Slow Schrödinger dynamics of gauged vortices

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Abstract

Multivortex dynamics in Manton's Schrödinger–Chern–Simons variant of the Landau–Ginzburg model of thin superconductors is studied within a moduli space approximation. It is shown that the reduced flow on M_N, the N vortex moduli space, is hamiltonian with respect to ω_{L^2}, the L^2 Kähler form on M_N. A purely hamiltonian discussion of the conserved momenta associated with the euclidean symmetry of the model is given, and it is shown that the euclidean action on $(\mathsf{M}_N, \omega_{L^2})$ is not hamiltonian. It is argued that the $N = 3$ flow is integrable in the sense of Liouville. Asymptotic formulae for ω_{L^2} and the reduced Hamiltonian for large intervortex separation are conjectured. Using these, a qualitative analysis of internal 3-vortex dynamics is given and a spectral stability analysis of certain rotating vortex polygons is performed. Comparison is made with the dynamics of classical fluid point vortices and geostrophic vortices.

1 Introduction

The Landau–Ginzburg theory of an idealized planar superconductor consists of a complex scalar field ϕ representing the electron pair condensate, and a $U(1)$ gauge potential A_i ($i = 1, 2$), interacting via the potential energy functional

$$V = \int \left(\frac{1}{2} B^2 + \frac{1}{2} \sum_i D_i\phi \overline{D_i\phi} + \frac{\mu^2}{8}(1 - |\phi|^2)^2 \right) d^2\mathbf{x}. \tag{1.1}$$

Here $D_i\phi = \partial_i\phi - iA_i\phi$ is the gauge covariant derivative, $B = \partial_1 A_2 - \partial_2 A_1$ is the magnetic field and μ is a coupling constant. The model admits topologically stable, spatially localized

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solutions (minimals of V) called vortices, the planar analogues of magnetic flux tubes in 3-dimensional superconductors. These have finite V, and so have a well-defined winding number at infinity, $N \in \mathbb{Z}$, which is interpreted physically as the net vortex number. The case of critical coupling, $\mu = 1$, is special because here, given any unordered choice of N complex numbers (not necessarily distinct) $\{z_1, \ldots, z_N\}$, there exists a minimal of V with N vortices located at $z = x_1 + ix_2 = z_r$, $r = 1, \ldots, N$, unique up to gauge equivalence [6]. More precisely the Higgs field ϕ of the solution has zeroes precisely at the prescribed points z_r. Physically, one says that static critically coupled vortices exert no net forces on one another. It follows that the moduli space of static N-vortex solutions is $\mathsf{M}_N \equiv \mathbb{C}^N$. Note that the vortex positions $\{z_r\}$ are not good global coordinates on M_N because we must identify solutions which differ by permutation of $\{z_r\}$. Good global coordinates are provided by the coefficients w_r of the unique monic polynomial in z whose roots are $\{z_r\}$ (counted with multiplicity), that is

$$z^N + w_1 z^{N-1} + \cdots + w_N := (z - z_1)\cdots(z - z_N). \tag{1.2}$$

To introduce dynamics to the theory, one must also define a kinetic energy functional, T, and many choices are possible. Manton has advocated the use of a Schrödinger–Chern–Simons functional linear in first time derivatives, explicitly,

$$T = \gamma \int \left(\frac{i}{2}(\overline{\phi} D_0 \phi - \phi \overline{D_0 \phi}) + BA_0 + E_2 A_1 - E_1 A_2 - A_0 \right) d^2\mathbf{x} \tag{1.3}$$

where A_0 is the temporal gauge field, $E_i = \partial_i A_0 - \partial_0 A_i$ is the electric field and γ is another coupling constant [8]. The resulting dynamics (governed by Lagrangian $L = T - V$) is first order in time and non-dissipative, and is hoped to give a good description of vortex dynamics in thin superconductors at very low temperatures.

In order to make progress on the problem of N-vortex dynamics in this model, one could resort to numerical solution of the field equations. Manton [8] has, however, pursued a different strategy, namely that of adiabatic approximation. One assumes that at each time the field configuration lies in M_N, the moduli space of static solutions of the $\mu = 1$ system, but that the field's position in M_N may vary slowly with time. This slow variation is assumed to be governed by the reduced Lagrangian, that is, $T - V$ restricted to M_N. The result is a finite dimensional dynamical system, in fact, a hamiltonian flow on M_N, as we shall see, which hopefully captures the important features of the full, infinite dimensional field dynamics. Of course if $\mu = 1$ exactly, the restricted potential is constant, and there is no dynamics — the vortices remain static. On the other hand, if μ differs greatly from 1 then approximating multivortex profiles by critical N-vortices is ill-justified. So the approximation is valid only in the near critical regime. This strategy has proved to be extremely successful in the study of vortex dynamics in the relativistic version of the Landau–Ginzburg model [15, 16], and has been justified by rigorous error estimates [19].

The present paper gives an analysis of vortex dynamics within the adiabatic approximation, following on from Manton's original paper [8]. Some of the results are anticipated by the study of the equivalent model on the two-sphere [14]. In section 2, for example, we prove that the reduced flow on M_N is hamiltonian with respect to the L^2 Kähler form, a result which can be understood from the results on S^2, at least formally, once the sphere's radius is taken to infinity. It is worthwhile giving independent proofs of the relevant results for two reasons, however. First, the planar model has far more direct physical significance, both as a model of extremely thin superconductors, and of the 3-dimensional case where translation symmetry is imposed. The spherical model is useful to study the quantum mechanics of the system, but its direct advantage is mathematical convenience ($\mathsf{M}_N \equiv \mathbb{CP}^N$ is then compact, hence the methods of geometric quantization are amenable to more explicit treatment) rather than physical applicability. Second, the formulation of the model on S^2 entails considerable technical subtleties absent in the planar model. When dealing with the planar model, therefore, many of the arguments simplify considerably.

Neither [8] nor [14] contains a quantitative discussion of classical vortex dynamics, and for good reason — neither the Kähler form nor the Hamiltonian on M_N are explicitly known. In the second half of this paper we will restrict attention to the regime of well-separated vortices, where explicit formulae can be inferred, and hence more progress is possible. The results turn out to be similar to the dynamics of so-called geostrophic vortices [12], of interest in meteorology, and, to a lesser extent, classical fluid point vortices [1]. We give a complete description of the internal 3-vortex dynamics and analyze the spectral stability properties of rigidly rotating regular polygons.

Lange and Schroers have studied the slow Schrödinger dynamics of *ungauged* Landau–Ginzburg vortices within an adiabatic approximation [7]. For them, there is no concept of critical coupling related to a self-duality structure, so no moduli space of static N-vortex solutions is available. Instead, they define M_2 to be the unstable manifold of the coincident double vortex, which they construct via numerical gradient flow. Their emphasis is very different from the present work, therefore. They use intensive numerical techniques to study two-vortex dynamics thoroughly, concentrating particularly on the case of overlapping vortices. Indeed, one of their main motivations was to provide the first (necessarily numerical) implementation of the unstable manifold method for solitons in spatial dimension exceeding one. In contrast, we focus on the opposite physical regime (large separation), for $N \geq 3$, where heavy numerical analysis is not required.

2 The reduced system

Let q^α, $\alpha = 1, \ldots, 2N$, be some choice of local coordinates on M_N (for example the real and imaginary parts of the vortex positions $z_r \in \mathbb{C}$). Then, since T depends only linearly on time derivatives, the Lagrangian restricted to M_N must take the form

$$L|_{\mathsf{M}_N} = \mathcal{A}_\alpha(q)\dot{q}^\alpha - V(q) \tag{2.1}$$

where $\mathcal{A}_\alpha$ are some functions of the coordinates $\{q^\beta\}$ which may naturally be interpreted as the components of a $U(1)$ connexion form $\mathcal{A} = \mathcal{A}_\alpha dq^\alpha$ on M_N, since changing $\mathcal{A}$ by an exact form $\mathcal{A} \mapsto \mathcal{A} + d\Phi$ changes $L|_{\mathsf{M}_N}$ by an irrelevant total time derivative [8]. The Euler–Lagrange equations resulting from $L|_{\mathsf{M}_N}$ are

$$\mathcal{B}_{\alpha\beta}\dot{q}^\alpha = -\frac{\partial V}{\partial q^\beta}, \tag{2.2}$$

where $\mathcal{B} = d\mathcal{A}$ is the curvature form of $\mathcal{A}$. Clearly, $\mathcal{B}$ is closed. Further, assuming that (2.2) is a well-defined flow on M_N, $\mathcal{B}$ must also be a *nondegenerate* bilinear form. So we may reinterpret $\mathcal{B}$ as a symplectic form on M_N. Given any function $F \in C^\infty(\mathsf{M}_N)$, let X_F denote its symplectic gradient, that is the unique vector field satisfying

$$\iota_{\mathsf{X}_F}\mathcal{B} = -dF \tag{2.3}$$

where ι denotes interior multiplication, $\iota_X\mathcal{B}(Y) := \mathcal{B}(X,Y)$. System (2.2) is then hamiltonian flow along X_V.

