ONE REMARK ON CONSTRUCTION OF SEPARATED FACTOR-SPACE

Yu.A. Neretin

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

GERMANY

Moscow State Institute of Electronic
and Mathematics
Moscow, Novozavodskaya 25/11-6-27

RUSSIA

MPI 96-26

ONE REMARK ON CONSTRUCTION OF SEPARATED FACTOR-SPACE

YU.A. NERETIN

Datum 31 december 1995

ABSTRACT. We discuss elementary constructions of boundaries of symmetric spaces.

Let M be a compact metric space. Let $M = \bigcup_{\alpha \in A} M_\alpha$ be a partition of M ($M_\alpha \cap M_\beta = \phi$ if $\alpha \neq \beta$). Then the factor-space A has canonical structure of a topological space . Recall that the set $P \subset A$ is closed if and only if $\bigcup_{\alpha \in P} M_\alpha$ is a closed subset in M. Let $a_1, a_2, \dots$ be a sequence in A. Let $a \in A$. Then $a_j \to a$ if there exist points $m_j \in M_{\alpha_j}, m \in M_\alpha$ such that $m_j \to m$ in M.

The space A is not need to be separated in Hausdorff sence. We are interested in the following question: how to construct separated analog of the factorspace A ?

1. PRELIMINARIES. HAUSDORFF CONVERGENCE

Let $N \subset M$ be a closed subset. Denote by M_ε the set of all points $m \in M$ satisfying the condition: there exist $n \in N$ such that $\rho(m, n) < \varepsilon$. Let [M] be the space of all closed subsets in M. *Hausdorff distance* $d(N, N')$ in [M] between N and N' is the infimum of $\varepsilon > 0$ such that $N \subset N'_\varepsilon$ and $N'_\varepsilon \subset N$.

Recall that the metric space [M] is compact. Recall also two simple facts on Hausdorff convergence. Denote by $\bar{S}$ the closure of the set S. Denote by $B_\varepsilon(m)$ the ball $\rho(m, n) < \varepsilon$.

Lemma 1. Let $N_j \in [M]$. Let K_σ ($\sigma \in \Sigma$) be all limit points of the sequence N_j. Then

a) $\overline{\cup_{\sigma \in \Sigma} K_\sigma}$ *coincides with the set of all $m \in M$ such that for all $\varepsilon > 0$ the set $N_j \cap B_\varepsilon(m)$ is nonempty for infinite number of j.*
b) $\cap_{\sigma \in \Sigma} K_\sigma$ *coincides with the set of all $m \in M$ such that for all $\varepsilon > 0$ the set $N_j \cap B_\varepsilon(m)$ is nonempty for sufficiently large j.*

Key words and phrases. Hausdorff distance, symmetric space,complete collineations, complete symmetric varieties, linear relation, Satake- Furstenberg boundary, Bruhat-Tits building.

2. Construction of separated factor-space

Let a partition $M = \underset{\alpha \in A}{\cup} M_\alpha$ satisfies the following condition

$*$) for each $B \subset A$ the set $\overline{\underset{\alpha \in B}{\cup} M_\alpha}$ is the union of elements of the partition.

Fix an open subset $\mathcal{A} \subset A$ such that factor-topology on $\mathcal{A}$ is separated. Denote by $\tilde{\mathcal{A}} \subset [M]$ the set of subsets $\overline{M}_\alpha$, $\alpha \in \mathcal{A}$. Let our data satify the condition

$**$) the map $\alpha \leftrightarrow \overline{M_\alpha}$ is a homeomorphism of the spaces $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

Definition. The *sepapated factor-space* [[A]] is the closure of $\tilde{\mathcal{A}}$ in Hausdorff metrics.

Remark. Of course the construction depends on the set $\mathcal{A} \subset A$.

3. Description of the set [[A]]

By lemma 1 and the condition $*$) the elements $N \in [[A]]$ are unions of elements M_α of the partition. Hence we associate to each $N \in [[A]]$ subset $S_N \subset A$ of all $\sigma \in A$ such that $M_\alpha \subset N$. Denote by [A] the set of all subsets S_N. By construction we have canonical bijection $[[A]] \leftrightarrow [A]$.

The following proposition is evident.

