basic properties of affine group schemes and of p-divisible groups. Their Subsections are numbered as A1 and A2 and as B1 to B9. The reader ought to refer to these Subsections only when they are quoted in the main text. Modulo few notations of Subsection 2.1, the two Appendices are independent of the main text.

1.9. On literature and Parts II to IV. Referring to Theorem 1.5 (a), all ordinary points of $\mathcal{N}_{k(v)}$ belong to $\mathcal{N}^{\mathrm{s}}_{k(v)}$ (cf. [No, Cor. 3.8]). If $(G,\mathcal{X})$ has compact factors and $\mathcal{N}^{\mathrm{s}} \neq \mathcal{N}$, then Theorem 1.5 (c) provides Néron models over $O_{(v)}$ which are not projective and thus which are not among the Néron models obtained in [Va3, Prop. 4.4.1]. For $p \geq 5$, the Corollary 1.7 (a) was first obtained in [Va1, Rm. 3.2.12, Thms. 5.1 and 6.4.1]. If $p=2$, one can use either [Va3, Section 3] or [Va6, Thm. 1.5] to provide plenty of situations in which all hypotheses of Main Corollary 1.7 hold and in which the adjoint Shimura pair $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ of $(G,\mathcal{X})$ is simple and of any one of the following Shimura types A_n, B_n, C_n, $D_n^{\mathbb{H}}$, and $D_n^{\mathbb{R}}$ defined in [De2]. The works [MFK], [Dr], [Mo], [Zi], [LR], [Ko], and [Va1] to [Va8] are the most relevant for the existence of good smooth integral models of Shimura varieties of Hodge type. See also [HT, Section 5] for a translation of part of [Dr] in terms of the existence of good smooth integral models in arbitrary ramified mixed characteristic $(0,p)$ of very simple unitary Shimura varieties. Theorems 1.5 and 1.6 are also key steps in proving the deep conjectures [Re1, Conjs. B 3.7 and B 3.12] and [Re2, Conj. 1.6].

In the next two paragraphs we assume that the adjoint Shimura pair $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ is simple and that $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$. The cases which are not covered by the Main Corollary 1.7 are of three disjoint types:

(2COMP) $p=2$, $(G,\mathcal{X})$ has compact factors, and the condition 1.6 (*) does not hold;

(PEL) all simple factors of $G^{\mathrm{ad}}_{\mathbb{R}}$ are non-compact and $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ is of either A_n (with $n \geq 1$) or C_n (with $n \geq 1$) or $D_n^{\mathbb{H}}$ (with $n \geq 4$) type;

(SPINNONCOMP) all simple factors of $G^{\mathrm{ad}}_{\mathbb{R}}$ are non-compact and $(G^{\mathrm{ad}},\mathcal{X}^{\mathrm{ad}})$ is of either B_n (with $n \geq 3$) or $D_n^{\mathbb{R}}$ (with $n \geq 4$) type.

The goal of the Part II of the paper is to show that the Main Corollary 1.7 continues to hold in the (2COMP) case (i.e., for $p=2$ one does not need to assume that the condition 1.6 (*) holds). Besides this Part I, in order to achieve the goals of Part II one is only left to provide concrete examples for which the Milne conjecture mentioned in Subsection 1.8 holds for $p=2$ (see Remark 5.6.1). In the (PEL) case, the Shimura pair $(G^{\mathrm{ad}}, \mathcal{X}^{\mathrm{ad}})$ is the adjoint of a Shimura pair of PEL type and thus in Part III of the paper we will show that this case can be handled as in [Va7] and [Va8] (no new ideas are required). Part IV of the paper will use [Va1, Prop. 6.6.2] and the formalism of smooth toroidal compactifications and elementary inductions, in order to handle the (SPINNONCOMP) case even for $p \leq 3$. Thus Parts II to IV will complete the proof of the existence of integral canonical models defined in [Va1] (i.e., the proof of Milne conjecture of [Mi2, Conj. 2.7] and [Va1, Conj. 3.2.5]) for all Shimura varieties of Hodge type.

Part I brings completely new ideas in order to: (i) shorten and simplify [Va1], and (ii) to extend many parts of [Va1] that were worked out only for $p \geq 5$ to the case of small primes $p \in \{2,3\}$. Corollary 1.7 (b) corrects for $p \geq 5$ in most cases an error in the proof of [Va1, Prop. 3.2.3.2 ii)] that invalidated [Va1, Rm. 6.4.1.1 2) and most of Subsubsection 6.4.11]. This correction was started in [Va3, Rm. 4.6 (b)] and [Va7, Thm. 5.1 (c) and App. E.8] and it is continued here; it will be completed in the Part IV of the paper.

2. Preliminaries

In Subsection 2.1 we include some conventions and notations to be used throughout the paper. In Subsection 2.2 we study the injective map $f:(G,\mathcal{X}) \hookrightarrow (\mathbf{GSp}(W,\psi), \mathcal{Y})$. In Subsection 2.3 we consider $\mathbb{C}$-valued points of $\mathrm{Sh}(G,\mathcal{X})$ and different realizations of Hodge cycles on abelian schemes over reduced $\mathbb{Q}$–schemes.

2.1. Conventions and notations. We recall that p is a prime and that k is a perfect field of characteristic p. Let $\sigma := \sigma_k$ be the Frobenius automorphism of k, $W(k)$, and $B(k) := W(k)[\frac{1}{p}]$. Let O, M, and F be as in the beginning of Section 1. If $*$ or $*_O$ is either a morphism or an object of the category of O-schemes and if R is a commutative O-algebra, let $*_R$ be the pull back of $*$ or $*_O$ to the category of R-schemes. Let $Z(F)$, F^{ad}, and F^{der} denote the center, the adjoint group scheme, and the derived group scheme (respectively) of F. We have $F^{\mathrm{ad}} = F/Z(F)$. The group schemes $\mathbf{SL}_{nO}$, etc., are over O. If $F_1 \hookrightarrow F$ is a closed embedding monomorphism of group schemes over O, then we identify F_1 with its image in F and we consider intersections of subgroups of $F_1(O)$ with subgroups of $F(O)$. By the essential tensor algebra of $M \oplus M^*$ we mean the O-module

$$\mathcal{T}(M) := \oplus_{s,t \in \mathbb{N} \cup \{0\}} M^{\otimes s} \otimes_O M^{*\otimes t}.$$

Let $F^1(M)$ be a direct summand of M. Let $F^0(M) := M$ and $F^2(M) := 0$. Let $F^1(M^*) := 0$, $F^0(M^*) := \{y \in M^* | y(F^1(M)) = 0\}$, and $F^{-1}(M^*) := M^*$. Let $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ be the tensor product filtration of $\mathcal{T}(M)$ defined by the exhaustive, separated filtrations $(F^i(M))_{i \in \{0,1,2\}}$ and $(F^i(M^*))_{i \in \{-1,0,1\}}$ of M and M^* (respectively). We refer to $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ as the filtration of $\mathcal{T}(M)$ defined by $F^1(M)$.

We identify naturally $\mathrm{End}(M) = M \otimes_O M^*$ and $\mathrm{End}(\mathrm{End}(M)) = M^{\otimes 2} \otimes_O M^{*\otimes 2}$. Let $x \in O$ be a non-divisor of 0. A family of tensors of $\mathcal{T}(M[\frac{1}{x}]) = \mathcal{T}(M)[\frac{1}{x}]$ is denoted $(u_\alpha)_{\alpha\in\mathcal{J}}$, with $\mathcal{J}$ as the set of indexes. Let M_1 be another free O-module of finite rank. Let $(u_{1\alpha})_{\alpha\in\mathcal{J}}$ be a family of tensors of $\mathcal{T}(M_1[\frac{1}{x}])$ indexed also by the set $\mathcal{J}$. By an isomorphism $(M, (u_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (M_1, (u_{1\alpha})_{\alpha\in\mathcal{J}})$ we mean an O-linear isomorphism $M \tilde{\to} M_1$ that extends naturally to an O-linear isomorphism $\mathcal{T}(M[\frac{1}{x}]) \tilde{\to} \mathcal{T}(M_1[\frac{1}{x}])$ which takes u_α to $u_{1\alpha}$ for all $\alpha \in \mathcal{J}$. We emphasize that we denote two tensors or bilinear forms in the same way, provided they are obtained one from another via either a reduction modulo some ideal or a scalar extension.

The notations r, N, $\mathcal{A}_{r,1,N}$, $\mathcal{M}_r$, $\mu_h : \mathbb{G}_{m\mathbb{C}} \to G_{\mathbb{C}}$, $(\mathbf{GSp}(W,\psi), \mathcal{Y})$, L, $K(N)$, K_p, $E(G,\mathcal{X}) \hookrightarrow \mathbb{C}$, $\mathrm{Sh}(G,\mathcal{X})$, $\mathrm{Sh}_C(G,\mathcal{X}) = \mathrm{Sh}(G,\mathcal{X})/C$, v, $k(v)$, $e(v)$, $O_{(v)}$, $f : (G,\mathcal{X}) \hookrightarrow (\mathbf{GSp}(W,\psi), \mathcal{Y})$, $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, $G_{\mathbb{Z}_{(p)}}$, $H = K_p \cap G_{\mathbb{Q}_p}(\mathbb{Q}_p) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, $f_0 : \mathrm{Sh}(G,\mathcal{X}) \to \mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})_{E(G,\mathcal{X})}$, $f_p : \mathrm{Sh}_H(G,\mathcal{X}) \to \mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi), \mathcal{Y})_{E(G,\mathcal{X})}$, $\mathcal{N}$, and $\mathcal{N}^{\mathrm{s}}$ will be as in Subsections 1.1, 1.3, and 1.4. Let $d := \dim_{\mathbb{C}}(\mathcal{X})$ and $l := \dim(G)$; we have $d, l \in \mathbb{N} \cup \{0\}$. Let $(\mathcal{A}, \lambda_{\mathcal{A}})$ be the principally polarized abelian scheme over $\mathcal{N}$ which is the natural pull back of the universal principally polarized abelian scheme over $\mathcal{M}_r$.

2.2. On the injective map f. Let $H^{(p)}$ be an arbitrary compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $H \times H^{(p)} \leqslant K(N)$. As the morphism $f_0 : \mathrm{Sh}(G,\mathcal{X}) \to \mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})_{E(G,\mathcal{X})}$ is a closed embedding, the induced morphisms $f_p : \mathrm{Sh}_H(G,\mathcal{X}) \to \mathrm{Sh}_{K_p}(\mathbf{GSp}(W,\psi), \mathcal{Y})_{E(G,\mathcal{X})}$ and $\mathrm{Sh}_{H\times H^{(p)}}(G,\mathcal{X}) \to \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W,\psi), \mathcal{Y})_{E(G,\mathcal{X})}$ are pro-finite and finite (respectively). Thus we can speak about the normalization $\mathcal{Q}$ of $\mathcal{A}_{r,1,NO_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_{H\times H^{(p)}}(G,\mathcal{X})$. We recall that every $O_{(v)}$-scheme of finite type is excellent (for instance, cf. [Ma, (34.A) and (34.B)]). The $O_{(v)}$-scheme $\mathcal{A}_{r,1,NO_{(v)}}$ is quasi-projective (cf. property 1.1 (i)) and thus it is also excellent. Therefore the $O_{(v)}$-scheme $\mathcal{Q}$ is normal, quasi-projective, faithfully flat, finite over $\mathcal{A}_{r,1,N}$, and has a relative dimension equal to $\dim(\mathrm{Sh}_{H\times H^{(p)}}(G,\mathcal{X})) = \dim_{\mathbb{C}}(\mathcal{X}) = d$. Let $\mathcal{Q}^{\mathrm{s}}$ be the smooth locus of $\mathcal{Q}$ over $O_{(v)}$; it is an open subscheme of $\mathcal{Q}$. As $\mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})$ is a pro-étale cover of $\mathcal{A}_{r,1,N\mathbb{Q}} = \mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})/K(N)$, the group $K(N)$ acts freely on $\mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})$. Thus the subgroup $H \times H^{(p)}$ of $K(N)$ acts freely on $\mathrm{Sh}(\mathbf{GSp}(W,\psi), \mathcal{Y})$ and therefore also on $\mathrm{Sh}(G,\mathcal{X})$. Thus $\mathcal{Q}_{E(G,\mathcal{X})} = \mathrm{Sh}_{H\times H^{(p)}}(G,\mathcal{X})$ is a smooth $E(G,\mathcal{X})$-scheme and therefore it is the open subscheme $\mathcal{Q}^{\mathrm{s}}_{E(G,\mathcal{X})}$ of $\mathcal{Q}^{\mathrm{s}}$.

2.2.1. Proposition. *The following three properties hold:*

(a) *The $O_{(v)}$-scheme $\mathcal{N}$ is a pro-étale cover of $\mathcal{Q}$ and $\mathcal{Q}$ is the quotient of $\mathcal{N}$ by $H^{(p)}$.*

(b) *The morphism $\mathcal{N} \to \mathcal{M}_r$ is finite.*

(d) *If Z is a regular, formally smooth scheme over a discrete valuation ring O which is of index of ramification 1 and is a faithfully flat $O_{(v)}$-algebra, then each morphism $Z_{E(G,\mathcal{X})} \to \mathcal{N}_{E(G,\mathcal{X})}$ extends uniquely to a morphism $Z \to \mathcal{N}$ of $O_{(v)}$-schemes.*

Proof: Let $N_2 := N$. Let $N_1 \in N\mathbb{N}$ be relatively prime to p. For $i \in \{1,2\}$ we write $K(N_i) = K_p \times K(N_i)^{(p)}$, where the group $K(N_i)^{(p)}$ is a compact, open subgroup of $\mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})$. The scheme $\mathcal{M}_r$ is a pro-étale cover of $\mathcal{M}_r/K(N_i)^{(p)} = \mathcal{A}_{r,1,N_i}$. Let H_i be a compact, open subgroup of $G(\mathbb{A}_f^{(p)}) \cap K(N_i)^{(p)}$ such that $\mathrm{Sh}(G,\mathcal{X})$ is a pro-étale cover of $\mathrm{Sh}_{H\times H_i}(G,\mathcal{X})$. The morphism $\mathrm{Sh}_{H\times H_i}(G,\mathcal{X})_{\mathbb{C}} \to \mathcal{A}_{r,1,N_i\mathbb{C}}$ is of finite type and a formally closed embedding at each $\mathbb{C}$-valued point of $\mathrm{Sh}_{H\times H_i}(G,\mathcal{X})_{\mathbb{C}}$. Let $\mathcal{Q}_i$ be the normalization of $\mathcal{A}_{r,1,N_iO_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_{H\times H_i}(G,\mathcal{X})$; it is a finite $\mathcal{A}_{r,1,N_iO_{(v)}}$-scheme and a normal, quasi-projective, faithfully flat $O_{(v)}$-scheme of relative dimension d.

As N_2 divides N_1, we have $K(N_1)^{(p)} \leqslant K(N_2)^{(p)}$. We assume that H_1 is a normal subgroup of H_2. The natural morphism $q_{12} : \mathcal{Q}_1 \to \mathcal{Q}_2 \times_{\mathcal{A}_{r,1,N_2O_{(v)}}} \mathcal{A}_{r,1,N_1O_{(v)}}$ of normal schemes is finite. We check that $q_{12E(G,\mathcal{X})}$ is an open closed embedding. As $q_{12E(G,\mathcal{X})}$ is a finite, étale morphism between normal $E(G,\mathcal{X})$-schemes of finite type, it is enough to check that the map $q_{12}(\mathbb{C}) : \mathcal{Q}_1(\mathbb{C}) \to \mathcal{Q}_2(\mathbb{C}) \times_{\mathcal{A}_{r,1,N_2O_{(v)}}(\mathbb{C})} \mathcal{A}_{r,1,N_1O_{(v)}}(\mathbb{C})$ is injective. We have

$$\mathrm{Sh}_{K_p\times H_i}(\mathbf{GSp}(W,\psi),\mathcal{Y})(\mathbb{C}) = \mathbf{GSp}(L,\psi)(\mathbb{Z}_{(p)})\backslash(\mathcal{Y}\times \mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})/H_i)$$

(for instance, cf. [Mi3, Prop. 4.11]). Also we have a natural disjoint union decomposition

$$\mathrm{Sh}_{H\times H_i}(G,\mathcal{X})(\mathbb{C}) = \cup_{[g_j]\in G(\mathbb{Q})\backslash G(\mathbb{Q}_p)/H} C_j\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)})/H_i),$$

where $g_j \in G(\mathbb{Q}_p)$ is a representative of the class $[g_j] \in G(\mathbb{Q})\backslash G(\mathbb{Q}_p)/H$ and where the group $C_j := G(\mathbb{Q}) \cap g_jHg_j^{-1}$ does not depend on $i \in \{1,2\}$. As we have an identity $\mathbf{GSp}(W,\psi)(\mathbb{Q}_p) = \mathbf{GSp}(W,\psi)(\mathbb{Q})K_p$ (cf. [Mi3, Lem. 4.9]), we can write $g_j = a_jh_j$, where $a_j \in \mathbf{GSp}(W,\psi)(\mathbb{Q})$ and $h_j \in K_p$. Thus

$$C_j \leqslant \mathbf{GSp}(W,\psi)(\mathbb{Q})\cap g_jK_pg_j^{-1} = \mathbf{GSp}(W,\psi)(\mathbb{Q})\cap a_jK_pa_j^{-1} = a_j\mathbf{GSp}(L,\psi)(\mathbb{Z}_{(p)})a_j^{-1} =: C_j^{\mathrm{big}}.$$

We have $C_j = G(\mathbb{Q}) \cap C_j^{\mathrm{big}}$. This is so as $g_jHg_j^{-1}$ is the group of $\mathbb{Z}_p$-valued points of the Zariski closure of G in $a_j\mathbf{GSp}(L,\psi)_{\mathbb{Z}_{(p)}}a_j^{-1}$.

To show that the map $q_{12}(\mathbb{C})$ is injective, it suffices to show that each one of the following commutative diagrams indexed by j

$$\begin{array}{ccc} C_j\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)})/H_1) & \xrightarrow{s_1} & \mathbf{GSp}(L,\psi)(\mathbb{Z}_{(p)})\backslash(\mathcal{Y}\times \mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})/H_1) \\ {\scriptstyle\pi_{12}}\downarrow & & \downarrow{\scriptstyle\pi_{12}^{\mathrm{big}}} \\ C_j\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)})/H_2) & \xrightarrow{s_2} & \mathbf{GSp}(L,\psi)(\mathbb{Z}_{(p)})\backslash(\mathcal{Y}\times \mathbf{GSp}(W,\psi)(\mathbb{A}_f^{(p)})/H_2), \end{array}$$

is such that the maps π_{12} and s_1 define an injective map of $C_j\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)})/H_1)$ into the fibre product of s_2 and π_{12}^{big}. Here the maps π_{12} and π_{12}^{big} are the natural projections. The maps s_1 and s_2 are defined by the rule: the equivalence class $[h,g]$, where $h \in \mathcal{X}$ and $g \in G(\mathbb{A}_f^{(p)})$, is mapped to the equivalence class $[a_j^{-1}h, a_j^{-1}g]$. Thus the fact that π_{12} and s_1 define an injective map of $C_j\backslash(\mathcal{X}\times G(\mathbb{A}_f^{(p)})/H_1)$ into the fibre product of s_2 and π_{12}^{big} is a direct consequence of the identity $C_j = G(\mathbb{Q}) \cap C_j^{\mathrm{big}}$. Thus $q_{12}(\mathbb{C})$ is injective.

Therefore $q_{12E(G,\mathcal{X})}$ is an open closed embedding. As q_{12} is also a finite morphism of normal, flat $O_{(v)}$-schemes of finite type, q_{12} itself is an open closed embedding. Thus

$\mathcal{Q}_1$ is an étale cover of $\mathcal{Q}_2$ that in characteristic 0 is an étale cover which (as $H_1 \lhd H_2$) induces Galois covers between connected components. Therefore $\mathcal{Q}_1$ is an étale cover of $\mathcal{Q}_2$ which induces Galois covers between connected components. By allowing H_1 to vary among the normal, open subgroups of H_2 and by a natural passage to limits, we get that $\mathcal{N}$ is a pro-étale cover of $\mathcal{Q}_2$ and that $\mathcal{Q}_2 = \mathcal{N}/H_2$. Thus by remarking that $H = H_2$ and $\mathcal{Q} = \mathcal{Q}_2$, we get that (a) holds.

As each morphism $q_{12} : \mathcal{Q}_1 \to \mathcal{Q}_2 \times_{\mathcal{A}_{r,1,N_2 O_{(v)}}} \mathcal{A}_{r,1,N_1 O_{(v)}}$ is an open closed embedding, by allowing H_1 to vary through all normal, open subgroups of H_2 we get that $\mathcal{N}$ is an open closed subscheme of $\mathcal{Q}_2 \times_{\mathcal{A}_{r,1,N_2 O_{(v)}}} \mathcal{M}_r$. Thus $\mathcal{N}$ is a finite $\mathcal{M}_r$-scheme i.e., (b) also holds.

To prove (c), we recall that Z is a healthy regular scheme in the sense of either [Va1, Def. 3.2.1 2)] or [Va2] (cf. [Va2, Thm. 1.3]). Thus (c) is implied by [Va1, Ex. 3.2.9 and Prop. 3.4.1], cf. the definitions [Va1, Def. 3.2.3 2), 3), and 6)]. $\square$

2.2.2. Lemma. *The scheme $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}$ and $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})}$. Moreover, if $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is a non-empty scheme, then $\mathcal{N}^{\mathrm{s}}$ together with the resulting action of $G(\mathbb{A}_f^{(p)})$ on it is a regular, formally smooth integral model of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$.*

Proof: As $\mathcal{N}$ is a pro-étale cover of the excellent, quasi-projective $O_{(v)}$-scheme $\mathcal{Q}$ (see Proposition 2.2.1 (a)), $\mathcal{N}^{\mathrm{s}}$ is a pro-étale cover of $\mathcal{Q}^{\mathrm{s}}$. Thus $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}$. As $\mathcal{Q}_{E(G,\mathcal{X})} = \mathcal{Q}^{\mathrm{s}}_{E(G,\mathcal{X})}$, we have $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})} = \mathcal{N}_{E(G,\mathcal{X})}$. The open subscheme $\mathcal{N}^{\mathrm{s}}$ of $\mathcal{N}$ is $G(\mathbb{A}_f^{(p)})$-invariant. As $G(\mathbb{A}_f^{(p)})$ acts continuously on $\mathcal{N}$, it also acts continuously on $\mathcal{N}^{\mathrm{s}}$. Thus if the scheme $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty, then $\mathcal{N}^{\mathrm{s}}$ together with the resulting continuous action of $G(\mathbb{A}_f^{(p)})$ on it is a regular, formally smooth integral model of $\mathrm{Sh}_H(G,\mathcal{X})$ over $O_{(v)}$. $\square$

2.2.3. Fact. *Suppose that there exists a simple factor G_1 of $G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$ which is an $\boldsymbol{SO}_{2n+1}$ group for some $n \in \mathbb{N}$. Let G_2 be the semisimple, normal subgroup of $G_{\overline{\mathbb{Q}}}$ whose adjoint is naturally identified with G_1. Then G_2 is a $\boldsymbol{Spin}_{2n+1}$ group.*

Proof: The representation of $\mathrm{Lie}(G_2)$ on $W \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is non-trivial and its irreducible subrepresentations are associated to the weight ϖ_n of the B_n Lie type, cf. [Mi3, p. 456]. This implies that G_2 is a $\mathbf{Spin}_{2n+1}$ group. $\square$

2.2.4. Lemma. *Suppose that $(G,\mathcal{X})$ has compact factors. Then $\mathcal{Q}$ is a projective $O_{(v)}$-scheme.*

Proof: Let G' be the smallest subgroup of G such that every element $h \in \mathcal{X}$ factors through $G'_{\mathbb{R}}$. It is a normal, reductive subgroup of G that contains G^{der}; thus we have $G'^{\mathrm{ad}} = G^{\mathrm{ad}}$. Let $h' \in \mathcal{X}$ be an element such that G' is the smallest subgroup of $\mathbf{GL}_W$ with the property that h' factors through $G'_{\mathbb{R}}$. We can assume that the $\mathbb{C}$-valued point $[h', 1_W] \in \mathrm{Sh}(G,\mathcal{X})/H \times H^{(p)}$ is definable over a number field (here 1_W is the identity element of $G(\mathbb{A}_f)/H^{(p)}$) and that ψ is a principal polarization of the Hodge $\mathbb{Z}$-structure on L defined by h'. Thus G' is the Mumford–Tate group of the principally polarized Hodge $\mathbb{Z}$–structure on L defined by h' and ψ and this principally polarized Hodge $\mathbb{Z}$–structure is associated naturally to a principally polarized abelian scheme over a number field.

Let $\mathcal{X}'$ be the $G'(\mathbb{R})$-conjugacy class of h'. The pair $(G',\mathcal{X}')$ is a Shimura pair whose reflex field and dimension are also $E(G,\mathcal{X})$ and d (respectively). Let $H' := H \cap G'(\mathbb{Q}_p)$

and $H'^{(p)} := H^{(p)} \cap G'(\mathbb{A}_f^{(p)})$. As $G'^{\mathrm{ad}} = G^{\mathrm{ad}}$, the Shimura pair $(G', \mathcal{X}')$ also has compact factors. Thus the normalization $\mathcal{Q}'$ of $\mathcal{A}_{r,1,NO_{(v)}}$ in $\mathrm{Sh}_{H' \times H'^{(p)}}(G', \mathcal{X}')$ is a projective $O_{(v)}$-scheme, cf. [Va3, Cor. 4.3].

The Shimura variety $\mathrm{Sh}(G', \mathcal{X}')$ is a closed subscheme of $\mathrm{Sh}(G, \mathcal{X})$ of dimension d and therefore it is an open closed subscheme of $\mathrm{Sh}(G, \mathcal{X})$. Thus each connected component of the normalization of $\mathcal{A}_{r,1,NO_{(v)}}$ (equivalently of $\mathcal{Q}$) in the ring of fractions of $\mathrm{Sh}(G, \mathcal{X})$ is a $G(\mathbb{A}_f)$-translation of a connected component of the normalization of $\mathcal{A}_{r,1,NO_{(v)}}$ (equivalently of $\mathcal{Q}'$) in the ring of fractions of $\mathrm{Sh}(G', \mathcal{X}')$.

As $\mathcal{Q}'$ of $\mathcal{A}_{r,1,N}$ is a projective $O_{(v)}$-scheme, from the last paragraph we get directly that $\mathcal{Q}$ is a projective $O_{(v)}$-scheme. $\square$

2.3. Tensors. Let $\mathfrak{T} : \mathrm{End}(W) \otimes_{\mathbb{Q}} \mathrm{End}(W) \to \mathbb{Q}$ be the trace form on $\mathrm{End}(W)$. If κ is a field of characteristic 0 and if $\dagger$ is a reductive subgroup of $\mathbf{GL}_{W \otimes_{\mathbb{Q}} \kappa}$, then the restriction of $\mathfrak{T}$ to $\mathrm{Lie}(\dagger)$ is non-degenerate (cf. A2 (b)). Let $\pi_\dagger$ be the projector of $\mathrm{End}(W \otimes_{\mathbb{Q}} \kappa)$ on $\mathrm{Lie}(\dagger)$ along the perpendicular on $\mathrm{Lie}(\dagger)$ with respect to $\mathfrak{T}$. If G_κ normalizes $\dagger$, then G_κ fixes $\pi_\dagger$.