No explicit formula for $\mathcal{B}$ or $\mathcal{A}$ exists. However, Manton has shown how $\mathcal{A}$ can be related to the analytic properties of the Higgs field ϕ near its zeroes. Let us assume that the vortex positions z_r are all distinct, and use these as local complex coordinates on M_N. Then it is known that in a neighbourhood of z_r

$$\log|\phi(z)|^2 = \log|z-z_r|^2 + a_r + \frac{1}{2}\overline{b}_r(z-z_r) + \frac{1}{2}b_r(\overline{z}-\overline{z}_r) + \cdots \tag{2.4}$$

where a_r and b_r are some unknown functions of the vortex positions z_s, a_r real, b_r complex. We have used here the conventions of [10]. Samols [15], adapting earlier work of Strachan [18], showed that the L^2 metric on M_N could be written entirely in terms of the unknown coefficients b_r, namely

$$g_{L^2} = \pi\sum_{r,s=1}^{N}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial z_r}\right)dz_r d\overline{z}_s. \tag{2.5}$$

This is a hermitian metric on $T\mathsf{M}_N$ which can be shown to be Kähler [15]. Using similar techniques, Manton has obtained [8] a similar formula for the connexion form $\mathcal{A}$; in our notation,

$$\mathcal{A} = i\pi\gamma\sum_{r=1}^{N}\left[(\overline{b}_r + \frac{1}{2}\overline{z}_r)dz_r - (b_r + \frac{1}{2}z_r)d\overline{z}_r\right]. \tag{2.6}$$

We will now establish the following proposition:

Proposition 1 *The connexion $\mathcal{A}$ has curvature form $\mathcal{B} = -2\gamma\omega_{L^2}$ where ω_{L^2} is the Kähler form corresponding to g_{L^2}.*

Proof: We will prove the formula on $\mathsf{M}_N\backslash\Delta_N$, where Δ_N is the measure zero set on which vortices coincide, and appeal to smoothness. Let us define the $(0,1)$-form

$$b := \sum_r b_r d\overline{z}_r. \tag{2.7}$$

Hermiticity of g_{L^2} and (2.5) imply

$$\frac{\partial b_s}{\partial z_r} \equiv \frac{\partial \overline{b}_r}{\partial \overline{z}_s} \quad \Rightarrow \quad \bar\partial\,\overline{b} = -\partial b. \tag{2.8}$$

Note that the connexion $\mathcal{A}$ and ω_{L^2} may be written, using (2.7) and (2.8)

$$\begin{aligned} \mathcal{A} &= i\pi\gamma\left[\frac{1}{2}\sum_r(\overline{z}_r dz_r - z_r d\overline{z}_r) + \overline{b} - b\right], \\ \omega_{L^2} &= \frac{i\pi}{2}\left(\sum_r dz_r\wedge d\overline{z}_r + \partial b - \bar\partial\,\overline{b}\right). \end{aligned} \tag{2.9}$$

It follows that

$$2\omega_{L^2} + \frac{1}{\gamma} d\mathcal{A} = i\pi(\partial b - \bar{\partial}\,\overline{b} + d\overline{b} - db) = i\pi(\partial\overline{b} - \bar{\partial}\,b). \tag{2.10}$$

Now $\partial\bar{\partial}b = -\bar{\partial}\partial b = \bar{\partial}^2\overline{b} = 0$ by (2.8), so $\bar{\partial}b$ is an antiholomorphic $(0,2)$-form. Hence its component functions are antiholomorphic functions on $\mathsf{M}_N\backslash\Delta_N$. But it is known [15] that the coefficients b_r, and their derivatives, decay exponentially fast at large vortex separation. Hence these antiholomorphic functions must in fact vanish. So $\bar{\partial}b = 0 = \partial\overline{b}$, and the proposition follows. □

For future reference, we note that $\bar{\partial}b = 0$ implies that

$$\frac{\partial b_r}{\partial \overline{z}_s} = \frac{\partial b_s}{\partial \overline{z}_r} \tag{2.11}$$

for all r, s.

So the adiabatic approximation to N-vortex dynamics is hamiltonian flow on $(\mathsf{M}_N, \omega_{L^2})$ with Hamiltonian

$$H = -\frac{1}{2\gamma} V|_{\mathsf{M}_N}. \tag{2.12}$$

Henceforth X_F will denote the symplectic gradient of F with respect to ω_{L^2}, rather than $\mathcal{B}$.

3 Conservation laws

We may now give a hamiltonian discussion of the conservation laws discovered by Manton and Nasir [9]. The natural action of the euclidean group $E(2) \cong U(1) \ltimes \mathbb{C}$ on the physical plane $\mathbb{C}$ induces a $E(2)$-action on M_N by $(e^{i\theta}, c) : \{z_r\} \mapsto \{e^{i\theta} z_r + c\}$ (strictly speaking, this only defines the action on $\mathsf{M}_N\backslash\Delta_N$, but one can use (1.2) to deduce a well defined action on the global coordinates w_r). This action is manifestly holomorphic. It is also isometric, since it leaves L^2-norms invariant. Hence the $E(2)$-action is symplectic. We would like to construct a moment map $\mu : \mathsf{M}_N \to \mathfrak{e}(2)^*$ for this action and identify its components with respect to a natural basis for $\mathfrak{e}(2)^*$ as the conserved momenta of the system, just as it has been done for the symplectic $SO(3)$-action in the S^2 model [14]. Unfortunately, no such moment map exists.

Certainly, given any $X \in \mathfrak{e}(2) \cong T_{(1,0)}E(2)$, the induced vector field $X^\sharp \in \Gamma(T\mathsf{M}_N)$ is hamiltonian, because all symplectic vector fields on M_N are hamiltonian (Y symplectic implies $\iota_Y\omega_{L^2}$ is closed; it is also exact since $H^1(\mathsf{M}_N) = 0$, whence Y is the symplectic gradient of some smooth function). Using the terminology of [11], the $E(2)$-action on M_N is *weakly hamiltonian*. So functions $P_j : \mathsf{M}_N \to \mathbb{R}$, $j = 0, 1, 2$ satisfying $-dP_j = \iota_{X_j}\omega_{L^2}$ must exist, where

$$X_0 = \frac{\partial^\sharp}{\partial\theta} = i\sum_r \left(z_r \frac{\partial}{\partial z_r} - \overline{z}_r \frac{\partial}{\partial \overline{z}_r} \right), \tag{3.1}$$

$$X_1 = \frac{\partial^\sharp}{\partial c_1} = \sum_r \left(\frac{\partial}{\partial z_r} + \frac{\partial}{\partial \overline{z}_r} \right), \tag{3.2}$$

$$X_2 = \frac{\partial^\sharp}{\partial c_2} = i\sum_r \left(\frac{\partial}{\partial z_r} - \frac{\partial}{\partial \overline{z}_r} \right) \tag{3.3}$$

generate rotations and translations in M_N. In fact, these functions are unique up to additive constants. The problem is that the P_j cannot be assembled into the components of an *equivariant* map $\mathsf{M}_N \to \mathfrak{e}(2)^*$. In other words, we have:

Proposition 2 *The symplectic $E(2)$-action on M_N is not hamiltonian.*

Proof: Assume, to the contrary, that a moment map μ exists. Each M_N contains a symplectic submanifold $(\mathbb{C}, \frac{i\pi N}{2} dZ \wedge d\overline{Z})$, namely the $E(2)$-orbit of any configuration of N coincident vortices (here Z denotes the common vortex position). The restriction $\hat{\mu}$ of μ to this orbit defines a moment map for the standard action of $E(2)$ on $\mathbb{C}$. Since $\hat{\mu}$ is equivariant, and the $SO(2)$ action fixes 0,

$$\hat{\mu}(0) = \hat{\mu}((e^{i\theta}, 0) \cdot 0) = Ad^*_{(e^{i\theta},0)}\hat{\mu}(0) \tag{3.4}$$

where Ad^* denotes the coadjoint action of $E(2)$ on $\mathfrak{e}(2)^*$. Hence $\hat{\mu}(0) = \alpha d\theta$ for some real constant α. But this completely determines $\hat{\mu}$ by transitivity of the translation action:

$$\hat{\mu}(Z) = \hat{\mu}((1, Z) \cdot 0) = Ad^*_{(1,Z)}\alpha d\theta = \alpha d\theta. \tag{3.5}$$

Thus, any equivariant map $\mathbb{C} \to \mathfrak{e}(2)^*$ is constant, and cannot generate a nontrivial group action. $\square$

Nonetheless, generating functions P_j for the X_j do exist, and one expects, given Manton and Nasir's Lagrangian analysis [9], they may be written locally (on $\mathsf{M}_N \backslash \Delta_N$) as

$$\begin{aligned} P_0 &= \frac{\pi}{2}\sum_r (|z_r|^2 + b_r\overline{z}_r + \overline{b}_r z_r) && (3.6) \\ P_1 &= i\frac{\pi}{2}\sum_r (z_r - \overline{z}_r) && (3.7) \\ P_2 &= \frac{\pi}{2}\sum_r (z_r + \overline{z}_r). && (3.8) \end{aligned}$$

Note that P_1, P_2 are easily globalized since $P_1 + iP_2 = -i\pi w_1$ in the global coordinates. A smooth globalization of P_0 must exist, but no obvious formula suggests itself.