Lemma 2. Let $S \subset A$. Then the following conditions are equivalent

a) $S \in [A]$

b) *There exist a sequence $a \in \mathcal{A}$ such that each limit point of a_j is an element of S and each element $s \in S$ is a limit of the sequence a_j in yhe factor-topology on A.*

Elements of $[A]$ we call *admissible subsets.*

4. Example: complete collineations

Let M be the Grassmann manifold Gr_n of all n-dimensional subspaces in $\mathbb{C}^n \oplus \mathbb{C}^n$. Let $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus 0$. Let $V \in Gr_n$. Define the subspace λV :

$$h \oplus p \in V \Leftrightarrow h \oplus \lambda p \in \lambda V$$

where $h \in \mathbb{C}^n \oplus 0$, $p \in 0 \oplus \mathbb{C}^n$. Consider the partition of Gr_n into $\mathbb{C}^*$ -orbits. Let $Op \subset Gr_n$ be the space of graphs of invertible operators. Of course the space Op coincide with the general linear group $GL_n(\mathbb{C})$. The factorspace $Op/\mathbb{C}^* = GL_n(\mathbb{C})/\mathbb{C}^*$ is the group $PGL_n(\mathbb{C})$ of invertible operators defined up to scalar multiplier.

We want to apply our construction to the space $M = Gr_n$ and $\mathcal{A} = PGL_n(\mathbb{C})$. We have to describe all admissible subsets in $Gr_n/\mathbb{C}^*$.

Example. Let $n = 2$. Consider the sequence $Q_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \in PGL_2(\mathbb{C})$. Then the set of limits of Q_n in $Gr_2/\mathbb{C}^*$ consists of points $V_1, \dots, V_5$ (= subspaces in $\mathbb{C}^2 \oplus \mathbb{C}^2$) enumerated below:

$$V_1 : (x, y; 0, 0)$$
$$V_2 : (x, y; 0, y)$$
$$V_3 : (x, 0; 0, y)$$
$$V_4 : (x, 0; x, y)$$
$$V_5 : (0, 0; x, y)$$

where $x, y \in V$. The subspaces V_1, V_3, V_5 are stable points of the group $\mathbb{C}^*$. The $\mathbb{C}^*$-orbits of V_2, V_4 are 1-dimensional complex curves.

Definition. Let $V \in Gr_n$. Then

a) *Kernel* $\operatorname{Ker} V = V \cap (\mathbb{C}^n \oplus 0)$

b) *Image* $\operatorname{Im} V$ is the projection of V to $0 \oplus \mathbb{C}^n$.

c) *Domain* $\operatorname{Dom} V$ is the projection of V to $\mathbb{C}^n \oplus 0$.

d) *Indefiniteness* $\operatorname{Indef} V = V \cap (0 \oplus \mathbb{C}^n)$.

Remark. Let $V \in Gr_n$ Then the subspace V induces by the obvious way the invertible operator

$$\mathrm{Dom}V/\mathrm{Ker}V \to \mathrm{Im}\ V/\mathrm{Indef}V$$

We denote this operator by $< V >$.

Definition. *Hinge* in $\mathbb{C}^n$ is a collection

$$\mathcal{P} = (Q_0, P_1, Q_1, P_2, Q_2, \dots, P_k, Q_k)$$

where Q_j, P_j are elements of Gr_n defined up to multiplier and

0.

$$Q_j = \operatorname{Ker} Q_j \oplus \operatorname{Indef} Q_j$$
$$P_j \neq \operatorname{Ker} P_j \oplus \operatorname{Indef} P_j$$

1. For each $j = 1, 2, \dots, k$

$$\operatorname{Ker} P_j = \operatorname{Ker} Q_j = \operatorname{Dom} P_{j+1}$$
$$\operatorname{Im} P_j = \operatorname{Im} Q_j = \operatorname{Indef} P_{j+1}$$

2.

$$Q_0 = \mathbb{C}^n \oplus 0 \ ; \ \operatorname{Dom} P_1 = \mathbb{C}^n$$
$$Q_k = 0 \oplus \mathbb{C}^n \ ; \ \operatorname{Im} P_k = \mathbb{C}^n.$$

Remark. Let P be the graph of an invertible operator $\mathbb{C}^n \to \mathbb{C}^n$. Then

$$(\mathbb{C}^n \oplus 0, P, 0 \oplus \mathbb{C}^n)$$

is a hinge.

Remark. The elements $Q_0, \dots, Q_{k+1}$ of a hinge are completely defined by the elements $P_1, \dots, P_k$. The subspaces Q_j are fixed points of the group $\mathbb{C}^*$. The $\mathbb{C}^*$ -orbits of P_j are 1-dimensional complex curves.