The image of each $h \in \mathcal{X}$ contains $Z(\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{R}})$. This implies that $Z(\mathbf{GL}_W) \leqslant G$. Thus each tensor of $\mathcal{T}(W^*)$ fixed by G belongs to the direct summand $\oplus_{u \in \mathbb{N} \cup \{0\}} W^{* \otimes u} \otimes_{\mathbb{Q}} W^{\otimes u}$ of $\mathcal{T}(W^*)$. Let

$$(v_\alpha)_{\alpha \in \mathcal{J}}$$

be a family of tensors in spaces of the form $W^{* \otimes u} \otimes_{\mathbb{Q}} W^{\otimes u} \subseteq \mathcal{T}(W^*)$ with $u \in \mathbb{N} \cup \{0\}$, that contains π_G, and that has the property (cf. [De3, Prop. 3.1 c)]) that G is the subgroup of $\mathbf{GL}_W$ which fixes v_α for all $\alpha \in \mathcal{J}$.

2.3.1. Complex manifolds. For a smooth $\mathbb{C}$-scheme Y, let Y^{an} be the complex manifold associated naturally to Y. It is well known that for every $u \in \mathbb{N}$ and for every abelian scheme $\pi_C : C \to Y$, we have a natural isomorphism

$$R^u \pi_{C^{\mathrm{an}}*}(\mathbb{C}) \xrightarrow{\sim} R^u \pi_{C^{\mathrm{an}}*}(\Omega^*_{C^{\mathrm{an}}/Y^{\mathrm{an}}})^{\nabla_C^{\mathrm{an}}} \tag{1}$$

of complex sheaves on Y^{an}. Here $\pi_{C^{\mathrm{an}}*} : C^{\mathrm{an}} \to Y^{\mathrm{an}}$ is the morphism of complex manifolds associated naturally to π_C and ∇_C^{an} is the connection on $R^u \pi_{C^{\mathrm{an}}*}(\Omega^*_{C^{\mathrm{an}}/Y^{\mathrm{an}}})$ induced by the Gauss–Manin connection on $R^u \pi_{C*}(\Omega^*_{C/Y})$.

2.3.2. Hodge cycles. We will use the terminology of [De3] on Hodge cycles on an abelian scheme B_X over a reduced $\mathbb{Q}$–scheme X. Thus we write each Hodge cycle v on B_X as a pair $(v_{\mathrm{dR}}, v_{\acute{e}t})$, where v_{dR} and $v_{\acute{e}t}$ are the de Rham and the étale component of v (respectively). The étale component $v_{\acute{e}t}$ as its turn has an l-component $v_{\acute{e}t}^l$, for each rational prime l.

In what follows we will be interested only in Hodge cycles on B_X that involve no Tate twists and that are tensors of different essential tensor algebras. Accordingly, if X is the spectrum of a field E, then in applications $v_{\acute{e}t}^p$ will be a suitable $\mathrm{Gal}(\overline{E}/E)$-invariant tensor of $\mathcal{T}(H^1_{\acute{e}t}(B_{\overline{X}}, \mathbb{Q}_p))$, where $\overline{X} := \mathrm{Spec}(\overline{E})$. If moreover $\overline{E}$ is a subfield of $\mathbb{C}$, then we will also use the Betti realization v_B of v: it is a tensor of $\mathcal{T}(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}, \mathbb{Q}))$ that corresponds to v_{dR} (resp. to $v_{\acute{e}t}^l$) via the canonical isomorphism that relates the Betti cohomology of $(B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}$ with $\mathbb{Q}$–coefficients with the de Rham (resp. the $\mathbb{Q}_l$

étale) cohomology of $B_{\overline{X}}$ (see [De3, Section 2]). We recall that v_B is also a tensor of the F^0-filtration of the Hodge filtration of $\mathcal{T}(H^1((B_X \times_X \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}}, \mathbb{C}))$.

2.3.3. On $\mathcal{A}_{E(G,\mathcal{X})}$. The choice of the $\mathbb{Z}$-lattice L of W and of the family of tensors $(v_\alpha)_{\alpha\in\mathcal{J}}$ allows a moduli interpretation of $\mathrm{Sh}(G,\mathcal{X})$ (see [De1], [De2], [Mi3], and [Va1, Subsection 4.1 and Lem. 4.1.3]). For instance, $\mathrm{Sh}(G,\mathcal{X})(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathcal{X} \times G(\mathbb{A}_f))$ is the set of isomorphism classes of principally polarized abelian varieties over $\mathbb{C}$ of dimension r, that carry a family of Hodge cycles indexed by $\mathcal{J}$, that have compatible level-N symplectic similitude structures for every $N \in \mathbb{N}$, and that satisfy some additional axioms. This interpretation endows the abelian scheme $\mathcal{A}_{E(G,\mathcal{X})}$ over $\mathcal{N}_{E(G,\mathcal{X})}$ with a family $(w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}}$ of Hodge cycles; all realizations of pulls back of $w_\alpha^{\mathcal{A}}$ via $\mathbb{C}$-valued points of $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})}$ correspond naturally to v_α.

2.3.4. Lemma. *Let $w \in \mathrm{Sh}(G,\mathcal{X})(\mathbb{C})$. We denote also by w the $\mathbb{C}$-valued point of $\mathcal{N}$ defined by w; thus we can define $(A_w, \lambda_{A_w}) := w^*((\mathcal{A}, \lambda_{\mathcal{A}}))$. Let u_α^w (resp. t_α^w) be the p-component of the étale component (resp. be the de Rham component) of the Hodge cycle $w^*(w_\alpha^{\mathcal{A}})$ on A_w. We have:*

(a) *There exist isomorphisms $(H^1_{\acute{e}t}(A_w, \mathbb{Z}_p), (u_\alpha^w)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, (v_\alpha)_{\alpha\in\mathcal{J}})$ that take the perfect bilinear form on $H^1_{\acute{e}t}(A_w, \mathbb{Z}_p)$ defined by λ_{A_w} to a $\mathbb{G}_{m\mathbb{Z}_p}(\mathbb{Z}_p)$-multiple of the perfect bilinear form ψ^* on $L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ defined by ψ.*

(b) *There exists isomorphisms $(H^1_{\mathrm{dR}}(A_w, \mathbb{C}), (t_\alpha^w)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha\in\mathcal{J}})$.*

Proof: We write $w = [h_w, g_w] \in \mathrm{Sh}(G,\mathcal{X})(\mathbb{C}) = G(\mathbb{Q})\backslash(\mathcal{X} \times G(\mathbb{A}_f))$, where $h_w \in \mathcal{X}$ and $g_w \in G(\mathbb{A}_f)$. From the standard moduli interpretation of $\mathrm{Sh}(G,\mathcal{X})(\mathbb{C})$ applied to $w \in \mathrm{Sh}(G,\mathcal{X})(\mathbb{C})$ we get (see [Di1], [Mi2], [Mi3], and [Va1, p. 454]) that the complex manifold A_w^{an} associated to A_w is $L_w\backslash W \otimes_{\mathbb{Q}} \mathbb{C}/F_w^{0,-1}$, where

(i) L_w is the $\mathbb{Z}$-lattice of W defined uniquely by the identity $L_w \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = g_w(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$;

(ii) $W \otimes_{\mathbb{Q}} \mathbb{C} = F_w^{0,-1} \oplus F_w^{-1,0}$ is the usual Hodge decomposition of the Hodge $\mathbb{Q}$–structure on W defined by $h_w \in \mathcal{X}$;

(iii) the principal polarization λ_{A_w} of A_w is defined naturally by a uniquely determined (non-zero) rational multiple of ψ;

(iv) under the canonical identifications $H^1_{\mathrm{dR}}(A_w/\mathbb{C}) = H^1_{\mathrm{dR}}(A_w^{\mathrm{an}}/\mathbb{C}) = W^* \otimes_{\mathbb{Q}} \mathbb{C}$, the tensor t_α^w gets identified with v_α for all $\alpha \in \mathcal{J}$.

Thus $(H^1_{\acute{e}t}(A_w, \mathbb{Z}_p), (u_\alpha^w)_{\alpha\in\mathcal{J}})$ is identified naturally with $(L_w^* \otimes_{\mathbb{Z}} \mathbb{Z}_p, (v_\alpha)_{\alpha\in\mathcal{J}})$ and therefore also with a $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$-conjugate of $(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, (v_\alpha)_{\alpha\in\mathcal{J}})$. Part (a) follows from this and from the existence of the rational multiple of ψ mentioned in the property (iii). Part (b) is implied by the property (iv). $\square$

2.3.5. Lemma. *Let $m \in \mathbb{N} \cup \{0\}$. Let $\mathcal{R}_1 := \mathbb{C}[[x_1, \ldots, x_m]]$, where $x_1, \ldots, x_m$ are independent variables. Let $\mathcal{I}_1 := (x_1, \ldots, x_m)$ be the maximal ideal of $\mathcal{R}_1$. Let $s \in \mathbb{N}$. Let $A_{w,s}$ be an abelian scheme over $\mathcal{R}_1/\mathcal{I}_1^s$ that is a deformation of A_w (i.e., we have $A_w = A_{w,s} \times_{\mathrm{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)} \mathrm{Spec}(\mathcal{R}_1/\mathcal{I}_1)$). Then there exists a unique isomorphism*

$$I_{w,s} : H^1_{\mathrm{dR}}(A_{w,s}/\mathcal{R}_1/\mathcal{I}_1^s) \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1/\mathcal{I}_1^s$$

that has the following two properties:

(i) *it lifts (i.e., modulo $\mathcal{I}_1/\mathcal{I}_1^s$ is) the identity automorphism of $H^1_{\mathrm{dR}}(A_w/\mathbb{C})$;*

(ii) *under it, the Gauss–Manin connection on $H^1_{\mathrm{dR}}(A_{w,s}/\mathcal{R}_1/\mathcal{I}_1^s)$ becomes isomorphic to the flat connection δ on $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R}_1/\mathcal{I}_1^s$ that annihilates $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes 1$.*

Proof: The uniqueness of $I_{w,s}$ is implied by the fact that $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes 1$ is the set of elements of $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R}_1/\mathcal{I}_1^s$ that are annihilated by δ. We consider an abelian scheme $\pi_{A_Y}: A_Y \to Y$ over a smooth $\mathbb{C}$-scheme Y which is a global deformation of $A_{w,s} \to \mathrm{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$. Let Z^{an} be a simply connected open submanifold of Y^{an} that contains the $\mathbb{C}$-valued point defined naturally by A_w. We identify naturally $\mathrm{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$ with a complex subspace of Y^{an} and thus also of Z^{an}. We apply formula (1) with $u=1$ and $C=A_Y$. The pull back of $R^1\pi_{A_Y^{\mathrm{an}}*}(\mathbb{C})$ to Z^{an} is a constant sheaf on Z^{an}. Thus by pulling back formula (1) to the complex subspace $\mathrm{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$ of Z^{an}, we get directly the existence of $I_{w,s}$. $\square$

2.3.6. Corollary. *Let m, $\mathcal{R}_1$, and $\mathcal{I}_1$ be as in Lemma 2.3.5 . Let $A_{w,\infty}$ be an abelian scheme over $\mathcal{R}_1$ that is a deformation of A_w. Then there exists a unique isomorphism*

$$I_{w,\infty}: H^1_{\mathrm{dR}}(A_{w,\infty}/\mathcal{R}_1) \xrightarrow{\sim} H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R}_1$$

that has the following two properties:

(i) *it lifts (i.e., modulo $\mathcal{I}_1$ is) the identity automorphism of $H^1_{\mathrm{dR}}(A_w/\mathbb{C})$;*

(ii) *under it, the Gauss–Manin connection on $H^1_{\mathrm{dR}}(A_{w,\infty}/\mathcal{R}_1)$ becomes isomorphic to the flat connection δ on $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes_{\mathbb{C}}\mathcal{R}_1$ that annihilates $H^1_{\mathrm{dR}}(A_w/\mathbb{C})\otimes 1$.*

If $w_\alpha^{\mathcal{R}_1}$ (resp. $\lambda_{A_{w,\infty}}$) is a Hodge cycle on (resp. a principal polarization of) $A_{w,\infty}$ that lifts the Hodge cycle $w^*(w_\alpha^{\mathcal{A}})$ on A_w (resp. lifts the principal polarization λ_{A_w} of A_w), then the isomorphism $I_{w,\infty}: \mathcal{T}(H^1_{\mathrm{dR}}(A_{w,\infty}/\mathcal{R}_1)) \xrightarrow{\sim} \mathcal{T}(H^1_{\mathrm{dR}}(A_w/\mathbb{C}))\otimes_{\mathbb{C}}\mathcal{R}_1$ induced naturally by $I_{w,\infty}$ takes the de Rham realization of $w_\alpha^{\mathcal{R}_1}$ (resp. of $\lambda_{A_{w,\infty}}$) to t_α^w (resp. to the de Rham realization of λ_{A_w}).

Proof: The existence and the uniqueness of $I_{w,\infty}$ follows from Lemma 2.3.5 by taking $s\to\infty$. We denote also by δ the flat connection on $\mathcal{T}(H^1_{\mathrm{dR}}(A_w/\mathbb{C}))\otimes_{\mathbb{C}}\mathcal{R}_1$ induced by δ (i.e., which annihilates $\mathcal{T}(H^1_{\mathrm{dR}}(A_w/\mathbb{C}))\otimes 1$). It is well known that each de Rham component of a Hodge cycle on $A_{w,\infty}$ is annihilated by the Gauss–Manin connection on $\mathcal{T}(H^1_{\mathrm{dR}}(A_{w,\infty}/\mathcal{R}_1))$. [Argument: this follows from [De3, Prop. 2.5] via a natural algebraization process]. Thus $I_{w,\infty}(w_\alpha^{\mathcal{R}_1})$ and t_α^w are tensors of $\mathcal{T}(H^1_{\mathrm{dR}}(A_w/\mathbb{C}))\otimes_{\mathbb{C}}\mathcal{R}_1$ which are annihilated by the flat connection δ on $\mathcal{T}(H^1_{\mathrm{dR}}(A_w/\mathbb{C}))\otimes_{\mathbb{C}}\mathcal{R}_1$ and which modulo $\mathcal{I}_1$ coincide. Thus the two tensors coincide i.e., we have $I_{w,\infty}(w_\alpha^{\mathcal{R}_1}) = t_\alpha^w$. A similar argument shows that $I_{w,\infty}$ takes $\lambda_{A_{w,\infty}}$ to the de Rham realization of λ_{A_w}. $\square$

3. Crystalline applications

Theorem 3.1 recalls a variant of the main result of [dJ2]. In Subsection 3.2 we first introduce several notations needed to prove Theorems 1.5 and 1.6 and then we apply the main result of [Va4] in the form recalled in B3. In Subsection 3.3 we apply the deformation theory of [Fa2, Section 7]. In Subsection 3.4 we prove Basic Theorem 1.5. In Subsection 3.5 we list few simple crystalline properties that are needed in Sections 4 and 5.

For (crystalline or de Rham) Fontaine comparison theory we refer to [Fo], [Fa2, Section 5], and [Va4]; see also B2 and B9. Let the field k be as in Subsection 2.1. As the Verschiebung maps of p-divisible groups will not be mentioned at all in what follows, we will use the terminology F-crystals (resp. filtered F-crystals) associated to p-divisible groups over k, $k[[x]]$, or $k((x))$ (resp. over $W(k)$ or $W(k)[[x]]$) instead of the terminology Dieudonné F-crystals (resp. filtered Dieudonné F-crystals) used in [BBM, Ch. 3], [BM, Chs. 2 and 3], or [dJ1].

Let x be an independent variable. The simplest form of [dJ2, Thm. 1.1] says:

3.1. Theorem (de Jong). *The natural functor from the category of non-degenerate F-crystals over Spec$(k[[x]])$ to the category of non-degenerate F-crystals over Spec$(k((x)))$ is fully faithful.*

For the notion non-degenerate crystal, [dJ2] refers to [Sa, 3.1.1, p. 331]. In this paper we only use the facts that F-crystals of abelian schemes over $\mathrm{Spec}(k[[x]])$ are non-degenerate and that non-degenerate F-crystals are stable under tensor products and duals.

3.2. Basic setting. From now on until the end, the field k will be assumed to be algebraically closed and we will use the notations of Subsection 2.1. Let $z \in \mathcal{N}(W(k))$. Let

$$(A, \lambda_A, (w_\alpha)_{\alpha\in\mathcal{J}}) := z^*(\mathcal{A}, \lambda_{\mathcal{A}}, (w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}}).$$

Let

$$(M, F^1, \phi, \psi_M)$$

be the principally quasi-polarized filtered F-crystal over k of the principally quasi-polarized p-divisible group (D, λ_D) of (A, λ_A). Thus ψ_M is a perfect alternating form on the free $W(k)$-module M of rank $2r$, F^1 is a maximal isotropic submodule of M with respect to ψ_M, the pair (M, ϕ) is a Dieudonné module, and for $a, b \in M$ we have $\psi_M(\phi(a) \otimes \phi(b)) = p\sigma(\psi_M(a\otimes b))$. The σ-linear automorphism ϕ of $M[\frac{1}{p}]$ acts on $M^*[\frac{1}{p}]$ by mapping $e \in M^*[\frac{1}{p}]$ to $\sigma \circ e \circ \phi^{-1} \in M^*[\frac{1}{p}]$ and it acts on $\mathcal{T}(M)[\frac{1}{p}]$ in the natural tensor product way.

Let t_α and u_α be the de Rham component of w_α and the p-component of the étale component of w_α (respectively). If $(F^i(\mathcal{T}(M)))_{i\in\mathbb{Z}}$ is the filtration of $\mathcal{T}(M)$ defined by F^1, then we have $t_\alpha \in F^0(\mathcal{T}(M))[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$. Let $\mathcal{G}$ be the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$.

It is known that w_α is a de Rham cycle i.e., t_α and u_α correspond to each other via de Rham and thus also the crystalline Fontaine comparison theory. If $A_{B(k)}$ is definable over a number field contained in $B(k)$, then this was known since long time (for instance, see [Bl, Thm. (0.3)]). The general case follows from loc. cit. and [Va1, Principle B of 5.2.16] (in [Va1, Subsection 5.2] an odd prime is used; however the proof of [Va1, Principle B of 5.2.16] applies to all primes). In particular, we have $\phi(t_\alpha) = t_\alpha$ for all $\alpha \in \mathcal{J}$.

Let $\mu : \mathbb{G}_{mW(k)} \to \mathbf{GL}_M$ be the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [Wi, p. 512]. The cocharacter μ acts on F^1 via the inverse of the identical character of $\mathbb{G}_{mW(k)}$ and it fixes a direct supplement F^0 of F^1 in M; therefore we have $M = F^1 \oplus F^0$. Moreover, μ fixes each tensor t_α (cf. the functorial aspects of [Wi, p. 513]). Thus μ factors through $\mathcal{G}$. Let

$$\mu : \mathbb{G}_{mW(k)} \to \mathcal{G}$$

be the resulting factorization. We emphasize that in connection to different Kodaira–Spencer maps, in what follows we will identity naturally $\mathrm{Hom}(F^1, F^0)$ with the direct summand $\{e \in \mathrm{End}(M) | e(F^0) = 0 \text{ and } e(F^1) \subseteq F^0\}$ of $\mathrm{End}(M)$.

3.2.1. Lemma. *The rank of the direct summand $Lie(\mathcal{G}_{B(k)}) \cap Hom(F^1, F^0)$ of $Lie(\mathcal{G}_{B(k)}) \cap End(M)$ is d.*

Proof: To prove the Lemma we can assume that k has countable transcendental degree; thus there exists an $O_{(v)}$-monomorphism $W(k) \hookrightarrow \mathbb{C}$. Let $\mathcal{F}_{B(k)}$ be the normalizer of $F^1[\frac{1}{p}]$ in $\mathcal{G}_{B(k)}$. The subgroup $\mathcal{F}_{B(k)}$ of $\mathcal{G}_{B(k)}$ is parabolic and its Lie algebra is equal to $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \{e \in \mathrm{End}(M)[\frac{1}{p}] | e(F^1[\frac{1}{p}]) \subseteq F^1[\frac{1}{p}]\}$. As μ factors through $\mathcal{G}$, we have a direct sum decomposition $\mathrm{Lie}(\mathcal{G}_{B(k)}) = \mathrm{Lie}(\mathcal{F}_{B(k)}) \oplus (\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1[\frac{1}{p}], F^0[\frac{1}{p}]))$ of $B(k)$-vector spaces. Thus the rank of $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)$ is $\dim_{B(k)}(\mathrm{Lie}(\mathcal{G}_{B(k)})) - \dim_{B(k)}(\mathrm{Lie}(\mathcal{F}_{B(k)}))$ and therefore it is also equal to $\dim(\mathcal{G}_{B(k)}/\mathcal{F}_{B(k)})$.

We will use the notations of the proof of Lemma 2.3.4 for a point $w \in \mathrm{Sh}(G, \mathcal{X})(\mathbb{C})$ that lifts the $\mathbb{C}$-valued point of $\mathcal{N}_{E(G,\mathcal{X})}$ defined naturally by $z_{B(k)}$ and by the $O_{(v)}$-monomorphism $W(k) \hookrightarrow \mathbb{C}$. Let $W^* \otimes_{\mathbb{Q}} \mathbb{C} = F_w^{1,0} \oplus F_w^{0,1}$ be the Hodge decomposition defined by $h_w \in \mathcal{X}$ (it is the dual of the Hodge decomposition of the property (ii) of the proof of Lemma 2.3.4). We have a natural isomorphism $(M \otimes_{W(k)} \mathbb{C}, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ that takes $F^1 \otimes_{W(k)} \mathbb{C}$ to $F_w^{1,0}$, cf. B9 and Lemma 2.3.4 (b). Thus we have an identity $\dim(\mathcal{G}_{B(k)}/\mathcal{F}_{B(k)}) = \dim(G_{\mathbb{C}}/P_w)$, where P_w is the parabolic subgroup of $G_{\mathbb{C}}$ which is the normalizer of $F_w^{1,0}$ (or of $F_w^{-1,0}$) in $G_{\mathbb{C}}$. But $G_{\mathbb{C}}/P_w$ is the compact dual of any connected component of $\mathcal{X}$ and thus has dimension d.

We conclude that the rank of $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)$ is d. □

3.2.2. Key Theorem. *If $p = 2$ we assume that $G_{\mathbb{Z}_{(p)}}$ is a torus. We have:*

(a) *There exist isomorphisms*

$$(M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}).$$

(b) *The group scheme $\mathcal{G}$ is isomorphic to $G_{W(k)} = G_{\mathbb{Z}_{(p)}} \times_{\mathrm{Spec}(\mathbb{Z}_{(p)})} Spec(W(k))$.*

Proof: The existence of an isomorphism $(M, (t_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}})$ follows from B3 applied to the pair $(D, (t_\alpha)_{\alpha \in \mathcal{J}})$. Thus it suffices to prove the Theorem under the extra assumption that k has a countable transcendental degree. This implies that there exists an $E(G, \mathcal{X})$-monomorphism $B(k) \hookrightarrow \mathbb{C}$. Let $w \in \mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})}(\mathbb{C})$ be the composite of the resulting morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(B(k))$ with the generic fibre of z. There exist isomorphisms $(H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$ (cf. Lemma 2.3.4 (a)) and thus (a) holds. Part (b) is implied by (a). □

3.3. Local deformation. Let $\mathcal{G}'$ be the universal smoothening of $\mathcal{G}$, cf. A1. Fontaine comparison theory implies that the group $\mathcal{G}_{B(k)} = \mathcal{G}'_{B(k)}$ is a form of $G_{B(k)}$ (see end of B6) and thus it is a reductive group over $B(k)$ of dimension l. Thus the relative dimension of $\mathcal{G}'$ over $W(k)$ is also l. Let R be the completion of the local ring of $\mathcal{G}'$ at the identity element of $\mathcal{G}'_k$. We choose an identification $R = W(k)[[x_1, \ldots, x_l]]$ such that the identity section of $\mathcal{G}'$ is defined by the identities $x_1 = \cdots = x_l = 0$. Let $g_{\mathrm{univ}} \in \mathcal{G}'(R)$ be the natural (universal) element.

Let $M_R := M \otimes_{W(k)} R$ and $F^1_R := F^1 \otimes_{W(k)} R$. Let Φ_R be the Frobenius lift of R that is compatible with σ and that takes x_i to x_i^p for all $i \in \{1, \ldots, l\}$. Let

$$\Phi := g_{\mathrm{univ}}(\phi \otimes \Phi_R) : M_R \to M_R;$$

it is a Φ_R-linear endomorphism of M_R. Let $\Omega^\wedge_{R/W(k)}$ be the p-adic completion of $\Omega_{R/W(k)}$; it is a free R-module that has $\{dx_1, \ldots, dx_l\}$ as an R-basis. Let $d\Phi_R : \Omega^\wedge_{R/W(k)} \to \Omega^\wedge_{R/W(k)}$ be the differential map of Φ_R. Let $\nabla : M_R \to M_R \otimes_R \Omega^\wedge_{R/W(k)}$ be the unique connection on M_R such that we have $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$, cf. B7. The connection ∇ is integrable and nilpotent modulo p, cf. B7. See B7.3 (i) to (iii) for the three main properties of ∇.

The $W(k)$-algebra R is complete in the $(x_1, \ldots, x_l)$-topology and moreover we have $\Phi_R((x_1, \ldots, x_l)) \subseteq (x_1, \ldots, x_l)^p$. This implies that each element of $\mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$ is of the form $\beta\Phi_R(\beta^{-1})$ for some element $\beta \in \mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$. As g_{univ} takes ψ_M to a $\mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$-multiple of ψ_M, we get that there exists a $\mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$-multiple ψ_{M_R} of the perfect alternating form ψ_M on M_R such that we have an identity

$$\psi_{M_R}(\Phi(a) \otimes \Phi(b)) = p\Phi_R(\psi_{M_R}(a \otimes b))$$

for all element $a, b \in M_R$. As 1 is the only element of $\mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$ fixed by Φ_R, this $\mathrm{Ker}(\mathbb{G}_{mW(k)}(R) \to \mathbb{G}_{mW(k)}(R/(x_1, \ldots, x_l)))$-multiple ψ_{M_R} of ψ_M is uniquely determined.

There exists a unique principally quasi-polarized p-divisible group (D_R, λ_{D_R}) over R that lifts (D, λ_D) and whose principally quasi-polarized filtered F-crystal over $R/2R$ is $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R})$, cf. B7.1 and B7.2.

Let (B_R, λ_{B_R}) be the principally polarized abelian scheme over R that lifts (A, λ_A) and whose principally quasi-polarized p-divisible group is (D_R, λ_{D_R}), cf. Serre–Tate deformation theory and Grothendieck algebraization theorem. Let

$$q_R : \mathrm{Spec}(R) \to \mathcal{M}_r$$

be the natural morphism that corresponds to (B_R, λ_{B_R}) and its level-N symplectic similitude structures which lift those of (A, λ_A) (here $N \geq 3$ is relatively prime to p). We have a canonical identification $H^1_{\mathrm{dR}}(B_R/R) = M_R = M \otimes_{W(k)} R$, cf. [Be, Ch. V, Subsection 2.3] and [BBM, Prop. 2.5.8]. Under this identification, the following two properties hold:

(i) the perfect form on M_R defined by the principal polarization λ_{B_R} of B_R gets identified with ψ_{M_R};

(ii) the p-adic completion of the Gauss–Manin connection on $H^1_{\mathrm{dR}}(B_R/R)$ defined by B_R gets identified with ∇ (cf. [Be, Ch. V, Prop. 3.6.4]).