It is a routine exercise to verify that $dP_j(Y) = \omega_{L^2}(Y, X_j)$ for all $Y \in \Gamma(T\mathsf{M}_N)$ as required, if one makes use of the identities

$$\begin{aligned} \sum_r b_r &\equiv 0 && (3.9) \\ \sum_r (b_r\overline{z}_r - \overline{b}_r z_r) &\equiv 0. && (3.10) \end{aligned}$$

Identity (3.9) is proved in [15], while (3.10) is proved for the spherical model in [14]. For the sake of completeness, we will derive it directly for the planar model.

Let $\phi \in \mathsf{M}_N$ be the Higgs field of the static solution with zeroes at $\{z_r\}$, and let $\phi^\theta \in \mathsf{M}_N$ be the field obtained from this by rotating all the vortex positions $z_r \mapsto e^{i\theta}z_r$. Then clearly $\phi^\theta(e^{i\theta}z) \equiv \phi(z)$, and hence (2.4) implies that

$$b_r(\{e^{i\theta}z_r\}) = e^{i\theta}b_r(\{z_r\}) \tag{3.11}$$

for all r. It follows that

$$X_0[b_r] = \frac{d}{d\theta}\bigg|_{\theta=0} b_r = ib_r, \quad \text{and} \quad X_0[\overline{b}_r] = -i\overline{b}_r, \tag{3.12}$$

since X_0 is real. But using our local expression for X_0, (3.1), we see that

$$X_0[b_s] = i\sum_r \left(z_r \frac{\partial b_s}{\partial z_r} - \overline{z}_r \frac{\partial b_s}{\partial \overline{z}_r} \right) = i\frac{\partial}{\partial \overline{z}_s} \sum_r (z_r\overline{b}_r - \overline{z}_r b_r) + ib_s \tag{3.13}$$

where we have used (2.8) and (2.11). Comparing (3.13) and its complex conjugate with (3.12) one sees that $\sum_r (b_r\overline{z}_r - \overline{b}_r z_r)$ is constant. That the constant vanishes follows from consideration of the limit where all N vortices coincide. In fact, constancy of the sum, rather than vanishing, is sufficient to prove that $\mathsf{X}_{P_0} = X_0$.

Our Hamiltonian $H = -\frac{1}{2\gamma} V|_{\mathsf{M}_N}$ is manifestly invariant under the $E(2)$-action, so

$$\{P_j, H\} = \omega_{L^2}(X_j, \mathsf{X}_H) = dH(X_j) = X_j[H] = 0 \tag{3.14}$$

and the momenta P_j are conserved by the flow of H.

The Lie subalgebra generated by the P_j in the Poisson algebra $C^\infty(\mathsf{M}_N)$ is determined by the relations

$$\{P_j, P_0\} \;=\; \epsilon_{jk} P_k, \qquad j = 1, 2 \tag{3.15}$$

$$\{P_1, P_2\} \;=\; N\pi, \tag{3.16}$$

where ϵ_{jk} denotes the antisymmetric tensor with $\epsilon_{12} = +1$. So we can see that it is a central extension $\mathfrak{e}(2) \oplus \mathbb{R}$ of the euclidean algebra. The nonvanishing of the Poisson bracket (3.16) is consistent with a field theory calculation of Hassaïne *et al.* [5]. In our context, given that $[X_1, X_2] = 0$, it provides another proof of Proposition 2. In fact, it implies that the $E(2)$-action on M_N corresponds to a nonzero 2-cocycle in the Lie algebra cohomology group $H^2(\mathfrak{e}(2); \mathbb{R}) \cong \mathbb{R}$; this group parametrizes obstructions of weakly hamiltonian $E(2)$-actions to be hamiltonian [11].

It is worthwhile to look at the content of Proposition 2 in the context of the results on the spherical model [14]. In the latter, spatial isometries are described by the group $SO(3)$, which is simple. Therefore, $H^2(\mathfrak{so}(3); \mathbb{R}) = 0$ and the $SO(3)$-action on $\mathsf{M}_N \equiv \mathbb{CP}^N$ is necessarily hamiltonian. Dualizing the corresponding moment map, we obtain an isomorphism between $\mathfrak{so}(3)$ and the Lie algebra of conserved angular momenta. The large radius limit $R \to \infty$ that relates the spherical model to the planar model determines a *contraction* of Lie algebras

$$\mathfrak{so}(3) \to \mathfrak{e}(2) \tag{3.17}$$

with parameter $\frac{1}{R}$. This is the well-known contraction of $\mathfrak{so}(3)$ to the graded Lie algebra associated to the filtration [4]

$$0 \subset \mathfrak{so}(2) \subset \mathfrak{so}(3), \tag{3.18}$$

where the choice of the $\mathfrak{so}(2)$ subalgebra corresponds to fixing an axis through the centre of the sphere. We can realize this contraction as a deformation of Lie algebras of vector fields, using the isometry actions on each M_N. At the level of the Poisson algebras, however, no homomorphism of Lie algebras from $\mathfrak{e}(2)$ to $C^\infty(\mathsf{M}_N)$ is available by Proposition 2, so it is not surprising that the closure of the generators P_j does not yield $\mathfrak{e}(2)$ in the planar model.

4 Well-separated vortices

Although the moduli space approximation has simplified the dynamical N-vortex problem in principle, we are still faced with the problem that neither the Kähler form ω_{L^2} nor the Hamiltonian $H = -\frac{1}{2\gamma}V$ are explicitly known. There is one physical regime in which explicit progress is possible, namely when all vortices are well separated from one another. In this situation, an asymptotic formula for the coefficients b_r, and hence ω_{L^2} is available [10]. By translation invariance, the formulae involve only $z_{rs} := z_r - z_s$. They are

$$b_r(z_1, \ldots, z_N) = \frac{q^2}{2\pi^2} \sum_{s \neq r} K_1(|z_{rs}|) \frac{z_{rs}}{|z_{rs}|} \tag{4.1}$$

$$\omega_{L^2} = i\frac{\pi}{2} \left[\sum_r dz_r \wedge d\overline{z}_r - \frac{q^2}{4\pi^2} \sum_r \sum_{s \neq r} K_0(|z_{rs}|) dz_{rs} \wedge d\overline{z}_{rs} \right], \tag{4.2}$$

where K_n denotes the modified Bessel's function of the second kind, and q is an unknown real constant which may be interpreted as the scalar monopole charge or magnetic dipole moment of a single vortex at critical coupling. Based on a string theoretic duality argument, Tong has conjectured that $q = -2\pi 8^{\frac{1}{4}}$, and this value is consistent with numerics [20, 17]. The formulae are believed to be correct up to linear order in exponentially small quantities (note $K_n(\rho)$ is exponentially small for large ρ). They have not been proved rigorously, but they have been derived via two independent physical arguments, one using matched asymptotic expansions and the other appealing to a point source formalism, and there is a good fit to known numerics. Similarly derived formulae for monopole moduli spaces have subsequently been proved rigorously, so one has good grounds for confidence.

