Periodicity Theorems and Conjectures in Hermitian K-theory

An appendix to the paper of R. Hazrat and N. Vavilov

Max Karoubi

The purpose of this appendix to [4], is to state some periodicity theorems and conjectures in an algebraic context which are related to Bak's work on the K-theory of forms [1].

Periodicity theorems in topological K-theory have already some history going back to Bott, Atiyah and others (see [5] for a survey). As we shall see, the analog of these theorems in a purely algebraic context uses the orthogonal or symplectic group rather than the general linear group. From this point of view, the K-theory of forms plays an important role.

1. Recall of basic definitions

1.1. The starting point, as in the main body of the paper [4], is a ring A with an anti-involution $a \to \overline{a}$ and a "sign of symmetry" $\varepsilon = \pm 1$. If E is a finitely generated projective (right) A-module, its dual E^* is the (right) A-module consisting of additive maps $f : E \to A$ such that $f(x\lambda) = \overline{\lambda} f(x)$, where λ belongs to A. There is an obvious natural isomorphism between E and its bidual E^{**}. Moreover, if $\alpha : E \to F$ is an A-linear map, its transpose ${}^t\alpha : F^* \to E^*$ is defined in the usual way, so that the correspondence $E \mapsto E^*$ defines a contravariant functor. A non-degenerate ε-hermitian form is simply an isomorphism $\phi : E \to E^*$ such that its transpose ${}^t\phi : E \cong E^{**} \to E^*$ coincides with $\varepsilon\phi$. It is well known and easy to show that an equivalent way of describing an ε-hermitian form is to give a $\mathbb{Z}$-bilinear map

$$\Phi : E \times E \to A$$

such that $\Phi(x\lambda, y\mu) = \overline{\lambda}\Phi(x, y)\mu$ and $\Phi(y, x) = \varepsilon\overline{\Phi}(x, y)$. However, the definition of non-degeneracy has built into it the notion of dual module.

1.2. Of particular importance are the *even* forms which may be written as

$$\phi = \phi_0 + \varepsilon\, {}^t\phi_0$$

(They are the only ones considered in this appendix, except in §4). The *unitary* group of (E, ϕ) is the group of automorphisms f of E such that $\phi = {}^tf.\phi.f$. If ϕ_0 is given, the *orthogonal* group of (E, ϕ_0) is the subgroup of the unitary group which elements f are such that ${}^tf.\phi_0.f$ may be written as $\phi_0 + v - \varepsilon\, {}^tv$ for some v. These two different groups are called respectively $\mathcal{O}^{\max}(E)$ and $\mathcal{O}^{\min}(E)$ in Bak's papers.

1.3. An illustrative example is $E = M \bigoplus M^* = H(M)$, called the hyperbolic module associated to M. The map

$$\phi_0 : M \bigoplus M^* \to M^* \bigoplus M^{**} \cong M \bigoplus M^*$$

is given by the matrix

$$\phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The elements of the unitary group $\mathcal{O}^{\max}(E)$ can be described as all block matrices

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $f^*f = ff^* = id$, where f^* is the "adjoint" matrix given by the formula

$$f^* = \begin{pmatrix} {}^t d & \varepsilon {}^t b \\ \varepsilon {}^t c & {}^t a \end{pmatrix}.$$

The matrix f belongs to the orthogonal group $\mathcal{O}^{\min}(E)$ if the additional assumption that the matrices ${}^t c.a$ and ${}^t d.b$ are of the type $u - \varepsilon {}^t u$ for some u is satisfied.

1.4. The importance of the hyperbolic modules comes essentially from the following fact (proved for instance in [7], p. 61): every quadratic or even hermitian module is a direct summand of an hyperbolic module and therefore of $H(A^n)$ for a certain n. Following again Bak's terminology, we shall denote by $\mathcal{O}^{\max}_{n,n}(A)$ (respectively $\mathcal{O}^{\min}_{n,n}(A)$) the unitary group (respectively the orthogonal group) associated to $H(A^n)$. The stabilized versions of these groups $\mathcal{O}^{\max}(A) = \operatorname{colim} \mathcal{O}^{\max}_{n,n}(A)$ and $\mathcal{O}^{\min}(A) = \operatorname{colim} \mathcal{O}^{\min}_{n,n}(A)$ will play an important role in the sequel. We shall simply write them as $\mathcal{O}(A)$ (instead of $\mathcal{O}^{\max}(A)$ or $\mathcal{O}^{\min}(A)$) for statements which apply to both situations.