From the property (ii) we get that for all $s \in \mathbb{N}$ we have::

(iii) the connection on $H^1_{\mathrm{dR}}(B_R/R)/(x_1,\dots,x_l)^s H^1_{\mathrm{dR}}(B_R/R) = M_R/(x_1,\dots,x_l)^s M_R$ induced by ∇ is the Gauss–Manin connection of $B_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R/(x_1,\dots,x_l)^s)$.

3.4. Proof of 1.5. In this Subsubsection we prove the Basic Theorem 1.5. Thus $e(v)=1$ i.e., the prime v of $E(G,\mathcal{X})$ is unramified over p. Let O be a faithfully flat $O_{(v)}$-algebra which is a discrete valuation ring of index of ramification 1. We will choose the field k such that we have an $O_{(v)}$-monomorphism $O \hookrightarrow W(k)$. Let Z be a regular, formally smooth O-scheme such that there exists a morphism $q_{Z_{E(G,\mathcal{X})}} : Z_{E(G,\mathcal{X})} \to \mathrm{Sh}_H(G,\mathcal{X}) = \mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})}$. Thus $q_{Z_{E(G,\mathcal{X})}}$ extends uniquely to a morphism $q_Z : Z \to \mathcal{N}$, cf. Proposition 2.2.1 (c). To prove Theorem 1.5 (a) we only need to show that q_Z factors through $\mathcal{N}^{\mathrm{s}}$. It suffices to check this under the extra assumptions that $O = W(k)$ and that $Z = \mathrm{Spec}(R_1)$, where $R_1 = W(k)[[x_1,\dots,x_m]]$ for some $m \in \mathbb{N}\cup\{0\}$. Let $z_Z \in Z(W(k))$ be the point defined by the $W(k)$-epimorphism $R_1 \twoheadrightarrow W(k)$ whose kernel is $(x_1,\dots,x_m)$. We will use the notations of Subsection 3.2 for the point

$$z := q_Z \circ z_Z \in \mathcal{N}(W(k)).$$

As $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}$ (cf. Lemma 2.2.2), to show that q_Z factors through $\mathcal{N}^{\mathrm{s}}$ it suffices to show that z factors through $\mathcal{N}^{\mathrm{s}}$.

Let $y : \mathrm{Spec}(k) \hookrightarrow \mathcal{N}_{W(k)}$ be the closed embedding defined naturally by the special fibre of $z \in \mathcal{N}(W(k))$. Let O^{big}_y and O_y be the completions of the local rings of y viewed as a k-valued point of $\mathcal{M}_{rW(k)}$ and $\mathcal{N}_{W(k)}$ (respectively). As $\mathcal{Q}$ is a normal, flat $O_{(v)}$-scheme of relative dimension d and as $\mathcal{N}$ is a pro-étale cover of $\mathcal{Q}$ (cf. Proposition 2.2.1 (a)), the local ring O_y is normal and has dimension $1+d$. The natural homomorphism $n_y : O^{\mathrm{big}}_y \to O_y$ is finite, cf. Proposition 2.2.1 (b). Let $h^{\mathrm{big}}_y : O^{\mathrm{big}}_y \to R$ be the $W(k)$-epimorphism defined naturally by q_R.

Let $S := W(k)[[x_1,\dots,x_d]]$. We consider a closed embedding $c_R : \mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ such that the following two properties hold (cf. B7.5 and Lemma 3.2.1):

(i) it is defined by a $W(k)$-epimorphism $h_R : R \twoheadrightarrow S$ with the property that $h_R((x_1,\dots,x_l)) \subseteq (x_1,\dots,x_d) \subseteq S$;

(ii) the pull back of $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R})$ via the closed embedding $\mathrm{Spec}(S/pS) \hookrightarrow \mathrm{Spec}(R/pR)$, is a principally quasi-polarized filtered F-crystal over S/pS whose Kodaira–Spencer map is injective and has an image equal to the direct summand $(\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1,F^0)) \otimes_{W(k)} S$ of $\mathrm{Hom}(F^1,F^0) \otimes_{W(k)} R \xrightarrow{\sim} \mathrm{Hom}(F^1, M/F^1) \otimes_{W(k)} S$.

From the property (ii) we get that the composite morphism $q_S := q_R \circ c_R : \mathrm{Spec}(S) \to \mathcal{M}_r$ is defined naturally by a $W(k)$-epimorphism $s^{\mathrm{big}}_y := h_R \circ h^{\mathrm{big}}_y : O^{\mathrm{big}}_y \twoheadrightarrow S$.

In order to show that there exists a $W(k)$-homomorphism $s_y : O_y \to S$ that makes the following diagram commutative

$$\begin{array}{ccc} O^{\mathrm{big}}_y & \xrightarrow{n_y} & O_y \\ {\scriptstyle h^{\mathrm{big}}_y}\downarrow & & \downarrow{\scriptstyle s_y} \\ R & \xrightarrow{h_R} & S, \end{array}$$

we will need to first recall a result of Faltings.

3.4.1. Proposition. *The tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}] = \mathcal{T}(M_R)[\frac{1}{p}]$ is the de Rham component of a Hodge cycle on $B_{R[\frac{1}{p}]}$.*

Proof: We recall that B_R is a deformation of A over R. As $t_\alpha \in \mathcal{T}(M)[\frac{1}{p}]$ is the de Rham component of the Hodge cycle w_α on $A_{B(k)}$ and due to B7.3 (ii), the Proposition is a result of Faltings. As the essence of this result is only outlined in [Va1, Rm. 4.1.5], we will include a complete proof of Faltings' result.

As $\mathcal{A}_{r,1,N}$ is a quasi-projective $\mathbb{Z}_{(p)}$-scheme and as the set $\mathcal{J}$ is countable, it suffices to prove the Proposition in the case when there exists a morphism $e_k : \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(W(k))$. We will view $\mathbb{C}$ as a $W(k)$-algebra via e_k. Let $\mathcal{R} := \mathbb{C}[[x_1, \ldots, x_l]]$ and $\mathcal{S} := \mathbb{C}[[x_1, \ldots, x_d]]$. Let $\mathcal{I} := (x_1, \ldots, x_l)$ be the maximal ideal of $\mathcal{R}$.

Let $(B_\mathcal{R}, \lambda_{B_\mathcal{R}}, (t_\alpha)_{\alpha\in\mathcal{J}})$ be the pull back of $(B_R, \lambda_{B_R}, (t_\alpha)_{\alpha\in\mathcal{J}})$ via the natural $W(k)$-monomorphism $R = W(k)[[x_1, \ldots, x_l]] \hookrightarrow \mathbb{C}[[x_1, \ldots, x_l]] = \mathcal{R}$. To prove the Proposition, it suffices to show that the tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} \mathcal{R} = \mathcal{T}(M_R \otimes_R \mathcal{R}) = \mathcal{T}(H^1_{\mathrm{dR}}(B_\mathcal{R}/\mathcal{R}))$ is the de Rham component of a Hodge cycle on $B_\mathcal{R}$.

Let $(C_\mathcal{S}, \lambda_{C_\mathcal{S}}, (w_\alpha^\mathcal{S})_{\alpha\in\mathcal{J}})$ be the pull back of $(\mathcal{A}, \lambda_\mathcal{A}, (w_\alpha^\mathcal{A})_{\alpha\in\mathcal{J}})$ via a formally étale morphism $\mathrm{Spec}(\mathcal{S}) \to \mathcal{N}^{\mathrm{s}}$ whose composite with the natural embedding $\mathrm{Spec}(\mathbb{C}) \hookrightarrow \mathrm{Spec}(\mathcal{S})$ is the point $e_k \circ z \in \mathcal{N}(\mathbb{C}) = \mathcal{N}^{\mathrm{s}}(\mathbb{C})$. Let $\mathcal{W} := H^1_{\mathrm{dR}}(C_\mathcal{S}/\mathcal{S})$. Let $\psi_\mathcal{W}$ be the perfect alternating form on $\mathcal{W}$ defined by $\lambda_{C_\mathcal{S}}$. Let $t_\alpha^\mathcal{S} \in \mathcal{T}(\mathcal{W})$ be the de Rham component of $w_\alpha^\mathcal{S}$. Let Δ be the Gauss–Manin connection on $\mathcal{W}$ defined by $C_\mathcal{S}$. We recall that ψ^* the alternating form on W^* (or $L_{(p)}$) defined naturally by ψ.

From Corollary 2.3.6 and (the proof of) Lemma 2.3.4 (b) we get that there exists $\varepsilon \in \mathbb{Q} \setminus \{0\}$ for which there exist isomorphisms

$$I : (\mathcal{W}, \psi_\mathcal{W}, (t_\alpha^\mathcal{S})_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_\mathbb{Q} \mathcal{S}, \varepsilon\psi^*, (v_\alpha)_{\alpha\in\mathcal{J}})$$

under which Δ becomes the flat connection on $W^* \otimes_\mathbb{Q} \mathcal{S}$ that annihilates $W^* \otimes 1$. But there exist isomorphisms of $(W^* \otimes_\mathbb{Q} \mathbb{C}, (v_\alpha)_{\alpha\in\mathcal{J}})$ that take ψ^* to $\varepsilon\psi^*$. Thus we can assume that $\varepsilon = 1$. We will fix such an isomorphism I and we view it as an identification. For each $\beta \in \mathbb{G}_{m\mathbb{C}}(\mathcal{S})$, there exist isomorphisms of $(W^* \otimes_\mathbb{Q} \mathcal{R}, (v_\alpha)_{\alpha\in\mathcal{J}})$ that take ψ^* to $\beta\psi^*$. Thus, based on the construction of M_R and on either Lemma 2.3.4 (b) or the proof of Lemma 3.2.1, we also get that there exist isomorphisms

$$I_A : (M_R \otimes_R \mathcal{R}, \psi_{M_R}, (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_\mathbb{Q} \mathcal{R}, \psi^*, (v_\alpha)_{\alpha\in\mathcal{J}}).$$

By induction on $s \in \mathbb{N}$ we show that there exists a unique morphism of $\mathbb{C}$-schemes

$$J_s : \mathrm{Spec}(\mathcal{R}/\mathcal{I}^s) \to \mathrm{Spec}(\mathcal{S})$$

that has the following two properties:

(i) *the kernel of the composite $\mathbb{C}$-homomorphism $\mathcal{S} \to \mathcal{R}/\mathcal{I}^s \twoheadrightarrow \mathcal{R}/\mathcal{I} = \mathbb{C}$ is the ideal $(x_1, \ldots, x_d)$ of $\mathcal{S}$;*

(ii) *there exists an isomorphism Q_s between the reduction of $(B_\mathcal{R}, \lambda_{B_\mathcal{R}}, (t_\alpha)_{\alpha\in\mathcal{J}})$ modulo $\mathcal{I}^s$ and $J_s^*((C_\mathcal{S}, \lambda_{C_\mathcal{S}}, (t_\alpha^\mathcal{S})_{\alpha\in\mathcal{J}}))$ which modulo $\mathcal{I}/\mathcal{I}^s$ is $1_{A_\mathbb{C}} = 1_{C_\mathcal{S} \times_\mathcal{S} \mathbb{C}} = 1_{B_\mathcal{R} \times_\mathcal{R} \mathbb{C}}$.*

As $\mathcal{N}^{\mathrm{s}}_{E(G,\mathcal{X})}$ is a finite, étale scheme over a closed subscheme of $\mathcal{M}_{rE(G,\mathcal{X})}$, the deformation $(C_{\mathcal{S}}, \lambda_{C_{\mathcal{S}}})$ of the principally polarized abelian variety $(A, \lambda_A)_{\mathbb{C}}$ is versal. Thus the Kodaira–Spencer map of Δ is injective and its image is a free $\mathcal{S}$-module of rank d. This implies the uniqueness of J_s. The existence of J_1 is obvious.

The passage from the existence of J_s to the existence of J_{s+1} goes as follows. Let $J'_{s+1} : \mathrm{Spec}(\mathcal{R}/\mathcal{I}^{s+1}) \to \mathrm{Spec}(\mathcal{S})$ be an arbitrary morphism of $\mathbb{C}$-schemes that lifts J_s. Let Δ_{s+1} be the connection on $\mathcal{W} \otimes_{\mathcal{S}} \mathcal{R}/\mathcal{I}^{s+1} = W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}$ which is the extension of Δ via J'_{s+1} (the last identification being defined by I). Let ∇_{s+1} be the Gauss–Manin connection on $H^1_{\mathrm{dR}}(B_R/R) \otimes_R \mathcal{R}/\mathcal{I}^{s+1} = M_R \otimes_R \mathcal{R}/\mathcal{I}^{s+1}$ defined by $B_R \times_{\mathrm{Spec}(R)} \mathrm{Spec}(\mathcal{R}/\mathcal{I}^{s+1})$; it is the extension of the connection ∇ on M_R (cf. property 3.3 (iii)) and therefore it annihilates each tensor $t_\alpha \in \mathcal{T}(M_R) \otimes_R \mathcal{R}/\mathcal{I}^{s+1}$ (cf. B7.3 (ii)). From Lemma 2.3.5 we get that:

(iii) there exists a unique isomorphism $I_{A,s+1} : M_R \otimes_R \mathcal{R}/\mathcal{I}^{s+1} \xrightarrow{\sim} W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}$ which lifts a fixed isomorphism between $(M_R \otimes_R \mathcal{R} \otimes_{\mathcal{R}} \mathcal{R}/\mathcal{I}, (t_\alpha)_{\alpha \in \mathcal{J}}) = (H^1_{\mathrm{dR}}(A_{\mathbb{C}}/\mathbb{C}), (t_\alpha)_{\alpha \in \mathcal{J}})$ and $(W^* \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ obtained as in Lemma 2.3.4 (b) and such that under it ∇_{s+1} becomes the flat connection δ_{s+1} on $W^* \otimes_{\mathbb{C}} \mathcal{R}/\mathcal{I}^{s+1}$ that annihilates $W^* \otimes 1$.

We denote also by $I_{A,s+1}$ the isomorphism $\mathcal{T}(M_R \otimes_R \mathcal{R}/\mathcal{I}^{s+1}) \xrightarrow{\sim} \mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{C})$ induced by $I_{A,s+1}$. As $I_{A,s+1}(t_\alpha)$ and v_α are two tensors of $W^* \otimes_{\mathbb{C}} \mathcal{R}/\mathcal{I}^{s+1}$ that are annihilated by δ_{s+1} and that coincide modulo $\mathcal{I}/\mathcal{I}^{s+1}$, we get that we have $I_{A,s+1}(t_\alpha) = v_\alpha$ for all $\alpha \in \mathcal{J}$. A similar argument shows that $I_{A,s+1}$ takes ψ_{M_R} to ψ^*. Thus we can choose I_A such that it lifts $I_{A,s+1}$. We will view the reduction $I_{A,s+1}$ of I_A modulo $\mathcal{I}^{s+1}$ as an identification. Thus we will also identify $\nabla_{s+1} = \delta_{s+1}$.

From the existence of I and the fact that $I_{A,s+1}$ is the reduction of I_A modulo $\mathcal{I}^{s+1}$, we get that there exists an isomorphism

$$D_{s+1} : J'^*_{s+1}((\mathcal{W}, \psi_{\mathcal{W}}, (t^{\mathcal{S}}_\alpha)_{\alpha \in \mathcal{J}})) = (W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}, \psi^*, (v_\alpha)_{\alpha \in \mathcal{J}}) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} (M_R \otimes_R \mathcal{R}/\mathcal{I}^{s+1}, \psi_{M_R}, (t_\alpha)_{\alpha \in \mathcal{J}}) = (W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}, \psi^*, (v_\alpha)_{\alpha \in \mathcal{J}})$$

with the properties that it lifts the identity automorphism of $W^* \otimes_{\mathbb{Q}} \mathbb{C}$ and that:

(iv) *it respects the Gauss–Manin connections i.e., it takes Δ_{s+1} to $\nabla_{s+1} = \delta_{s+1}$.*

From the uniqueness part of the property (iii) we also get that

(v) *D_{s+1} modulo $\mathcal{I}^s$ is the isomorphism defined by Q_s;*

Let $F^1_{A,s+1}$ and $F^1_{C,s+1}$ be the Hodge filtrations of $W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}$ defined naturally by $B_{\mathcal{R}}$ and $J'^*_{s+1}(C_{\mathcal{S}})$ (respectively) via the above identifications. The direct summands $F^1_{A,s+1}$ and $D_{s+1}(F^1_{C,s+1})$ of $W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}$ coincide modulo $\mathcal{I}^s/\mathcal{I}^{s+1}$, cf. property (v). Moreover, there exist direct sum decompositions

$$W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1} = F^1_{A,s+1} \oplus F^0_{A,s+1} = F^1_{C,s+1} \oplus F^0_{C,s+1}$$

defined naturally by cocharacters $\mu_{A,s+1}$ and $\mu_{C,s+1}$ of the reductive subgroup scheme $G_{\mathcal{R}/\mathcal{I}^{s+1}}$ of $\mathbf{GL}_{W^* \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^{s+1}}$. Argument: the existence of $\mu_{A,s+1}$ is a direct consequence of the existence of the cocharacter $\mu : \mathbb{G}_{mW(k)} \to \mathcal{G}$ (see paragraph before Lemma 3.2.1) and of the definition of F_R^1 (see Subsection 3.3) while the existence of $\mu_{C,s+1}$ is well known. As $F^1_{A,s+1}$ and $D_{s+1}(F^1_{C,s+1})$ coincide modulo $\mathcal{I}^s/\mathcal{I}^{s+1}$, we can choose $\mu_{A,s+1}$ and $\mu_{C,s+1}$ such that $D_{s+1}^{-1}\mu_{A,s+1}D_{s+1}$ and $\mu_{C,s+1}$ coincide modulo $\mathcal{I}^s/\mathcal{I}^{s+1}$. Thus based on [DG, Vol. II, Exp. IX, Thm. 3.6], there exists $g_{s+1} \in \mathrm{Ker}(G(\mathcal{R}/\mathcal{I}^{s+1}) \to G(\mathcal{R}/\mathcal{I}^s))$ such that we have $D_{s+1}^{-1}\mu_{A,s+1}D_{s+1} = g_{s+1}\mu_{C,s+1}g_{s+1}^{-1}$. Thus $D_{s+1}(g_{s+1}(F^1_{C,s+1})) = F^1_{A,s+1}$.[1]

As the morphism $\mathrm{Spec}(\mathcal{S}) \to \mathcal{N}^{\mathrm{s}}$ is formally étale, the Kodaira–Spencer map $\mathfrak{K}$ of Δ is injective and its image is a free $\mathcal{S}$-module that has rank d and that is equal to the image of $\mathrm{Lie}(G_{\mathcal{S}})$ into the codomain of $\mathfrak{K}$. Thus we can replace J'_{s+1} by another morphism $J_{s+1} : \mathrm{Spec}(\mathcal{R}/\mathcal{I}^{s+1}) \to \mathrm{Spec}(\mathcal{S})$ lifting J_s and such that under it and I the Hodge filtration $F^1_{C,s+1}$ gets replaced by $g_{s+1}(F^1_{C,s+1})$. Thus D_{s+1} becomes the de Rham realization of an isomorphism Q_{s+1} which is between the reduction of $(B_{\mathcal{R}}, \lambda_{B_{\mathcal{R}}}, (t_\alpha)_{\alpha\in\mathcal{J}})$ modulo $\mathcal{I}^{s+1}$ and $J^*_{s+1}((C_{\mathcal{S}}, \lambda_{C_{\mathcal{S}}}, (t^{\mathcal{S}}_\alpha)_{\alpha\in\mathcal{J}}))$ and which lifts Q_s. Thus the morphism J_{s+1} has the desired properties. This ends the induction.

Let $J_\infty : \mathrm{Spec}(\mathcal{R}) \to \mathrm{Spec}(\mathcal{S})$ be the morphism defined by J_s's ($s \in \mathbb{N}$). The isomorphism Q_s is uniquely determined by properties (i) and (ii) and this implies that Q_{s+1} lifts Q_s. Thus we get the existence of an isomorphism

$$Q_\infty : (B_{\mathcal{R}}, \lambda_{B_{\mathcal{R}}}, (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} J^*_\infty((C_{\mathcal{S}}, \lambda_{C_{\mathcal{S}}}, (t^{\mathcal{S}}_\alpha)_{\alpha\in\mathcal{J}}))$$

which modulo $\mathcal{I}$ is defined by $1_{A_{\mathbb{C}}}$. Thus for each $\alpha \in \mathcal{J}$, the tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} \mathcal{R}$ is the de Rham component of the Hodge cycle $Q_\infty^{-1}(J^*_\infty(w^{\mathcal{S}}_\alpha))$ on $B_{\mathcal{R}}$. $\square$

3.4.2. End of the proof of 1.5. The existence of the isomorphism Q_∞ implies that the morphism $q_R : \mathrm{Spec}(R) \to \mathcal{M}$ factors through $\mathcal{N}$ in such a way that modulo the ideal $(x_1, \ldots, x_l)$ of R it defines the point $z \in \mathcal{N}(W(k))$. Therefore the $W(k)$-epimorphism s_y^{big} : $O_y^{\mathrm{big}} \twoheadrightarrow S$ (see paragraph before Proposition 3.4.1) factors through $n_y : O_y^{\mathrm{big}} \to O_y$. By reasons of dimensions of local, normal rings, we get that the resulting $W(k)$-epimorphism $s_y : O_y \twoheadrightarrow S$ is an isomorphism. Thus $\mathcal{N}_{W(k)}$ is formally smooth at z and therefore z factors through $\mathcal{N}^{\mathrm{s}}$. Thus Theorem 1.5 (a) holds and y is a k-valued point of $\mathcal{N}^{\mathrm{s}}_{W(k)}$.

As s_y is an isomorphism, the $W(k)$-homomorphism $n_y : O_y^{\mathrm{big}} \to O_y$ is onto. This implies that the natural $W(k)$-morphism $\mathcal{N}^{\mathrm{s}}_{W(k)} \to \mathcal{M}_{rW(k)}$ is a formally closed embedding at $y \in \mathcal{N}^{\mathrm{s}}_{W(k)}(k)$. As the morphism q_Z of the beginning of Subsection 3.4 was arbitrary, the role of $z \in \mathcal{N}(W(k))$ is that of an arbitrary $W(k)$-valued of $\mathcal{N}$ (and thus cf. Theorem 1.5 (a)) of $\mathcal{N}^{\mathrm{s}}$. Thus the $W(k)$-morphism $\mathcal{N}^{\mathrm{s}}_{W(k)} \to \mathcal{M}_{rW(k)}$ is a formally closed embedding at every k-valued point of $\mathcal{N}^{\mathrm{s}}_{W(k)}$. Thus Theorem 1.5 (b) also holds.

We check that the statement 1.5 (c) holds. Let Z be a smooth $O_{(v)}$-scheme such that we have a morphism $q_{Z_{E(G,\mathcal{X})}} : Z_{E(G,\mathcal{X})} \to \mathrm{Sh}_{H\times H^{(P)}}(G, \mathcal{X})$. From Proposition 2.2.1 (b) and Lemma 2.2.4 we get that $\mathcal{N}/H^{(p)}$ has an étale cover which is projective. This implies that $\mathcal{N}/H^{(p)}$ is a proper $O_{(v)}$-scheme. From this and the valuative criterion of properness, we get that there exists an open subscheme U_Z of Z such that it contains $Z_{E(G,\mathcal{X})}$, the complement of U_Z in Z has codimension in Z at least 2, and the morphism $q_{Z_{E(G,\mathcal{X})}}$ extends

1 The original approach of Faltings used the strictness of filtrations of morphisms between Hodge $\mathbb{R}$-structures in order to get the existence of the element g_{s+1}.

uniquely to a morphism $q_{U_Z} : U_Z \to \mathcal{N}/H^{(p)}$. From the classical purity theorem of Nagata and Zariski (see [Gr, Exp. X, Thm. 3.4 (i)]) we get that the étale cover $U_Z \times_{\mathcal{N}/H^{(p)}} \mathcal{N} \to U_Z$ extends uniquely to an étale cover $Z_\infty \to Z$. From this and Theorem 1.5 (a) we get that the natural morphism $U_Z \times_{\mathcal{N}/H^{(p)}} \mathcal{N} \to \mathcal{N}$ extends uniquely to a morphism $Z_\infty \to \mathcal{N}$. This implies that the morphism $q_{Z_{E(G,\mathcal{X})}}$ extends uniquely to a morphism $q_Z : Z \to \mathcal{N}/H^{(p)}$. Thus $\mathcal{N}/H^{(p)}$ is a Néron model of its generic fibre $\mathrm{Sh}_{H\times H^{(p)}}(G,\mathcal{X})$ over $O_{(v)}$ i.e., Theorem 1.5 (c) holds. This ends the proof of Basic Theorem 1.5. $\square$

3.5. Simple properties. We denote also by q_R the factorization of $q_R : \mathrm{Spec}(R) \to \mathcal{M}_r$ through either $\mathcal{N}$ or (cf. Theorem 1.5 (a)) $\mathcal{N}^{\mathrm{s}}$ which modulo $(x_1, \ldots, x_l)$ is the $W(k)$-valued point $z \in \mathcal{N}(W(k)) = \mathcal{N}^{\mathrm{s}}(W(k))$. As $s_y : O_y \to S$ is a $W(k)$-isomorphism and as we have a $W(k)$-epimorphism $h_R : R \twoheadrightarrow S$, the morphism $q_R : \mathrm{Spec}(R) \to \mathcal{N}^{\mathrm{s}}$ is formally smooth. Under the canonical identification $H^1_{\mathrm{dR}}(B_R/R) = M_R = M \otimes_{W(k)} R$, the pull back of $w_\alpha^{\mathcal{A}}$ via the morphism $\mathrm{Spec}(R[\frac{1}{p}]) \to \mathcal{N}_{E(G,\mathcal{X})} = \mathrm{Sh}_H(G,\mathcal{X})$ defined by q_R, is a Hodge cycle on $B_{R[\frac{1}{p}]}$ whose de Rham component is $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$. This follows either from the existence of Q_∞ or (in Faltings' approach) from the fact that there exists no non-trivial tensor of $\mathcal{T}(M) \otimes_{W(k)} (x_1, \ldots, x_l)[\frac{1}{p}]$ fixed by Φ.

3.5.1. The open subscheme $\mathcal{N}^{\mathrm{m}}$. For $p > 2$ let $\mathcal{N}^{\mathrm{m}} := \mathcal{N}^{\mathrm{s}}$. If $p = 2$ let $\mathcal{N}^{\mathrm{m}}$ be the maximal open subscheme of $\mathcal{N}^{\mathrm{s}}$ with the property that for every algebraically closed field k of characteristic p and for every $z \in \mathcal{N}^{\mathrm{m}}(W(k))$, the statement 3.2.2 (a) (and thus also 3.2.2 (b)) holds. Thus regardless of the parity of p, for every such field k and for every $z \in \mathcal{N}^{\mathrm{m}}(W(k))$, the statement 3.2.2 (a) holds. We now check the following two properties:

(i) Always $\mathcal{N}^{\mathrm{m}}$ is a $G(\mathbb{A}_f^{(p)})$-invariant, open subscheme of $\mathcal{N}^{\mathrm{s}}$.