It is not so straightforward to obtain an asymptotic formula for H. It is known [8] that

$$V|_{\mathsf{M}_N} = N\pi + \frac{\mu^2 - 1}{8} \int (1 - |\phi|^2)^2 d^2\mathbf{x}, \tag{4.3}$$

but no obvious method of estimating the integral for well-separated vortices suggests itself. We shall instead proceed by an indirect physical argument. For μ close to 1, one expects that $V|_{\mathsf{M}_N}$ is close to V_N^μ, the Landau–Ginzburg energy of N well-separated coupling μ vortices held at positions $\{z_r\}$, and for this an asymptotic formula *has* been found [17]:

$$V_N^\mu = \frac{1}{2} \sum_r \sum_{s \neq r} \tilde{\mathcal{U}}(|z_{rs}|) \tag{4.4}$$

where $\tilde{\mathcal{U}}$ is the asymptotic 2-vortex interaction potential,

$$\tilde{\mathcal{U}}(\rho) := \frac{1}{2\pi} [m(\mu)^2 K_0(\rho) - q(\mu)^2 K_0(\mu\rho)], \tag{4.5}$$

and we have discarded an irrelevant constant (N times the vortex mass). The factor of $\frac{1}{2}$ is included in (4.4) since the sum is over all *ordered* distinct vortex pairs. Note that the scalar monopole charge $q(\mu)$ and magnetic dipole moment $m(\mu)$ vary with the coupling, and are not

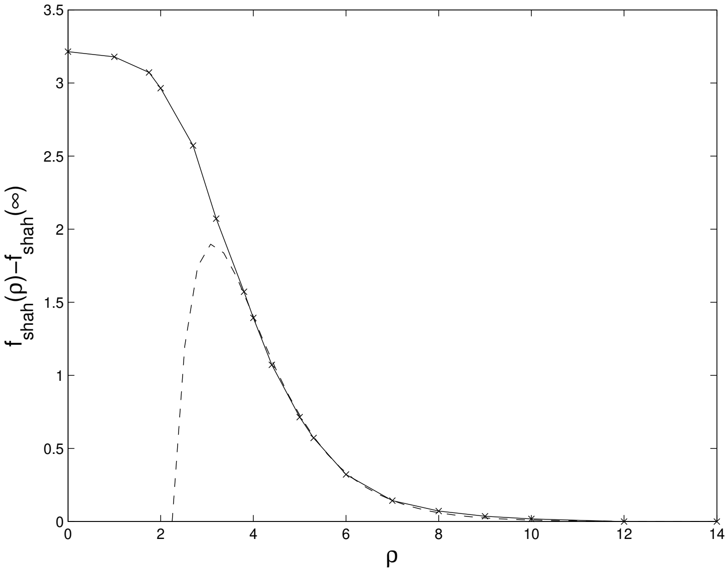

Figure 1: *The quantity* $\int(1-|\phi|^2)^2 d^2\mathbf{x}$ *for a vortex pair, as a function of the vortex separation* ρ*. Solid line and crosses: Shah's numerical data, taken from* [16]*; dashed line: our conjectured asymptotic formula, as in equation* (4.8).

equal except when $\mu = 1$. Note also that, at $\mu = 1$, $\tilde{\mathcal{U}} \equiv 0$, as one expects. Now let $\mu = 1+\delta\mu$, $\delta\mu$ small, and assume that, at sufficiently large vortex separation, $V|_{\mathsf{M}_N} = V_N^\mu$ to leading order in $\delta\mu$. Again discarding additive constants, one is led to conjecture that

$$\begin{aligned} V|_{\mathsf{M}_N} &= \frac{1}{2}\sum_r \sum_{s\neq r} \mathcal{U}(|z_{rs}|) \\ \mathcal{U}(\rho) &:= \frac{q^2}{4\pi}(\mu^2-1)\left[\rho K_1(\rho) - \nu K_0(\rho)\right], \end{aligned} \tag{4.6}$$

where $\nu = 2(q'(1) - m'(1))/q \approx 2.7060$, using the tabulated data for $m(\mu)$ and $q(\mu)$ in [17]. One way to test this conjecture is to deduce from it an asymptotic formula for the integral

$$f_{\mathrm{Shah}}(\rho) = \int (1-|\phi|^2)^2 d^2\mathbf{x} \tag{4.7}$$

for the case of critically coupled $N = 2$ static vortices with intervortex separation ρ (note that $E(2)$ symmetry implies the integral may only depend on ρ in this case). The function f_{Shah} has been evaluated numerically by Shah [16]. Formula (4.6) implies that

$$f_{\mathrm{Shah}}(\rho) - f_{\mathrm{Shah}}(\infty) = \frac{2q^2}{\pi}\left[\rho K_1(\rho) - \nu K_0(\rho)\right] \tag{4.8}$$

asymptotically at large ρ. A comparison of Shah's numerical data with our conjectured asymptotic form is given in figure 1. The fit for $\rho > 4$ is very good.

The N-vortex equations of motion in the asymptotic regime are obtained by extracting the leading asymptotic term in the symplectic gradient of H. Since H is already exponentially small, an exponential correction to ω_{L^2} makes no contribution at this order, so we can actually set $\omega_{L^2} = \pi\omega_0$, where

$$\omega_0 = \frac{i}{2}\sum_r dz_r \wedge d\overline{z}_r \tag{4.9}$$

is the canonical symplectic form on $\mathbb{C}^N$. We find, to leading order,

$$\dot{z}_r = i\sum_{s\neq r} F(|z_{rs}|)z_{rs}, \quad F(\rho) := -\frac{\mathcal{U}'(\rho)}{2\pi\gamma\rho}. \tag{4.10}$$

Note that system (4.10) is similar to the equations of motion for a system of identical fluid point vortices, or geostrophic vortices, of vorticity $1/\gamma$, which would result from replacing $\mathcal{U}$ by

$$\mathcal{U}_{\text{fluid}}(\rho) = \log\rho, \quad \mathcal{U}_{\text{geo}}(\rho) = K_0(\rho) \tag{4.11}$$

respectively [1, 12]. The system with $\mathcal{U}_{\text{fluid}}$ is particularly well studied. We shall see that our system behaves rather more like the geostrophic vortex system, which has not been so heavily studied, though there are still significant differences.

Two-vortex dynamics in the moduli space approximation is almost trivial: the vortices orbit their centre of mass at constant speed and separation. This is a special case of the rotating polygon solutions whose stability properties we will analyze in section 6. We turn now to a discussion of 3-vortex dynamics.

5 The dynamics of three vortices

Novikov has given a thorough treatment of the internal dynamics of identical vortex triples interacting via $\mathcal{U}_{\text{fluid}}$ in [13]. His method can be adapted readily to deal with our system in its asymptotic form, (4.10). The basic idea is to identify trajectories in the internal phase space of the system with level curves of the quantities conserved by flow (4.10). Since Novikov's method relies on exploiting the conservation laws enjoyed by the system, it is more satisfactory to apply it directly to the momenta conserved by the full flow, in asymptotic form, rather than the momenta conserved by the asymptotic flow (4.10), and this is how we shall proceed. Of course, either approach yields the same results.

To begin with, let N be general. Let us define the *centroid* of an N-vortex configuration by

$$Z := \frac{1}{i\pi N}(P_1 + iP_2) = \frac{1}{N}\sum_r z_r \tag{5.1}$$

and note that

$$\sum_r |z_r|^2 = N|Z|^2 + \frac{1}{2N}\sum_r\sum_{s\neq r}|z_{rs}|^2. \tag{5.2}$$

Since Z and P_0 are conserved, it follows that

$$\begin{aligned} Q &:= \frac{2N}{\pi}P_0 - N^2|Z|^2 \\ &= \sum_r \left(N(b_r\overline{z}_r + \overline{b}_r z_r) - \frac{1}{2}\sum_{s\neq r}|z_{rs}|^2 \right) \end{aligned} \tag{5.3}$$

is conserved also. The corresponding hamiltonian vector field can be readily computed as

$$\begin{aligned}\mathsf{X}_Q &= \frac{2N}{\pi}X_0 - \frac{1}{\pi^2}(2P_1X_1 + 2P_2X_2)\\ &= \frac{2Ni}{\pi}\sum_r \left((z_r - Z)\frac{\partial}{\partial z_r} - (\overline{z}_r - \overline{Z})\frac{\partial}{\partial \overline{z}_r}\right),\end{aligned} \tag{5.4}$$

and this describes rigid rotations of vortex configurations about their centroids. Using (2.8), (2.11) and (3.10), one finds

$$\{P_j, Q\} = 0, \qquad j = 1, 2. \tag{5.5}$$

For a moment, let us consider $N = 3$. Equation (5.5) implies that the set of three conserved quantities $\{P_1, Q, H\}$ is in involution. If the corresponding hamiltonian vector fields are linearly independent at all points of an open dense subset $\mathsf{M}_3 \setminus S \subset \mathsf{M}_3$, the dynamics of three vortices is Liouville integrable [3] on the six-dimensional phase space M_3. This manifold is then foliated by invariant tori [2], some of them possibly degenerate. Clearly, X_1 is independent of both X_Q and X_H everywhere in M_N. Hence, the question of Liouville integrability reduces to the condition of linear independence of X_Q and X_H. Our intuition about the dynamics of $N = 2$ vortices [8] leads us to claim that this condition is generic on M_3. Actually, we expect X_Q and X_H to be linearly dependent exactly at the points of a subset $S \subset \mathsf{M}_3$ consisting of configurations with special symmetry: z_r either at the vertices of an equilateral triangle, at the ends and midpoint of a line segment, or on the locus of coincidence Δ_3; notice that all these conditions are algebraic in the coordinates z_r (and their complex conjugates), so that S is indeed closed with dense complement. We shall see in the following that this picture is consistent with what we can learn about the dynamics of 3-vortices in the asymptotic regime of large separation. The configurations in S that correspond to well-separated 3-vortices are special cases of the vortex polygons that we shall discuss in section 6.