1.5. The theories "max" and "min" are the same if we assume the existence of an element λ in the centre of A such that $\lambda + \overline{\lambda} = 1$. To see this, it suffices to show that if μ is either a $(-\varepsilon)$-hermitian form or an element of A with the property that $\mu = -\varepsilon\overline{\mu}$, respectively, then either $\mu = \eta - \varepsilon\eta^*$ for some sesquilinear η or $\mu = \eta - \varepsilon\overline{\eta}$ for some $\eta \in A$, respectively. But $\mu = -\varepsilon\mu^*$ implies $\mu = (\lambda + \overline{\lambda})\mu = \lambda\mu - \overline{\lambda}\varepsilon\lambda^* = (\lambda\mu) - \varepsilon(\lambda\mu)^*$. Similarly, $\mu = -\varepsilon\overline{\mu}$ implies $\mu = (\lambda\mu) - \varepsilon(\overline{\lambda\mu})$.

2. Negative KQ-groups

2.1. We define $KQ_0^{\max}(A)$, (respectively $KQ_0^{\min}(A)$) as the Grothendieck group built out of even hermitian modules (respectively quadratic modules) provided with the orthogonal direct sum operation. We define $KQ_1^{\max}(A)$ and $KQ_1^{\min}(A)$ as the quotients of $\mathcal{O}^{\max}(A)$ and $\mathcal{O}^{\min}(A)$, respectively, by their commutator subgroups. In his book ([1], p.190-191), Bak proves a fundamental exact sequence relating the "max" and the "min" groups (in fact a more general one with different form parameters), where KQ means always ${}_\varepsilon KQ$

$$KQ_1^{\min}(A) \longrightarrow KQ_1^{\max}(A) \longrightarrow \Theta_0(A) \longrightarrow KQ_0^{\max}(A) \longrightarrow KQ_0^{\min}(A) \longrightarrow 0.$$

The group $\Theta_0(A)$ has an explicit description. It is the quotient

$$(\Gamma/\Lambda \otimes_A \Gamma/\Lambda)/[a \otimes b - b \otimes a, a \otimes b - a \otimes ba\bar{b}],$$

where $\Gamma = \Gamma(A)$ is the group of elements σ in A such that $\overline{\sigma} = \varepsilon\sigma$ and $\Lambda = \Lambda(A)$ is the group of elements σ in A which may be written as $\eta + \varepsilon\overline{\eta}$.

2.2. The negative K and KQ-groups are usually defined using the suspension of the ring A (see [7] for instance). In this way, we define $KQ_{-1}(A) = KQ(SA), KQ_{-2}(A) = KQ_{-1}(SA) = KQ(S^2A)$, etc. On the other hand, SA is the quotient of the cone CA of the ring A which is "flabby" and the obvious map $\Theta_0(CA) \to \Theta_0(SA)$ is onto. The group $\Theta_0(CA)$ fits into Bak's exact sequence (where again KQ means ${}_\varepsilon KQ$)

$$KQ_1^{\min}(CA) \longrightarrow KQ_1^{\max}(CA) \longrightarrow \Theta_0(CA) \longrightarrow KQ_0^{\max}(CA) \longrightarrow KQ_0^{\min}(CA).$$

Since the KQ-groups of CA are reduced to 0, it follows that $\Theta_0(CA)$ and hence $\Theta_0(SA)$ are also reduced to 0. This implies the following theorem:

Theorem 2.1. *The obvious map*

$$KQ_n^{\min}(A) \to KQ_n^{\max}(A)$$

is an isomorphism for $n < 0$.

2.3. These negative K and KQ-groups play an important role in the proof of the periodicity theorem in Hermitian K-theory as it was emphasized in [6]. We shall sketch this periodicity statement in the next sections.

3. The periodicity theorem in Hermitian K-theory

3.1. As a standard notation, let us call $\mathcal{K}(A)$ the classifying space of algebraic K-theory (its homotopy groups are Quillen's K-groups). We shall also call ${}_\varepsilon\mathcal{KQ}(A)$ the classifying space of Hermitian K-theory whose homotopy groups are ${}_\varepsilon KQ_n(A)$. Strictly speaking, in our context, one should distinguish between the "max" and the "min" categories, in which case we should write ${}_\varepsilon KQ_n^{\max}(A)$ or ${}_\varepsilon KQ_n^{\min}(A)$.

3.2. The forgetful functor induces a continuous map

$${}_\varepsilon\mathcal{KQ}(A) \to \mathcal{K}(A),$$

whereas the hyperbolic functor induces a map backwards

$${}_{-\varepsilon}\mathcal{K}(A) \to \mathcal{KQ}(A),$$

(note the change of symmetry of ε which is justified below). Let us call ${}_\varepsilon\mathcal{V}(A)$ (respectively ${}_{-\varepsilon}\mathcal{U}(A)$) the homotopy fibers of these maps. As it was detailed in [5, 6], the periodicity theorem (also called the fundamental theorem in Hermitian K-theory) states that there is a natural homotopy equivalence between the spaces ${}_\varepsilon\mathcal{V}(A)$ and the loop space $\Omega_{-\varepsilon}\mathcal{U}(A)$. However, this theorem has been proved in [6] only with the additional assumption that there is an element λ in the centre of A such that $\lambda + \overline{\lambda} = 1$, in which case there is no need to distinguish between our favorite categories "max" and "min".