(ii) If the statement 3.2.2 (a) holds for $z \in \mathcal{N}^{\mathrm{s}}(W(k))$, then $z \in \mathcal{N}^{\mathrm{m}}(W(k))$.

To check (i) and (ii) we can assume that $p = 2$. The right translations of z by elements of $G(\mathbb{A}_f^{(2)})$ corresponds to passages to isogenies prime to 2 of the abelian scheme A. Thus the triple $(M, \phi, (t_\alpha)_{\alpha\in\mathcal{J}})$ depends only on the $G(\mathbb{A}_f^{(2)})$-orbit of z. Thus if statement 3.2.2 (a) holds for z, then the statement 3.2.2 (a) also holds for every point in the $G(\mathbb{A}_f^{(2)})$-orbit of z. This implies (i).

Let $\mathcal{Q}$ and $\mathcal{Q}^{\mathrm{s}}$ be as in Subsection 2.2. By enlarging N we can assume that the triple $(\mathcal{A}, \lambda_{\mathcal{A}}, (w_\alpha^{\mathcal{A}})_{\alpha\in\mathcal{J}})$ is the pull back of an analogue triple $\mathcal{T}$ over $\mathcal{Q}$. Let $\mathrm{Spec}(V)$ be an affine, open subscheme of $\mathcal{Q}^{\mathrm{s}}$ such that z maps to $\mathrm{Spec}(V)$. Let $(M_V, \psi_{M_V}, (t_\alpha^V)_{\alpha\in\mathcal{J}})$ be the de Rham realization of the pull back of $\mathcal{T}$ to $\mathrm{Spec}(V)$. By shrinking $\mathrm{Spec}(V)$, we can assume that M_V is a free V-module of rank $2r$. The existence of the formally smooth morphism $q_R : \mathrm{Spec}(R) \to \mathcal{N}^{\mathrm{s}}$ implies that we have isomorphisms (cf. the beginning of Subsection 3.5 and the fact that the statement 3.2.2 (a) holds for $z \in \mathcal{N}^{\mathrm{s}}(W(k))$)

$$(M_V \otimes_V R, (t_\alpha^V)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (M_R, (t_\alpha)_{\alpha\in\mathcal{J}}) = (M \otimes_{W(k)} R, (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} R, (v_\alpha)_{\alpha\in\mathcal{J}}).$$

From this and Artin approximation theorem (see [BLR, Ch. 3, 3.6, Thm. 16]) we get that there exists a smooth, affine morphism $\mathrm{Spec}(V') \to \mathrm{Spec}(V)$ through which z factors and such that we have an isomorphism $(M_V \otimes_V V', (t_\alpha^V)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} V', (v_\alpha)_{\alpha\in\mathcal{J}})$. Let $\mathcal{V}$

be the open subscheme of $\mathrm{Spec}(V)$ which is the image of $\mathrm{Spec}(V')$ in $\mathrm{Spec}(V)$. The pull back of $\mathcal{V}_0$ to $\mathcal{N}^{\mathrm{s}}$ is an open subscheme of $\mathcal{N}^{\mathrm{m}}$ that contains the point $z \in \mathcal{N}^{\mathrm{s}}(W(k))$. Thus (ii) also holds.

We end this Section with a Lemma which will be needed in Section 5.

3.5.2. Lemma. *Let $\mu : \mathbb{G}_{mW(k)} \to \mathcal{G}$ and $M = F^1 \oplus F^0$ be as in Subsection 3.2. Let $y \in \mathcal{N}^{\mathrm{s}}(W(k))$ be defined by $z \in \mathcal{N}^s(W(k)) = \mathcal{N}(W(k))$. Let $\mu_1 : \mathbb{G}_{mW(k)} \to \mathcal{G}$ be another cocharacter such that we have a direct sum decomposition $M = F_1^1 \oplus F_1^0$ with the property that for each $i \in \{0, 1\}$, μ_1 acts on F_1^i via the $-i$-th power of the identity character of $\mathbb{G}_{mW(k)}$. If the triple (M, F_1^1, ϕ) is a filtered F-crystal over k, then there exists a point $z_1 \in \mathcal{N}^{\mathrm{s}}(W(k)) = \mathcal{N}(W(k))$ that lifts $y \in \mathcal{N}^{\mathrm{s}}(k)$ and such that the principally quasi-polarized filtered F-crystal over k of $z_1^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is precisely (M, F_1^1, ϕ, ψ_M).*

Proof: For $n \in \mathbb{N}$ let $W_n(k) := W(k)/p^nW(k)$. We have $F^1/pF^1 = F_1^1/pF_1^1$. By induction on $n \in \mathbb{N}$ we show that there exists a point $z(n) \in \mathcal{N}^{\mathrm{s}}(W(k)) = \mathcal{N}(W(k))$ that has the following three properties:

(i) it lifts $y \in \mathcal{N}^{\mathrm{s}}(k)$;

(ii) for $n \geq 2$ the $W_{n-1}(k)$-valued points of $\mathcal{N}^{\mathrm{s}}$ defined by $z(n-1)$ and $z(n)$ coincide;

(iii) the principally quasi-polarized filtered F-crystal over k of $z(n)^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is of the form $(M, F_1^1(n), \phi, \psi_M)$, where $F_1^1(n)$ is congruent to F_1^1 modulo p^n.

Let $z(1) := z$; obviously the base of the induction for $n = 1$ holds. The passage from n to $n+1$ goes as follows. Not to introduce extra notations by replacing z with $z(n)$, we can assume that $z(n) = z$; thus we have $F^1/p^nF^1 = F_1^1/p^nF_1^1$. Let U_{big} be the smooth, unipotent, closed subgroup scheme of $\mathbf{GL}_M$ defined by the rule: if $\ddagger$ is a commutative $W(k)$-algebra, then $U_{\mathrm{big}}(\ddagger) := 1_{M\otimes_{W(k)}\ddagger} + \mathrm{Hom}(F^1, F^0) \otimes_{W(k)} \ddagger$.

As $F^1/p^nF^1 = F_1^1/p^nF_1^1$, there exists a unique element $u \in \mathrm{Ker}(U_{\mathrm{big}}(W(k)) \to U_{\mathrm{big}}(W_n(k)))$ such that we have an identity $F_1^1 = u(F^1)$. We write $u = 1_M + v$, where $v \in p^n \mathrm{Hom}(F^1, F^0) = p^n\mathrm{Lie}(U_{\mathrm{big}})$. Let $\mathcal{T}(M) = \oplus_{i\in\mathbb{Z}}\tilde{F}^i(\mathcal{T}(M))$ be the direct sum decomposition such that $\mathbb{G}_{mW(k)}$ acts on $\tilde{F}^i(\mathcal{T}(M))$ through μ as the $-i$-th power of its identity character. The filtration $(F^i(\mathcal{T}(M)))_{i\in\mathbb{Z}}$ of $\mathcal{T}(M)$ defined by F^1 satisfies for all $i \in \mathbb{Z}$ the following identity $F^i(\mathcal{T}(M)) = \oplus_{j\,\geq\, i}\tilde{F}^j(\mathcal{T}(M))$. As μ and μ_1 are two cocharacters of $\mathcal{G}$, they fix each t_α. In particular, we have $t_\alpha \in \tilde{F}^0(\mathcal{T}(M))[\frac{1}{p}]$ and the tensor $u^{-1}(t_\alpha) = (1_M - v)(t_\alpha)$ belongs to $F^0(\mathcal{T}(M))[\frac{1}{p}]$. As $v \in \mathrm{Hom}(F^1, F^0) \subseteq \tilde{F}^{-1}(\mathcal{T}(M))$, the component of $(1_M - v)(t_\alpha)$ in $\tilde{F}^{-1}(\mathcal{T}(M))[\frac{1}{p}]$ is $-v(t_\alpha)$. As this component must be 0, we get hat v annihilates t_α for all $\alpha \in \mathcal{J}$. Thus $v \in \mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{End}(M)$. We conclude that

$$v \in p^n[\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)]. \tag{2}$$

As the image of the Kodaira–Spencer map of ∇ is $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0) \otimes_{W(k)} R$ (cf. B7.3 (iii)) and as the morphism $q_R : \mathrm{Spec}(R) \to \mathcal{N}^{\mathrm{s}}$ is formally smooth, from (2) we get that there exists a lift $z(n+1)$ of $z(n)$ modulo p^n such that the principally quasi-polarized filtered F-crystal over k of $z(n+1)^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is $(M, F_1^1(n+1), \phi, \psi_M)$, where $F_1^1(n+1)$ is congruent to $u(F^1) = F_1^1$ modulo p^{n+1}. This ends the induction.

From the property (ii) we get that there exists a point $z_1 \in \mathcal{N}^{\mathrm{s}}(W(k))$ that lifts $z(n)$ modulo p^n for all $n \in \mathbb{N}$. Thus z_1 also lifts y, cf. property (i). From the property (iii) we get that the principally quasi-polarized filtered F-crystal over k of $z_1^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is (M, F_1^1, ϕ, ψ_M). $\square$

4. Applications to integral models

Lemma 4.1 presents a simple criterion on when the $k(v)$-scheme $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty. In Subsection 4.2 we apply Theorem 1.5 (a) and Lemma 4.1 to prove the existence of good integral models of $\mathrm{Sh}_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ for a large class of maximal compact subgroups $\tilde{H}$ of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Corollary 4.2.3 can be viewed as a complete solution to the conjecture of Langlands of [La, p. 411] for Shimura varieties of Hodge type.

4.1. Lemma. *We assume that one of the following two conditions holds:*

(i) *there exists a smooth, affine group scheme $G^{\mathrm{g}}_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ that extends G (i.e., it has G as its generic fibre), that has a special fibre $G^{\mathrm{g}}_{\mathbb{F}_p}$ of the same rank as G, and that has the property that there exists a homomorphism $G^{\mathrm{g}}_{\mathbb{Z}_{(p)}} \to G_{\mathbb{Z}_{(p)}}$ which extends the identity automorphism of G;*

(ii) *we have $e(v) = 1$ and the group scheme $G_{\mathbb{Z}_{(p)}}$ is quasi-reductive for $(G, \mathcal{X}, v)$ in the sense of Definition 1.3.2 (b).*

Then $e(v) = 1$ and the $k(v)$-scheme $\mathcal{N}^{\mathrm{m}}_{k(v)}$ (and thus also $\mathcal{N}^{\mathrm{s}}_{k(v)}$) is non-empty.

Proof: Suppose that (i) holds. Each torus of $G^{\mathrm{g}}_{\mathbb{F}_p}$ lifts to a torus of $G^{\mathrm{g}}_{\mathbb{Z}_p}$, cf. [DG, Vol. II, Exp. XII, Cor. 1.10]. Thus $G^{\mathrm{g}}_{\mathbb{Z}_p}$ has tori of rank equal to the rank of G. Let $T^{\mathrm{g}}_{\mathbb{Z}_{(p)}}$ be a torus of $G^{\mathrm{g}}_{\mathbb{Z}_{(p)}}$ of the same rank as G and such that there exists $h \in \mathcal{X}$ which factors through $T^{\mathrm{g}}_{\mathbb{R}}$. Its existence is implied by [Ha, Lem. 5.5.3]. The pair $(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ is a Shimura pair. Each prime of $E(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ that divides v is unramified over p (cf. [Mi3, Prop. 4.6 and Cor. 4.7]) and thus we have $e(v) = 1$. The intersection $H^{\mathrm{g}} := H \cap T^{\mathrm{g}}_{\mathbb{Z}_{(p)}}(\mathbb{Q}_p)$ is the unique hyperspecial subgroup $T^{\mathrm{g}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ of $T^{\mathrm{g}}_{\mathbb{Z}_{(p)}}(\mathbb{Q}_p)$. Thus there exists an integral model $\mathcal{Z}^{\mathrm{g}}$ of $\mathrm{Sh}_{H^{\mathrm{g}}}(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ over the normalization of $O_{(v)}$ which is a pro-étale cover of $O_{(v)}$, cf. either [Mi2, Rm. 2.16] or [Va1, Ex. 3.2.8]. In particular, $\mathcal{Z}^{\mathrm{g}}$ is a regular, formally étale, faithfully flat $O_{(v)}$-scheme. The functorial morphism $\mathrm{Sh}_{H^{\mathrm{g}}}(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\}) \to \mathrm{Sh}_H(G, \mathcal{X})$ of $E(G, \mathcal{X})$-schemes extends uniquely to a morphism $\mathcal{Z}^{\mathrm{g}} \to \mathcal{N}^{\mathrm{s}}$ of $O_{(v)}$-schemes, cf. Theorem 1.5 (a). There exist points $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$. Let $(v_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{g}}}$ be a family of tensors of $\mathcal{T}(W^*)$ such that $T^{\mathrm{g}}_{\mathbb{Q}}$ is the subgroup of $\mathbf{GL}_{W^*}$ that fixes v_α for all $\alpha \in \mathcal{J}^{\mathrm{g}}$. We can assume that $\mathcal{J} \subseteq \mathcal{J}^{\mathrm{g}}$ and that for each $\alpha \in \mathcal{J}$, the tensor v_α is the tensor introduced in Subsection 2.3. We will use the notations of Subsection 3.2 for $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$. From Theorem 3.2.2 (a) applied to the point $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$ we get that there exists an isomorphism $(M, (t_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{g}}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{g}}})$ (each t_α with $\alpha \in \mathcal{J}^{\mathrm{g}}$, is the de Rham realization of the Hodge cycle on $A_{B(k)}$ that corresponds naturally to v_α). Thus as $\mathcal{J} \subseteq \mathcal{J}^{\mathrm{g}}$, the statement 3.2.2 (a) holds for the $W(k)$-valued point of $\mathcal{N}^{\mathrm{s}}$ defined by z. From this and the property 3.5.1 (ii) we get that this last point factors through $\mathcal{N}^{\mathrm{m}}$. Thus the $k(v)$-scheme $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is non-empty.

We now assume that (ii) holds; thus $e(v) = 1$. Let $G^{\mathrm{r}}_{\mathbb{Z}_p}$ and μ_v be as in Definition 1.3.2 (b). Let $T^{\mathrm{r}}_{\mathbb{F}_p}$ be a maximal torus of $G^{\mathrm{r}}_{\mathbb{F}_p}$. Due to the existence of μ_v, $T^{\mathrm{r}}_{\mathbb{F}_p}$ has positive rank. The torus $T^{\mathrm{r}}_{\mathbb{F}_p}$ lifts to a torus $T^{\mathrm{r}}_{\mathbb{Z}_p}$ of $G^{\mathrm{r}}_{\mathbb{Z}_p}$, cf. [DG, Vol. II, Exp. XII, Cor. 1.10]. Let $T^{\mathrm{g}}_{0\mathbb{Q}_p}$ be a maximal torus of $G_{\mathbb{Q}_p}$ which has $T^{\mathrm{r}}_{\mathbb{Q}_p}$ as a subtorus. Let T^{g} be a maximal torus of G such that there exists an element $h \in \mathcal{X}$ which factors through $T^{\mathrm{g}}_{\mathbb{R}}$ and moreover $T^{\mathrm{g}}_{\mathbb{Q}_p}$ is $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$-conjugate to $T^{\mathrm{g}}_{0\mathbb{Q}_p}$. Again, the existence of T^{g} is implied by [Ha, Lem. 5.5.3]. Thus (up to $G_{\mathbb{Z}_p}(\mathbb{Z}_p)$-conjugation) we can assume that we have $T^{\mathrm{g}}_{0\mathbb{Q}_p} = T^{\mathrm{g}}_{\mathbb{Q}_p}$.

The intersection $H^{\mathrm{g}} := H \cap T^{\mathrm{g}}(\mathbb{Q}_p)$ is not necessarily the maximal compact, open subgroup of $T^{\mathrm{g}}(\mathbb{Q}_p)$ and the subgroup $T^{\mathrm{g}}(\mathbb{Q})H^{\mathrm{g}}$ of $T^{\mathrm{g}}(\mathbb{Q}_p)$ is not necessarily $T^{\mathrm{g}}(\mathbb{Q}_p)$ itself. However, the intersection $T^{\mathrm{r}}_{\mathbb{Z}_p}(\mathbb{Q}_p) \cap H$ is the unique hyperspecial subgroup $T^{\mathrm{r}}_{\mathbb{Z}_p}(\mathbb{Z}_p)$ of $T^{\mathrm{r}}_{\mathbb{Z}_p}(\mathbb{Q}_p)$. We fix an $O_{(v)}$-monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ as in Definition 1.3.2 (b). As μ_h and $\mu_{v\mathbb{C}}$ are $G(\mathbb{C})$-conjugate and as $G^{\mathrm{r}}_{\mathbb{C}}$ is a normal subgroup of $G_{\mathbb{C}}$, μ_h factors through the intersection $T^{\mathrm{g}}_{\mathbb{C}} \cap G^{\mathrm{r}}_{\mathbb{C}}$ and thus through $T^{\mathrm{r}}_{\mathbb{C}} = T^{\mathrm{r}}_{\mathbb{Z}_p} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{C})$. Thus as $T^{\mathrm{r}}_{\mathbb{Z}_p}$ splits over a finite, unramified extension of $\mathbb{Z}_p$, we get that the field of definition $E(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ of μ_h is a number subfield of $\mathbb{C}$ that contains $E(G, \mathcal{X})$ and that is unramified over v. From class field theory (see [Lan, Th. 4 of p. 220]) and the reciprocity map of [Mi2, pp. 163–164] we easily get that each connected component of $\mathrm{Sh}_{H^{\mathrm{g}}}(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})_{\mathbb{C}}$ is the spectrum of an abelian extension of $E(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ unramified over all primes of $E(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ that divide v. Thus there exists an integral model $\mathcal{Z}^{\mathrm{g}}$ of $\mathrm{Sh}_{H^{\mathrm{g}}}(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ over the normalization of $O_{(v)}$ in $E(T^{\mathrm{g}}_{\mathbb{Q}}, \{h\})$ which has the same properties as above. Let $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$.

Let $(v_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{r}}}$ be a family of tensors of $\mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ such that $T^{\mathrm{r}}_{\mathbb{Q}}$ is the subgroup of $\mathbf{GL}_{W^* \otimes_{\mathbb{Q}} \mathbb{Q}_p}$ that fixes v_α for all $\alpha \in \mathcal{J}^{\mathrm{r}}$. We can assume that $\mathcal{J} \subseteq \mathcal{J}^{\mathrm{r}}$ and that for each $\alpha \in \mathcal{J}$, the tensor v_α is the tensor introduced in Subsection 2.3.

We will use the notations of Subsection 3.2 for $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$ and for k of countable transcendental degree. Let $\rho_D : \mathrm{Gal}(B(k)) \to \mathbf{GL}_{H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Q}_p)} \xrightarrow{\sim} \mathbf{GL}_{L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_p}$ be the p-adic Galois representation associated to the p-divisible group D of A. Let $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$ be the Zariski closure of $\mathrm{Im}(\rho_D)$ in $\mathbf{GL}_{L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_p}$; it is a connected group (cf. B1) which is a subgroup of $T^{\mathrm{g}}_{\mathbb{Q}_p}$. As the groups $T^{\mathrm{g}}_{\mathbb{Q}_p}$ and $T^{\mathrm{r}}_{\mathbb{Q}_p}$ are normalized by $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$, we can speak about the subgroups $\mathcal{T}^{\mathrm{r}}_{B(k)}$ and $\mathcal{T}^{\mathrm{g}}_{B(k)}$ of $\mathcal{G}_{B(k)}$ that correspond to $T^{\mathrm{r}}_{\mathbb{Q}_p}$ and $T^{\mathrm{g}}_{\mathbb{Q}_p}$ (respectively) via Fontaine comparison theory for D (cf. B6). The generic fibre of μ factors through $\mathcal{T}^{\mathrm{g}}_{B(k)}$, cf. Subsection 3.2 applied in the context of $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$. Under the canonical and natural identifications $M \otimes_{W(k)} \mathbb{C} = H^1_{\mathrm{dR}}(A/W(k)) \otimes_{W(k)} \mathbb{C} = H^1(A_{\mathbb{C}}, \mathbb{C}) \xrightarrow{\sim} W^* \otimes_{\mathbb{Q}} \mathbb{C}$ (see B9 and Lemma 2.3.4 (b)), the cocharacter μ_h gets identified with $\mu_{\mathbb{C}}$ (cf. B9.1). As μ_h factors through $T^{\mathrm{r}}_{\mathbb{C}}$, we get that $\mu_{B(k)}$ factors through $\mathcal{T}^{\mathrm{r}}_{B(k)}$. From this and B6 (ii) we get that $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$ is a subgroup of $T^{\mathrm{r}}_{\mathbb{Q}_p}$. This implies that each v_α with $\alpha \in \mathcal{J}^{\mathrm{r}}$ defines naturally an étale Tate-cycle u_α on $D_{B(k)}$.

As $T^{\mathrm{r}}_{\mathbb{Z}_p}$ is a torus, (even for $p = 2$) from B3 applied to the pair $(D, (u_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{r}}})$ we get that there exist isomorphisms $(M, (t_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{r}}}) \xrightarrow{\sim} (H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{r}}})$ $\xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), (v_\alpha)_{\alpha \in \mathcal{J}^{\mathrm{r}}})$ (each $t_\alpha \in \mathcal{T}(M[\frac{1}{p}])$ with $\alpha \in \mathcal{J}^{\mathrm{r}}$, corresponds to u_α via Fontaine comparison theory for D). As $\mathcal{J} \subseteq \mathcal{J}^{\mathrm{r}}$, we get that the image of $z \in \mathcal{Z}^{\mathrm{g}}(W(k))$ in $\mathcal{N}^s(W(k))$ belongs to $\mathcal{N}^{\mathrm{m}}(W(k))$. Thus the $k(v)$-scheme $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is non-empty. □

4.2. Integral models for maximal compact, open subgroups. Let $\tilde{H}$ be a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $\tilde{G}_{\mathbb{Z}_p}$ be a smooth, affine group scheme over $\mathbb{Z}_p$ that extends $G_{\mathbb{Q}_p}$ and such that $\tilde{H} = \tilde{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$, cf. [Ti, p. 52]. Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the smooth, affine group scheme over $\mathbb{Z}_{(p)}$ that extends G and whose extension to $\mathbb{Z}_p$ is $\tilde{G}_{\mathbb{Z}_p}$, cf. [Va1, Claim 3.1.3.1]. Let $\tilde{L}_{(p)}$ be a $\mathbb{Z}_{(p)}$-lattice of W such that the monomorphism $G \hookrightarrow \mathbf{GL}_W$ extends to a homomorphism $\tilde{G}_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_{\tilde{L}_{(p)}}$, cf. [Ja, Part I, 10.9].

4.2.1. Lemma. *We can modify the $\mathbb{Z}$-lattice L of W and the injective map $f : (G, \mathcal{X}) \hookrightarrow (\boldsymbol{GSp}(W, \psi), \mathcal{Y})$, such that we have an identity $H = \tilde{H}$ and $L_{(p)}$ is a $\tilde{G}_{\mathbb{Z}_{(p)}}$-module.*[1]

Proof: Let $\tilde{L}$ be the $\mathbb{Z}$-lattice of W such that we have $\tilde{L}[\frac{1}{p}] = L[\frac{1}{p}]$ and $\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \tilde{L}_{(p)}$. If ψ induces a perfect form on $\tilde{L}$, then by replacing L with $\tilde{L}$ we get that $H = \tilde{H}$. [Argument: as $\tilde{H}$ is a maximal compact subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, the monomorphism $\tilde{H} \hookrightarrow G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathbf{GL}_{\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p}(\mathbb{Z}_p)$ is an isomorphism.] If ψ does not induces a perfect form on $\tilde{L}$, then we will need to modify f as follows.

Let $L'_1 := \tilde{L} \oplus \tilde{L}^*$. Let $W_1 := L'_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L'_{1(p)} := L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let ψ'_1 be a perfect alternating form on L'_1 such that the group scheme $\mathbf{SL}_{\tilde{L}}$, when viewed naturally as a subgroup scheme of $\mathbf{SL}_{L'_1}$, is in fact a subgroup scheme of $\mathbf{Sp}(L'_1, \psi'_1)$. Thus $\tilde{L}$ and $\tilde{L}^*$ are both maximal isotropic $\mathbb{Z}$-lattices of W_1 with respect to ψ'_1. Let G^0 be the identity component of the intersection $G \cap \mathbf{Sp}(W, \psi)$ (one can easily check that in fact we have $G^0 = G \cap \mathbf{Sp}(W, \psi)$). Let $\tilde{G}^0_{\mathbb{Z}_{(p)}}$ be the Zariski closure in $\tilde{G}_{\mathbb{Z}_{(p)}}$ of G^0; it is a closed subgroup scheme of $\mathbf{SL}_{\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}}$ and thus also of $\mathbf{GSp}(L'_{1(p)}, \psi'_1)$. The subgroup scheme of $\mathbf{GSp}(L'_{1(p)}, \psi'_1)$ generated by $Z(\mathbf{GL}_{L'_{1(p)}})$ and $\tilde{G}^0_{\mathbb{Z}_{(p)}}$ is a group scheme which is naturally identified with $\tilde{G}_{\mathbb{Z}_{(p)}}$ itself.

Let $h \in \mathcal{X}$. Let $\mathfrak{A}$ be the free $\mathbb{Z}_{(p)}$-module of alternating forms on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ that are fixed by $\tilde{G}^0_{\mathbb{Z}_{(p)}}$. There exist elements of $\mathfrak{A} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R}$ that define polarizations of the Hodge $\mathbb{Q}$–structure on W_1 defined by h, cf. [De2, Cor. 2.3.3]. Thus the real vector space $\mathfrak{A} \otimes_{\mathbb{Z}_{(2)}} \mathbb{R}$ has a non-empty, open subset of such polarizations, cf. [De2, Subsubsection 1.1.18 (a)]. A standard application to $\mathfrak{A}$ of the approximation theory for independent valuations, implies the existence of an alternating form ψ_1 on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ that is fixed by $\tilde{G}^0_{\mathbb{Z}_{(p)}}$, that is congruent to ψ'_1 modulo p, and that defines a polarization of the Hodge $\mathbb{Q}$–structure on W_1 defined by h. Thus there exists an injective map $f_1 : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W_1, \psi_1), \mathcal{Y}_1)$ of Shimura pairs.

As ψ_1 is congruent to ψ'_1 modulo p, it is a perfect, alternating form on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let L_1 be a $\mathbb{Z}$-lattice of W_1 such that ψ_1 induces a perfect alternating form on L_1 and we have $L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. As above we argue that $\tilde{H} = G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathbf{GL}_{L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p}(\mathbb{Z}_p)$. $\square$

4.2.2. Corollary. *Let $\tilde{H}$ be a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be a smooth, affine group scheme over $\mathbb{Z}_{(p)}$ that has G as its generic fibre and such that*

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1 We emphasize that the resulting homomorphism $\tilde{G}_{\mathbb{Z}_{(p)}} \to \mathbf{GL}_{L_{(p)}}$ of smooth group schemes over $\mathbb{Z}_{(p)}$, is not necessarily a closed embedding.