When all $|z_{rs}|$ are large, the angular momentum becomes

$$P_0 = \frac{\pi}{2}\left[\sum_r |z_r|^2 + \frac{q^2}{2\pi^2}\sum_r\sum_{s\neq r}|z_{rs}|K_1(|z_{rs}|)\right], \tag{5.6}$$

whereas

$$Q = \frac{1}{2}\sum_r\sum_{s\neq r}\left[|z_{rs}|^2 + \frac{Nq^2}{\pi^2}|z_{rs}|K_1(|z_{rs}|)\right]. \tag{5.7}$$

Note that, unlike Z and P_0, Q and H depend only on the relative separations of the vortices $|z_{rs}|$, to this order.

If we now set $N = 3$ and define $\mathbf{x} = (|z_{23}|, |z_{31}|, |z_{12}|)$, we see that $\mathbf{x}(t)$ is confined to a level curve of $(\hat{H}, \hat{Q})$ in $\mathbb{R}^3$, where

$$\begin{aligned}\hat{H} &= \mathcal{U}(x_1) + \mathcal{U}(x_2) + \mathcal{U}(x_3),\\ \hat{Q} &= g(x_1) + g(x_2) + g(x_3), \quad g(x) := x^2 + \frac{Nq^2}{\pi^2}xK_1(x).\end{aligned} \tag{5.8}$$

Our strategy is, then, to construct such level curves. This almost completely determines the internal dynamics of the triple (meaning the dynamics up to rigid rotations), up to reparametrization of time. The only thing not determined is a discrete variable, the *orientation* of the triple. Two orientations are possible: traversing the triangle with vertices at z_r clockwise can order the vertices 123 or 132.

To construct a level curve of $(\hat{H}, \hat{Q})$, we may choose some initial data $\mathbf{x}(0) = \mathbf{x}_0$ and then solve the first order system

$$\dot{\mathbf{x}}(s) = \frac{\nabla \hat{H} \times \nabla \hat{Q}}{|\nabla \hat{H} \times \nabla \hat{Q}|} \tag{5.9}$$

to yield a level curve in arc-length parametrization ($|\dot{\mathbf{x}}(s)| \equiv 1$). Since we work only to linear order in exponentially small quantities, and $\hat{H}$ is small, we may keep only the leading term in $\hat{Q}$, that is, take

$$\hat{Q} = |\mathbf{x}|^2 \tag{5.10}$$

in (5.9). So level sets of $\hat{Q}$ may be approximated by concentric spheres of radius $\hat{Q}^{\frac{1}{2}}$ in $\mathbb{R}^3$, and the level curves of $(\hat{H}, \hat{Q})$ are simply the intersection curves between level sets of $\hat{H}$ and these spheres. Alternatively, given a radius $\hat{Q}^{\frac{1}{2}}$, we may seek level curves of $H^{\hat{Q}} : S^2 \to \mathbb{R}$,

$$H^{\hat{Q}}(\mathbf{y}) = \mathcal{U}(\hat{Q}^{\frac{1}{2}} y_1) + \mathcal{U}(\hat{Q}^{\frac{1}{2}} y_2) + \mathcal{U}(\hat{Q}^{\frac{1}{2}} y_3), \tag{5.11}$$

where $\mathbf{x} = \hat{Q}^{\frac{1}{2}} \mathbf{y}$ and S^2 is the unit 2-sphere in $\mathbb{R}^3$. It is straightforward to generate contour plots of $H^{\hat{Q}}$ for given $\hat{Q}$ within any convenient coordinate system on S^2, and hence deduce the corresponding internal trajectories. Note that $\hat{Q}$ determines the root-mean-square of the triangle side lengths, x_1, x_2, x_3. It turns out that the level curves are not very sensitive to $\hat{Q}$, provided we choose it to be fairly large (which it should be, in the asymptotic regime), so let us imagine that $\hat{Q}$ has been fixed.

Not every $\mathbf{y} \in S^2$ represents a valid vortex triple. Each y_i must be non-negative, and since they represent the side lengths of a euclidean triangle, none may exceed the sum of the other two. So $\mathbf{y}(t)$ is confined to a triangular region in S^2, bounded by great-circular arcs (the intersection of the sphere with the colinearity planes $y_1 = y_2 + y_3$, $y_2 = y_3 + y_1$, $y_3 = y_1 + y_2$), and centred on $\mathbf{y}_0 = (1, 1, 1)/3^{\frac{1}{2}}$. This central point represents an equilateral triple. The orthogonal projection of this region onto the tangent plane $T_{\mathbf{y}_0} S^2$ is depicted in figure 2. The boundaries project to congruent elliptic arcs. Any point on the boundary represents a colinear vortex triple, and the boundaries intersect at coincidence points (where a vortex pair coalesces). Since our analysis is restricted to the well-separated regime, level curves which approach these corners closely are of suspect validity.

Figure 2 depicts level curves for the choice $\hat{Q} = 441$ and several values of $\hat{H}$. Our discussion is qualitatively similar to Novikov's [13], so will be kept brief. The minimum possible value of $H^{\hat{Q}}$ is

$$H^{\hat{Q}}_{\min} = H^{\hat{Q}}(\mathbf{y}_0) = 3\mathcal{U}(\sqrt{\hat{Q}/3}) \tag{5.12}$$

attained by the equilateral triple. The level "curve" for this value consists of the single point $\mathbf{y}_0$, so the internal configuration of this triple remains constant. In fact the triangle rotates about its centre at constant speed, as will be discussed in section 6. Increasing $H^{\hat{Q}}$, we obtain closed curves centred on $\mathbf{y}_0$. In this low energy regime, the shape of the triangle varies periodically in time as the triangle spins, but the triangle's orientation remains constant. The

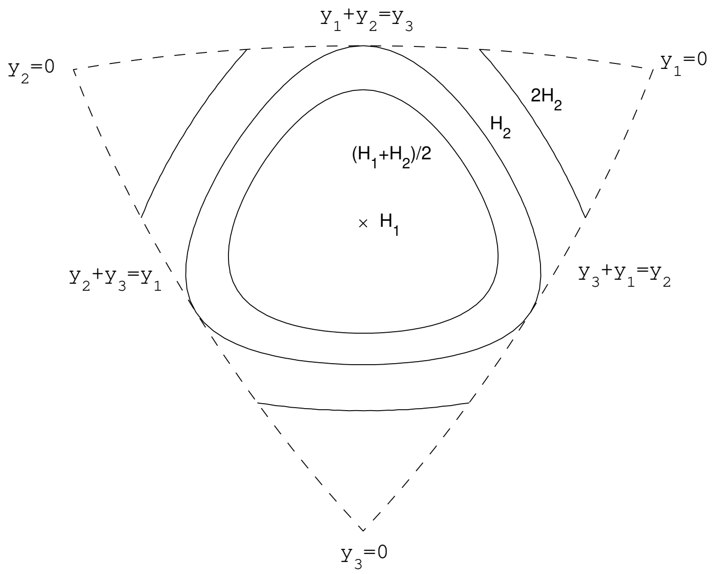

Figure 2: *Level curves of $\hat{H}^{\hat{Q}}$ for $\hat{Q} = 441$. Here $H_1 = \hat{H}^{\hat{Q}}_{\rm min}$ and $H_2 = \hat{H}^{\hat{Q}}_{\rm crit}$. The dashed curves represent the colinearity boundaries, which intersect at coincidence points.*

periods of shape change and rotation are generically incommensurate. Clearly, these solutions are all stable.

