Generalized Riemannian structures

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June 26, 2007

1 Introduction

"Generalized geometry" is an approach to differential geometric structures which seems remarkably well-adapted to some of the concepts in String Theory and Supergravity, for example: 3-form flux, gauged sigma-models, D-branes. It implicitly assumes the existence of a background gerbe.

The fundamental idea is to take a manifold M of dimension n and replace its tangent bundle by the direct sum $T \oplus T^*$ of the tangent bundle and its dual. This has a natural inner product of signature (n, n) defined by $(X + \xi, X + \xi) = i_X \xi$ and for any 2-form B, $X + \xi \mapsto X + \xi + i_X B$ is an isometry.

The other feature is the Courant bracket

$$[u, v] = [X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$

on sections of $T \oplus T^*$, which replaces the Lie bracket on vector fields. When B is closed the transformation $X + \xi \mapsto X + \xi + i_X B$ preserves the Courant bracket.

We can twist $T \oplus T^*$ using a 1-cocycle with values in closed two-forms to get a bundle $0 \to T^* \to E \xrightarrow{\pi} T \to 0$ with an inner product and Courant bracket. A connective structure on a gerbe gives this information.

2 Generalized metrics

A generalized metric is defined as a rank n subbundle of E on which the inner product is positive definite. It defines a splitting of the sequence above, as does its orthogonal complement $V^\perp$. The average of these is an isotropic splitting which defines a curving of the gerbe, and yields a three-form curvature H.

The splittings V and $V^\perp$ define two liftings X^+, X^- of a vector field X and we can define two affine Riemannian connections ∇^+, ∇^- by

$$2g\nabla^+_X Y = [X^-, Y^+] - [X, Y]^-, \qquad 2g\nabla^-_X Y = [X^+, Y^-] - [X, Y]^+$$

which have skew torsion H and $-H$ respectively. The simplest example is to take $V \subset T \oplus T^*$ as the graph of a metric $g : T \to T^*$ to get the familiar formula for the Levi-Civita connection:

$$\left[\frac{\partial}{\partial x_i} - g_{ik}dx_k, \frac{\partial}{\partial x_j} + g_{jk}dx_k\right] - \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]^- = \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right) dx_k = 2g_{\ell k}\Gamma^\ell_{ij} dx_k.$$

Connections with skew torsion should always be viewed in pairs like this: the curvature tensors R^+, R^- satisfy a modified Bianchi identity: $R^+(X, Y, Z, W) = R^-(Z, W, X, Y)$.

3 Generalized Kähler metrics

Definition 1 *A generalized complex structure on a manifold M of dimension $2m$ with bundle E is an automorphism $J : E \to E$ such that*

- $J^2 = -1$
- $(Ju, v) + (u, Jv) = 0$
- *if $Ju = iu, Jv = iv$ then $J[u, v] = i[u, v]$ using the Courant bracket.*

Here we have imitated the definition of a Kähler metric, but replaced T by E, a metric by the natural inner product, and the Lie bracket by the Courant bracket. The linear algebra data consists of a reduction of structure group of the bundle E to $U(m, m) \subset SO(2m, 2m)$.

The interesting feature about this notion is that it includes both symplectic and complex manifolds: for a complex manifold with $E = T \oplus T^*$ we take

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

and for a symplectic manifold

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

A Kähler manifold is a manifold with both a complex structure and a symplectic structure and a compatibility condition between the two. Both of these structures can be encoded as generalized complex structures, and it turns out that compatibility means they commute. Thus our generalized geometry definition is:

Definition 2 *A generalized Kähler structure consists of two commuting generalized complex structures J_1, J_2 such that the quadratic form $(J_1 J_2 u, u)$ is positive definite.*

A theorem of Marco Gualtieri [2] is:

Theorem 1 *A generalized Kähler structure on a manifold M defines*

- *a generalized metric*
- *two integrable complex structures I_+, I_- on M such that the metric g is Hermitian with respect to both*
- *the connections ∇^+, ∇^- of the generalized metric preserve I^+, I^- respectively.*

Conversely, up to the action of a closed B-field, this data determines a generalized Kähler structure on M.