$\tilde{H} = \tilde{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ *(see beginning of Subsection 4.2). We assume that one of the following two conditions holds:*

(i) *the special fibre $\tilde{G}_{\mathbb{F}_p}$ of $\tilde{G}_{\mathbb{Z}_p}$ has a torus of the same rank as G;*

(ii) *we have $e(v) = 1$ and the group scheme $\tilde{G}_{\mathbb{Z}_{(p)}}$ is quasi-reductive for $(G, \mathcal{X}, v)$.*

Then there exists a unique regular, formally smooth integral model $\tilde{\mathcal{N}}^{\mathrm{s}}$ of $\mathrm{Sh}_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of index of ramification 1 and is a faithfully flat $O_{(v)}$-algebra, then each morphism $Z_{E(G,\mathcal{X})} \to \tilde{\mathcal{N}}^{\mathrm{s}}_{E(G,\mathcal{X})}$ extends uniquely to a morphism $Z \to \tilde{\mathcal{N}}^{\mathrm{s}}$ between $O_{(v)}$-schemes.

Proof: We can assume that the injective map $f : (G, \mathcal{X}) \to (\mathbf{GSp}(W, \psi), \mathcal{Y})$ of Shimura pairs is such that $\tilde{H} = H$ and $L_{(p)}$ is a $\tilde{G}_{\mathbb{Z}_{(p)}}$-module, cf. Lemma 4.2.1. If (i) holds, then the condition 4.1 (i) holds. If (ii) holds, let $\tilde{G}^{\mathrm{r}}_{\mathbb{Z}_p}$ be a reductive, normal, closed subgroup scheme of $\tilde{G}_{\mathbb{Z}_p}$ such that there exists a cocharacter $\mu_v : \mathbb{G}_{mW(k(v))} \to \tilde{G}^{\mathrm{r}}_{W(k(v))}$ with the property that the extension of μ_v to $\mathbb{C}$ via an (any) $O_{(v)}$-monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ defines a cocharacter of $G_{\mathbb{C}}$ that is $G(\mathbb{C})$-conjugate to the cocharacters μ_h ($h \in \mathcal{X}$) introduced in the beginning of Subsection 1.3. The group $G^{\mathrm{der}}_{\mathbb{C}}$ has no simple factors that are $\mathbf{SO}_{2n+1}$ groups for some $n \in \mathbb{N}$, cf. Fact 2.2.3. Thus the natural homomorphism $\tilde{G}^{\mathrm{r}}_{\mathbb{Z}_p} \to \mathbf{GL}_{L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p}$ is a closed embedding, cf. [Va5, Prop. 2.5.2 (c)]. Thus $\tilde{G}^{\mathrm{r}}_{\mathbb{Z}_p}$ is naturally a closed subgroup scheme of $G_{\mathbb{Z}_p}$. This implies that the group scheme $G_{\mathbb{Z}_{(p)}}$ is also quasi-reductive for $(G, \mathcal{X}, v)$. Thus if (ii) holds, then the condition 4.1 (ii) holds.

As one of the two conditions 4.1 (i) and (ii) holds, the $k(v)$-scheme $\mathcal{N}^{\mathrm{s}}_{k(v)}$ is non-empty (cf. Lemma 4.1). Based on Theorem 1.5 (a) and the fact that $\tilde{H} = H$, we get that as $\tilde{\mathcal{N}}^{\mathrm{s}}$ we can take $\mathcal{N}^{\mathrm{s}}$ itself. □

4.2.3. Corollary. *Let $(G, \mathcal{X})$ be a Shimura pair of Hodge type. Let v a prime of the reflex field $E(G, \mathcal{X})$ that divides a prime p with the property that the group $G_{\mathbb{Q}_p}$ is unramified. Then for each hyperspecial subgroup $\tilde{H}$ of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, there exists a unique regular, formally smooth integral model $\tilde{\mathcal{N}}^{\mathrm{s}}$ of $\mathrm{Sh}_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of index of ramification 1 and is a faithfully flat $O_{(v)}$-algebra, then each morphism $Z_{E(G,\mathcal{X})} \to \tilde{\mathcal{N}}^{\mathrm{s}}_{E(G,\mathcal{X})}$ extends uniquely to a morphism $Z \to \tilde{\mathcal{N}}^{\mathrm{s}}$ between $O_{(v)}$-schemes.*

Proof: As $\tilde{H}$ is a hyperspecial subgroup, we can assume that the group scheme $\tilde{G}_{\mathbb{Z}_p}$ is reductive. This implies that $\tilde{G}_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$. Thus the condition 4.2.2 (i) holds. Thus the Corollary follows from Corollary 4.2.2. □

5. Proof of the Main Theorem

In this section we take k to be a field extension of $k(v)$ that is algebraically closed and has a countable transcendental degree. Let the notations $(v_\alpha)_{\alpha \in \mathcal{J}}$, $(w^{\mathcal{A}}_\alpha)_{\alpha \in \mathcal{J}}$, and $\pi_\dagger$ be as in Subsection 2.3. For a point $z \in \mathcal{N}^{\mathrm{s}}(W(k)) = \mathcal{N}(W(k))$, the following notations $(A, \lambda_A, (w_\alpha)_{\alpha \in \mathcal{J}})$, $(M, F^1, \phi, \psi_M, (t_\alpha)_{\alpha \in \mathcal{J}})$, $M = F^1 \oplus F^0$, and $\mu : \mathbb{G}_{mW(k)} \to \mathcal{G}$ are as in

Subsection 3.2. In Subsections 5.1 to 5.7 we prove the Main Theorem 1.6. In Subsection 5.8 we prove the Main Corollary 1.7.

Let $R_0 := W(k)[[x]]$, where x is an independent variable. Let Φ_{R_0} be the Frobenius lift of R_0 that is compatible with σ and that takes x to x^p. Let $Z_0 := \mathrm{Spec}(R_0)$.

5.1. Basic notations and facts. We begin the proof of the Main Theorem 1.6 by introducing some basic notations and facts. We have $e(v) = 1$ and $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for $(G, \mathcal{X}, v)$. We recall that $\mathcal{N}^{\mathrm{m}}$ is an open subscheme of $\mathcal{N}^{\mathrm{s}}$ (cf. Subsubsection 3.5.1) and therefore also of $\mathcal{N}$ (cf. Lemma 2.2.2). Thus $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is also an open subscheme of $\mathcal{N}_{k(v)}$. Moreover, the open embedding $\mathcal{N}^{\mathrm{m}} \hookrightarrow \mathcal{N}$ is a pro-étale cover of an open embedding between quasi-projective $O_{(v)}$-schemes (cf. Proposition 2.2.1 (a) and the property 3.5.1 (i)) and the $k(v)$-scheme $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is non-empty (cf. Lemma 4.1). Thus to show that $\mathcal{N}^{\mathrm{m}}_{k(v)}$ is a non-empty, open closed subscheme of $\mathcal{N}_{k(v)}$, we only need to show that for each commutative diagram of the following type

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[[x]]) & \longleftarrow & \mathrm{Spec}(k((x))) \\ \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_{k((x))}} \\ \mathcal{N} & \longleftarrow & \mathcal{N}_{k(v)} & \longleftarrow & \mathcal{N}^{\mathrm{m}}_{k(v)}, \end{array} \tag{3}$$

the morphism $y : \mathrm{Spec}(k) \to \mathcal{N}$ factors through the open subscheme $\mathcal{N}^{\mathrm{m}}$ of $\mathcal{N}$. All the horizontal arrows of the diagram (3) are natural embeddings. Until Subsection 5.5 we study different properties of the diagram (3) that are needed to prove Theorems 1.6 (a) to (c) in Subsections 5.5 to 5.7 (respectively).

We consider the principally quasi-polarized filtered F-crystal

$$(M_0, \Phi_0, \nabla_0, \psi_{M_0})$$

over $k[[x]]$ of $q^*((\mathcal{A}, \lambda_{\mathcal{A}}) \times_{\mathcal{N}} \mathcal{N}_{k(v)})$. Thus M_0 is a free R_0-module of rank $2r$, Φ_0 is a Φ_{R_0}-linear endomorphism of M_0, and ∇_0 is an integrable and nilpotent modulo p connection on M_0 such that we have $\nabla_0 \circ \Phi_0 = (\Phi_0 \otimes d\Phi_{R_0}) \circ \nabla_0$.

Let O be the unique local ring of R_0 that is a discrete valuation ring of mixed characteristic $(0, p)$. Let $\mathcal{O}$ be the completion of O. Let $\Phi_{\mathcal{O}}$ be the Frobenius lift of $\mathcal{O}$ defined by Φ_{R_0} via a natural localization and completion. Let $k_1 := \overline{k((x))}$. Let

$$\mathrm{Spec}(W(k_1)) \to Z_0$$

be the Teichmüller lift with respect to Φ_{R_0}; under it $W(k_1)$ gets naturally the structure of a $*$-algebra, where $* \in \{R_0, O, \mathcal{O}\}$.

As the $O_{(v)}$-scheme $\mathcal{N}^{\mathrm{m}}$ is formally smooth, there exists a lift $\tilde{z}_1 : \mathrm{Spec}(\mathcal{O}) \to \mathcal{N}^{\mathrm{m}}$ of the morphism $q_{k((x))} : \mathrm{Spec}(k((x))) \to \mathcal{N}^{\mathrm{m}}$ defined naturally by $q_{k((x))}$ and denoted in the same way. Let

$$(\tilde{A}_1, \lambda_{\tilde{A}_1}, (w_{1\alpha})_{\alpha \in \mathcal{J}}) := \tilde{z}_1^*(\mathcal{A}, \lambda_{\mathcal{A}}, (w_\alpha)_{\alpha \in \mathcal{J}}).$$

Let $t_{1\alpha}$ be the de Rham realization of $w_{1\alpha}$. We identify canonically $M_0 \otimes_{R_0} \mathcal{O} = H^1_{\mathrm{dR}}(\tilde{A}_1/\mathcal{O})$ (cf. [Be, Ch. V, Subsection 2.3]) and thus we view each $t_{1\alpha}$ as a tensor of $\mathcal{T}(M_0 \otimes_{R_0} \mathcal{O})[\frac{1}{p}]$.

For $\alpha \in \mathcal{J}$ let $n(\alpha) \in \mathbb{N} \cup \{0\}$ be the unique number such that we have $v_\alpha \in W^{*\otimes n(\alpha)} \otimes_{\mathbb{Q}} W^{\otimes n(\alpha)} \subseteq \mathcal{T}(W^*)$, cf. the definition of v_α in Subsection 2.3. Let $n_\alpha \in \mathbb{N} \cup \{0\}$ be the smallest number such that

$$p^{n_\alpha} t_{1\alpha} \in (M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n_\alpha}) \otimes_{R_0} \mathcal{O} \subseteq \mathcal{T}(M_0 \otimes_{R_0} \mathcal{O}).$$

5.1.1. Proposition. *For all $\alpha \in \mathcal{J}$ we have $p^{n_\alpha} t_{1\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n(\alpha)} \subseteq \mathcal{T}(M_0)$.*

Proof: The tensor $p^{n_\alpha} t_{1\alpha}$ is fixed by the natural σ_{k_1}-linear automorphism of $\mathcal{T}(M_0 \otimes_{R_0} B(k_1))$ defined by Φ_0 (see Subsection 3.2). Thus (as $\mathrm{Spec}(W(k_1)) \to Z_0$ is a Teichmüller lift) the tensor $p^{n_\alpha} t_{1\alpha}$ is also fixed by the natural $\Phi_{\mathcal{O}}$-linear endomorphism of $\mathcal{T}(M_0 \otimes_{R_0} \mathcal{O})[\frac{1}{p}]$ defined by Φ_0.

The field $k((x))$ has $\{x\}$ as a p-basis i.e., $\{1, x, \ldots, x^{p-1}\}$ is a basis of $k((x))$ over $k((x))^p = k((x^p))$. Thus the p-adic completion of the $\mathcal{O}$-module $\Omega_{\mathcal{O}/W(k)}$ of relative differentials is naturally isomorphic to $\mathcal{O}dx$, cf. [BM, Prop. 1.3.1]. Let $\nabla_0 : M_0 \otimes_{R_0} \mathcal{O} \to M_0 \otimes_{R_0} \mathcal{O}dx$ be the connection which is the natural extension of the connection ∇_0 on M_0.

The de Rham component of $w_\alpha^{\mathcal{A}}$ is annihilated by the Gauss–Manin connection of $\mathcal{A}$ (this is a property of Hodge cycles, for instance it follows from [De3, Prop. 2.5] applied in the context of a quotient of $\mathrm{Sh}_H(G, \mathcal{X})$ by a small compact, open subgroup of $G(\mathbb{A}_f^{(p)})$). Thus the tensor $p^{n_\alpha} t_{1\alpha}$ is annihilated by the Gauss–Manin connection on $\mathcal{T}(H^1_{\mathrm{dR}}(\tilde{A}_1/\mathcal{O})) = \mathcal{T}(M_0 \otimes_{R_0} \mathcal{O})$ of $\tilde{A}_1$ and thus also by the p-adic completion of this last connection. In other words, $p^{n_\alpha} t_{1\alpha}$ is annihilated by the connection $\nabla_0 : M_0 \otimes_{R_0} \mathcal{O} \to M_0 \otimes_{R_0} \mathcal{O}dx$ (cf. [Be, Ch. V, Prop. 3.6.4]).

As the field $k((x))$ has a p-basis, each F-crystal over $k((x))$ is uniquely determined by its evaluation at the thickening naturally associated to the closed embedding $\mathrm{Spec}(k((x))) \hookrightarrow \mathrm{Spec}(\mathcal{O})$ (cf. [BM, Prop. 1.3.3]). Thus the natural identification

$$(M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n_\alpha}) \otimes_{R_0} \mathcal{O} = \mathrm{End}(M_0^{\otimes n(\alpha)} \otimes_{R_0} \mathcal{O})$$

allows us to view $p^{n_\alpha} t_{1\alpha}$ as an endomorphism of the F-crystal over $k((x))$ defined by the tensor product of $n(\alpha)$-copies of $(M_0 \otimes_{R_0} \mathcal{O}, \Phi_0 \otimes \Phi_{\mathcal{O}}, \nabla_0)$. From this and Theorem 3.1 we get that $p^{n_\alpha} t_{1\alpha}$ is (the crystalline realization of) an endomorphism of the F-crystal over $k[[x]]$ defined by the tensor product of $n(\alpha)$-copies of (M_0, Φ_0, ∇_0). This implies that $p^{n_\alpha} t_{1\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{*\otimes n(\alpha)} \subseteq \mathcal{T}(M_0)$. $\square$

5.1.2. Group schemes. Next we introduce several notations that pertain to group schemes. Let $G^{\mathrm{r}}_{\mathbb{Z}_p}$ be a reductive, normal, closed subgroup scheme of $G_{\mathbb{Z}_p}$ as in Definition 1.3.2 (b); we emphasize that in general it is not the pull back to $\mathrm{Spec}(\mathbb{Z}_p)$ of a closed subgroup scheme of $G_{\mathbb{Z}_{(p)}}$. Let

$$G^{\mathrm{rad}}_{\mathbb{Q}_p} = \prod_{i \in I^{\mathrm{r}}} G^{\mathrm{r}}_{i\mathbb{Q}_p}$$

be the product decomposition into $\mathbb{Q}_p$-simple, adjoint groups. Let $G^{\mathrm{rder}}_{i\mathbb{Q}_p}$ be the normal, semisimple subgroup of $G^{\mathrm{rder}}_{\mathbb{Q}_p}$ whose adjoint is $G^{\mathrm{r}}_{i\mathbb{Q}_p}$.

Let $\pi^{\mathrm{r}} \in \mathrm{End}(M_0 \otimes_{R_0} B(k_1))$ (resp. $\pi_i^{\mathrm{r}} \in \mathrm{End}(M_0 \otimes_{R_0} B(k_1))$) be the tensor that corresponds to the projector $\pi_{G^{\mathrm{r}}_{\mathbb{Q}_p}}$ (resp. to $\pi_{G^{\mathrm{rder}}_{i\mathbb{Q}_p}}$) of Subsection 2.3 via Fontaine comparison theory for (the p-divisible group of) $\tilde{A}_{1W(k_1)}$, cf. B6. By enlarging the family $(v_\alpha)_{\alpha\in\mathcal{J}}$, we can assume that the projectors $\pi_{G^{\mathrm{r}}_{\mathbb{Q}_p}}$ and $\pi_{G^{\mathrm{rder}}_{i\mathbb{Q}_p}}$ with $i \in I^{\mathrm{r}}$ are $\mathbb{Q}_p$-linear combinations of the v_α's (this is so as these projectors are fixed by $G_{\mathbb{Q}_p}$). Thus π^{r} and π_i^{r}'s are linear combinations of $t_{1\alpha}$'s. From this and Proposition 5.1.1 we get that in fact we have $\pi^{\mathrm{r}}, \pi_i^{\mathrm{r}} \in \mathrm{End}(M_0[\frac{1}{p}])$. Thus there exists $n^{\mathrm{r}} \in \mathbb{N}\cup\{0\}$ such that both $p^{n^{\mathrm{r}}}\pi^{\mathrm{r}}$ and $p^{n^{\mathrm{r}}}\pi_i^{\mathrm{r}}$ belong to $\mathrm{End}(M_0)$ and are $\mathbb{Z}_p$-linear combinations of the $t_{1\alpha}$'s with $\alpha \in \mathcal{J}$.

By enlarging the family $(v_\alpha)_{\alpha\in\mathcal{J}}$, we can also assume that each element of $\mathrm{End}(L^*_{(p)}) = L^*_{(p)} \otimes L_{(p)}$ fixed by $G_{\mathbb{Z}_{(p)}}$ is v_{α_0} for some $\alpha_0 \in \mathcal{J}$. Let $Z^0(G^{\mathrm{r}}_{\mathbb{Z}_p})$ be the maximal subtorus of $Z(G^{\mathrm{r}}_{\mathbb{Z}_p})$. Let $Z^{\mathrm{r}}_{\mathbb{Z}_p}$ be the center of the centralizer of $Z^0(G^{\mathrm{r}}_{\mathbb{Z}_p})$ in $\mathbf{GL}_{L_{(p)}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p}$; it is a torus over $\mathbb{Z}_p$ that contains $Z^0(G^{\mathrm{r}}_{\mathbb{Z}_p})$. Let $\mathcal{B}^{\mathrm{r}}$ be the commutative, semisimple $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p)$ whose elements are the elements of $\mathrm{Lie}(Z^{\mathrm{r}}_{\mathbb{Z}_p})$. Each element $e \in \mathcal{B}^{\mathrm{r}}$ is a $\mathbb{Z}_p$-linear combination of endomorphisms of $L^*_{(p)}$ fixed by $G_{\mathbb{Z}_{(p)}}$ and thus it defines naturally a $\mathbb{Z}_p$-endomorphism e of $\mathcal{A}$. For simplicity we denote also by $e \in \mathrm{End}(M_0)$ the crystalline realization of the $\mathbb{Z}_p$-endomorphism $q^*(e)$ of $q^*(\mathcal{A} \times_{\mathcal{N}} \mathcal{N}_{k(v)})$.

Let η be the field of fractions of R_0. Let $\mathcal{G}_{0\eta}$ be the subgroup of $\mathbf{GL}_{M_0\eta}$ that fixes $p^{n_\alpha}t_{1\alpha}$ for all $\alpha \in \mathcal{J}$ (this definition makes sense due to Proposition 5.1.1). The group $G_{0B(k_1)}$ corresponds to $G_{\mathbb{Q}_p}$ via Fontaine comparison theory for (the p-divisible group of) $\tilde{A}_{1B(k_1)}$. This implies that $\mathcal{G}_{0\eta}$ is a reductive group.

5.1.3. Lemma. *There exists (resp. for $i \in I^{\mathrm{r}}$ there exists) a unique reductive (resp. semisimple) subgroup $\mathcal{G}^{\mathrm{r}}_{0\eta}$ (resp. $\mathcal{G}^{\mathrm{rder}}_{0i\eta}$) of $\mathcal{G}_{0\eta}$ whose Lie algebra is Im(π^{r}) (resp. is Im(π_i^{r})). The subgroup $\mathcal{G}^{\mathrm{r}}_{0\eta}$ (resp. $\mathcal{G}^{\mathrm{rder}}_{0i\eta}$) of $\mathcal{G}_{0\eta}$ is normal. Moreover each geometric pull back of $\mathcal{G}^{\mathrm{rder}}_{0\eta}$ has no normal subgroup which is an $\boldsymbol{SO}_{2n+1}$ group for some $n \in \mathbb{N}$.*

Proof: We will prove the Lemma only for $\mathcal{G}^{\mathrm{r}}_{0\eta}$, as the arguments for $\mathcal{G}^{\mathrm{rder}}_{0i\eta}$ are the same. From Fontaine comparison theory for (the p-divisible group of) $\tilde{A}_{1W(k_1)}$ we get that there exists a unique reductive subgroup $\mathcal{G}^{\mathrm{r}}_{0B(k_1)}$ of $\mathbf{GL}_{M_0\otimes_{R_0}B(k_1)}$ whose Lie algebra is $\mathrm{Im}(\pi^{\mathrm{r}})\otimes_\eta B(k_1)$, cf. B6 (i). From A2 (a) applied with $(W, \mathcal{L}, \eta, \eta_1) = (M_0 \otimes_{R_0} \eta, \mathrm{Im}(\pi^{\mathrm{r}}), \eta, B(k_1))$, we get that there exists a unique reductive subgroup $\mathcal{G}^{\mathrm{r}}_{0\eta}$ of $\mathbf{GL}_{M_0\otimes_{R_0}\eta}$ whose Lie algebra is $\mathrm{Im}(\pi^{\mathrm{r}})$. The group $\mathcal{G}^{\mathrm{r}}_{0\eta}$ is a subgroup of $\mathcal{G}_{0\eta}$, as this holds after extension to $B(k_1)$. Thus the first part of the Lemma holds.

But π^{r} is fixed by $\mathcal{G}_{0\eta}$ (as this holds after tensorization with $B(k_1)$, cf. B6) and thus $\mathrm{Im}(\pi^{\mathrm{r}})$ is a $\mathcal{G}_{0\eta}$-submodule of $\mathrm{Lie}(\mathcal{G}_{0\eta})$. From this and the uniqueness part of the Lemma, we get that $\mathcal{G}^{\mathrm{r}}_{0\eta}$ is a subgroup of $\mathcal{G}_{0\eta}$ normalized by $\mathcal{G}_{0\eta}(\eta)$ and thus also by $\mathcal{G}_{0\eta}$. As $\mathcal{G}^{\mathrm{r}}_{0B(k_1)}$ corresponds to the normal subgroup $G^{\mathrm{r}}_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ via Fontaine comparison theory for (the p-divisible group of) $\tilde{A}_{1W(k_1)}$, from Fact 2.2.3 we get that each geometric pull back of $\mathcal{G}^{\mathrm{rder}}_{0\eta}$ has no normal subgroup which is an $\mathbf{SO}_{2n+1}$ group for some $n \in \mathbb{N}$. $\square$

5.2. Key Theorem. *Let $\mathcal{G}^{\mathrm{r}}_0$ be the Zariski closure of $\mathcal{G}^{\mathrm{r}}_{0\eta}$ in $\boldsymbol{GL}_{M_0}$. Then the closed subscheme $\mathcal{G}^{\mathrm{r}}_0$ of $\boldsymbol{GL}_{M_0}$ is a reductive subgroup scheme.*

Proof: We check that this Theorem is only a particular case of [Va5, Thm. 6.3 (b)]. Let $\mathfrak{C}_0$ be the F-crystal over $k[[x]]$ defined by (M_0, Φ_0, ∇_0). Let $\mathfrak{C}_{0k((x))}$ be the F-crystal over $k((x))$ which is the natural pull back of $\mathfrak{C}_0$. Let $(\tilde{t}_\alpha)_{\alpha\in\mathcal{J}_p}$ be the family of tensors formed by $\mathbb{Z}_p$-linear combinations of the tensors of the family $(p^{n_\alpha} t_{1\alpha})_{\alpha\in\mathcal{J}}$. We can assume that there exists a natural number $s \geq 4$ such that we have $s \geq n_\alpha$ for all $\alpha \in \mathcal{J}$. From Proposition 5.1.1 we get that $(\tilde{t}_\alpha)_{\alpha\in\mathcal{J}_p}$ is a family of endomorphism of $\mathfrak{D}_1 := \oplus_{l=1}^s \mathfrak{C}_{0k((x))}^{\otimes l}$, where $\mathfrak{C}_{0k((x))}^{\otimes l}$ is the tensor product of l copies of $\mathfrak{C}_{0k((x))}$. Let $\mathcal{K} := \mathcal{O}[\frac{1}{p}]$. We check that the six axioms of [Va5, Subsection 6.2] hold in the context of $(\tilde{t}_\alpha)_{\alpha\in\mathcal{J}_p}$ and of the subgroup $\mathcal{G}_{0\mathcal{K}}^{\mathrm{r}}$ of $\mathcal{G}_{0\mathcal{K}}$.

As $\tilde{z}_1 \in \mathcal{N}^{\mathrm{m}}(\mathcal{O})$, there exists isomorphisms $(M_0 \otimes_{R_0} W(k_1), (t_{1\alpha})_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L_{(p)}^* \otimes_{\mathbb{Z}_{(p)}} W(k_1), (v_\alpha)_{\alpha\in\mathcal{J}})$. This implies that the Zariski closure of $\mathcal{G}_{0B(k_1)}^{\mathrm{r}}$ in $\mathbf{GL}_{M_0\otimes_{R_0} W(k_1)}$ is isomorphic to $G_{W(k_1)}^{\mathrm{r}}$ and thus it is a reductive group scheme over $W(k_1)$. Therefore the Zariski closure $\mathcal{G}_{0\mathcal{O}}^{\mathrm{r}}$ of $\mathcal{G}_{0\mathcal{K}}^{\mathrm{r}}$ in $\mathbf{GL}_{M_0\otimes_{R_0}\mathcal{O}}$ is a reductive group scheme over $\mathcal{O}$. Thus the axiom [Va5, 6.2 (i)] holds.

Let $\mathcal{J}_c$ be the subset of $\mathcal{J}_p$ such that $\{\tilde{t}_\alpha|\alpha \in \mathcal{J}_c\}$ corresponds to $\mathrm{Lie}(Z_{\mathbb{Z}_p}^{\mathrm{r}})$ via Fontaine comparison theorem for (the p-divisible group of) $\tilde{A}_{1W(k_1)}$. Thus we can identify naturally $\mathcal{B}^{\mathrm{r}}$ with a $\mathbb{Z}_p$-subalgebra of $\mathrm{End}(M_0)$. The centralizer of $\mathcal{B}^{\mathrm{r}} \otimes_{\mathbb{Z}_p} \mathcal{O}$ in $\mathbf{GL}_{M_0\otimes_{R_0}\mathcal{O}}$ is a torus over $\mathcal{O}$ which contains $Z^0(\mathcal{G}_{0\mathcal{O}}^{\mathrm{r}})$ (as one can easily check this over $W(k_1)$). Thus the axiom [Va5, 6.2 (ii)] holds.