Increasing $H^{\hat{Q}}$ further, the level curve eventually becomes tangent to the colinearity boundaries, at the critical value

$$H^{\hat{Q}}_{\rm crit} = H^{\hat{Q}}((1,1,2)/6^{\frac{1}{2}}) = 2\mathcal{U}(\sqrt{\hat{Q}/6}) + \mathcal{U}(2\sqrt{\hat{Q}/6}). \tag{5.13}$$

This level curve should not be interpreted as a closed orbit. Rather, we have three fixed points, $(1,1,2)/\sqrt{6}$, $(1,2,1)/\sqrt{6}$, $(2,1,1)/\sqrt{6}$, joined by 3 heteroclinic orbits. The fixed points represent colinear triples with the middle vortex equidistant from the outer vortices, a special case of the filled rotating polygons which will be discussed in section 6. This is stationary (but not static) and clearly unstable: small perturbations lead to divergent dynamical regimes.

For $H^{\hat{Q}} > H^{\hat{Q}}_{\rm crit}$, the level curve splits into 3 disjoint pieces. The actual trajectory stays on one segment of this curve – it reverses direction each time it hits a colinearity boundary, with a corresponding change in the triangle's orientation. At these points, the triangle collapses and turns itself inside out, so for larger energies, the triple's internal motion consists of a periodic "flip-flopping". As the energy grows unbounded, the trajectory approaches a coincidence point, and hence escapes the presumed region of validity of our approximation.

It is interesting to note that *every* trajectory intersects the isosceles lines (the straight lines containing $\mathbf{y}_0$ and the coincidence points), so every 3-vortex motion passes, at least once, through an isosceles configuration.

6 Rigidly rotating polygons

If N vortices are placed at the vertices of a regular N-gon, they will rotate at constant speed about their centroid, which remains fixed [8]. This behaviour is familiar from both the fluid and geostrophic vortex systems. Given the asymptotic equations of motion (4.10), it is straightforward to derive frequency-radius relations and spectral stability properties of these solutions, valid for large polygon side length. Note that a change in the coupling constants μ, γ may always be absorbed into a rescaling and/or reversal of time, so the stability properties will be independent of our choice of parameters.

It is convenient to transform to a co-rotating frame by defining new coordinates $y_r(t) \in \mathbb{C}$,

$$z_r(t) =: y_r(t)e^{i\Omega t} \tag{6.1}$$

for some fixed choice of frequency $\Omega \in \mathbb{R}$, so that frequency Ω solutions are static in the new frame. System (4.10) in these coordinates is

$$\dot{y}_r = i\sum_{s\neq r} F(|y_{rs}|)y_{rs} - i\Omega y_r, \tag{6.2}$$

where $y_{rs} := y_r - y_s$. Note that (6.2) is hamiltonian flow on $(\mathbb{C}^N, \omega_0)$ with respect to the modified Hamiltonian

$$H_\Omega = \sum_r \sum_{s\neq r} \mathcal{U}(|y_{rs}|) - \Omega \sum_r |y_r|^2. \tag{6.3}$$

Let $\lambda := e^{2\pi i/N}$. Then $y_r = \lambda^r \sigma$, $\sigma \in (0, \infty)$, is a static solution of (6.2) provided

$$\Omega = \sum_{j=1}^{N-1} F(|1 - \lambda^j|\sigma). \tag{6.4}$$

This is the frequency-radius relation for a rotating N-gon.

We turn now to stability properties. Let $y \in \mathbb{C}^N$ be a fixed point of the co-rotating flow. Then we may define the linearization of the flow about y,

$$\Lambda_y : T_y\mathbb{C}^N \to T_y\mathbb{C}^N, \qquad \Lambda_y : X \mapsto \nabla_X \mathsf{X}_{H_\Omega}, \tag{6.5}$$

where ∇ is *any* affine connexion on $\mathbb{C}^N$, and X_{H_Ω} is the symplectic gradient of H_Ω. The fixed point y is spectrally stable if Λ_y has no eigenvalues with positive real part. Now, since X_{H_Ω} is symplectic, it follows that $\Lambda_y \in \mathfrak{aut}(T_y\mathbb{C}^N, \omega_0)$, that is,

$$\omega_0(\Lambda_y X, Y) + \omega_0(X, \Lambda_y Y) \equiv 0 \tag{6.6}$$

for all $X, Y \in T_y\mathbb{C}^N$. It follows that, if η is an eigenvalue of Λ_y, so are $-\eta$, $\overline{\eta}$ and $-\overline{\eta}$. In order that y be spectrally stable, therefore, $\operatorname{spec}\Lambda_y$ must be purely imaginary.

In the case of the rotating N-gon, $y = \tilde{y} = (\lambda\sigma, \lambda^2\sigma, \ldots, \sigma)$, one can find $\operatorname{spec}\Lambda_{\tilde{y}}$ explicitly due to the cyclic symmetry of the configuration [12]. Let $\pi : \mathbb{Z} \to \mathbb{Z}_N$ be the group homomorphism with $\pi(1) = 1$. Then a $N \times N$ real matrix A is *right circulant* if there exists $\hat{A} \in \mathbb{R}^N$ such that

$$A_{ij} = \hat{A}_{\pi(j-i+1)} \tag{6.7}$$

for all i, j. In order to exploit this symmetry, it is convenient to define a twisted complex basis $e_{(i)} \in \mathbb{C}^N$, $i = 1, \ldots, N$, such that

$$e_{(i)j} = \lambda^i \delta_{ij} \tag{6.8}$$

there being no summation implied. If we write down the matrix representing $\Lambda_{\tilde{y}}$ with respect to the associated basis for $\mathbb{R}^N$, that is, $\{\mathrm{Re}\, e_{(1)}, \ldots, \mathrm{Re}\, e_{(n)}, \mathrm{Im}\, e_{(1)}, \ldots, \mathrm{Im}\, e_{(n)}\}$, we find that

$$\Lambda_{\tilde{y}} = \begin{bmatrix} A & B \\ C & A \end{bmatrix} \tag{6.9}$$

where each of the real $N \times N$ blocks is right circulant. The explicit formulae for $\hat{A}, \hat{B}, \hat{C} \in \mathbb{R}^N$ are rather complicated, and are presented in the appendix. Of more interest are the following symmetries possessed by these coefficients:

$$\left.\begin{array}{rrcr} \hat{A}_1 = 0, & \hat{A}_{N+2-s} & = & -\hat{A}_s \\ & \hat{B}_{N+2-s} & = & \hat{B}_s \\ & \hat{C}_{N+2-s} & = & \hat{C}_s \end{array}\right\} s = 2, \ldots, N. \tag{6.10}$$

Right circulancy is a powerful symmetry because the eigenvectors of a right circulant matrix are independent of its entries. For each $j \in \{1, \ldots, N\}$, let

$$\mathbf{x}_j = (1, \lambda^j, \lambda^{2j}, \ldots, \lambda^{(N-1)j}) \in \mathbb{C}^N. \tag{6.11}$$

Then $\mathbf{x}_j$ is simultaneously an eigenvector of A, B and C, with eigenvalues

$$\alpha_j = \hat{A} \cdot \mathbf{x}_j, \quad \beta_j = \hat{B} \cdot \mathbf{x}_j, \quad \gamma_j = \hat{C} \cdot \mathbf{x}_j. \tag{6.12}$$

From (6.10), (6.11) and (6.12), one sees that α_j is purely imaginary, while β_j and γ_j are real. Note that $\overline{\mathbf{x}}_j = \mathbf{x}_{N-j}$ for all $1 \leq j \leq N-1$, and that $\overline{\mathbf{x}}_N = \mathbf{x}_N$. It follows that

$$\alpha_{N-j} = \overline{\alpha}_j = -\alpha_j, \quad \beta_{N-j} = \overline{\beta}_j = \beta_j, \quad \gamma_{N-j} = \overline{\gamma}_j = \gamma_j, \tag{6.13}$$

for $1 \leq j \leq N-1$, and that $\alpha_N = \overline{\alpha}_N = -\alpha_N$, so $\alpha_N = 0$. Short calculations using the detailed forms of $\hat{B}$, $\hat{C}$ (see appendix) establish that $\beta_N = 0$ and $\gamma_1 = -\beta_1$.