3.3. We should not list all the applications of this periodicity theorem which are detailed in [5, 6]. We should say however that a nice consequence is an exact sequence (which is part of a 12 terms exact sequence detailed in [6], p. 278)

$$k_1(A) \longrightarrow {}_{-\varepsilon}WQ_2(A) \longrightarrow {}_{-\varepsilon}WQ'_0(A) \longrightarrow k'_1(A).$$

Here ${}_{-\varepsilon}WQ'_0(A)$ is the "coWitt group", i.e., the kernel of the forgetful functor ${}_{\varepsilon}KQ_0(A) \to K_0(A)$. The group ${}_{-\varepsilon}WQ_2(A)$ is the cokernel of the hyperbolic map $K_2(A) \to {}_{-\varepsilon}KQ_2(A)$. Finally, $k_1(A)$ (respectively $k'_1(A)$) is the 0 (respectively the 1st) Tate cohomology group of $\mathbb{Z}/2$ acting on the Bass group $K_1(A)$. This exact sequence has also been proved by R. Sharpe in a wider context ([8], see also [1], p. 227).

3.4. Let us now look at the general case (i.e., we don't assume the existence of such a λ as in 3.2). A closer look of the proof of the periodicity theorem [6] shows that there is a well defined map (we don't need λ) for that):

$$\beta : {}_{\varepsilon}\mathcal{V}^{\max}(A) \to \Omega_{-\varepsilon}\mathcal{U}^{\min}(A).$$

It is essentially given by a cup-product with a remarkable element u_2 in ${}_{-1}KQ_2^{\max}(\mathbb{Z})$ (cf. [6], p.273, line 5 and the reference [9] in this paper, p. 249).

Conjecture 1. *The map β defined above is an homotopy equivalence.*

3.5. A strategy for a proof is to define a map backwards

$$\beta' : \Omega_{-\varepsilon}\mathcal{U}^{\min}(A) \to {}_{\varepsilon}\mathcal{V}^{\max}(A)$$

following the methods in [6]. This can be done by using the ideas of Clauwens [3] about almost symmetric forms. Hopefully, in a future paper, we shall prove that β and β' are homotopy equivalences mutually inverse.

3.6. **Example.** Let A be a field of characteristic 2. Since $WQ_0'^{\max}(A) = 0$, the conjecture implies that

$$\begin{aligned}\operatorname{coker}(K_2(A) \to {}_{-1}KQ_2^{\min}(A)) &= \\ &= \operatorname{coker}(H_2([GL(A), GL(A)]; \mathbb{Z}) \to H_2([O(A), O(A)]; \mathbb{Z})) = 0.\end{aligned}$$

Note that $[O(A), O(A)]$ is the kernel of the homomorphism

$$O(A) \to KQ_1^{\min}(A) \cong \mathbb{Z}/2 \times A^*/A^{*2},$$

(see [7], p. 81 for instance). Moreover, if A is finite, $K_2(A) = 0$ and $A^*/A^{*2} = 0$, as A^* is of odd order, which makes the above formulas simpler.

4. Generalization and conjectures with form parameters

4.1. The Conjecture 1 does not answer completely the periodicity problem. One would like to have as target of β the group with the form parameter "max" instead of "min" for instance. One way to proceed is to fully exploit form parameters as in [1]. Most of the statements here are conjectural for the moment.

More precisely, if Λ is a form parameter, one can consider not only the category of Λ-*quadratic modules* of sign ε but also the category of Λ-*hermitian modules* of the same sign, as in [1], p. 31. We just put the extra condition that the hermitian form ϕ has the property that $\phi(x,x)$ belongs to λ. However, it is not true anymore that any hermitian module is a direct summand of an hyperbolic module, except if $\Lambda = \min$, which is the case of even hermitian forms.

We now follow essentially the notations of [1] in order to emphasize the role of the form parameter Λ. For instance, we denote by ${}_\varepsilon KH_0(A,\Lambda)$ the Grothendieck group of the category of ε-*hermitian* modules such that $\phi(x,x)$ belongs to Λ as above. In the same way, we write ${}_\varepsilon\mathcal{KH}(A,\Lambda)$ for the classifying space of this category. We also write ${}_\varepsilon\mathcal{VH}(A,\Lambda)$ for the homotopy fiber of the forgetful map ${}_\varepsilon\mathcal{KH}(A,\Lambda) \to \mathcal{K}(A)$, etc [1]. We adopt the same notation for the theory KQ, i.e., ${}_\varepsilon KQ(A,\Lambda)$ instead of ${}_\varepsilon KQ(A)$, etc.