Such structures form the target spaces for the nonlinear sigma model with $(2,2)$ supersymmetry [1].

4 Poisson structures

On any generalized Kähler manifold there is a 2-form $\Phi(X,Y) = g([I^+, I^-]X, Y)$. which vanishes identically for a Kähler metric, where $I^+ = -I^-$. Using the metric it defines the real part of a holomorphic section of $\Lambda^2 T$ which is a complex Poisson structure [3]. There is one for I^+ and one for I^-. On a complex surface we can replace these by meromorphic 2-forms $\omega + i\omega', \omega + i\omega''$. The hermitian form for I^+ is the $(1,1)$ part (with respect to I^+) of ω''.

A concrete example is the Hopf surface $S = \mathbf{C}^2\backslash\{0\}/\mathbf{Z}$ where the $\mathbf{Z}$-action is $(z_1, z_2) \mapsto (\lambda z_1, \lambda z_2), \lambda > 1$ and the meromorphic form is $dz_1 dz_2/z_1 z_2$. The diffeomorphism $\varphi(z_1, z_2) = (z_2, \bar{z}_1)/r^2$ is symplectic with respect to ω, the real part of this form, and

$$\varphi^*(\omega + i\omega') = \frac{1}{r^2}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) + \frac{\bar{z}_1}{z_2 r^2} dz_1 dz_2 + \frac{z_2}{\bar{z}_1 r^2} d\bar{z}_1 d\bar{z}_2.$$

We can see here that the $(1,1)$ part of $\omega'' = \varphi^*\omega'$ is the product metric on $S^1 \times S^3$.

A more interesting class of examples is to take a Del Pezzo surface with a choice of hermitian metric on K^* whose curvature is positive. If σ, a section of K^*, is the Poisson structure for $I^+ = I$, then we use the Hamiltonian flow of the function $\log \|\sigma\|^2$ to define a diffeomorphism φ_t which extends across the zero of σ. Then $\omega + i\omega'' = \varphi_t(\omega + i\omega')$ defines a generalized Kähler structure for small enough deformation parameter t [4].

5 Curving of gerbes

In the case of complex surfaces, $\omega' - \omega''$ is singular where $I_+ = -I_-$ and $\omega' + \omega''$ where $I_+ = I_-$. If the anticanonical divisor is connected then one of these is non-zero, say $\omega' - \omega'' = \rho$. This is a symplectic form and in fact, up to a B-field, defines one of the two generalized complex structures. Then the three-form H satisfies

$$H = \frac{1}{2i} d((\sigma - \bar{\sigma})\rho^2) = dB$$

where $\sigma\rho^2 \in \Omega^{0,2}$ is obtained by contraction. This is an explicit curving on the trivial gerbe.

On the Hopf surface the divisor has two components so we get ρ_0 and ρ_1 smooth on one and with a singularity on the other. Covering the surface with the two complements of the curves, these two forms define similarly a curving of a connective structure on a non-trivial gerbe, if λ or the Poisson tensor are chosen appropriately.

References

[1] S.J.Gates, C.M.Hull & M.Roček, *Twisted multiplets and new supersymmetric non-linear σ-models*, Nuclear Phys. **B 248** (1984) 157–186.

[2] M.Gualtieri, *Generalized complex geometry*, math.DG/0401221.

[3] N.J.Hitchin, *Instantons, Poisson structures and generalized Kähler geometry*, Commun. Math. Phys. **265** (2006) 131 – 317.

[4] N.J.Hitchin, *Bihermitian metrics on Del Pezzo surfaces*, **math.DG/0608213**, to appear in Journal of Symplectic Geometry.

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