Now, relative to the basis $\{\mathbf{x}_1^+, \mathbf{x}_1^-, \ldots, \mathbf{x}_N^+, \mathbf{x}_N^-\}$, where $\mathbf{x}_j^\pm := (\mathbf{x}_j, \pm\mathbf{x}_j) \in \mathbb{C}^{2N}$, $\Lambda_{\tilde{y}}$ is block diagonal, with 2×2 blocks

$$\Lambda_j = \begin{bmatrix} \alpha_j + \frac{1}{2}(\beta_j + \gamma_j) & \frac{1}{2}(\beta_j + \gamma_j) \\ -\frac{1}{2}(\beta_j - \gamma_j) & \alpha_j - \frac{1}{2}(\beta_j + \gamma_j) \end{bmatrix}. \tag{6.14}$$

It follows that the eigenvalues of $\Lambda_{\tilde{y}}$ are

$$\eta_j^\pm = \alpha_j \pm \sqrt{\beta_j \gamma_j}. \tag{6.15}$$

Owing to the symmetry properties of α_j, β_j, γ_j, (6.13), we see that

$$\eta_{N-j}^\pm \equiv -\eta_j^\mp, \tag{6.16}$$

N	2	3	4	5	6	7	≥ 8
gauged	1	1	1	1	0	0	0
fluid	1	1	1	1	1	1	0
geostrophic	1	1	1	1	0	0	0

Table 1: *Comparison of spectral stability properties for rotating vortex N-gons of various types, 0 =unstable, 1 =stable. In the gauged and geostrophic cases, the entries refer to stability at large polygon radius σ. There may be windows of anomalous (in)stability at small σ, but these lie outside the regime of validity of the calculations. Entries for geostrophic and fluid vortices are taken from* [12].

so that eigenvalues generically come in quartets. Note also that $\eta_N^\pm = 0$ (since $\alpha_N = \beta_N = 0$) and that $\eta_1^+ = -\eta_{N-1}^- = i\Omega$. These eigenvalues originate from the $E(2)$ symmetry enjoyed by system (6.2), as will be shown shortly.

Since $\alpha_j \in i\mathbb{R}$ for all j, $\tilde{y}$ is spectrally stable provided that $\beta_j\gamma_j \leq 0$ for all j. But $\beta_1 \equiv -\gamma_1$, and $\beta_{N-j}\gamma_{N-j} \equiv \beta_j\gamma_j$ for $2 \leq j \leq N-1$, so there are in fact only $k-1$ stability criteria, where $N = 2k$ or $N = 2k+1$, namely

$$\beta_j\gamma_j \leq 0, \qquad j = 2, \ldots, k. \tag{6.17}$$

For $N = 2, 3$ stability is automatic, for $N = 4, 5$ there is one criterion, for $N = 6, 7$ two criteria, and so on.

In the first nontrivial case, $N = 4$, $\lambda = i$ and the single stability criterion is

$$(\beta_2\gamma_2)(\sigma) = -8\sqrt{2}\sigma^2 F'(\sqrt{2}\sigma)F'(2\sigma) \leq 0. \tag{6.18}$$

Since F has no critical points in the well-separated (large σ) regime, the rotating square is spectrally stable for all σ sufficiently large. As N increases, the expressions for $\beta_j\gamma_j$ become increasingly complicated so that it is not feasible to check their sign by hand. It is straightforward to check the criteria graphically by plotting the sign of $\beta_j\gamma_j$ against σ, however. The results are summarized and compared with the previously studied cases of fluid and geostrophic vortices in table 1.

We may give a similar analysis of the case of $N-1$ vortices on the vertices of a regular polygon orbiting another vortex at the polygon's centre. Let $\lambda = e^{2\pi i/(N-1)}$. Then $\hat{y} = (0, \lambda\sigma, \lambda^2\sigma, \ldots, \sigma) \in \mathbb{C}^N$ is a fixed point of the flow (6.2), provided that

$$\Omega = F(\sigma) + \sum_{j=1}^{N-2} F(|1-\lambda^j|\sigma)(1-\lambda^j). \tag{6.19}$$

Once again the linearization $\Lambda_{\hat{y}} \in \mathfrak{aut}(T_{\hat{y}}\mathbb{C}^N, \omega_0)$, so $\hat{y}$ is spectrally stable if and only if $\operatorname{spec}\Lambda_{\hat{y}} \subset i\mathbb{R}$. The presence of the centre vortex destroys the right circulant symmetry of Λ, however, so we must resort to a numerical algorithm to generate $\operatorname{spec}\Lambda_{\hat{y}}$. By plotting $\max\{|\mathrm{Re}\,\eta| : \eta \in \operatorname{spec}\Lambda_{\hat{y}}\}$ against σ, we can determine which of the $(N-1)$-gons are spectrally stable in the well separated regime. The results, which were generated using Matlab's eigenvalue finder, are summarized in table 2.

$N-1$	2	3	4	5	6	7	8	9	≥ 10
gauged	0	1	1	1	1	1	0	0	0
fluid	0	1	1	1	1	1	1	1	0
geostrophic	0	0	1	1	1	1	0	0	0

Table 2: *Comparison of spectral stability properties for vortex $(N-1)$-gons rotating about a single central vortex for various vortex types, key and comments as for table 1.*

These results have been derived in the approximation of large σ. In the full adiabatic approximation, they will receive corrections due to the subleading terms in ω_{L^2} and H, and we are assuming that these corrections will be very small when σ is large. What reason do we have to believe that the seemingly delicate property of spectral stability, $\operatorname{spec}\Lambda \subset i\mathbb{R}$, will not be destroyed by these perturbations? The spectra found exactly (for $\Lambda_{\tilde{y}}$), or numerically (for $\Lambda_{\hat{y}}$), had no accidental degeneracies for σ sufficiently large, in the stable cases. All the eigenvalues were simple, with the exception of $\eta = 0$, which we will shortly return to. No simple imaginary eigenvalue can be perturbed off the imaginary axis because, due to the reflexion symmetries enjoyed by $\operatorname{spec}\Lambda$, it would have to split in two, which is impossible by conservation of multiplicity. Further, the 0 eigenvalue, which turns out to have multiplicity 2, is fixed at 0 by symmetry considerations, as we shall now see.

We are free to transform to the co-rotating frame in the full adiabatic flow, by redefining our Hamiltonian

$$H \mapsto H_\Omega = H - \Omega P_0. \tag{6.20}$$

Then $\tilde{y}$ and $\hat{y}$ will still be fixed points of X_{H_Ω}, provided σ is chosen suitably. In each case ($y = \tilde{y}$ or $\hat{y}$) we may study the spectrum of the linearization $\Lambda_y : X \mapsto \nabla_X \mathsf{X}_{H_\Omega} \in \mathfrak{aut}(T_y\mathsf{M}_N, \omega_{L^2})$, which will be a slightly perturbed version of the asymptotic spectrum already considered. Recall that X_0, X_1, X_2 denote the vector fields generating rotations and translations (3.1)–(3.3). Now for any fixed point $y \in \mathsf{M}_N$,

$$\Lambda_y X_j = \nabla_{X_j}\mathsf{X}_{H_\Omega} = [X_j, \mathsf{X}_{H_\Omega}] = -\Omega[X_j, X_0] \tag{6.21}$$

since $[X_j, \mathsf{X}_H] = \mathsf{X}_{\{P_j, H\}} = 0$ by the $E(2)$ symmetry. Hence

$$\Lambda_y X_0 = 0, \quad \Lambda_y X_1 = -\Omega X_2, \quad \Lambda_y X_2 = \Omega X_1, \tag{6.22}$$

whence it follows that $\{0, i\Omega, -i\Omega\} \subset \operatorname{spec}\Lambda_y$. Hence, the double 0 eigenvalue is fixed under any perturbation maintaining $E(2)$ symmetry, so we conclude that the simple stability analysis given above is structurally stable at large σ.

7 Conclusion

In this paper we have confirmed the expectation raised by [14] that the adiabatic approximation to N-vortex dynamics in Manton's model of a planar superconductor amounts to a natural hamiltonian flow on $(\mathsf{M}_N, \omega_{L^2})$. A hamiltonian account of the conserved momenta descending from the symplectic but nonhamiltonian action of $E(2)$ on M_N has been given, and based on it we argued that the dynamics of three vortices is integrable in the sense of Liouville. We have derived an asymptotic formula for the 2-vortex interaction potential close to critical coupling, namely

$$\mathcal{U}(\rho) = \frac{q^2}{4\pi}(\mu^2 - 1)[\rho K_1(\rho) - \nu K_0(\rho)], \qquad q = -2\pi 8^{\frac{1}{4}}, \quad \nu \approx 2.7060, \tag{7.1}$$

and used this to analyze internal 3-vortex dynamics at large separation. We have studied the spectral stability properties of rigidly rotating vortex polygons, both with and without a central vortex, and compared our results with the more familiar cases of fluid and geostrophic vortices.