If $\mathcal{J}_d := \{\alpha \in \mathcal{J}_p|\tilde{t}_\alpha \in \{p^{n_{\mathrm{r}}}\pi_i^{\mathrm{r}}|i \in I^{\mathrm{r}}\}\}$, then $\mathrm{Lie}(\mathcal{G}_{0i\mathcal{K}}^{\mathrm{rder}}) = p^{n_{\mathrm{r}}}\pi_i^{\mathrm{r}}(\mathrm{End}(M_0 \otimes_{R_0} \mathcal{K}))$ and therefore the axiom [Va5, 6.2 (iii)] holds. As for all $i \in I^{\mathrm{r}}$ the adjoint group $G_{i\mathbb{Q}_p}^{\mathrm{r}}$ is simple, the Killing form on $\mathrm{Lie}(G_{i\mathbb{Q}_p}^{\mathrm{rder}}) = \mathrm{Lie}(G_{i\mathbb{Q}_p}^{\mathrm{r}})$ is a non-zero rational multiple of the restriction to $\mathrm{Lie}(G_{i\mathbb{Q}_p}^{\mathrm{rder}}) = \mathrm{Lie}(G_{i\mathbb{Q}_p}^{\mathrm{r}})$ of the trace form $\mathfrak{T}$ on $\mathrm{End}(W) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. This implies that the Killing form on $\mathrm{Lie}(\mathcal{G}_{0i\mathcal{K}}^{\mathrm{rder}})$ is a non-zero rational multiple of the restriction to $\mathrm{Lie}(\mathcal{G}_{0i\mathcal{K}}^{\mathrm{rder}})$ of the trace form on $\mathrm{End}(M_0) \otimes_{R_0} \mathcal{K}$. Thus the axiom [Va5, 6.2 (iv)] holds. As $G_{\mathbb{Q}_p}^{\mathrm{rad}} = \prod_{i\in I^{\mathrm{r}}} G_{i\mathbb{Q}_p}^{\mathrm{r}}$, it is easy to see that we have a natural isogeny $\prod_{i\in I^{\mathrm{r}}} \mathcal{G}_{0i\mathcal{K}}^{\mathrm{rder}} \to \mathcal{G}_{0\mathcal{K}}^{\mathrm{rder}}$. Thus the axiom [Va5, 6.2 (v)] holds. The fact that the axiom [Va5, 6.2 (vi)] holds follows from the last part of Lemma 5.1.3.

As axioms [Va5, 6.2 (i) to (vi)] hold, the Theorem follows from [Va5, Thm. 6.3 (b)].□

5.3. Applying 5.2. Let $(A_1, \lambda_{A_1}) := z_1^*(\mathcal{A}, \lambda_{\mathcal{A}}) = (\tilde{A}_1, \lambda_{\tilde{A}_1})_{W(k_1)}$. Let $(M_1, F_1^1, \phi_1, \psi_{M_1})$ be the principally quasi-polarized filtered F-crystal over k_1 of (A_1, λ_{A_1}). Let $\mathcal{G}_1$ and $\mu_1 : \mathbb{G}_{mW(k_1)} \to \mathcal{G}_1$ be the analogues of $\mathcal{G}$ and $\mu : \mathbb{G}_{mW(k)} \to \mathcal{G}$ but obtained working with $z_1 \in \mathcal{N}^{\mathrm{s}}(W(k_1))$ instead of some $z \in \mathcal{N}(W(k))$. We can identify naturally $M_1 = M_0 \otimes_{R_0} W(k_1)$. Thus we can view each tensor $t_{1\alpha}$ as a tensor of $\mathcal{T}(M_1)[\frac{1}{p}]$ and we can also view the reductive group scheme $\mathcal{G}_{0W(k_1)}^{\mathrm{r}}$ as a normal, closed subgroup scheme of the Zariski closure $\mathcal{G}_1$ of $\mathcal{G}_{0B(k_1)}$ in $\mathbf{GL}_{M_1}$.

We fix an $O_{(v)}$-monomorphism $W(k_1) \hookrightarrow \mathbb{C}$. We have canonical isomorphisms

$$\rho_{1\mathbb{C}} : (M_0 \otimes_{R_0} W(k_1) \otimes_{W(k_1)} \mathbb{C}, (t_{1\alpha})_{\alpha\in\mathcal{J}}) = (M_1 \otimes_{W(k_1)} \mathbb{C}, (t_{1\alpha})_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (W^* \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha\in\mathcal{J}})$$

such that $F_1^1 \otimes_{W(k_1)} \mathbb{C}$ is mapped to the Hodge filtration of $W^* \otimes_{\mathbb{Q}} \mathbb{C}$ defined by a cocharacter $\mu_h : \mathbb{G}_{m\mathbb{C}} \to G_{\mathbb{C}}$ introduced in Subsection 1.1 (see B9 and Lemma 2.3.4 (b)). We know that $\mu_{1\mathbb{C}}$ is $\mathcal{G}_1(\mathbb{C})$-conjugate to some (any) μ_h, cf. B9.1. From this and the Definition 1.3.2 (b) we get that μ_1 factors through $\mathcal{G}_{0W(k_1)}^{\mathrm{r}}$.

Let $\bar{F}_0^1$ be the kernel of Φ_0 modulo p; it is a free module over $k[[x]] = R_0/pR_0$ of rank r. As the cocharacter μ_1 factors through $\mathcal{G}^{\mathrm{r}}_{0W(k_1)}$, the normalizer of $\bar{F}_0^1 \otimes_{k[[x]]} k_1$ in $\mathcal{G}^{\mathrm{r}}_{0k_1}$ is a parabolic subgroup of $\mathcal{G}^{\mathrm{r}}_{0k_1}$ and thus (as $\bar{F}_0^1 \otimes_{k[[x]]} k_1$ is defined over $k((x))$) it is also the natural pull back of a parabolic subgroup $\mathcal{F}^{\mathrm{r}}_{0k((x))}$ of $\mathcal{G}^{\mathrm{r}}_{0k((x))}$. The $k[[x]]$-scheme of parabolic subgroup schemes of $\mathcal{G}^{\mathrm{r}}_{0k[[x]]}$ is projective, cf. [DG, Vol. III, Exp. XXVI, Cor. 3.5]. Thus the Zariski closure $\mathcal{F}^{\mathrm{r}}_{0k[[x]]}$ of $\mathcal{F}^{\mathrm{r}}_{0k((x))}$ in $\mathcal{G}^{\mathrm{r}}_{0k[[x]]}$ is a parabolic subgroup scheme of $\mathcal{G}^{\mathrm{r}}_{0k[[x]]}$. As $\mathcal{G}^{\mathrm{r}}_0$ is a split reductive group scheme and as μ_{1k_1} factors through $\mathcal{G}^{\mathrm{r}}_{0k_1}$, there exists a cocharacter $\mu_{0k[[x]]} : \mathbb{G}_{mk[[x]]} \to \mathcal{G}^{\mathrm{r}}_{0k[[x]]}$ that factors through $\mathcal{F}^{\mathrm{r}}_{0k[[x]]}$ and that produces a direct sum decomposition $M_0/pM_0 = \bar{F}_0^1 \oplus \bar{F}_0^0$ with the property that for each $i \in \{0,1\}$, every element $\beta \in \mathbb{G}_{mk[[x]]}(k[[x]])$ acts through $\mu_{0k[[x]]}$ on $\bar{F}_0^i$ via the multiplication with β^{-i}.

We choose a cocharacter

$$\mu_0 : \mathbb{G}_{mR_0} \to \mathcal{G}^{\mathrm{r}}_0$$

that lifts $\mu_{0k[[x]]}$, cf. [DG, Vol. II, Exp. IX, Thms. 3.6 and 7.1]. Let $M_0 = F_0^1 \oplus F_0^0$ be the direct sum decomposition with the property that for each $i \in \{0,1\}$, every element $\beta \in \mathbb{G}_{mR_0}(R_0)$ acts through μ_0 on F_0^i via the multiplication with β^{-i}; the notations match i.e., we have $F_0^i/pF_0^i = \bar{F}_0^i$.

We consider the $W(k)$-epimorphism $R_0 \twoheadrightarrow W(k)$ whose kernel is the ideal (x). Let

$$(M, F^1, \phi, \mathcal{G}, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M) := (M_0, F_0^1, \Phi_0, \mathcal{G}_0, (t_{1\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_0}) \otimes_{R_0} W(k).$$

5.4. Extra crystalline applications. If $p > 2$ or if $p = 2$ and (M, ϕ) has no integral slopes, then there exists a unique p-divisible group D over $W(k)$ whose filtered F-crystal over k is (M, F^1, ϕ) (cf. [Va4, Prop. 2.2.4]); due to the uniqueness part, ψ_M is the crystalline realization of a (unique) principal quasi-polarization λ_D of D. If $p = 2$ and (M, ϕ) has integral slopes, we consider an arbitrary principally quasi-polarized p-divisible group (D, λ_D) over $W(k)$ whose principally quasi-polarized filtered F-crystal over k is (M, F^1, ϕ) (cf. B5.1).

Let $(D_{R_0}, \lambda_{D_{R_0}})$ be the principally quasi-polarized p-divisible group over R_0 that lifts (D, λ_D) and whose principally quasi-polarized F-crystal over R_0/pR_0 is $(M_0, F_0^1, \Phi_0, \nabla_0, \psi_{M_0})$, cf. B7.1 and B7.2. Let

$$q_{R_0} : Z_0 \to \mathcal{M}_r$$

be the morphism that (i) lifts the composite of y with the morphism $\mathcal{N}^{\mathrm{s}} \to \mathcal{M}_r$ and that (ii) has the property that the principally quasi-polarized p-divisible group of the pull back of the universal principally polarized abelian scheme over $\mathcal{M}_r$ via q_{R_0} is $(D_{R_0}, \lambda_{D_{R_0}})$. Let

$$z_2 : \mathrm{Spec}(W(k_1)) \to \mathcal{M}_r$$

be the composite of the Teichmüller lift $\mathrm{Spec}(W(k_1)) \to Z_0$ of Subsection 5.1 with q_{R_0}.

Let (A_2, λ_{A_2}) be the principally polarized abelian scheme over $W(k_1)$ that is the pull back through z_2 of the universal principally polarized abelian scheme over $\mathcal{M}_r$. The principally quasi-polarized filtered F-crystal of (A_2, λ_{A_2}) is canonically identified with $(M_1, F_2^1, \phi_1, \psi_{M_1})$, where F_2^1 is a direct summand of M_1 of rank r. Let $(F_2^i(\mathcal{T}(M_1)))_{i\in\mathbb{Z}}$ be the filtration of $\mathcal{T}(M_1)$ defined by F_2^1 and let $(F_0^i(\mathcal{T}(M_0)))_{i\in\mathbb{Z}}$ be the filtration of $\mathcal{T}(M_0)$ defined by F_0^1. For each $\alpha \in \mathcal{J}$, the tensor $t_{1\alpha} \in \mathcal{T}(M_0)[\frac{1}{p}]$ is annihilated by ∇_0, is fixed by Φ_0, and belongs to $F_0^0(\mathcal{T}(M_0))[\frac{1}{p}]$. This implies that we have $t_{1\alpha} \in F_2^0(\mathcal{T}(M_1))[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$. Thus as before Lemma 3.2.1 we argue that the canonical split cocharacter of (M_1, F_2^1, ϕ_1) defined in [Wi, p. 512] factors through the closed subgroup scheme $\mathcal{G}_1 = \mathcal{G}_{0W(k_1)}$ of $\mathbf{GL}_{M_1}$; let $\mu_2 : \mathbb{G}_{mW(k_1)} \to \mathcal{G}_1$ be the resulting factorization. Due to Lemma 3.5.2 applied to $z_1 \in \mathcal{N}^{\mathrm{s}}(W(k_1))$ and to $\mu_2 : \mathbb{G}_{mW(k_1)} \to \mathcal{G}_1$, there exists a point $z_3 \in \mathcal{N}^{\mathrm{s}}(W(k_1))$ that lifts the k-valued point of $\mathcal{N}^{\mathrm{m}}$ defined by either z_1 or z_2 and such that the filtered F-crystal of $(A_3, \lambda_{A_3}) := z_3^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is precisely $(M_1, F_2^1, \phi_1, \psi_{M_1})$. Let (D_3, λ_{D_3}) be the principally quasi-polarized p-divisible group of (A_3, λ_{A_3}).

5.5. Proof of 1.6 (a). If $p = 2$ and the condition 1.6 (*) holds, then the 2-rank of A_{1k_1} is 0. Accordingly, in this Subsection we assume that either $p > 2$ or $p = 2$ and the 2-rank of A_{1k_1} is 0. Due to our assumptions on p and A_{1k}, the p-divisible groups D_2 and D_3 are the same lift of the p-divisible group of A_{1k_1} (cf. [Va4, Prop. 2.2.4]). This implies that the $W(k_1)$-valued points of $\mathcal{M}_r$ defined by z_2 and z_3 coincide. From this and Theorem 1.5 (b) we get that z_2 is the $W(k_1)$-valued point of $\mathcal{M}_r$ defined by z_3. Thus z_2 factors through $\mathcal{N}^{\mathrm{s}}$. This implies that q_{R_0} factors through $\mathcal{N}$. From this and Theorem 1.5 (a) we get that q_{R_0} factors through $\mathcal{N}^{\mathrm{s}}$. Let $z \in \mathcal{N}^{\mathrm{s}}(W(k))$ be the point that is the composite of the factorization $\mathrm{Spec}(R_0) \to \mathcal{N}^{\mathrm{s}}$ of q_{R_0} with the Teichmüller section $\mathrm{Spec}(W(k)) \hookrightarrow Z_0$. Our notations match with the ones of Subsection 3.2 i.e., (D, λ_D) is the principally quasi-polarized p-divisible group of $(A, \lambda_A) := z^*(\mathcal{A}, \lambda_{\mathcal{A}})$ and the principally quasi-polarized filtered F-crystal of (D, λ_D) is (M, F^1, ϕ, ψ_M).

From the proof of the property 3.5.1 (ii) we get that there exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (M_1, (t_{1\alpha})_{\alpha\in\mathcal{J}})$. Thus as the statement 3.2.2 (a) holds for $z_1 \in \mathcal{N}^{\mathrm{m}}(W(k_1))$, we get that there exist isomorphisms (see Subsection 3.2 for u_α's)

$$(M\otimes_{W(k)}W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (H^1_{\acute{e}t}(A_{B(k)}, \mathbb{Z}_p)\otimes_{\mathbb{Z}_p}W(k_1), (u_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}W(k_1), (v_\alpha)_{\alpha\in\mathcal{J}}).$$

From this and B4 we get that there exist isomorphisms $(M, (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (L^*_{(p)}\otimes_{\mathbb{Z}_{(p)}}W(k), (v_\alpha)_{\alpha\in\mathcal{J}})$. From this and Lemma 2.3.4 (a) we get that the statement 3.2.2 (a) holds for $z \in \mathcal{N}^{\mathrm{s}}(W(k))$. Thus we have $z \in \mathcal{N}^{\mathrm{m}}(W(k))$, cf. property 3.5.1 (ii). This implies that the morphism $y : \mathrm{Spec}(k) \to \mathcal{N}$ factors through $\mathcal{N}^{\mathrm{m}}$. This ends the proof of Theorem 1.6 (a).

5.6. Proof of 1.6 (b). If $p > 2$ or if $p = 2$ and the condition 1.6 (*) holds, then Theorem 1.6 (b) is implied by Theorems 1.6 (a) and 1.5 (b). Thus to prove Theorem 1.6 (b), we can assume that $p = 2$ and that the condition 1.6 (*) does not hold. Not to introduce extra notations, we can assume that the point $y \in \mathcal{N}(k)$ of the diagram (3) is the image of an arbitrary k-valued point y of $\mathcal{P}^{\mathrm{m}}$. If 2-rank of A_{1k_1} is 0, then from Subsection 5.5 we get that $y \in \mathcal{N}^{\mathrm{m}}(k) \subseteq \mathcal{N}^{\mathrm{s}}(k) \subseteq \mathcal{N}(k)$. We easily get that the k-scheme $\mathcal{P}^{\mathrm{m}}_k$ is regular at y and that the natural morphism $\mathcal{P}^{\mathrm{m}}_k \to \mathcal{M}_{rk}$ is a formally closed embedding at $y \in \mathcal{P}^{\mathrm{m}}(k)$. Thus to prove Theorem 1.6 (b) we can assume that $p = 2$ and that the 2-rank of A_{1k_1} is positive. The 2-divisible groups D_2 and D_3 over $W(k_1)$ might not be the same lift of the 2-divisible group of A_{1k_1} and thus below we will have to use an approach different from the one of Subsection 5.5.

Let the quintuple $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R})$ be as in Subsection 3.3. As in Subsections 3.3 and 3.4 we can speak about two morphisms $q_R : \mathrm{Spec}(R) \to \mathcal{M}_r$ and $q_S : \mathrm{Spec}(S) \to \mathcal{M}_r$ and about a closed embedding $c_R : \mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ such that the following three properties hold:

(i) we have $q_S = q_R \circ c_R$ and the $W(k)$-homomorphism $s^{\mathrm{big}}_y : O^{\mathrm{big}}_y \to S$ that defines q_S is onto (see Subsection 3.4 for O^{big}_y);

(ii) the morphism $y : \mathrm{Spec}(k) \to \mathcal{M}_r$ defined naturally by y, factors through q_S;

(iii) the principally quasi-polarized F-crystal over R/pR of the pull back through q_R of the universal principally quasi-polarized abelian scheme over $\mathcal{M}_r$, is $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R})$.

Let $\bar{O}^{\mathrm{m}}_y$ be the completion of the local ring of y in $\mathcal{P}^{\mathrm{m}}_k$. We consider a morphism $j_{Z_0} : Z_0 \to \mathrm{Spec}(S)$ such that $(M_0, F^1_0, \Phi_0, \nabla_0, \psi_{M_0}, (t_{1\alpha})_{\alpha\in\mathcal{J}})$ is the pull back of $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R}, (t_\alpha)_{\alpha\in\mathcal{J}})$ via $c_R \circ j_{Z_0} : Z_0 \to \mathrm{Spec}(R)$, cf. B7.4 and B7.5. As a principally quasi-polarized 2-divisible group over $R_0/2R_0$ is uniquely determined by its principally quasi-polarized F-crystal over $R_0/2R_0$ (cf. [BM, Thm. 4.1.1]), the existence of j_{Z_0} implies that the morphism $q_{k[[x]]} : \mathrm{Spec}(k[[x]]) \to \mathcal{M}_r$ defined by $q_{R_0} : \mathrm{Spec}(R_0) \to \mathcal{M}_r$ (equivalently by the morphism q of diagram (3)) factors through the morphism $q_{S/2S} : \mathrm{Spec}(S/2S) \to \mathcal{M}_r$ defined by q_S. As this property holds for every morphism $q_{k[[x]]} : \mathrm{Spec}(k[[x]]) \to \mathcal{N}$ that factors through $\mathcal{P}^{\mathrm{m}}$ in such a way that its generic fibre $q_{k((x))} : \mathrm{Spec}(k((x))) \to \mathcal{N}$ factors through $\mathcal{N}^{\mathrm{m}}$, the natural k-homomorphism $O^{\mathrm{big}}_y \to \bar{O}^{\mathrm{m}}_y$ factors through the $W(k)$-epimorphism $O^{\mathrm{big}}_y \twoheadrightarrow S/2S$ defined by $q_{S/2S}$. Thus we have natural k-homomorphisms $O^{\mathrm{big}}_y/2O^{\mathrm{big}}_y \twoheadrightarrow S/2S \to \bar{O}^{\mathrm{m}}_y$. As $\mathcal{N}_{E(G,\mathcal{X})}$ is a closed subscheme of $\mathcal{M}_{rE(G,\mathcal{X})}$ (cf. hypotheses), $\mathcal{N}$ is the normalization of a flat, closed subscheme of $\mathcal{M}_{rO_{(v)}}$ that extends $\mathcal{N}_{E(G,\mathcal{X})}$. Thus the ring $\bar{O}^{\mathrm{m}}_y$ is a local ring of the normalization of a reduced quotient of $O^{\mathrm{big}}_y/2O^{\mathrm{big}}_y$ and therefore it is a local ring of the normalization of a reduced quotient of $S/2S$. As $\bar{O}^{\mathrm{m}}_y$ has dimension d (as $\mathcal{Q}$ of Subsection 2.2 has relative dimension d), by reasons of dimensions we get that this reduced quotient of $S/2S$ is $S/2S$ itself. Thus the k-homomorphism $S/2S \to \bar{O}^{\mathrm{m}}_y$ is a k-isomorphism. Thus we have k-epimorphisms

$$O^{\mathrm{big}}_y/2O^{\mathrm{big}}_y \twoheadrightarrow S/2S \xrightarrow{\sim} \bar{O}^{\mathrm{m}}_y. \tag{4}$$

As y was an arbitrary k-valued point of $\mathcal{P}^{\mathrm{m}}$, from the k-isomorphism part of (4) we get that the $k(v)$-scheme $\mathcal{P}^{\mathrm{m}}$ is regular and formally smooth. Moreover, from the k-epimorphism part of (4) we get that the morphism $\mathcal{P}^{\mathrm{m}}_k \to \mathcal{M}_{rk}$ is a formally closed embedding at all k-valued points of $\mathcal{P}^{\mathrm{m}}_k$. This ends the proof of Theorem 1.6 (b).

5.6.1. Remark. As $(M_0, F^1_0, \Phi_0, \nabla_0, \psi_{M_0}, (t_{1\alpha})_{\alpha\in\mathcal{J}})$ is the pull back of $(M_R, F^1_R, \Phi, \nabla, \psi_{M_R}, (t_\alpha)_{\alpha\in\mathcal{J}})$ via $c_R \circ j_{Z_0} : Z_0 \to \mathrm{Spec}(R)$ and as $M_R = M \otimes_{W(k)} R$, there exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (M_0 \otimes_{R_0} W(k_1), (t_{1\alpha})_{\alpha\in\mathcal{J}}) = (M_1, (t_{1\alpha})_{\alpha\in\mathcal{J}})$. As $z_1 \in \mathcal{N}^{\mathrm{m}}(W(k_1))$, there exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}})$. Thus there exist isomorphisms $(M, (t_\alpha)_{\alpha\in\mathcal{J}}) \xrightarrow{\sim} (L^*_{(p)} \otimes_{\mathbb{Z}_{(p)}} W(k), (t_\alpha)_{\alpha\in\mathcal{J}})$, cf. proof of B4. It is easy to check that the point $y \in \mathcal{N}(k)$ belongs to $\mathcal{N}^{\mathrm{m}}(k)$ if and only if in the first paragraph of Subsection 5.4 we can choose (D, λ_D) such that for D there exists an isomorphism r_D as in B3. Thus the very root of the Part II of the paper is to solve the Problem of B5.

5.7. Proof of 1.6 (c). Let $\mathcal{Q}$ and $\mathcal{Q}^{\mathrm{s}}$ be as in Subsection 2.2. As $(G,\mathcal{X})$ has compact factors, $\mathcal{Q}$ is a projective $O_{(v)}$-scheme (cf. Lemma 2.2.4). From the property 3.5.1 (i) we get that $\mathcal{N}^{\mathrm{m}}$ is the pull back of a smooth, open subscheme $\mathcal{Q}^{\mathrm{m}}$ of $\mathcal{Q}$. To prove Theorem 1.6 (c) it suffices to show that if $\mathcal{C}$ is a connected component of $\mathcal{Q}_{W(k)}$, then either $\mathcal{C}\subseteq\mathcal{Q}^{\mathrm{m}}$ or $\mathcal{C}\cap\mathcal{Q}^{\mathrm{m}}_{W(k)}=\mathcal{C}_{B(k)}$. It suffices to show that if $\mathcal{Q}^{\mathrm{m}}_{W(k)}$ contains points of the special fibre of $\mathcal{C}$, then $\mathcal{C}\subseteq\mathcal{Q}^{\mathrm{m}}_{W(k)}$. The $W(k)$-scheme $\mathcal{C}$ is integral (being connected and normal). As the k-scheme $\mathcal{Q}^{\mathrm{m}}_k\cap\mathcal{C}$ is non-empty and as $\mathcal{Q}^{\mathrm{m}}$ is smooth, there exist $W(k)$-valued points of $\mathcal{C}$. Thus the ring of global functions of $\mathcal{C}$ is $W(k)$. From [Har, Ch. III, Cor. 11.3] applied to the projective $W(k)$-morphism $\mathcal{C}\to\mathrm{Spec}(W(k))$ we get that the special fibre $\mathcal{C}_k$ of $\mathcal{C}$ is connected. But $\mathcal{Q}^{\mathrm{m}}_k\cap\mathcal{C}_k$ is an open closed subscheme of $\mathcal{C}_k$, cf. Theorem 1.6 (a). From the last two sentences we get that $\mathcal{Q}^{\mathrm{m}}_k\cap\mathcal{C}_k=\mathcal{C}_k$. Thus $\mathcal{Q}^{\mathrm{m}}_{W(k)}\cap\mathcal{C}=\mathcal{C}$. This ends the proof of Theorem 1.6 (c) and thus also of the Main Theorem 1.6.

5.8. The proof of 1.7. As $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, H is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Thus the connected components of $\mathrm{Sh}_H(G,\mathcal{X})_{\mathbb{C}}$ are permuted transitively by $G(\mathbb{A}_f^{(p)})$, cf. [Va1, Lem. 3.3.2]. As the group $G(\mathbb{A}_f^{(p)})$ acts on $\mathcal{N}^{\mathrm{m}}$ (cf. property 3.5.1 (i)) and as $\mathcal{N}^{\mathrm{m}}$ contains a non-empty open closed subscheme of $\mathcal{N}$ (cf. Theorem 1.6 (c)), we get $\mathcal{N}^{\mathrm{m}}=\mathcal{N}$. As $\mathcal{N}^{\mathrm{m}}\subseteq\mathcal{N}^{\mathrm{s}}\subseteq\mathcal{N}$, we get $\mathcal{N}^{\mathrm{m}}=\mathcal{N}^{\mathrm{s}}=\mathcal{N}$. Thus Corollary 1.7 (a) follows from [Va1, Cor. 3.4.4].

We check the Corollary 1.7 (b). We know that $\mathcal{Q}$ is a normal, quasi-projective $O_{(v)}$-scheme which is the quotient of $\mathcal{N}=\mathcal{N}^{\mathrm{s}}$ through $H^{(p)}$ and that the quotient morphism $\mathcal{N}^{\mathrm{s}}\to\mathcal{Q}$ is a pro-étale cover, cf. the beginning of Subsection 2.2 and Proposition 2.2.1 (a). Thus $\mathcal{Q}=\mathcal{Q}^{\mathrm{s}}$ is smooth over $O_{(v)}$. Moreover, $\mathcal{Q}$ is a Néron model of its generic fibre $\mathrm{Sh}_K(G,\mathcal{X})$ over $O_{(v)}$ (cf. Theorem 1.5 (c)). As $\mathcal{Q}$ is also a projective $O_{(v)}$-scheme (cf. Lemma 2.2.4), we get that Corollary 1.7 (b) holds. This ends the proof of the Main Corollary 1.7.

Appendix A: On affine group schemes

Let $p\in\mathbb{N}$ be a prime. Let k be an algebraically closed field of characteristic p. Let $W(k)$ be the ring of Witt vectors with coefficients in k. Let $B(k):=W(k)[\frac{1}{p}]$.