Theorem. *The space $[PGL_n]$ of all admissible subsets in $Gr_n/\mathbb{C}^*$ coincides with the space of all hinges.*

The space $[PGL_n]$ coincide with the *complete collineation* space constructed by Semple (see [2]). It is a smooth algebraic variety and the group PGL_n is an open dense subset in $[PGL_n]$. On equivalence of these two constructions see see [8]. Complete collineations is a partial case of complete symmetric varieties, see De Concini, Procesi [3].

5. Example. Furstenberg-Satake compactification of Riemannian symmetric space

We will only discuss the case $PGL_n(\mathbb{R})/SO(n)$. Consider the space $\mathbb{R}^n \oplus \mathbb{R}^n$ provided by a skew-symmetric bilinear form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\mathcal{L}$ be the grassmannian of all Lagrangian subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$. Denote by $\mathbb{R}^*$ the multiplicative group of real positive numbers. This group acts on $\mathcal{L}$ by multiplications of linear relations on scalars.

Denote by R the open subset in $\mathcal{L}$ consisting of graphs of operators $S : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to see that

$$\{\text{matrix } S \text{ is symmetric}\} \Leftrightarrow \{\text{the grapf of } S \text{ is an element of } \mathcal{L}\ \}$$

The group $GL_n(\mathbb{R})$ acts on R by the formula $g : S \mapsto g^t S g$. The stabilizer of the point $S = E$ is the orthogonal group $O(n)$. Hence $GL_n(\mathbb{R})$-orbit X of E is a homogeneous space $GL_n(\mathbb{R})/O(n)$. Points of X correpond to positive definite matrices S.

Now we apply the construction of the sections 2-3 to the space $\mathcal{L}$ and to the open subset $X = GL_n(\mathbb{R})/O(n)$. Then the completion consists of hinges

$$P = (Q_0, P_1, Q_1, \dots, P_k, Q_k)$$

such that $P_j \in \mathcal{L}, Q_j \in \mathcal{L}$ and the operators $< P_j >$ (see section 4) are positive definite.

6. Example. Boundary of Bruhat-Tits building

Let Q_p be a p-adic field. Let M be the space of all $\mathbb{Z}_p$ -submodules in $\mathbb{Q}_p$. Let $B \subset M$ be the space of all lattices. The group $\mathbb{Q}_p^*$ act on M in a natural way. Then the corresponding separated factor-space consists of collections

$$(R_0, T_1, R_1, \dots, T_k, R_k)$$

where $0 = R_0 \subset T_1 \subset R_1 \subset T_2 \dots \subset R_k = \mathbb{Q}_p^n$ are elements of M defined up to multiplier, R_j are subspaces and images of T_j in R_j/R_{j-1} are lattices.

I thanks C. De Concini, S.L.Tregub and E.B.Vinberg for discussion of this subject.

Bibliography

1. E. Study, *Über die Geometrie der Kegelschnitte, insbesondere deren charakteristische Probleme*, Math. Ann., 27 (1886), 51-58.
2. I.G. Semple, *The variety whose points represent complete collineations of S_r on S'_r*, Rend. Math. 10 (1951), 201-280.
3. Fürstenberg, H., *A Poisson formula for semisimple Lie groups*, Ann. of Math. (2) 77 (1963), 335-386.
4. Satake, I., *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math. (2) 71 (1960), 77-110.
5. Alguneid, A.R., *Complete quadrics primals in four dimensional space*, Proc. Math. Phys. Soc. Egypt, 4 (1952), 93-104.
6. C. De Concini, C. Prochesi, *Complete symmetric varieties*, Lect. Notes Math., 996, 1-44.
7. Oshima, T., Sekiguchi, I., *Eigenspaces of invariant differential operators on an affine symmetric space*, Invent. Math. 57 (1980), 1-81.
8. Neretin, Yu. A., *On universal completions of complex classical groups*, Funct.Anal.Appl.,26:1.
9. Vinberg E.B., *On reductive algebraic semigroups*, Adv.Sov.Math., volume dedicated to E.B.Dynkin, to appear.
10. Neretin Yu.A., *Hinges and Study-Semple-Furstenberg-Satake-De Concini-Procesi- Oshima boundary*, to appear.

Adress Moscow State Institute of electronic and mathematics
Current address: Max-Planck-Institute fur Mathematik, Bonn
E-mail address: neretin@ mpim-bonn.mpg.de ; neretin@ matan.miemstu.msk.su