Now that we have some quantitative dynamical predictions from the adiabatic approximation, it would be interesting to test its validity numerically. There are really two separate issues. First, does the hamiltonian flow give a good account of low-energy vortex dynamics near critical coupling? Second, do our asymptotic formulae for $\mathcal{U}$ and ω_{L^2} give a good approximation to this flow at large separation? Both questions may be addressed by numerical analysis of 2-vortex dynamics. According to the adiabatic approximation, two vortices orbit one another at constant radius and angular frequency indefinitely. Is this behaviour seen in numerical simulations of the full field theory? One would expect the orbiting pair to gently radiate energy and hence slowly drift either together or apart (depending on the choice of μ, γ), but hopefully this happens on a much longer time scale than the period of their orbit. If their separation ρ is large, then their angular frequency should be close to

$$\Omega(\rho) = F(\rho) = \frac{q^2}{8\pi\gamma}(\mu^2 - 1)\left(1 + \frac{\nu}{\rho}\right) K_0(\rho). \tag{7.2}$$

This is very different from the inverse square law found for ungauged Landau–Ginzburg vortices [7]. It would be interesting to test this formula numerically in terms of both its ρ and μ dependence.

Acknowledgements

The authors wish to thank Nick Manton for useful discussions and correspondence.

Appendix

The simplest way to construct the matrix representing $\Lambda_{\tilde{y}}$ with respect to the twisted basis $e_{(i)}$ defined in equation (6.8) is to incorporate the twisting into the co-rotating coordinate system, by defining $x_r(t)$ such that $y_r(t) = \lambda^r x_r(t)$. The equations of motion (6.2) become

$$\dot{x}_r = i \sum_{s \neq r} F(|x_r - \lambda^{s-r} x_s|)(x_r - \lambda^{s-r} x_s) - i\Omega x_r. \tag{A1}$$

Then $\tilde{x} = (\sigma, \sigma, \ldots, \sigma)$ is a fixed point of (A1) provided (6.4) holds. The linearization of (A1) about $\tilde{x}$ is

$$\delta\dot{x}_r = iR(\sigma)\delta x_t - iS(\sigma)\delta\overline{x}_r - i\sum_{s \neq r} \lambda^{s-r} Q(|1 - \lambda^{s-r}|\sigma)\delta x_s + i\sum_{s \neq r} P(|1 - \lambda^{s-r}|\sigma)\delta\overline{x}_s, \tag{A2}$$

where

$$
\begin{aligned}
P(\sigma) &:= \frac{1}{2}\sigma F'(\sigma), \\
Q(\sigma) &:= F(\sigma) + P(\sigma), \\
R(\sigma) &:= \sum_{j=1}^{N-1}[\lambda^j F(|1-\lambda^j|\sigma) + P(|1-\lambda^j|\sigma)], \\
S(\sigma) &:= \sum_{j=1}^{N-1}\lambda^j P(|1-\lambda^j|\sigma).
\end{aligned}
\tag{A3}
$$

Note that P, Q, R, S are all real. Decomposing (A2) into real and imaginary parts, with $\delta x_r = \delta x_r^1 + i\delta x_r^2$, one finds that

$$
\begin{bmatrix} \delta\dot{x}_1^1 \\ \vdots \\ \delta\dot{x}_N^1 \\ \delta\dot{x}_1^2 \\ \vdots \\ \delta\dot{x}_N^2 \end{bmatrix}
=
\begin{bmatrix} A & B \\ C & A \end{bmatrix}
\begin{bmatrix} \delta x_1^1 \\ \vdots \\ \delta x_N^1 \\ \delta x_1^2 \\ \vdots \\ \delta x_N^2 \end{bmatrix}
\tag{A4}
$$

where A, B, C are the $N \times N$ right circulant matrices generated by $\hat{A}$, $\hat{B}$, $\hat{C} \in \mathbb{R}^N$:

$$
\begin{array}{lll}
\hat{A}_1 = 0 & \hat{A}_s = \mathrm{Im}\,(\lambda^{s-1})Q(|1-\lambda^{s-1}|\sigma) & s = 2,\dots,N \\
\hat{B}_1 = -R(\sigma) - S(\sigma) & \hat{B}_s = \mathrm{Re}\,(\lambda^{s-1})Q(|1-\lambda^{s-1}|\sigma) + P(|1-\lambda^{s-1}|\sigma) & s = 2,\dots,N \\
\hat{C}_1 = R(\sigma) - S(\sigma) & \hat{C}_s = -\mathrm{Re}\,(\lambda^{s-1})Q(|1-\lambda^{s-1}|\sigma) + P(|1-\lambda^{s-1}|\sigma) & s = 2,\dots,N
\end{array}
\tag{A5}
$$

The matrix in equation (A4) is $\Lambda_{\tilde{y}}$ in this basis. The symmetries of the components of $\hat{A}, \hat{B}, \hat{C}$ claimed in (6.10) follow immediately, once we note that $\lambda^N = 1$ and $\lambda^{-1} = \overline{\lambda}$ so that $\lambda^{(N+2-s)-1} = \overline{\lambda}^{s-1}$ when $2 \le s \le N$. Note that

$$
\begin{aligned}
\beta_N &= \sum_{s=1}^{N}\hat{B}_s = -R(\sigma) - S(\sigma) + \sum_{s=2}^{N}\left[\mathrm{Re}\,(\lambda^{s-1})Q(|1-\lambda^{s-1}|\sigma) + P(|1-\lambda^{s-1}|\sigma)\right] \\
&= \sum_{j=1}^{N-1}\left[(\mathrm{Re}\,(\lambda^j) - \lambda^j)(F(|1-\lambda^j|\sigma) + P(|1-\lambda^j|\sigma)\right] = 0,
\end{aligned}
\tag{A6}
$$

as previously claimed. Note also that

$$
\begin{aligned}
\beta_1 + \gamma_1 &= \hat{B}_1 + \hat{C}_1 + \sum_{j=1}^{N-1}(\hat{B}_{j+1} + \hat{C}_{j+1})\lambda^j \\
&= -2S(\sigma) + 2\sum_{j=1}^{N-1}\lambda^j P(|1-\lambda^j|\sigma) = 0
\end{aligned}
\tag{A7}
$$

so that $\gamma_1 = -\beta_1$, as was claimed.

References

[1] H. Aref, "Point vortex motions with a center of symmetry" *Phys. Fluids* **25** (1982) 2183–2187.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd edition, New York, Springer-Verlag, 1989.

[3] M. Audin, *Spinning Tops*, Cambridge, Cambridge University Press, 1996.

[4] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge, Cambridge University Press, 1984.

[5] M. Hassaïne, P.A. Horváthy and J.-C. Yera, "Non-relativistic Maxwell–Chern–Simons vortices" *Ann. Phys.* **263** (1998) 276–294, `hep-th/9706188`.

[6] A. Jaffe and C. Taubes, *Vortices and Monopoles*, Boston, Birkhäuser, 1980.

[7] O. Lange and B.J. Schroers, "Unstable manifolds and Schrödinger dynamics of Ginzburg-Landau vortices" *Nonlinearity* **15** (2002) 1471–1488, `nlin.PS/0201047`.

[8] N.S. Manton, "First-order vortex dynamics" *Ann. Phys.* **256** (1997) 114–131, `hep-th/9701027`.

[9] N.S. Manton and S.M. Nasir, "Conservation laws in a first-order dynamical system of vortices" *Nonlinearity* **12** (1998) 851–865, `hep-th/9809071`.

[10] N.S. Manton and J.M. Speight, "Asymptotic interactions of critically coupled vortices" *Commun. Math. Phys.* **236** (2003) 535–555, `hep-th/0205307`.

[11] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd edition, Oxford, Clarendon Press, 1998.

[12] G.K. Morikawa and E.V. Swenson, "Interacting motion of rectilinear geostrophic vortices" *Phys. Fluids* **14** (1971) 1058–1073.

[13] E.A. Novikov, "Dynamics and statistics of a system of vortices" *Sov. Phys. JETP* **41** (1975) 937–943.

[14] N.M. Romão, "Quantum Chern–Simons vortices on a sphere" *J. Math. Phys.* **42** (2001) 3445–3469, `hep-th/0010277`.

[15] T.M. Samols, "Vortex scattering" *Commun. Math. Phys.* **145** (1992) 149–179,

[16] P.A. Shah, "Vortex scattering at near-critical coupling" *Nucl. Phys.* **B294** (1994) 259–276, `hep-th/9402075`.

[17] J.M. Speight, "Static intervortex forces" *Phys. Rev.* **D55** (1997) 3830–3835, `hep-th/9603155`.

[18] I.A.B. Strachan, "Low-velocity scattering of vortices in a modified abelian Higgs model" *J. Math. Phys.* **33** (1992) 102–110.

[19] D. Stuart, "Dynamics of abelian Higgs vortices in the near Bogomolny regime" *Commun. Math. Phys.* **159** (1994) 51–91.

[20] D. Tong, "NS5-branes, T-duality and worldsheet instantons" *JHEP* **07** (2002) 013–036, `hep-th/0204186`.