A1. Canonical dilatations. Let $\mathcal{G}$ be an affine, flat group scheme over $W(k)$. Let $a\in\mathcal{G}(W(k))$. The Néron measure of the defect of smoothness $\delta(a)\in\mathbb{N}\cup\{0\}$ of $\mathcal{G}$ at a is the length of the torsion part of $a^*(\Omega_{\mathcal{G}/\mathrm{Spec}(W(k))})$. As $\mathcal{G}$ is a group scheme over $W(k)$, the value of $\delta(a)$ does not depend on $a\in\mathcal{G}(W(k))$ and therefore we denote it by $\delta(\mathcal{G})$. We have $\delta(\mathcal{G})\in\mathbb{N}$ if and only if $\mathcal{G}$ is not smooth, cf. [BLR, Ch. 3, 3.3, Lem. 1]. Let $\mathcal{F}_k$ be the Zariski closure in $\mathcal{G}_k$ of all special fibres of $W(k)$-valued points of $\mathcal{G}$; it is a reduced subgroup of $\mathcal{G}_k$. We write $\mathcal{F}_k=\mathrm{Spec}(R_{\mathcal{G}}/J_{\mathcal{G}})$, where $\mathcal{G}=\mathrm{Spec}(R_{\mathcal{G}})$ and where $J_{\mathcal{G}}$ is the ideal of $R_{\mathcal{G}}$ that defines $\mathcal{F}_k$. By the *canonical dilatation* of $\mathcal{G}$ we mean the affine $\mathcal{G}$-scheme $\mathcal{G}_1=\mathrm{Spec}(R_{\mathcal{G}_1})$, where $R_{\mathcal{G}_1}$ is the $R_{\mathcal{G}}$-subalgebra of $R_{\mathcal{G}}[\frac{1}{p}]$ generated by $\frac{x}{p}$ with $x\in J_{\mathcal{G}}$.

The $W(k)$-scheme $\mathcal{G}_1$ has a canonical group scheme structure and the morphism $\mathcal{G}_1 \to \mathcal{G}$ is a homomorphism of group schemes over $W(k)$, cf. [BLR, Ch. 3, 3.2, Prop. 2 (d)]. Moreover the $W(k)$-morphism $\mathcal{G}_1 \to \mathcal{G}$ has the following universal property: each $W(k)$-morphism $Z \to \mathcal{G}$ of flat $W(k)$-schemes whose special fibre factors through the closed embedding $\mathcal{F}_k \hookrightarrow \mathcal{G}_k$, factors uniquely through $\mathcal{G}_1 \to \mathcal{G}$ (cf. [BLR, Ch. 3, 3.2, Prop. 1 (b)]). If $\mathcal{G}$ is smooth, then $\mathcal{F}_k = \mathcal{G}_k$ and therefore $\mathcal{G}_1 = \mathcal{G}$.

Either $\mathcal{G}_1$ is smooth or we have $0 < \delta(\mathcal{G}_1) < \delta(\mathcal{G})$, cf. [BLR, Ch. 3, 3.3, Prop. 5]. Thus by using at most $\delta(\mathcal{G})$ canonical dilatations (the first one of $\mathcal{G}$, the second one of $\mathcal{G}_1$, etc.), we get the existence of a unique smooth, affine group scheme $\mathcal{G}'$ over $W(k)$ endowed with a homomorphism $\mathcal{G}' \to \mathcal{G}$ whose fibre over $B(k)$ is an isomorphism and which has the following universal property: each $W(k)$-morphism $Z \to \mathcal{G}$, with Z a smooth $W(k)$-scheme, factors uniquely through $\mathcal{G}' \to \mathcal{G}$. One calls $\mathcal{G}'$ the *universal smoothening* of $\mathcal{G}$.

A2. Lemma. *Let η be a field of characteristic* 0*. Let W be a finite dimensional vector space over η. Let $\mathcal{L}$ be a Lie subalgebra of $End(W)$. Suppose that there exists a field extension η_1 of η such that $\mathcal{L} \otimes_\eta \eta_1$ is the Lie algebra of a connected (resp. reductive) subgroup $\mathcal{F}_{\eta_1}$ of $\boldsymbol{GL}_{W\otimes_\eta \eta_1}$. We have:*

(a) *there exists a unique connected (resp. reductive) subgroup $\mathcal{F}$ of $\boldsymbol{GL}_W$ whose Lie algebra is $\mathcal{L}$ (the notations match i.e., its extension to η_1 is $\mathcal{F}_{\eta_1}$);*

(b) *if $\mathcal{F}$ is a reductive subgroup of $\boldsymbol{GL}_W$, then the restriction of the trace form on $End(W)$ to $\mathcal{L}$ is non-degenerate.*

Proof: We prove (a). The uniqueness part is implied by [Bo, Ch. I, 7.1]. Loc cit. also implies that if $\mathcal{F}$ exists, then its extension to η_1 is indeed $\mathcal{F}_{\eta_1}$. It suffices to prove (a) for the case when $\mathcal{F}$ is connected. We consider commutative η-algebras κ such that there exists a closed subgroup scheme $\mathcal{F}_\kappa$ of $\mathbf{GL}_{W\otimes_\eta \kappa}$ whose Lie algebra is $\mathcal{L} \otimes_\eta \kappa$. Our hypotheses imply that as κ we can take η_1. Thus as κ we can also take a finitely generated η-subalgebra of η_1. By considering the reduction modulo a maximal ideal of this last η-algebra, we can assume that κ is a finite field extension of η. Even more, (as η has characteristic 0) we can assume that κ is a finite Galois extension of η. By replacing $\mathcal{F}_\kappa$ with its identity component, we can assume that $\mathcal{F}_\kappa$ is connected. Due to the mentioned uniqueness part, the Galois group $\mathrm{Gal}(\kappa/\eta)$ acts naturally on the connected subgroup $\mathcal{F}_\kappa$ of $\mathbf{GL}_{W\otimes_\eta \kappa}$. As $\mathcal{F}_\kappa$ is an affine scheme, the resulting Galois descent on $\mathcal{F}_\kappa$ with respect to $\mathrm{Gal}(\kappa/\eta)$ is effective (cf. [BLR, Ch. 6, 6.1, Thm. 5]). This implies the existence of a subgroup $\mathcal{F}$ of $\mathbf{GL}_W$ whose extension to κ is $\mathcal{F}_\kappa$. As $\mathrm{Lie}(\mathcal{F}) \otimes_\eta \kappa = \mathrm{Lie}(\mathcal{F}_\kappa) = \mathcal{L} \otimes_\eta \kappa$, we have $\mathrm{Lie}(\mathcal{F}) = \mathcal{L}$. The group $\mathcal{F}$ is connected as $\mathcal{F}_\kappa$ is so. Therefore $\mathcal{F}$ exists. Thus (a) holds.

To check (b) we can assume that η is algebraically closed. Using isogenies, it suffices to prove (b) in the case when $\mathcal{F}$ is either $\mathbb{G}_{m\eta}$ or a semisimple group whose adjoint is simple. If $\mathcal{F}$ is $\mathbb{G}_{m\eta}$, then the $\mathcal{F}$-module W is a direct sum of one dimensional $\mathcal{F}$-modules. We easily get that there exists an element $x \in \mathcal{L} \setminus \{0\}$ which is a semisimple element of $\mathrm{End}(W)$ whose eigenvalues are integers. The trace of x^2 is a non-trivial sum of squares of natural numbers and thus it is non-zero. Thus (b) holds if $\mathcal{F}$ is $\mathbb{G}_{m\eta}$. If $\mathcal{F}$ is a semisimple group whose adjoint is simple, then $\mathcal{L}$ is a simple Lie algebra over η. From Cartan solvability criterion we get that the restriction of the trace form on $\mathrm{End}(W)$ to $\mathcal{L}$ is non-zero and therefore (as $\mathcal{L}$ is a simple Lie algebra) it is non-degenerate. Thus (b) holds. $\square$

Appendix B: Complements on p-divisible groups

Let p, k, $W(k)$, and $B(k)$ be as in Appendix A. Let $\sigma := \sigma_k$ be the Frobenius automorphism of k, $W(k)$, and $B(k)$. We fix an algebraic closure $\overline{B(k)}$ of $B(k)$. Let $\mathrm{Gal}(B(k)) := \mathrm{Gal}(\overline{B(k)}/B(k))$. Let D be a p-divisible group over $W(k)$. Let (M, ϕ) be the contravariant Dieudonné module of D_k. Thus M is a free $W(k)$-module of rank equal to the height of D and $\phi : M \to M$ is a σ-linear endomorphism such that we have $pM \subseteq \phi(M)$. Let F^1 be the direct summand of M that is the Hodge filtration defined by D. We have $\phi(M + \frac{1}{p}F^1) = M$. The rank of F^1 is the dimension of D_k. Let $M^* := \mathrm{Hom}(M, W(k))$. Let $\mathcal{T}(M)$ and its filtration $(F^i(\mathcal{T}(M)))_{i\in\mathbb{Z}}$ defined by F^1, be as in Subsection 2.1. For $f \in M^*[\frac{1}{p}]$ let $\phi(f) := \sigma \circ f \circ \phi^{-1} \in M^*[\frac{1}{p}]$. Thus ϕ acts in the usual tensor product way on $\mathcal{T}(M[\frac{1}{p}])$.

B1. Galois modules. Let D^{t} be the Cartier dual of D. Let $H^1(D) := T_p(D^{\mathrm{t}}_{B(k)})(-1)$ be the dual of the Tate-module $T_p(D_{B(k)})$ of $D_{B(k)}$. Thus $H^1(D)$ is a free $\mathbb{Z}_p$-module of the same rank as M and $\mathrm{Gal}(B(k))$ acts on it. Let $F^0(H^1(D)) := H^1(D)$ and $F^1(H^1(D)) := 0$. Let

$$\rho_D : \mathrm{Gal}(B(k)) \to \mathbf{GL}_{H^1(D)}(W(k))$$

be the natural Galois representation associated to $D_{B(k)}$. Let $\mathcal{D}^{\acute{e}t}$ be the Zariski closure in $\mathbf{GL}_{H^1(D)}$ of $\mathrm{Im}(\rho_D)$. From [Wi, Prop. 4.2.3] one gets that the generic fibre $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$ is connected. See Subsection 2.1 for $\mathcal{T}(H^1(D))$; it is naturally a $\mathrm{Gal}(B(k))$-module. By an *étale Tate-cycle* on $D_{B(k)}$ we mean a tensor of $\mathcal{T}(H^1(D[\frac{1}{p}])) = \mathcal{T}(H^1(D))[\frac{1}{p}]$ that is fixed by $\mathrm{Gal}(B(k))$ (equivalently by $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$). In what follows we will fix a family $(v_\alpha)_{\alpha\in\mathcal{J}}$ of étale Tate-cycles on $D_{B(k)}$. Let $G^{\acute{e}t}$ be the Zariski closure in $\mathbf{GL}_{H^1(D)}$ of the subgroup of $\mathbf{GL}_{H^1(D)[\frac{1}{p}]}$ that fixes v_α for all $\alpha \in \mathcal{J}$. The group scheme $\mathcal{D}^{\acute{e}t}$ is a subgroup scheme of $G^{\acute{e}t}$.

B2. Fontaine comparison theory. We refer to [Fo], [Fa2], and [Va4] for the following review on *Fontaine comparison theory*. This theory provides us with three rings $B^+_{\mathrm{crys}}(W(k))$, $B_{\mathrm{crys}}(W(k))$, and $B_{\mathrm{dR}}(W(k))$ that have the following six properties:

(i) the rings are integral $W(k)$-algebras that are equipped with exhaustive and decreasing filtrations and with a Galois action; moreover $B_{\mathrm{dR}}(W(k))$ is a field;

(ii) we have $W(k)$-monomorphisms $B^+_{\mathrm{crys}}(W(k)) \hookrightarrow B_{\mathrm{crys}}(W(k)) \hookrightarrow B_{\mathrm{dR}}(W(k))$;

(iii) the ring $B^+_{\mathrm{crys}}(W(k))$ is faithfully flat over $W(k)$ and has a natural Frobenius lift that is compatible with σ and that also extends to an endomorphism of $B_{\mathrm{crys}}(W(k))$;

(iv) there exists a $B^+_{\mathrm{crys}}(W(k))$-linear monomorphism

$$i^+_D : M \otimes_{W(k)} B^+_{\mathrm{crys}}(W(k)) \hookrightarrow H^1(D) \otimes_{\mathbb{Z}_p} B^+_{\mathrm{crys}}(W(k))$$

that respects the tensor product filtrations, the Galois actions, and the Frobenius endomorphisms (the Frobenius endomorphism of $H^1(D)$ being $1_{H^1(D)}$);

(v) the $B_{\mathrm{dR}}(W(k))$-linear map $i_D := i_D^+ \otimes 1_{B_{\mathrm{dR}}(W(k))}$ is a bijection that induces naturally a $B_{\mathrm{dR}}(W(k))$-linear isomorphism denoted in the same way

$$i_D : \mathcal{T}(M) \otimes_{W(k)} B_{\mathrm{dR}}(W(k)) \tilde{\to} \mathcal{T}(H^1(D)) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}(W(k));$$

(vi) each étale Tate-cycle v_α defines a tensor $t_\alpha := i_D(v_\alpha) \in F^0(\mathcal{T}(M))[\frac{1}{p}] \subseteq \mathcal{T}(M)[\frac{1}{p}]$ that is fixed by ϕ.

Let $\mathcal{G}$ be the Zariski closure in $\mathbf{GL}_M$ of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$. It is a flat, closed subgroup scheme of $\mathbf{GL}_M$ such that we have $\phi(\mathrm{Lie}(\mathcal{G}_{B(k)})) = \mathrm{Lie}(\mathcal{G}_{B(k)})$. Let $\mu : \mathbb{G}_m \to \mathcal{G}$ be a cocharacter that produces a direct sum decomposition $M = F^1 \oplus F^0$ such that for each $i \in \{0, 1\}$, every element $\beta \in \mathbb{G}_m(W(k))$ acts through μ on F^i as the multiplication with β^{-i}. For instance, we can take μ to be the factorization through $\mathcal{G}$ of the inverse of the canonical split cocharacter $\mu_{\mathrm{can}} : \mathbb{G}_m \to \mathbf{GL}_M$ of (M, F^1, ϕ) defined in [Wi, p. 512] (the cocharacter μ_{can} fixes each tensor t_α, cf. the functorial aspects of [Wi, p. 513]).

B3. Theorem (see [Va4, Thm. 1.2]). *If $p = 2$, we assume that $G_{\acute{e}t}$ is a torus. Then there exists an isomorphism $r_D : (M, (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha\in\mathcal{J}})$ (in the sense of Subsection 2.1).*

B4. Lemma. *Let k_1 be an algebraically closed field that contains k. Suppose that there exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (H^1(D) \otimes_{\mathbb{Z}_p} W(k_1), (v_\alpha)_{\alpha\in\mathcal{J}})$. Then there exists an isomorphism $r_D : (M, (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha\in\mathcal{J}})$.*

Proof: To check the existence of r_D we can assume that we have $t_\alpha \in \mathcal{T}(M)$ and $v_\alpha \in H^1(D)$ for all $\alpha \in \mathcal{J}$. Thus we an speak about the affine $W(k)$-scheme of finite type $\mathfrak{P}$ that parameterizes isomorphisms between $(M, (t_\alpha)_{\alpha\in\mathcal{J}})$ and $(H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha\in\mathcal{J}})$. We know that $\mathfrak{P}$ has a $W(k_1)$-valued point. As the monomorphism $W(k) \hookrightarrow W(k_1)$ is of ramification index one, from [BLR, Ch. 3, 3.6, Prop. 4] we get that there exists a morphism $\mathfrak{P}' \to \mathfrak{P}$ of $W(k)$-schemes such that $\mathfrak{P}'$ is smooth over $W(k)$ and has a $W(k_1)$-valued point. Thus the special fibre $\mathfrak{P}'_k$ is non-empty. As $\mathfrak{P}'$ is smooth over $W(k)$ and has a non-empty special fibre, it has $W(k)$-valued points. Thus $\mathfrak{P}$ also has $W(k)$-valued points and therefore the isomorphism r_D exists. $\square$

B5. On p=2. The following problem will play a key role in the Part II of the paper.

Problem. Suppose that $p = 2$ and that $\mathcal{G}$ is a reductive group scheme over $W(k)$. Show that there exists a 2-divisible group $\tilde{D}$ over $W(k)$ which lifts D_k, whose filtered F-crystal over k is as well the triple (M, F^1, ϕ), and for which there exists an isomorphism $r_{\tilde{D}} : (M, (t_\alpha)_{\alpha\in\mathcal{J}}) \tilde{\to} (H^1(\tilde{D}) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha\in\mathcal{J}})$. Here $v_\alpha \in \mathcal{T}(H^1(\tilde{D}))[\frac{1}{p}] = \mathcal{T}(H^1(D))[\frac{1}{p}]$ is the tensor that corresponds to t_α via Fontaine comparison theory for either $\tilde{D}$ or D.

We have a natural principally quasi-polarized variant of the above Problem. Next we will solve the most particular case of this variant.

B5.1. Lifting polarizations in mixed characteristic $(0,2)$. Suppose that $p=2$ and that D_k has a principal quasi-polarization λ_{D_k}. Let $r\in\mathbb{N}$ be such that the height of D is $2r$. Let ψ_M be the perfect, alternating form on M that is the crystalline realization of λ_{D_k}. Let $\tilde F^1$ be a direct summand of M that lifts $F^1/2F^1$ and that is isotropic with respect to ψ_M (i.e., and such that $\psi_M(\tilde F^1,\tilde F^1)=0$); the triple $(M,\tilde F^1,\phi,\psi_M)$ is a principally quasi-polarized F-crystal over k. We recall the essence of the argument that there exists a principally quasi-polarized 2-divisible group $(\tilde D,\lambda_{\tilde D})$ over $W(k)$ which lifts (D_k,λ_{D_k}) and whose principally quasi-polarized F-crystal over k is $(M,\tilde F^1,\phi,\psi_M)$.

Let $W_2(k):=W(k)/4W(k)$. Based on Grothendieck–Messing deformation theory, it suffices to show that there exists a lift $(\tilde D_{W_2(k)},\lambda_{\tilde D_{W_2(k)}})$ of (D_k,λ_{D_k}) to $W_2(k)$ such that the Hodge filtration of $M/4M$ defined by $\tilde D_{W_2(k)}$ is $\tilde F^1/4\tilde F^1$. Let $(D^0_{W_2(k)},\lambda_{D^0_{W_2(k)}})$ be an arbitrary principally quasi-polarized 2-divisible group over $W_2(k)$ that lifts (D_k,λ_{D_k}). The lifts of (D_k,λ_{D_k}) to $W_2(k)$ are parameterized by the k-valued points of an affine space $\mathbb{A}_{\text{def}}^{\frac{r(r+1)}{2}}$, the origin corresponding to $(D^0_{W_2(k)},\lambda_{D^0_{W_2(k)}})$. The lifts of $F^1/2F^1$ to direct summands of $M/4M$ which are isotropic with respect to ψ_M, are parameterized by the k-valued points of an affine space $\mathbb{A}_{\text{lifts}}^{\frac{r(r+1)}{2}}$, the origin corresponding to the Hodge filtration of $M/4M$ defined by $D^0_{W_2(k)}$.

We have a natural morphism $h_{\text{fil}}:\mathbb{A}_{\text{def}}^{\frac{r(r+1)}{2}}\to\mathbb{A}_{\text{lifts}}^{\frac{r(r+1)}{2}}$ over k that at the level of k-valued points takes a lift $(D^1_{W_2(k)},\lambda_{D^1_{W_2(k)}})$ of (D_k,λ_{D_k}) to $W_2(k)$ to the Hodge filtration of $M/4M$ defined by $D^1_{W_2(k)}$. The morphism h_{fil} is finite and surjective on k-valued points, cf. proof of [Va6, Prop. 6.4.5]. Thus there exists a principally quasi-polarized 2-divisible group $(\tilde D_{W_2(k)},\lambda_{\tilde D_{W_2(k)}})$ over $W_2(k)$ that lifts (D_k,λ_{D_k}) and whose Hodge filtration is $\tilde F^1/4\tilde F^1$. This ends the argument for the existence $(\tilde D,\lambda_{\tilde D})$.

B6. Group correspondences. Let $F^{\acute{e}t}_{\mathbb{Q}_p}$ be a reductive, closed subgroup of $G^{\acute{e}t}_{\mathbb{Q}_p}$. The restriction to $\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})$ of the trace form on $\mathrm{End}(H^1(D)[\frac{1}{p}])$ is non-degenerate, cf. A2 (b). Let $\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})^\perp$ be the perpendicular on $\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})$ with respect to the trace form on $\mathrm{End}(H^1(D)[\frac{1}{p}])$; we have a direct sum decomposition of $\mathbb{Q}_p$-vector spaces

$$\mathrm{End}(H^1(D)[\frac{1}{p}])=\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})\oplus\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})^\perp.$$

Let $\pi^{\acute{e}t}$ be the projector of $\mathrm{End}(H^1(D)[\frac{1}{p}])$ on $\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})$ along $\mathrm{Lie}(F^{\acute{e}t}_{\mathbb{Q}_p})^\perp$; it is an idempotent of $\mathrm{End}(H^1(D)[\frac{1}{p}])$ fixed by each subgroup of $\mathbf{GL}_{H^1(D)[\frac{1}{p}]}$ that normalizes $F^{\acute{e}t}_{\mathbb{Q}_p}$.

Suppose that $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$ normalizes $F^{\acute{e}t}_{\mathbb{Q}_p}$ (for instance, this holds if $F^{\acute{e}t}_{\mathbb{Q}_p}$ is a normal subgroup of $G^{\acute{e}t}_{\mathbb{Q}_p}$). Thus $\pi^{\acute{e}t}$ is fixed by $\mathcal{D}^{\acute{e}t}_{\mathbb{Q}_p}$ and therefore also by $\mathrm{Im}(\rho_D)$.

Let $\pi^{\text{crys}}\in\mathrm{End}(M[\frac{1}{p}])$ be the projector that corresponds to $\pi^{\acute{e}t}$ via Fontaine comparison theory. We have the following two properties:

(i) There exists a unique reductive subgroup $\mathcal{F}_{B(k)}$ of $\mathcal{G}_{B(k)}$ whose Lie algebra is $\mathrm{Im}(\pi^{\text{crys}})$.

(ii) If the generic fibre of μ_{can} factors through $\mathcal{F}_{B(k)}$, then $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ is a subgroup of $F_{\mathbb{Q}_p}^{\acute{e}t}$.

We check (i). As i_D^{-1} is a $B_{\text{dR}}(W(k))$-linear isomorphism that takes $\pi^{\acute{e}t}$ to π^{crys}, the group $i_D^{-1}(F_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B_{\text{dR}}(W(k)))i_D$ is a subgroup of $i_D^{-1}(G_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B_{\text{dR}}(W(k)))i_D = \mathcal{G}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k))$ whose Lie algebra is $\text{Im}(\pi^{\text{crys}}) \otimes_{B(k)} B_{\text{dR}}(W(k))$. Thus as $B_{\text{dR}}(W(k))$ is a field, from A2 (a) applied with $(W, \mathcal{L}, \eta, \eta_1) = (M[\frac{1}{p}], \text{Im}(\pi^{\text{crys}}), B(k), B_{\text{dR}}(W(k)))$, we get that there exists a unique reductive subgroup $\mathcal{F}_{B(k)}$ of $\mathbf{GL}_{M[\frac{1}{p}]}$ whose Lie algebra is $\text{Im}(\pi^{\text{crys}})$. As $\mathcal{F}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k))$ is a subgroup of $\mathcal{G}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k))$, the group $\mathcal{F}_{B(k)}$ is in fact a subgroup of $\mathcal{G}_{B(k)}$. Thus (i) holds.

We check (ii). Let l_{can} be the Lie algebra of the image of the generic fibre of μ_{can}. As π^{crys} is fixed by ϕ, the Lie algebra $\text{Lie}(\mathcal{F}_{B(k)}) = \text{Im}(\pi^{\text{crys}})$ is normalized by ϕ. Let $\mathcal{D}_{B(k)}$ be the smallest connected subgroup of $\mathcal{F}_{B(k)}$ with the property that $\text{Lie}(\mathcal{D}_{B(k)})$ contains $\phi^m(l_{\text{can}})$ for all $m \in \mathbb{Z}$. From [Bo, Ch. I, 7.1] we get that all conjugates of the generic fibre of μ_{can} through integral powers of ϕ factor through $\mathcal{D}_{B(k)}$ (in fact, $\mathcal{D}_{B(k)}$ is the smallest subgroup of $\mathcal{F}_{B(k)}$ for which this property holds). This implies that $\mathcal{D}_{B(k)}$ corresponds to $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ via Fontaine comparison theory (cf. [Wi, Prop. 4.2.3]) i.e., we have an identity

$$\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B_{\text{dR}}(W(k)) = i_D(\mathcal{D}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k)))i_D^{-1}$$

of subgroups of $\mathbf{GL}_{H^1(D)\otimes_{\mathbb{Z}_p} B_{\text{dR}}(W(k))}$. As $\mathcal{D}_{B(k)}$ is a subgroup of $\mathcal{F}_{B(k)}$ and as $F_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B_{\text{dR}}(W(k)) = i_D(\mathcal{F}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k)))i_D^{-1}$, we get that (ii) holds.

As we also have $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B_{\text{dR}}(W(k)) = i_D(\mathcal{G}_{B(k)} \times_{B(k)} B_{\text{dR}}(W(k)))i_D^{-1}$, the groups $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\mathbb{Q}_p} B(k)$ and $\mathcal{G}_{B(k)}$ are forms of each other.

B7. Faltings deformation theory. Let $l \in \mathbb{N} \cup \{0\}$. Let $R = W(k)[[x_1, \ldots, x_l]]$ be a formal power series in l variables with coefficients in $W(k)$. Let Φ_R be the Frobenius lift of R that is compatible with σ and that takes x_i to x_i^p for all $i \in \{1, \ldots, l\}$. Let $\Omega_{R/W(k)}^\wedge = \oplus_{i=1}^l R dx_i$ be the p-adic completion of the R-module of relative differentials $\Omega_{R/W(k)}$. Let $d\Phi_R : \Omega_{R/W(k)}^\wedge \to \Omega_{R/W(k)}^\wedge$ be the differential map of Φ_R.

Let (M_R, F_R^1, Φ) be a triple such that the following four axioms hold:

(i) M_R is a free R-module of rank equal to the height of D;

(ii) F_R^1 is a direct summand of M_R of rank equal to the rank of F^1;

(iii) $\Phi : M_R \to M_R$ is a Φ_R-linear endomorphism that induces an R-linear isomorphism $(M_R + \frac{1}{p}F_R^1) \otimes_{R\ \Phi_R} R \xrightarrow{\sim} M_R$;

(iv) the reduction of (M_R, F_R^1, Φ) modulo the ideal $(x_1, \ldots, x_l)$ is canonically identified with (M, F^1, ϕ).

It is known that there exists a unique connection $\nabla : M_R \to M_R \otimes_R \Omega_{R/W(k)}^\wedge$ such that we have an identity $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$, cf. [Fa2, Thm. 10]. Loc. cit. also shows that ∇ is integrable and nilpotent modulo p. Let Φ act in the natural tensor way on $\mathcal{T}(M_R)[\frac{1}{p}]$; thus if $e \in M_R^* := \text{Hom}(M_R, R)$, then $\Phi(e) \in M_R^*[\frac{1}{p}]$ is the unique element such that we have $\Phi(e)(\Phi(a)) = \Phi_R(e(a)) \in R$ for all $a \in M_R$.