4.2. If A and B are two rings with form parameters Λ and Γ, we now define an important "cup-product"

$${}_\varepsilon KH_0(A,\Lambda) \times {}_\eta KQ_0(B,\Gamma) \to {}_{\varepsilon\eta} KQ_0(A \otimes B, \Lambda \otimes \Gamma).$$

In order to do this, we first make the following remark. Let τ_0 be a sesquilinear form on a C-module defining a quadratic form (with respect to a form parameter Σ and a sign σ) and let us call τ the associated σ-hermitian form. Then we may write

$$\tau_0(u,v) + \tau_0(v,u) = \tau_0(u,v) + \sigma\overline{\tau}_0(v,u) + \tau_0(v,u) - \sigma\overline{\tau}_0(v,u),$$

which shows that $\tau_0(u,v) + \tau_0(v,u) = \tau(u,v) \bmod \Sigma$.

We apply this remark to the following situation; Φ is a $A - \Lambda$ hermitian form of sign ε and Ψ_0 is a $B - \Gamma$ quadratic form of sign η. Then we claim that $\tau_0 = \Phi \otimes \Psi_0$ is a $A \otimes B - \Lambda \otimes \Gamma$ quadratic form of sign $\sigma = \varepsilon\eta$, independent of the choice of Ψ_0.

First, it is clear that $\tau = \Phi \otimes \Psi$ (where Ψ is the η-hermitian form associated to Ψ_0) is a σ-hermitian form. Secondly, we have to show that $\tau_0(u,u)$ is well defined in $(A \otimes B)/\Lambda \otimes \Gamma$, i.e., is independent of Ψ_0. For this we write $u = \sum_1^n x_i \otimes y_i$ and

$$\tau_0(u,u) = \sum_{i \neq j} (\Phi \otimes \Psi_0)[(x_i \otimes y_i), (x_j \otimes y_j)] + \sum_i (\Phi \otimes \Psi_0)[(x_i \otimes y_i), (x_i \otimes y_i)].$$

According to the previous remark, the first sum is

$$\sum_{i<j} (\Phi \otimes \Psi_0)[(x_i \otimes y_i), (x_j \otimes y_j)] \bmod \Lambda \otimes \Gamma,$$

[1] It is not clear however how to deloop these spaces. Therefore, our statements or conjectures are considered for spaces and not for spectra.

whereas the second sum is
$$\sum_i \Phi(x_i, x_i) \otimes \Psi_0(y_i \otimes y_i).$$
Since $\Phi(x_i, x_i)$ belongs to Λ, the last sum makes sense in $\Lambda \otimes B/\Gamma$ and therefore $\tau_0(u, u)$ is well defined in $(A \otimes B)/\Lambda \otimes \Gamma$ and is independent of Ψ_0.

4.3. The cup-product defined above can be extended to the categorical level and defines a pairing between the associated classifying spaces
$${}_\varepsilon\mathcal{KH}(A, \Lambda) \times {}_\eta\mathcal{KQ}(B, \Gamma) \longrightarrow {}_{\varepsilon\eta}\mathcal{KQ}(A \otimes B, \Lambda \otimes \Gamma).$$

An important example for us is $B = \mathbb{Z}$, $\Gamma = \max$ and $\eta = -1$. Then the same method as in [6] enables us to define a map (note the intertwining between hermitian and quadratic modules: The theory $\mathcal{V}$ is hermitian, whereas the theory $\mathcal{U}$ is quadratic)
$$\beta : {}_\varepsilon\mathcal{VH}(A, \Lambda) \to \Omega_{-\varepsilon}\mathcal{UQ}(A, \Lambda).$$

Our second conjecture can now be stated as follows:

Conjecture 2. *The map β defined above is an homotopy equivalence.*

4.4. We remark first that this conjecture implies the previous one for $\Lambda = \min$, since ${}_\varepsilon\mathcal{VH}(A, \min)$ is just ${}_\varepsilon\mathcal{V}^{\max}(A)$, whereas ${}_{-\varepsilon}\mathcal{UQ}(A, \min)$ is ${}_{-\varepsilon}\mathcal{U}^{\min}(A)$, with our previous notations in 3.4.

Secondly, we remark that a consequence of Conjecture 2 is proved in the book of Bak[1], p. 277, Lemma 11.30, and is related to the results of R. Sharpe [8] (for $\Lambda = \min$), already quoted in 3.3 (with less generality).

Finally, this conjecture seems related to the results of Barge and Lannes [2] for A a commutative ring and $\Lambda = \max$ (at least for lower homotopy groups).

For the moment this second conjecture is widely open, although, as we mentioned, some evidence may be found in [8, 1] and [2].

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Université Paris 7, Denis Diderot, 2 Place Jussieu, 75251 Paris, France
E-mail address: max.karoubi@gmail.com