B7.1. Lemma. *There exists a unique p-divisible group D_R over R that lifts D and such that its filtered F-crystal over R/pR is $(M_R, F^1_R, \Phi, \nabla)$.*

Proof: Let J be an ideal of R such that R is complete in the J-adic topology (for instance, J could be (p), $(x_1, \ldots, x_l)$, or $p(x_1, \ldots, x_l)$). Let $\mathrm{Spf}(R)$ be the formal scheme which is the formal completion of $\mathrm{Spec}(R)$ along $\mathrm{Spec}(R/J)$. The categories of p-divisible groups over $\mathrm{Spec}(R)$ and respectively over $\mathrm{Spf}(R)$ are canonically isomorphic, cf. [dJ1, Lem. 2.4.4]; below we will use this fact without any extra comment. The existence of D_R is implied by [Fa2, Thm. 10]. The uniqueness of the fibre $D_{R/pR}$ of D_R over $\mathrm{Spec}(R/pR)$ is implied by [BM, Thm. 4.1.1]. As the ideal $p(x_1, \ldots, x_l)$ of $R/(x_1, \ldots, x_l)^m$ has a natural nilpotent divided power structure for all $m \in \mathbb{N}$, from the Grothendieck–Messing deformation theory we get that D_R is the unique p-divisible group over R that lifts both D and $D_{R/pR}$ and whose filtered F-crystal is $(M_R, F^1_R, \Phi, \nabla)$. $\square$

Until B8 we will assume that D has a principal quasi-polarization λ_D. Let ψ_M be the perfect, alternating form on M that is the crystalline realization of λ_D; for $a, b \in M$ we have $\psi_M(\phi(a) \otimes \phi(b)) = p\sigma(\psi_M(a \otimes b))$. We also assume that there exists a perfect, alternating form ψ_{M_R} on M_R that lifts ψ_M (i.e., which modulo $(x_1, \ldots, x_l)$ is ψ_M) and such that for $a, b \in M_R$ we have $\psi_{M_R}(\Phi(a) \otimes \Phi(b)) = p\Phi_R(\psi_M(a \otimes b))$.

B7.2. Lemma. *There exists a unique principal quasi-polarization λ_{D_R} of D_R that lifts λ_D and whose crystalline realization is ψ_{M_R}.*

Proof: Let $(M^{\mathrm{t}}_R, F^{1\mathrm{t}}_R, \Phi^{\mathrm{t}}, \nabla^{\mathrm{t}})$ be the filtered F-crystal over R/pR of the Cartier dual D^{t}_R of D_R. The form ψ_{M_R} defines naturally an isomorphism $(M^{\mathrm{t}}_R, F^{1\mathrm{t}}_R, \Phi^{\mathrm{t}}) \xrightarrow{\sim} (M_R, F^1_R, \Phi)$. As the connections ∇ and ∇^{t} are uniquely determined by (M_R, F^1_R, Φ) and $(M^{\mathrm{t}}_R, F^{1t}_R, \Phi^{\mathrm{t}})$ (respectively), the last isomorphism extends to an isomorphism

$$\theta : (M^{\mathrm{t}}_R, F^{1t}_R, \Phi^{\mathrm{t}}, \nabla^{\mathrm{t}}) \xrightarrow{\sim} (M_R, F^1_R, \Phi, \nabla)$$

of filtered F-crystals over R/pR.

The ring R/pR has a finite p-basis $\{x_1, \ldots, x_l\}$ in the sense of [BM, Def. 1.1.1]. Thus from the fully faithfulness part of [BM, Thm. 4.1.1] we get that there exists a unique principal quasi-polarization $\lambda_{D_{R/pR}}$ of $D_{R/pR}$ whose crystalline realization is θ; it lifts the special fibre of λ_D. As the ideal $p(x_1, \ldots, x_l)$ of $R/(x_1, \ldots, x_l)^m$ has a natural nilpotent divided power structure for all $m \in \mathbb{N}$, from the Grothendieck–Messing deformation theory we get that there exists a unique principal quasi-polarization λ_{D_R} of D_R that lifts both $\lambda_{D_{R/pR}}$ and λ_D and whose crystalline realization is ψ_{M_R}. $\square$

B7.3. Construction. Let $M = F^1 \oplus F^0$ be the direct sum decomposition such that the cocharacter $\mu : \mathbb{G}_{mW(k)} \to \mathbf{GL}_M$ acts trivially on F^0. We naturally identify $\mathrm{Hom}(F^1, F^0)$ with the direct summand $\{e \in \mathrm{End}(M) | e(F^0) = 0 \text{ and } e(F^1) \subseteq F^0\}$ of $\mathrm{End}(M)$. Let $\mathcal{G}'$ be the universal smoothening of $\mathcal{G}$, cf. A2 (a). Suppose that $\mathcal{G}$ is a closed subgroup scheme of $\mathbf{GSp}(M, \psi_M)$ and that $R = W(k)[[x_1, \ldots, x_l]]$ is the local ring of the completion of $\mathcal{G}'$ along the identity section. Thus the relative dimension of $\mathcal{G}$ over $W(k)$ is l. We take

$$(M_R, F^1_R, \psi_{M_R}) := (M, F^1, \psi_M) \otimes_{W(k)} R \text{ and } \Phi := g_{\mathrm{univ}}(\phi \otimes \Phi_R),$$

where $g_{\mathrm{univ}} \in \mathcal{G}'(R)$ is the universal element. Let

$$\mathfrak{C}_{\mathrm{univ}} := (M_R, F^1_R, \Phi, \nabla, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M).$$

We have the following three properties (see [Va4, Subsection 3.3 1)] for (i) and (ii) and see proof of [Va4, Thm. 5.2] for (iii)):

(i) *the connection* ∇ *is of the form* $\delta+\gamma$, *where* δ *is the flat connection on* $M_R = M\otimes_{W(k)} R$ *that annihilates* $M\otimes 1$ *and where* $\gamma \in (\mathit{Lie}(\mathcal{G}_{B(k)})\cap \mathit{End}(M))\otimes_{W(k)} \Omega^\wedge_{R/W(k)}$*;*

(ii) *the connection on* $\mathcal{T}(M_R) = \mathcal{T}(M)\otimes_{W(k)} R$ *induced naturally by* ∇ *annihilates the tensor* $t_\alpha \in \mathcal{T}(M)\otimes_{W(k)} R[\frac{1}{p}]$ *for all* $\alpha\in\mathcal{J}$*;*

(iii) *the connection* ∇ *is versal and its Kodaira–Spencer map has an image which is the direct summand* $(\mathit{Lie}(\mathcal{G}_{B(k)})\cap \mathit{Hom}(F^1,F^0))\otimes_{W(k)} R$ *of* $\mathit{Hom}(F^1,F^0)\otimes_{W(k)} R \xrightarrow{\sim} \mathit{Hom}(F^1, M/F^1)\otimes_{W(k)} R$*.*

Let $m\in\mathbb{N}\cup\{0\}$, $R_1 := W(k)[[x_1,\ldots,x_m]]$, and $Z := \mathrm{Spec}(R_1)$. Let Φ_{R_1} be the Frobenius lift of R_1 that is compatible with σ and that takes x_i to x_i^p for all $i\in\{1,\ldots,m\}$. Let $\mathfrak{C}_1 := (M_1, F^1_1, \Phi_1, \nabla_1, (t_{1\alpha})_{\alpha\in\mathcal{J}}, \psi_{M_1})$ be a principally quasi-polarized filtered F-crystal over $\mathrm{Spec}(R_1/pR_1)$ endowed with a family of tensors $(t_{1\alpha})_{\alpha\in\mathcal{J}}$ of $\mathcal{T}(M_1)[\frac{1}{p}]$ such that the following three axioms hold:

(iv) Φ_1 *induces an* R_1*-linear isomorphism* $(M_1+\frac{1}{p}F^1_1)\otimes_{R_1\ \Phi_{R_1}} R_1 \xrightarrow{\sim} M_1$*;*

(v) *each tensor* $t_{1\alpha}$ *is fixed by* Φ_1, *is annihilated by* ∇_1, *and belongs to* $F^0(\mathcal{T}(M_1))[\frac{1}{p}]$ *(here* $(F^i(\mathcal{T}(M_1)))_{i\in\mathbb{Z}}$ *is the filtration of* $\mathcal{T}(M_1)$ *defined by* F^1_1, *cf. Subsection 2.1);*

(vi) *its reduction modulo the ideal* $I_1 := (x_1,\ldots,x_m)$ *is* $(M, F^1, \phi, (t_\alpha)_{\alpha\in\mathcal{J}}, \psi_M)$*.*

The R_1-module M_1 is free of rank equal to the rank of M, cf. property (vi). Let $\mathcal{T}(M_1)$ be as in Subsection 2.1. Let $z_Z : \mathrm{Spec}(W(k)) \hookrightarrow Z$ be the closed embedding defined by the ideal I_1 of R_1.

B7.4. Theorem. *There exists a morphism* $i_Z : Z \to \mathit{Spec}(R)$ *of* $W(k)$*-schemes such that* $g_{\mathrm{univ}}\circ i_Z\circ z_Z$ *is the identity section of* $\mathcal{G}'$ *and* $i_Z^*(\mathfrak{C}_{\mathrm{univ}})$ *is isomorphic to* $\mathfrak{C}_1$ *under an isomorphism which modulo the ideal* I_1 *becomes the identity automorphism of* 1_M*.*

Proof: If $\mathcal{G}$ is smooth, then the Theorem is a particular case of [Fa2, Thm. 10 and Rm. iii) after it]. To prove the Theorem in the general case, we follow the proof of [Va4, Thm. 5.2]. Let (D_Z, λ_{D_Z}) be the unique principally quasi-polarized p-divisible group over Z that lifts (D,λ_D) and whose principally quasi-polarized filtered F-crystal over R_1/pR_1 is $(M_1, F^1_1, \Phi_1, \nabla_1, \psi_{M_1})$, cf. B7.2.

By induction on $s\in\mathbb{N}$ we show that there exists a morphism $i_{Z,s} : \mathrm{Spec}(R_1/I_1^s) \to \mathrm{Spec}(R)$ of $W(k)$-schemes which at the level of rings maps $(x_1\ldots,x_l)$ to I_1/I_1^s and such that $i^*_{Z,s}((D_R,\lambda_{D_R}))$ is isomorphic to (D_Z,λ_{D_Z}) modulo I_1^s under a unique isomorphism $\mathcal{D}_s$ that has the following two properties:

(i) *it lifts the identity automorphism of* (D,λ_D)*;*

(ii) *it defines an isomorphism* $\mathcal{E}_s$ *between* $\mathfrak{C}_1$ *modulo* I_1^s *and* $i^*_{Z,s}(\mathfrak{C}_{\mathrm{univ}})$ *which modulo* I_1/I_1^s *is the identity automorphism of* 1_M*.*

As $\Phi_{R_1}(I_1) \subseteq I_1^p$ and R_1 is I_1-adically complete, such an isomorphism $\mathcal{E}_s$ is unique. If $s = 1$ we take $i_{Z,s}$ to be defined by the $W(k)$-epimorphism $R \twoheadrightarrow R/(x_1, \ldots, x_l) = W(k) = R_1/I_1$ and we take $\mathcal{D}_1$ and $\mathcal{E}_1$ to be defined by the identity automorphism of (D, λ_D) and by 1_M (respectively). Thus the existence and the uniqueness of $i_{Z,1}$ and $\mathcal{D}_1$ are obvious.

The passage from s to $s+1$ goes as follows. We endow the ideal $J_s := I_1^s/I_1^{s+1}$ of R_1/I_1^{s+1} with the trivial divided power structure; thus $J_s^{[2]} = 0$. The uniqueness of $\mathcal{D}_{s+1}$ is implied by the uniqueness of $\mathcal{D}_s$ and of $\mathcal{E}_{s+1}$, cf. Grothendieck–Messing deformation theory. Thus to end the induction we are left to show that we can choose $i_{Z,s+1}$ such that $\mathcal{D}_{s+1}$ and $\mathcal{E}_{s+1}$ exist.

Let $\tilde{i}_{Z,s+1} : \mathrm{Spec}(R_1/I_1^{s+1}) \to \mathrm{Spec}(R)$ be an arbitrary morphism of $W(k)$-schemes through which $i_{Z,s}$ factors naturally. We write

$$\tilde{i}^*_{Z,s+1}(M_R, F_R, \Phi, \nabla, \psi_{M_R}) = (M \otimes_{W(k)} R_1/I_1^{s+1}, F^1 \otimes_{W(k)} R_1/I_1^{s+1}, {}_{s+1}\Phi, {}_{s+1}\nabla, \psi_M).$$

Due to the existence of $\mathcal{D}_s$, there exists (cf. Grothendieck–Messing deformation theory) a direct summand ${}_{s+1}F^1$ of $M \otimes_{W(k)} R_1/I_1^{s+1}$ that lifts $F^1 \otimes_{W(k)} R_1/I_1^s$ and such that the quintuple $(M_1, F_1, \Phi_1, \nabla_1, \psi_{M_1})$ modulo I_1^{s+1} is isomorphic to the quintuple $(M \otimes_{W(k)} R_1/I_1^{s+1}, {}_{s+1}F^1, {}_{s+1}\Phi, {}_{s+1}\nabla, \psi_M)$ under an isomorphism $\tilde{\mathcal{E}}_{s+1}$ that lifts the one defined by $\mathcal{E}_s$. Let $t_{1\alpha,s+1} \in \mathcal{T}(M \otimes_{W(k)} R_1/I_1^{s+1})$ be the image under $\tilde{\mathcal{E}}_{s+1}$ of $t_{1\alpha}$. As $t_{1\alpha}$ is fixed by Φ_1, the tensor $t_{1\alpha,s+1}$ is fixed by ${}_{s+1}\Phi$. As $\tilde{\mathcal{E}}_{s+1}$ lifts $\mathcal{E}_s$, the reductions modulo J_s of t_α and $t_{1\alpha,s+1}$ coincide. As ${}_{s+1}\Phi(\mathcal{T}(M) \otimes_{W(k)} J_s) = 0$, inside $\mathcal{T}(M) \otimes_{W(k)} R_1/I_1^{s+1}$ we have

$$t_{1\alpha,s+1} - t_\alpha = {}_{s+1}\Phi(t_{1\alpha,s+1} - t_\alpha) \in {}_{s+1}\Phi(\mathcal{T}(M) \otimes_{W(k)} J_s) = 0.$$

Thus we have $t_{1\alpha,s+1} = t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R_1/I_1^{s+1}$ for all $\alpha \in \mathcal{J}$.

The remaining part of the inductive argument is as in the last four paragraphs of the proof of [Va4, Thm. 5.2]. Briefly, let U_{big} and U be the smooth, unipotent, closed subgroup schemes of $\mathbf{GL}_M$ and $\mathcal{G}$ (respectively) defined by the rule: if $\diamond$ is a commutative $W(k)$-algebra, then $U_{\mathrm{big}}(\diamond) := 1_{M \otimes_{W(k)} \diamond} + \mathrm{Hom}(F^1, F^0) \otimes_{W(k)} \diamond$ and

$$U(\diamond) := 1_{M \otimes_{W(k)} \diamond} + (\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)) \otimes_{W(k)} \diamond.$$

Let $v_{s+1} \in \mathrm{Lie}(U_{\mathrm{big}}) \otimes_{W(k)} J_s$ be the unique element such that we have

$$(1_{M \otimes_{W(k)} R_1/I_1^{s+1}} + v_{s+1})(F^1 \otimes_{W(k)} R_1/I_1^{s+1}) = {}_{s+1}F^1.$$

As in the proof of [Va4, Thm. 5.2] we argue that $v_{s+1} \in \mathrm{Lie}(U) \otimes_{W(k)} J_s$. The image of the Kodaira–Spencer map of ∇ is the direct summand $(\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)) \otimes_{W(k)} R = \mathrm{Lie}(U) \otimes_{W(k)} R$ of $\mathrm{Hom}(F^1, F^0) \otimes_{W(k)} R = \mathrm{Lie}(U_{\mathrm{big}}) \otimes_{W(k)} R$, cf. B7.4 (vi). Thus as in loc. cit. we can replace $\tilde{i}_{Z,s+1}$ by another morphism $i_{Z,s+1} : \mathrm{Spec}(R_1/I_1^{s+1}) \to \mathrm{Spec}(R)$ through which $i_{Z,s}$ factors and for which ${}_{s+1}F^1$ gets replaced by (i.e., becomes) $F^1 \otimes_{W(k)} R_1/I_1^{s+1}$. From Grothendieck–Messing deformation theory we get that $i^*_{Z,s+1}((D_R, \lambda_{D_R}))$ is isomorphic to (D_Z, λ_{D_Z}) modulo I_1^{s+1} under an isomorphism $\mathcal{D}_{s+1}$ which lifts $\mathcal{D}_s$ and which defines an isomorphism $\mathcal{E}_{s+1}$ between $\mathfrak{C}_1$ modulo I_1^{s+1} and $i^*_{Z,s+1}(\mathfrak{C}_{\mathrm{univ}})$. As $\mathcal{D}_{s+1}$

lifts $\mathcal{D}_s$, the uniqueness of $\mathcal{E}_s$ implies that $\mathcal{E}_{s+1}$ lifts $\mathcal{E}_s$ and thus also $\mathcal{E}_1$. This ends the induction.

We take $i_Z : Z \to \operatorname{Spec}(R)$ such that it lifts $i_{Z,s}$ for all $s \in \mathbb{N}$. From the very definition of $i_{Z,1}$ we get that $g_{\text{univ}} \circ i_Z \circ z_Z$ is the identity section of $\mathcal{G}'$. Moreover, $i_Z^*(\mathfrak{C}_{\text{univ}})$ is isomorphic to $\mathfrak{C}_1$ under an isomorphism that lifts $\mathcal{E}_s^{-1}$ for all $s \in \mathbb{N}$. $\square$

B7.5. Variant. Let $d \in \mathbb{N} \cup \{0\}$ be the rank of $\operatorname{Lie}(\mathcal{G}_{B(k)}) \cap \operatorname{Hom}(F^1, F^0) = \operatorname{Lie}(U)$. Let $S := W(k)[[x_1, \ldots, x_d]]$. We consider a closed embedding $\operatorname{Spec}(S) \hookrightarrow \operatorname{Spec}(R)$ such that the following two properties hold:

(i) at the level of $W(k)$-algebras, the ideal $(x_1, \ldots, x_l)$ of R maps to the ideal $(x_1, \ldots, x_d)$ of S;

(ii) the pull back $\mathfrak{D}_{\text{univ}}$ of $\mathfrak{C}_{\text{univ}}$ via the closed embedding $\operatorname{Spec}(S) \hookrightarrow \operatorname{Spec}(R)$, has a Kodaira–Spencer map which is injective and whose image equals to the direct summand $(\operatorname{Lie}(\mathcal{G}_{B(k)}) \cap \operatorname{Hom}(F^1, F^0)) \otimes_{W(k)} S$ of $\operatorname{Hom}(F^1, F^0) \otimes_{W(k)} S \xrightarrow{\sim} \operatorname{Hom}(F^1, M/F^1) \otimes_{W(k)} S$.

The proof of B7.4 applies to give us that there exists a morphism $j_Z : Z \to \operatorname{Spec}(S)$ of $W(k)$-schemes such that $j_Z^*(\mathfrak{D}_{\text{univ}})$ is isomorphic to $\mathfrak{C}_1$ under an isomorphism which modulo I_1 becomes the identity automorphism of 1_M. As the Kodaira–Spencer map of $\mathfrak{D}_{\text{univ}}$ is injective, the morphism j_Z is unique. In simpler words, we can choose $i_Z : Z \to \operatorname{Spec}(R)$ to factor through the closed embedding $\operatorname{Spec}(S) \hookrightarrow \operatorname{Spec}(R)$ and the resulting factorization is our morphism $j_Z : Z \to \operatorname{Spec}(S)$.

B8. On abelian schemes. Suppose that D is the p-divisible group of an abelian scheme A over $W(k)$. It is well known that we have two canonical and functorial identifications:

(i) $H^1_{\text{dR}}(A/W(k)) = M$ of $W(k)$-modules (see [Be, Ch. V, Subsection 2.3] and [BBM, Prop. 2.5.8]);

(ii) $H^1(D) = H^1_{\acute{e}t}(A_{\overline{B(k)}}, \mathbb{Z}_p)$ of $\operatorname{Gal}(B(k))$-modules.

The crystalline conjecture (see [Fa1] and [Fo]) provides a $B_{\text{crys}}(W(k))$-linear isomorphism

$$i_A : H^1_{\text{dR}}(A/W(k)) \otimes_{W(k)} B_{\text{crys}}(W(k)) \xrightarrow{\sim} H^1_{\acute{e}t}(A_{\overline{B(k)}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}}(W(k))$$

that is compatible with the tensor product filtrations, with the $\operatorname{Gal}(B(k))$-actions, and with the Frobenius endomorphisms. See [Va1, Subsubsection 5.2.15] for a proof of the following property (strictly speaking, the paragraphs before loc. cit. work with a prime $p \geq 3$ but the arguments of loc. cit. work for all primes):

(iii) under the identifications of (i) and (ii), we have $i_A = i_D^+ \otimes 1_{B_{\text{crys}}(W(k))}$.

B9. On Hodge cocharacters. In this Subsection we assume that we have a monomorphism $W(k) \hookrightarrow \mathbb{C}$ and that D is the p-divisible group of an abelian scheme A over $W(k)$.

We recall that we have canonical identifications

$$M \otimes_{W(k)} \mathbb{C} = H^1_{\text{dR}}(A/W(k)) \otimes_{W(k)} \mathbb{C} = H^1_{\text{dR}}(A_{\mathbb{C}}/\mathbb{C}) = F^{1,0} \oplus F^{0,1}, \tag{5}$$

where the last identity is the usual Hodge decomposition. Under (5) we can identify

$$F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}.$$

Let $A^{\mathrm{an}}_{\mathbb{C}}$ be the analytic space associated to $A_{\mathbb{C}}$. Let $W := H_1(A^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$ be the first Betti homology group of $A^{\mathrm{an}}_{\mathbb{C}}$ with rational coefficients. Let $W^* := \mathrm{Hom}(W, \mathbb{Q})$. We identify naturally $W^* \otimes_{\mathbb{Q}} \mathbb{C}$ with the first Betti cohomology group $H^1(A^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C})$ and thus also with $H^1_{\mathrm{dR}}(A_{\mathbb{C}}/\mathbb{C}) = M \otimes_{W(k)} \mathbb{C}$. We consider also the Hodge decomposition

$$W \otimes_{\mathbb{Q}} \mathbb{C} = H_1(A^{\mathrm{an}}_{\mathbb{C}}, \mathbb{C}) = F^{-1,0} \oplus F^{0,-1} \tag{6}$$

that is the dual of the Hodge decomposition $W^* \otimes_{\mathbb{Q}} \mathbb{C} = H^1_{\mathrm{dR}}(A_{\mathbb{C}}/\mathbb{C}) = F^{1,0} \oplus F^{0,1}$. Let $\mu_A : \mathbb{G}_{m\mathbb{C}} \to \mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{C}}$ be the Hodge cocharacter that fixes $F^{0,-1}$ and that acts on $F^{-1,0}$ via the identity character of $\mathbb{G}_{m\mathbb{C}}$. We also view μ_A as a cocharacter (denoted in the same way) $\mu_A : \mathbb{G}_{m\mathbb{C}} \to \mathbf{GL}_{W^* \otimes_{\mathbb{Q}} \mathbb{C}} = \mathbf{GL}_{M \otimes_{W(k)} \mathbb{C}} = \mathbf{GL}_{H^1_{\mathrm{dR}}(A_{\mathbb{C}}/\mathbb{C})}$ that fixes $F^{0,1}$ and that acts on $F^{1,0}$ via the inverse of the identity character of $\mathbb{G}_{m\mathbb{C}}$.

B9.1. Lemma. *Let the cocharacter $\mu : \mathbb{G}_m \to \mathcal{G}$ be as in B2. Suppose that for every $\alpha \in \mathcal{J}$ the tensor $t_\alpha \in \mathcal{T}(M)[\frac{1}{p}] = \mathcal{T}(H^1_{\mathrm{dR}}(A/W(k)))[\frac{1}{p}]$ is the de Rham component of a Hodge cycle on $A_{B(k)}$. We also assume that $\mathcal{G}_{B(k)}$ is a reductive group. Then the cocharacter $\mu_A : \mathbb{G}_{m\mathbb{C}} \to \boldsymbol{GL}_{M \otimes_{W(k)} \mathbb{C}}$ factors through $\mathcal{G}_{\mathbb{C}}$ and is $\mathcal{G}(\mathbb{C})$-conjugate with $\mu_{\mathbb{C}}$. Thus if $\mathcal{G}_{B(k)}$ is a torus, then $\mu_A = \mu_{\mathbb{C}}$.*

Proof: Let $v^B_\alpha \in \mathcal{T}(W^*)$ be the Betti realization of t_α; it is fixed by μ_A. The identity $W^* \otimes_{\mathbb{Q}} \mathbb{C} = M \otimes_{W(k)} \mathbb{C}$ gives birth to an identity $\mathcal{T}(W^* \otimes_{\mathbb{Q}} \mathbb{C}) = \mathcal{T}(M \otimes_{W(k)} \mathbb{C})$ under which the tensors t_α and v^B_α are as well identified. Thus the cocharacter $\mu_A : \mathbb{G}_{m\mathbb{C}} \to \mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{C}}$ fixes t_α for all $\alpha \in \mathcal{J}$ and therefore it factors through $\mathcal{G}_{\mathbb{C}}$. Let $\mathcal{P}_{\mathbb{C}}$ be the parabolic subgroup of $\mathcal{G}_{\mathbb{C}}$ that normalizes $F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}$. Both the cocharacters $\mu_A : \mathbb{G}_{m\mathbb{C}} \to \mathbf{GL}_{M \otimes_{W(k)} \mathbb{C}}$ and $\mu_{\mathbb{C}}$ factor through $\mathcal{P}_{\mathbb{C}}$ and thus a $\mathcal{P}_{\mathbb{C}}(\mathbb{C})$-conjugate $\mu'_{\mathbb{C}}$ of $\mu_{\mathbb{C}}$ commutes with μ_A. As the commuting cocharacters $\mu'_{\mathbb{C}}$ and μ_A of $\mathcal{P}_{\mathbb{C}}$ act on $F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}$ and on $M \otimes_{W(k)} \mathbb{C}/(F^1 \otimes_{W(k)} \mathbb{C}) = H^1_{\mathrm{dR}}(A_{\mathbb{C}}/\mathbb{C})/F^{1,0}$ in the same way, we have $\mu'_{\mathbb{C}} = \mu_A$. Thus the cocharacters $\mu_{\mathbb{C}}$ and μ_A are $\mathcal{P}_{\mathbb{C}}(\mathbb{C})$-conjugate and therefore they are also $\mathcal{G}(\mathbb{C})$-conjugate.$\square$

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Adrian Vasiu
Department of Mathematical Sciences
Binghamton University
Binghamton, New York 13902-6000
e-mails:adrian@math.arizona.edu adrian@math.binghamton.edu
fax: 1-607-777-2450